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# UNIFORM LOWER BOUNDS FOR RANDOMLY STOPPED BANACH SPACE VALUED RANDOM SUMS[1]

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Let $\{X_n\}$ be a sequence of independent Banach space valued random elements with partial sums $S_n = X_1 + \cdots + X_n$. Then let $T$ be any possibly randomized stopping time based on $\{X_n\}$. Fix any $\alpha > 0$ and let $\Phi(\cdot)$ be any nondecreasing continuous function on $[0, \infty)$ with $\Phi(0) = 0$ such that $\Phi(cx) \leq c^\alpha \Phi(x)$ for all $x \geq 0$, $c \geq 2$. Put $S_n^* = \max_{1 \leq k \leq n} \|S_k\|$ and $a_n^* = E\Phi(S_n^*)$. It is proved that there exists a universal constant $c_\alpha^* < \infty$ depending only on $\alpha$ [and otherwise independent of $(B, \|\cdot\|)$, $\{X_n\}$, $T$ and $\Phi(\cdot)$] such that $Ea_T^* \leq c_\alpha^* E\Phi(S_T^*)$. As a consequence, $E\Phi(S_{T_c}^*) = \infty$ whenever $P(T_c < \infty) = 1$ and $c \geq c_\alpha^*$, where

$$T_c = \begin{cases} \text{first } n: c\Phi(S_n^*) < a_n^*, \\ \infty, \quad \text{if no such } n \text{ exists.} \end{cases}$$

In fact, $Ea_{T_c}^* = \infty$, too. An upper bound for $c_\alpha^*$ is constructed.

**1. Summary.** Let $X_1, X_2, \ldots$ be independent random elements taking values in a Banach space $(B, \|\cdot\|)$. Let $T$ be any (possibly randomized) stopping time with respect to $\{X_n\}$ and then let $\Phi(\cdot)$ be any nondecreasing continuous function on $[0, \infty)$ with $\Phi(0) = 0$ such that for some fixed $\alpha > 0$,

$$\Phi(cx) \leq c^\alpha \Phi(x) \quad \text{for all } x \geq 0, c \geq 2. \tag{1.1}$$

Denote the family of such $\Phi$ by $\mathscr{F}_\alpha$. Put $S_k = X_1 + \cdots + X_k$ and

$$a_n^* = E \max_{1 \leq k \leq n} \Phi(\|S_k\|). \tag{1.2}$$

We prove that there exists a universal constant $c_\alpha^*$ depending only on $\alpha > 0$ [and otherwise independent of $(B, \|\cdot\|)$, $\{X_n\}$, $T$ and $\Phi(\cdot)$] such that

$$Ea_T^* \leq c_\alpha^* E \max_{1 \leq n \leq T} \Phi(\|S_n\|). \tag{1.3}$$

Equivalently, if $\tilde{T}$ has the same marginal distribution as $T$ but is independent of $\{X_n\}$, then

$$E \max_{1 \leq n \leq \tilde{T}} \Phi(\|S_n\|) \leq c_\alpha^* E \max_{1 \leq n \leq T} \Phi(\|S_n\|). \tag{1.4}$$

An upper bound for $c_\alpha^*$ is constructed. Clearly, $c_\alpha^* \geq 1$. The complementary reverse inequality bound is proved in Klass (1988). Hence, despite how $T$ is constructed, the order of magnitude of $E \max_{1 \leq n \leq T} \Phi(\|S_n\|)$ is the same as it would be if $T$ were independent of $\{X_n\}$!

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Let

$$T_c = \begin{cases} \min\left\{n \geq 1: c \max_{1 \leq k \leq n} \Phi(\|S_k\|) < a_n^*\right\}, \\ \infty, \quad \text{if no such } n \text{ exists.} \end{cases} \tag{1.5}$$

As a consequence of (1.3), for each $c \geq c_\alpha^*$, either

$$P(T_c = \infty) > 0 \tag{1.6}$$

or else

$$E \max_{1 \leq n \leq T_c} \Phi(\|S_n\|) = \infty. \tag{1.7}$$

Letting

$$T_c^* = \begin{cases} \min\left\{n \geq 1: \max_{1 \leq k \leq n} \Phi(\|S_k\|) > c a_n^*\right\}, \\ \infty, \quad \text{if no such } n \text{ exists,} \end{cases} \tag{1.8}$$

the related analog of (1.6) and (1.7) also holds (due in part to de la Peña) and may be found in Klass (1988). Of course, whenever (1.7) (or its related analog) holds and $P(T_c < \infty) = 1$ $[P(T_c^* < \infty) = 1]$ then $Ea_{T_c}^* = \infty$ as well.

Specializing to the case in which the $X_n$'s are i.i.d. standard normal random variables and noting that $a_n^* \approx \Phi(\sqrt{n})$, inequality (1.3) combined with the reverse bound mentioned above together yield the fact that

$$E \max_{1 \leq n \leq T} \Phi(|S_n|) / E\Phi(\sqrt{T})$$

is uniformly bounded away from 0 and $\infty$ by finite positive constants depending only on $\alpha > 0$. The continuous-time Brownian motion analog was proved by Burkholder and Gundy (1970). Their result in turn extends or relates to a long line of time-honored results dating back to Khintchine (1923), Paley (1932), Marcinkiewicz and Zygmund (1937, 1938), Burkholder (1966), Millar (1968) and others.

The bounds given for $E \max_{1 \leq n \leq T} \Phi(\|S_n\|)$ depend on approximation of $a_n^* = E \max_{1 \leq k \leq n} \Phi(|S_n|)$. In the real-valued case, order-of-magnitude results are obtained in Klass (1981). For the Banach space case, consult Klass (1985).

**2. Sketch of the derivation of the main theorem.** Our derivation of (1.3) makes repeated use of the following lemma, a special case of a very important one due to Burkholder and Gundy (1970). Its key role is predicated on the effective manner in which it exploits *dependence* among random variables.

LEMMA 2.1 [Burkholder and Gundy (1970) and Burkholder (1973)]. *Let $U$ and $V$ be nonnegative random variables. Suppose there exist positive reals $\beta, \delta, \gamma$ such that $\beta^{-1} - \gamma > 0$ and*

$$P(U \geq \beta y, V \leq \delta y) \leq \gamma P(U \geq y) \quad \text{for all } y > 0. \tag{2.1}$$

*Then*

$$EU \leq (\beta^{-1} - \gamma)^{-1} \delta^{-1} EV. \tag{2.2}$$

As pointed out in the Appendix, this lemma can be strictly improved.

Ideally, to establish (1.3), one would like to use Lemma 2.1 with $U = a_T^*$ and $V = \max_{1 \le n \le T} \Phi(\|S_n\|)$. However, being dependent only on $T$, $a_T^*$ is not sufficiently responsive to the constraint $\{\max_{1 \le n \le T} \Phi(\|S_n\|) \le \delta y\}$. Therefore $a_T^*$ must be replaced by something more closely correlated with $\max_{1 \le n \le T} \Phi(\|S_n\|)$. Wald's identity $E\sum_{j=1}^{T} EY_j = E\sum_{j=1}^{T} Y_j$ suggests a means of creating such correlated variates, substituting random quantities $Y_j$ for constants $EY_j$. To adopt this approach, we must construct $Y_j$'s for which $a_T^*$ is approximated by $\sum_{j=1}^{T} EY_j$.

Define

$$S_n^* = \max_{1 \le k \le n} \|S_k\|, \tag{2.3}$$

$$S_{(m,n)}^* = S_{(m,n-1]}^* = \max_{m < j < n} \|S_j - S_m\|. \tag{2.4}$$

Then let $m_0 = 0$ and $m_1 =$ first $m > m_0$: $P(S_m^* > 0) > 0$. We assume that $m_1 < \infty$ exists. Having defined $m_0, m_1, \dots, m_j$, let

$$m_{j+1} = \begin{cases} \text{first } n > m_j\text{: } E\Phi\big(S_{(m_j,n]}^*\big) \ge \gamma_\alpha E\Phi\big(S_{(m_{j-1},m_j]}^*\big), \\ \infty, \quad \text{if no such } j \text{ exists}, \end{cases} \tag{2.5}$$

where

$$\gamma_\alpha = 1 + 16^{-\alpha-1}. \tag{2.6}$$

In this way we construct a sequence of constants $m_0, m_1, \dots$. Technically, one might observe that $m_j = \infty$ for all $j \ge L$, where

$$L = \sup\{j\colon m_{j-1} < \infty\}. \tag{2.7}$$

(Note that $L$ is a nonrandom constant.) Put

$$b_j = E\Phi\big(S_{(m_{j-1},m_j]}^*\big). \tag{2.8}$$

We might want to let $Y_j = \Phi(S_{(m_{j-1},m_j]}^*)$. For this to be reasonable, it must be shown that $a_T^*$ and $\sum_{j=1}^{\infty} b_j I(T \ge m_j)$ are comparable. [Since

$$\begin{aligned} \sum_{j=1}^{\infty} b_j I(T \ge m_j) &= \sum_{j=1}^{\infty} (b_1 + b_2 + \cdots + b_j) I(m_j \le T < m_{j+1}) \\ &\approx \sum_{j=1}^{\infty} b_j I(m_j \le T < m_{j+1}), \end{aligned}$$

it suffices to verify that $a_T^*$ and $b_k$ are comparable whenever $m_k \le T < m_{k+1}$. This is done in Theorem 4.2.]

On more careful examination, however, it is seen that $Y_j$ and $I(T \ge m_j)$ are not independent, so that Wald's identity does not apply. Moreover, if we attempt to look at $\sum_{j=1}^{\infty} b_j I(T > m_{j-1})$ instead, we recover use of Wald's identity but may now lose control over the magnitude of $\sum_{j=1}^{\infty} b_j I(T > m_{j-1})$ as compared with that of $a_T^*$. This will occur whenever $m_{k-1} < T < m_k$ and $b_{m_k} \gg b_{m_{k-1}}$. Hence it is quite possible that $E\sum_{j=1}^{\infty} b_j I(T > m_{j-1})$ could be $+\infty$ even though $Ea_T^* < \infty$. The problem that arises is due to the potential hugeness of $E\Phi(\|X_{m_k}\|)$ when $m_{k-1} < T < m_k$. Put

$$\Phi_j = \Phi\big(S^*_{(m_{j-1}, m_j)}\big). \tag{2.9}$$

By construction of $\{m_j\}$,

$$E\Phi_j < \gamma_\alpha b_{j-1} \quad (\le b_j). \tag{2.10}$$

Since

$$\begin{aligned} \Phi(\|x+y\|) &\le \Phi(\|x\| + \|y\|) \\ &\le \Phi(2\|x\|) + \Phi(2\|y\|) \\ &\le 2^\alpha(\Phi(\|x\|) + \Phi(\|y\|)), \end{aligned} \tag{2.11}$$

we also have

$$b_j \le 2^\alpha\Big(E\Phi_j + E\Phi\big(\|X_{m_j}\|\big)\Big). \tag{2.12}$$

Furthermore, since $\|X_{m_j}\| \le 2S^*_{(m_{j-1}, m_j]}$,

$$E\Phi\big(\|X_{m_j}\|\big) \le 2^\alpha b_j. \tag{2.13}$$

Hence

$$\begin{cases} \sum_{j=1}^{\infty} b_j I(T \ge m_j) \quad \text{and} \\ \sum_{j=2}^{\infty} (E\Phi_j) I(T > m_{j-1}) + \sum_{j=1}^{\infty} E\Phi\big(\|X_{m_j}\|\big) I(T \ge m_j) \end{cases} \tag{2.14}$$

have comparable orders of magnitude. (Note that $\Phi_1 \equiv 0$.) Moreover, Wald's identity does apply to each of the two sums in the second part of (2.14). It therefore suffices to produce constants $c_{\alpha 1}$ and $c_{\alpha 2}$ such that

$$E\sum_{j=1}^{\infty} \Phi\big(\|X_{m_j}\|\big) I(T \ge m_j) \le c_{\alpha 1} E\Phi(S_T^*) \tag{2.15}$$

and

$$E\sum_{j=2}^{\infty} \Phi_j I(T > m_{j-1}) \le c_{\alpha 2} E\Phi(S_T^*). \tag{2.16}$$

Try as one might, it seems impossible to establish either (2.15) or (2.16) by means of the Burkholder–Gundy lemma. From computations that would follow its use, the difficulty appears to stem from the fact that $\{\Phi(S_T^*) \le \delta y\}$ does not effectively restrict the size of $T$. What is to be done?

Notice that if the random quantities $\Phi(\|X_{m_j}\|)$ or $\Phi_j$ were growing at the rate of their expected sum $b_j$ (or more), then the constraint $\{\Phi(S_T^*) \le \delta y\}$ would prevent there being too many such terms prior to $T$ and in particular it would effectively inhibit $T$ from exceeding some $m_{k_y}$ for some $b_{m_{k_y}}$ roughly equal to $\delta y$ (or some multiple thereof).

By altering $\Phi(\|X_{m_j}\|)$ and $\Phi_j$ somewhat, we can virtually ensure that this guaranteed growth rate occurs (unless the altered quantities are 0!), without changing the order of magnitude of the resulting total expectation. The feasibility of this derives from the fact that for any $W \geq 0$,

$$EWI\left(W \geq \tfrac{1}{2}EW\right) \geq \tfrac{1}{2}EW, \tag{2.17}$$

whence $W$ and $WI(W \geq \frac{1}{2}EW)$ have comparable expectations. As an amusing consequence, it becomes important to truncate away *small* values of the random variables while retaining the large ones—something totally contrary to traditional practice.

Let

$$\Phi_j' = \Phi_j I\left(\Phi_j \geq 2^{-\alpha-2} b_j\right) \tag{2.18}$$

and

$$X_{m_j}' = X_{m_j} I\left(\Phi\left(\|X_{m_j}\|\right) \geq 2^{-\alpha-2}\varepsilon_\alpha b_j\right), \tag{2.19}$$

for some $0 < \varepsilon_\alpha \ll 1$, to be specified later [see (4.13)]. In Lemma 3.1 (to follow) it is proved that

$$\begin{aligned} \sum_{j=1}^{\infty} b_j P(T \geq m_j) \leq {} & 2^{\alpha+1} E \sum_{j=2}^{\infty} \Phi_j' I(T > m_{j-1}) \\ & + 2^{\alpha+1} E \sum_{j=1}^{\infty} \Phi\left(\|X_{m_j}'\|\right) I(T \geq m_j). \end{aligned} \tag{2.20}$$

The Burkholder–Gundy lemma does indeed apply to the second quantity on the right-hand side above, i.e., we can show by direct application of it that (see Lemma 3.2) there exists $c_{\alpha 1}' < \infty$ such that

$$E \sum_{j=1}^{\infty} \Phi\left(\|X_{m_j}'\|\right) I(T \geq m_j) \leq c_{\alpha 1}' E\Phi(S_T^*). \tag{2.21}$$

Similarly, it is a fairly straightforward matter (see Lemma 3.3) to show that there exists $c_{\alpha 2}' < \infty$ such that

$$E \sum_{j=2}^{\infty} \Phi_j' I(T \geq m_j) \leq c_{\alpha 2}' E\Phi(S_T^*). \tag{2.22}$$

Therefore the quantity $E\sum_{j=2}^{\infty}\Phi_j' I(m_{j-1} < T < m_j)$ remains to be upper-bounded. Since

$$S_{(m_{j-1}, m_j)}^* \leq \max\left\{4S_T^*, 2S_{(T, m_j)}^*\right\}$$

whenever $m_{j-1} < T < m_j$, it follows that

$$\begin{aligned} E\Phi_j' I(m_{j-1} < T < m_j) \leq {} & 4^\alpha E\Phi(S_T^*) I(m_{j-1} < T < m_j) \\ & + 2^\alpha E\Phi\left(S_{(T, m_j)}^*\right) I(m_{j-1} < T < m_j). \end{aligned} \tag{2.23}$$

Noting that

$$
\begin{aligned}
E\Phi\big(S^*_{(T,m_j)}\big)I(m_{j-1} < T < m_j) &= \sum_{m_{j-1}<n<m_j} E\Phi\big(S^*_{(n,m_j)}\big)P(T=n) \\
&\le \sum_{m_{j-1}<n<m_j} E\Phi\big(2S^*_{(m_{j-1},m_j)}\big)P(T=n) \\
&\le \sum_{m_{j-1}<n<m_j} 2^\alpha E\Phi_j P(T=n) \\
&= 2^\alpha E\Phi_j P(m_{j-1} < T < m_j),
\end{aligned}
$$

we may conclude that

$$
\begin{aligned}
&E\sum_{j=2}^{\infty} \Phi_j' I(m_{j-1} < T < m_j) \\
&\quad \le 4^\alpha E\Phi(S_T^*) + 4^\alpha \sum_{j=2}^{\infty} E\Phi_j P(m_{j-1} < T < m_j).
\end{aligned}
\tag{2.24}
$$

Now if only $\sum_{j=2}^{\infty} E\Phi_j P(m_{j-1} < T < m_j)$ could be upper-bounded in terms of $E\Phi(S_T^*)$, our theorem would be complete.

By Lemma 4.4, there exists a finite constant $c'_{\alpha,3}$ depending only on $\alpha > 0$ such that

$$
E\Phi_{j+1} \le 2^{-3\alpha-2}(b_1 + \cdots + b_j) + 2^{-3\alpha-2} c'_{\alpha,3} \sum_{i=1}^{j} E\Phi(\|X'_{m_i}\|). \tag{2.25}
$$

The above inequality holds subject to the further assumption that

$$
\Phi(cx) \le c^\alpha \Phi(x) \quad \text{for all } x \ge 0 \text{ and all } c \ge 1. \tag{2.26}
$$

With this additional proviso,

$$
\begin{aligned}
2\sum_{j=1}^{\infty} b_j P(T \ge m_j) &\le 2^{\alpha+2} E \sum_{j=1}^{\infty} \Phi(\|X'_{m_j}\|) I(T \ge m_j) \\
&\quad + 2^{\alpha+2} E \sum_{j=2}^{\infty} \Phi_j' I(T > m_{j-1}) \quad [\text{by } (2.20)] \\
&\le 2^{\alpha+2}(c'_{\alpha 1} + c'_{\alpha 2} + 4^\alpha) E\Phi(S_T^*) \\
&\quad + 2^{3\alpha+2} \sum_{j=2}^{\infty} E\Phi_j P(m_{j-1} < T < m_j) \\
&\qquad [\text{by } (2.21), (2.22) \text{ and } (2.24)].
\end{aligned}
$$

Invoking (2.25),

$$2^{3\alpha+2}\sum_{j=2}^{\infty} E\Phi_j P(m_{j-1} < T < m_j)$$

$$\le \sum_{j=1}^{\infty} (b_1 + \cdots + b_j) P(m_j \le T < m_{j+1})$$

$$+ c'_{\alpha,3}\sum_{j=1}^{\infty}\sum_{i=1}^{j} E\Phi(\|X'_{m_i}\|) P(m_j \le T < m_{j+1})$$

$$= \sum_{j=1}^{\infty} b_j P(T \ge m_j) + c'_{\alpha,3}\sum_{j=1}^{\infty} E\Phi(\|X'_{m_j}\|) P(T \ge m_j)$$

(summing by parts)

$$= \sum_{j=1}^{\infty} b_j P(T \ge m_j) + c'_{\alpha,3} E\sum_{j=1}^{\infty} \Phi(\|X'_{m_j}\|) I(T \ge m_j)$$

(by Wald's identity)

$$\le \sum_{j=1}^{\infty} b_j P(T \ge m_j) + c'_{\alpha,1}c'_{\alpha,3} E\Phi(S_T^*) \quad [\text{by } (2.21)].$$

Hence

$$2\sum_{j=1}^{\infty} b_j P(T \ge m_j) \le \left(2^{\alpha+2}c'_{\alpha,1} + 2^{\alpha+2}c'_{\alpha,2} + 2^{3\alpha+2} + c'_{\alpha,1}c'_{\alpha,3}\right) E\Phi(S_T^*) + \sum_{j=1}^{\infty} b_j P(T \ge m_j).$$

Solving,

$$\sum_{j=1}^{\infty} b_j P(T \ge m_j) \le \left(2^{\alpha+2}c'_{\alpha,1} + 2^{\alpha+2}c'_{\alpha,2} + 2^{3\alpha+2} + c'_{\alpha,1}c'_{\alpha,3}\right) E\Phi(S_T^*). \tag{2.27}$$

Finally, since

$$a_T^* I(m_j \le T < m_{j+1}) \le c_\alpha b_j I(m_j \le T < m_{j+1}) \tag{2.28}$$

by Theorem 4.2, a summation and trivial upper bound of the right-hand side reveals that

$$E a_T^* \le c_\alpha \sum_{j=1}^{\infty} b_j P(T \ge m_j). \tag{2.29}$$

Combining (2.27) and (2.29), it is clear that (1.3) holds modulo proviso (2.26) and the establishment of (2.20), (2.21), (2.22), (2.25) and (2.28). In fact, (2.26) *is not restrictive*; for suppose (1.3) holds whenever (2.26) holds and take any

$\Phi(\cdot) \in \mathscr{F}_\alpha$ [so that (1.1) holds]. Putting

$$\Phi_a(x) = \frac{1}{x}\int_0^x \Phi(y)\,dy, \tag{2.30}$$

we claim that $\Phi_a(\cdot) \in \mathscr{F}_{\alpha_a}$ and satisfies (2.26) with $\alpha_a = 2^{2\alpha+1} - 1$ and that

$$\Phi_a(x) \le \Phi(x) \le 2^{\alpha+1}\Phi_a(x) \quad \text{for all } x \ge 0. \tag{2.31}$$

(The validity of these facts is proved in the Appendix.)

Armed with these results, let

$$a_{n1}^* = E \max_{1\le k\le n} \Phi_a(\|S_k\|) \tag{2.32}$$

and notice that

$$a_n^* \le 2^{\alpha+1}a_{n1}^* \quad \text{and} \quad E\Phi_a(S_T^*) \le E\Phi(S_T^*).$$

Applying (1.3), it follows that

$$Ea_T^* \le 2^{\alpha+1}Ea_{T1}^* \le 2^{\alpha+1}c_{\alpha_a}^* E\Phi_a(S_T^*) \le 2^{\alpha+1}c_{\alpha_a}^* E\Phi(S_T^*).$$

Therefore (1.3) holds modulo (1.1) if it holds modulo (2.26). Consequently, for the remainder of the paper we will assume that $\Phi(\cdot)$ satisfies (2.26). It remains to establish inequalities (2.20), (2.21), (2.22), (2.25) and (2.28). We will prove each of the first three of these in the next section. The last two are somewhat more involved. They will be proved in Section 4, together with two supplementary lemmas. For convenience, the main theorem and principal corollary are restated in Section 5.

**3. Proof of inequalities (2.20) to (2.22).** Inequality (2.20) permits us to upper-bound a sum of constants in terms of an expectation of a sum of random quantities. In order to achieve this bound, certain events ($\{T \ge m_j\}$) must be slightly enlarged (to $\{T > m_{j+1}\}$). The loss thereby incurred is remedied by the analysis presented at the end of Section 2 from inequality (2.23) onward.

LEMMA 3.1 [also inequality (2.20)].

$$\begin{aligned}\sum_{j=1}^{\infty} b_j P(T \ge m_j) \le{}& 2^{\alpha+1}E\sum_{j=2}^{\infty}\Phi_j' I(T > m_{j-1})\\ &+ 2^{\alpha+1}E\sum_{j=1}^{\infty}\Phi(\|X_{m_j}'\|)I(T \ge m_j).\end{aligned} \tag{3.1}$$

[Actually the sum on the left-hand side of (3.1) extends only up to $L - 1$, where $L$ is defined in (2.7), and hence each of the sums on the right could also be restricted to this range. In particular, this follows because for $j \ge L$ we have $P(T \ge m_j) = P(T \ge \infty) = 0$.]

PROOF. Observe first that

$$b_j \le 2^{\alpha+1}\Big(E\Phi_j' + E\Phi\big(\|X_{m_j}'\|\big)\Big). \tag{3.2}$$

To see this, note that

$$\begin{aligned}
b_j &\le E\Big\{\Phi\big(2S^*_{(m_{j-1},m_j)}\big) + \Phi\big(2\|X_{m_j}\|\big)\Big\} \le 2^\alpha E\Phi_j + 2^\alpha E\Phi\big(\|X_{m_j}\|\big) \quad \text{[see (2.12)]}\\
&\le 2^\alpha E\Phi_j I\big(\Phi_j < 2^{-\alpha-2}b_j\big) + 2^\alpha E\Phi_j' + 2^\alpha E\Phi\big(\|X_{m_j}\|\big) I\Big(\Phi\big(\|X_{m_j}\|\big) < 2^{-\alpha-2}b_j\Big)\\
&\quad + 2^\alpha E\Phi\big(\|X'_{m_j}\|\big)\\
&\le 2^{-2}b_j + 2^\alpha E\Phi_j' + 2^{-2}b_j + 2^\alpha E\Phi\big(\|X'_{m_j}\|\big).
\end{aligned}$$

Solving for $b_j$, (3.2) follows. Therefore

$$\begin{aligned}
\sum_{j=1}^{\infty} b_j P(T \ge m_j) &= E\sum_{j=1}^{L-1} b_j I(T \ge m_j)\\
&\le 2^{\alpha+1}E\sum_{j=1}^{L-1}\Big(E\Phi_j' + E\Phi\big(\|X'_{m_j}\|\big)\Big)I(T \ge m_j)\\
&\le 2^{\alpha+1}E\sum_{j=2}^{L-1}(E\Phi_j')I(T > m_{j-1})\\
&\quad + 2^{\alpha+1}E\sum_{j=1}^{L-1}\Big(E\Phi\big(\|X'_{m_j}\|\big)\Big)I(T \ge m_j) \quad (\text{since } \Phi_1 \equiv 0)\\
&= 2^{\alpha+1}E\sum_{j=2}^{L-1}\Phi_j' I(T > m_{j-1}) + 2^{\alpha+1}E\sum_{j=1}^{L-1}\Phi\big(\|X'_{m_j}\|\big)I(T \ge m_j)
\end{aligned}$$

(by linearity and independence; or alternatively, by Wald's identity).

If desired, one can then extend these sums to $+\infty$. □

Each of the summands on the right-hand side of (3.1) needs to be upper-bounded in terms of $E\Phi(S_T^*)$.

LEMMA 3.2 [also, inequality (2.21)]. *Take any $\beta > 1$ and $0 < \delta < 2^{-\alpha}(\beta - 1)$ so small that*

$$c_{\beta\delta} \equiv \delta^{-1}\Big(\beta^{-1} - \delta 2^{3\alpha+2}(\varepsilon_\alpha)^{-1}\big(1 - (\gamma_\alpha)^{-1}\big)^{-1}(\beta - 1 - \delta 2^\alpha)^{-1}\Big)^{-1} > 0.$$

*Then*

$$E\sum_{j=1}^{\infty}\Phi\big(\|X'_{m_j}\|\big)I(T \ge m_j) \le c'_{\alpha 1}E\Phi(S_T^*), \tag{3.3}$$

*where*

$$c'_{\alpha 1} = \inf\{\textit{such } c_{\beta\delta}\}. \tag{3.4}$$

PROOF. For each $y > 0$, let

$$\tau_y = \begin{cases} \text{first } k\colon \sum_{j=1}^{k}\Phi\big(\|X'_{m_j}\|\big) \ge y, & \text{if such } m_k \le T \text{ exists},\\ \infty, & \text{otherwise.}\end{cases}$$

Let $k_y = $ last $k < L$: $2^{-\alpha-2}\varepsilon_\alpha b_k \le 2^\alpha \delta y$. Observe that $2^{-\alpha-2}\varepsilon_\alpha b_k$ represents the lower bound on $\Phi(\|X'_{m_k}\|)$ when it is positive. Whenever $\Phi(S_T^*) \le \delta y$ we have $\Phi(\|X_{m_j}\|) \le 2^\alpha \delta y$ for $m_j \le T$ and so we must have $\Phi(\|X'_{m_j}\|) = 0$ for $m_{k_y} < m_j \le T$. Curiously, we have here a rare instance in which the *small values* of a variate need to be truncated away while *large values* are retained. The grip this affords us on certain expectations will presently become apparent. Note that

$$
\begin{aligned}
E\sum_{j=1}^{k_y} \Phi(\|X'_{m_j}\|) &\le E\sum_{j=1}^{k_y} \Phi\big(2S^*_{(m_{m-1}, m_j]}\big) \\
&\le 2^\alpha \sum_{j=1}^{k_y} b_j \le 2^\alpha b_{k_y} \sum_{j=1}^{k_y} (\gamma_\alpha)^{j-k_y} \\
&< 2^\alpha\big(1-(\gamma_\alpha)^{-1}\big)^{-1} b_{k_y} \\
&\le 2^{3\alpha+2}(\varepsilon_\alpha)^{-1}\big(1-(\gamma_\alpha)^{-1}\big)^{-1}\delta y.
\end{aligned} \tag{3.5}
$$

Hence, using our old tool (Burkholder and Gundy's),

$$
\begin{aligned}
&P\left(\sum_{j=1}^{\infty} \Phi(\|X'_{m_j}\|) I(T \ge m_j) \ge \beta y,\ \Phi(S_T^*) \le \delta y\right) \\
&= P\left(\sum_{j=1}^{\tau_y - 1} \Phi(\|X'_{m_j}\|) + \Phi\big(\|X'_{m_{\tau_y}}\|\big)\right. \\
&\qquad \left. + \sum_{\{j:\ \tau_y < j \le k_y,\ m_j \le T\}} \Phi(\|X'_{m_j}\|) \ge \beta y,\ \tau_y < \infty,\ \Phi(S_T^*) \le \delta y\right) \\
&\le P\left(\sum_{\{j:\ m_{\tau_y} < m_j \le m_{k_y} \wedge T\}} \Phi(\|X'_{m_j}\|) \ge (\beta - 1 - \delta 2^\alpha) y\right) \\
&\le (\beta - 1 - \delta 2^\alpha)^{-1} y^{-1} E \sum_{j=2}^{k_y} \Phi(\|X'_{m_j}\|) I(j > \tau_y, T \ge m_j) \\
&\hspace{20em} \text{(Markov's inequality)} \\
&= (\beta - 1 - \delta 2^\alpha)^{-1} y^{-1} \sum_{j=2}^{k_y} E\Phi(\|X'_{m_j}\|) P(j > \tau_y, T \ge m_j) \\
&\le (\beta - 1 - \delta 2^\alpha)^{-1} y^{-1} \sum_{j=2}^{k_y} E\Phi(\|X'_{m_j}\|) P(m_{\tau_y} < T) \\
&\le 2^{3\alpha+2}(\varepsilon_\alpha)^{-1}(\beta - 1 - \delta 2^\alpha)^{-1}\big(1-(\gamma_\alpha)^{-1}\big)^{-1}\delta \\
&\qquad \times P\left(\sum_{j=1}^{\infty} \Phi(\|X'_{m_j}\|) I(T \ge m_j) \ge y\right) \quad [\text{by (3.5)}].
\end{aligned}
$$

Now (3.3) results from application of Lemma 2.1. □

We treat part of the other series next.

LEMMA 3.3 [also, inequality (2.22)]. *Take any* $\beta > 1$ *and* $0 < \delta < 2^{-\alpha}(\beta - 1)$ *so small that* $\tilde{c}_{\beta\delta} \equiv \delta^{-1}(\beta^{-1} - \delta 4^{\alpha+1}(\beta - 1 - \delta 2^{\alpha})^{-1}(1 - (\gamma_\alpha)^{-1})^{-1})^{-1} > 0$. *Let* $c'_{\alpha 2} = \inf\{$*such* $\tilde{c}_{\beta\delta}\}$. *Then*

$$E\sum_{j=2}^{\infty} \Phi'_j I(T \geq m_j) \leq c'_{\alpha 2} E\Phi(S_T^*). \tag{3.6}$$

(The summation on $j$ actually extends only up to $L - 1$.)

PROOF. This time let (for $y > 0$)

$$\tau_y = \begin{cases} \text{first } k\colon \sum_{j=2}^{k} \Phi'_j \geq y, & \text{if such } m_k \leq T \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Let $k_y =$ last $k < L$: $2^{-\alpha-2} b_k \leq 2^\alpha \delta y$. Noting that $E\Phi'_j \leq b_j \leq b_{k_y}(\gamma_\alpha)^{j-k_y}$ for $j \leq k_y$,

$$\sum_{2 \leq j \leq k_y} E\Phi'_j \leq \delta 4^{\alpha+1} y\left(1 - (\gamma_\alpha)^{-1}\right)^{-1}. \tag{3.7}$$

Next, we claim that whenever $\tau_y < \infty$ and $\Phi(S_T^*) \leq \delta y$ we have

$$\Phi'_j I(T \geq m_j) = 0 \quad \text{for } j > k_y \tag{3.8}$$

and

$$\Phi'_{\tau_y} \leq 2^\alpha \delta y. \tag{3.9}$$

To see this, first observe that whenever $m_j \leq T$, we have

$$S^*_{(m_{j-1}, m_j)} \leq 2S_T^*,$$

and so for such $j$,

$$\Phi_j \leq \Phi(2S_T^*) \leq 2^\alpha \Phi(S_T^*).$$

Second, if $\Phi'_j > 0$, $m_j \leq T$, and $\Phi(S_T^*) \leq \delta y$, then

$$2^{-\alpha-2} b_j \leq \Phi'_j \leq \Phi_j \leq 2^\alpha \delta y,$$

whence $j \leq k_y$ and both (3.8) and (3.9) hold.

Continuing to assume that $\tau_y < \infty$ and $\Phi(S_T^*) \leq \delta y$,

$$\begin{aligned} U &\equiv \sum_{j=1}^{\infty} \Phi'_j I(T \geq m_j) \\ &= \sum_{j=2}^{\infty} \Phi'_j I(T \geq m_j) \\ &\leq \sum_{j=2}^{\tau_y - 1} \Phi'_j + \Phi'_{\tau_y} + \sum_{\{j\colon \tau_y < j \leq k_y\}} \Phi'_j I\left(T > m_{\tau_y}\right) \quad [\text{by } (3.8)] \\ &\leq y + 2^\alpha \delta y + \sum_{\{\tau_y < j \leq k_y\}} \Phi'_j I\left(T > m_{\tau_y}\right) \left[\text{by the definition of } \tau_y \text{ and by } (3.9)\right]. \end{aligned}$$

We may now conclude that

$$
\begin{aligned}
P(U \geq \beta y, \Phi(S_T^*) \leq \delta y) &\leq P\Bigg( \sum_{\tau_y < j \leq k_y} \Phi_j' I\big(T > m_{\tau_y}\big) \geq (\beta - 1 - \delta 2^\alpha) y \Bigg) \\
&\leq (\beta - 1 - \delta 2^\alpha)^{-1} y^{-1} E \sum_{\tau_y < j \leq k_y} \Phi_j' I\big(T > m_{\tau_y}\big) \\
&\leq (\beta - 1 - \delta 2^\alpha)^{-1} y^{-1} \\
&\quad \times \sum_{n=1}^{k_y - 1} P(\tau_y = n, T > m_n) \sum_{n < j \leq k_y} E\Phi_j' \\
&\leq \delta 4^{\alpha+1} (\beta - 1 - \delta 2^\alpha)^{-1} \big(1 - (\gamma_\alpha)^{-1}\big)^{-1} \\
&\quad \times \sum_{n=1}^{k_y - 1} P(\tau_y = n, T > m_n) \quad [\text{by } (3.7)] \\
&\leq \delta 4^{\alpha+1} (\beta - 1 - \delta 2^\alpha)^{-1} \big(1 - (\gamma_\alpha)^{-1}\big)^{-1} P(U \geq y) \\
&\qquad \big[\text{since } P(\tau_y < \infty) = P(U \geq y)\big].
\end{aligned}
$$

Now invoke Lemma 2.1 to deduce (3.6). □

**4. Proofs of (2.28) and (2.25).** With verification of inequalities (2.28) and (2.25), the proof of the principal result of this paper, (1.3), will be complete. These inequalities in turn depend on two additional lemmas. Since (2.28) utilizes only one of them, we address it and the requisite lemma first.

LEMMA 4.1. *Fix any $\alpha > 0$. Suppose $\Phi$: $[0, \infty) \to [0, \infty)$ is a nondecreasing continuous function such that $\Phi(cx) \leq c^\alpha \Phi(x)$ for all $x \geq 0$, $c \geq 1$. Let $U$ and $V$ be random elements taking values in a Banach space $(B, \|\cdot\|)$. Suppose that for some $y \geq 0$ and $\lambda > 0$,*

$$E\Phi(\|U\|) \leq y \tag{4.1}$$

*and*

$$E\Phi(\|V\|) \leq \lambda y. \tag{4.2}$$

*Then*

$$E \max\{\Phi(\|U\|), \Phi(\|U + V\|)\} \leq g_\alpha(\lambda) y, \tag{4.3}$$

*where*

$$g_\alpha(x) = (1 + \sqrt{x})(1 + x^{1/2\alpha})^\alpha. \tag{4.4}$$

PROOF. Let $c = \lambda^{1/2\alpha}$. Then

$$\begin{aligned}E\max\{\Phi(\|U\|),\Phi(\|U+V\|)\} &\le E\Phi(\|(1+c)U\|)I(\|V\|\le c\|U\|)\\ &\quad + E\Phi(\|(1+c^{-1})V\|)I(\|V\|>c\|U\|)\\ &\le (1+c)^{\alpha}E\Phi(\|U\|)+(1+c^{-1})^{\alpha}E\Phi(\|V\|)\\ &\le \left((1+c)^{\alpha}+c^{2\alpha}(1+c^{-1})^{\alpha}\right)y\\ &= (1+c)^{\alpha}(1+c^{\alpha})y\\ &= g_{\alpha}(\lambda)y. \qquad \square\end{aligned}$$

REMARK. The preceding lemma is needed to establish (4.10). However, its full strength is not required. All that is needed is that the lemma hold for nonnegative real-valued $U$ and $V$.

THEOREM 4.2 [also known as inequality (2.28)]. *Take any $m_j \le m < m_{j+1}$. Then there exists a constant $c_\alpha < \infty$ depending only on $\alpha$ [and otherwise independent of $(B, \|\cdot\|)$, $\Phi$, $\{X_n\}$, $m_j$ and $m$] such that*

$$2^{-\alpha}b_j \le E\Phi(S_m^*) \le c_\alpha b_j. \tag{4.5}$$

PROOF. $S^*_{(m_{j-1}, m_j]} \le 2S^*_{m_j} \le 2S^*_m$, whence $b_j \le E\Phi(2S^*_m) \le 2^\alpha E\Phi(S^*_m)$. Theright-hand side is more complicated. Let $j^* =$ first $i$: $E\Phi(S^*_{m_i}) \ge 2^{-1}E\Phi(S^*_{m_j})$. Now if

$$E\Phi\left(S^*_{m_{j^*-1}}\right) \le 2^{-\alpha-1}E\Phi\left(S^*_{m_{j^*}}\right), \tag{4.6}$$

then we must have a large increment $b_{j^*}$ comparable to $E\Phi(S^*_{m_{j^*}})$ and hence also to $E\Phi(S^*_{m_j})$. To see this, note that since

$$\begin{aligned}E\Phi\left(S^*_{m_{j^*}}\right) &\le E\left(\Phi\left(2S^*_{m_{j^*-1}}\right) + \Phi\left(2S^*_{(m_{j^*-1}, m_{j^*}]}\right)\right)\\ &\le 2^\alpha E\Phi\left(S^*_{m_{j^*-1}}\right) + 2^\alpha b_{j^*},\end{aligned}$$

we must have $b_{j^*} \ge 2^{-\alpha-1}E\Phi(S^*_{m_{j^*}})$. Therefore

$$\begin{aligned}E\Phi(S^*_m) &\le E\left(\Phi\left(2S^*_{m_j}\right) + \Phi\left(2S^*_{(m_j, m_{j+1})}\right)\right)\\ &< 2^\alpha E\Phi\left(S^*_{m_j}\right) + 2^\alpha E\phi_{j+1}\\ &\le 2^{\alpha+1}E\Phi\left(S^*_{m_{j^*}}\right) + 2^\alpha\gamma_\alpha b_j\\ &\le 2^{2\alpha+2}b_{j^*} + 2^\alpha\gamma_\alpha b_j\\ &\le \left(2^{2\alpha+2} + 2^\alpha\gamma_\alpha\right)b_j \quad (\text{since } j \ge j^*),\end{aligned}$$

which was to be proved. We may therefore suppose that

$$E\Phi\left(S^*_{m_{j^*-1}}\right) > 2^{-\alpha-1}E\Phi\left(S^*_{m_{j^*}}\right). \tag{4.7}$$

There exists $\varepsilon_\alpha^* > 0$ such that

$$\prod_{i=0}^{\infty} g_\alpha\left(\varepsilon_\alpha^*(\gamma_\alpha)^{-i}\right) = 2, \tag{4.8}$$

where $g_\alpha(y) = (1 + \sqrt{y})(1 + y^{1/2\alpha})^\alpha$.

To obtain a contradiction, suppose also that the $b_j$ increment is so small that

$$b_j \le 2^{-\alpha-2}\varepsilon_\alpha^* E\Phi\left(S_{m_j}^*\right). \tag{4.9}$$

Then for $j^* \le k \le j$,

$$\begin{aligned} b_k &\le (\gamma_\alpha)^{k-j} b_j \\ &\le (\gamma_\alpha)^{k-j} 2^{-\alpha-2}\varepsilon_\alpha^* E\Phi\left(S_{m_j}^*\right) \quad \text{[by (4.9)]} \\ &\le (\gamma_\alpha)^{k-j} 2^{-\alpha-1}\varepsilon_\alpha^* E\Phi\left(S_{m_{j^*}}^*\right) \quad \text{(by definition of } j^*) \\ &< (\gamma_\alpha)^{k-j}\varepsilon_\alpha^* E\Phi\left(S_{m_{j^*-1}}^*\right) \quad \text{[by (4.7)]} \\ &\le \varepsilon_\alpha^*(\gamma_\alpha)^{k-j} E\Phi\left(S_{m_{k-1}}^*\right). \end{aligned}$$

And so the $b_k$ increments are relatively even smaller. Using this fact, we invoke Lemma 4.1 to obtain

$$E\Phi\left(S_{m_k}^*\right) \le g_\alpha\left(\varepsilon_\alpha^*(\gamma_\alpha)^{k-j}\right) E\Phi\left(S_{m_{k-1}}^*\right). \tag{4.10}$$

Using (4.10) repeatedly,

$$\begin{aligned} E\Phi\left(S_{m_j}^*\right) &\le E\Phi\left(S_{m_{j^*-1}}^*\right) \prod_{k=j^*}^{j} g_\alpha\left(\varepsilon_\alpha^*(\gamma_\alpha)^{k-j}\right) \\ &< 2E\Phi\left(S_{m_{j^*-1}}^*\right) \quad \text{[by (4.8)]} \\ &< E\Phi\left(S_{m_j}^*\right) \quad \text{(by construction of } j^*). \end{aligned}$$

This yields the desired contradiction. Therefore

$$E\Phi(S_m^*) \le 2^\alpha E\Phi\left(S_{m_j}^*\right) + 2^\alpha\gamma_\alpha b_j < 2^{2\alpha+2}(\varepsilon_\alpha^*)^{-1} b_j + 2^\alpha\gamma_\alpha b_j$$

[since (4.9) fails], whence the right-hand side of (4.5) again holds. □

The next lemma is rather trivial. We state it for easy reference and ease of exposition.

LEMMA 4.3. *For any* $1 \le k \le j$,

$$b_k \le (\gamma_\alpha - 1)\left(\gamma_\alpha^{j-k+1} - 1\right)^{-1}(b_1 + \cdots + b_j). \tag{4.11}$$

PROOF. Since $b_{n+i} \geq (\gamma_\alpha)^i b_n$ for all $i \geq 0$ and $n \geq 1$,

$$(b_1 + \cdots + b_j) \geq b_k + \cdots + b_j \geq b_k \sum_{i=0}^{j-k} (\gamma_\alpha)^i$$
$$= b_k(\gamma_\alpha^{j-k+1} - 1)(\gamma_\alpha - 1)^{-1}. \qquad \square$$

To proceed, note that if

$$(4.12) \qquad d_\alpha = 1 + [16^{\alpha+1}]$$

then (since $\gamma_\alpha = 1 + 16^{-\alpha-1}$)

$$(\gamma_\alpha)^{d_\alpha+1}(\gamma_\alpha^{d_\alpha+1} - 1)^{-1}(\gamma_\alpha - 1) = (1 - \gamma_\alpha^{-1-d_\alpha})^{-1}16^{-\alpha-1}$$
$$< (1 - \tfrac{1}{2})^{-1}16^{-\alpha-1} < 2^{-3(\alpha+1)}.$$

Since $g_\alpha(\lambda)$ is strictly increasing to $+\infty$ and $g_\alpha(0) = 1$, there exists a unique $0 < \varepsilon_\alpha < \infty$ such that

$$(4.13) \quad (\gamma_\alpha)^{d_\alpha+1}(g_\alpha(\varepsilon_\alpha/\gamma_\alpha))^{d_\alpha}(\gamma_\alpha - 1)((\gamma_\alpha)^{d_\alpha+1} - 1)^{-1} = 2^{-3\alpha-2},$$

where $g_\alpha(\lambda)$ is as defined in (4.4).

It is a matter of trivial verification that

$$(4.14) \qquad \varepsilon_\alpha < 2 - \gamma_\alpha.$$

The next result establishes inequality (2.25).

LEMMA 4.4. *Let $\gamma_\alpha$, $d_\alpha$, $g_\alpha(\lambda)$ and $\varepsilon_\alpha$ be as just described. Suppose* (2.26) *holds. Then there exists $c'_{\alpha,3} < \infty$ depending only on $\alpha$ [and not otherwise depending on $(B, \|\cdot\|)$, $\Phi$, $\{X_n\}$ or $j$, etc.] such that*

$$(4.15) \quad E\Phi_{j+1} \leq 2^{-3\alpha-2}(b_1 + \cdots + b_j) + 2^{-3\alpha-2}c'_{\alpha,3} \sum_{i=1}^{j} E\Phi(\|X'_{m_i}\|).$$

PROOF. We may assume $\Phi(x)$ is not identically 0. Let $x_* = \inf\{x > 0: \Phi(x) > 0\}$. By continuity, $\Phi(x_*) = 0$. If $x_* > 0$ then for any $x \geq 2x_*$,

$$0 < \Phi(x) = \Phi\left(\frac{x}{x_*}x_*\right) \leq \left(\frac{x}{x_*}\right)^\alpha \Phi(x_*) = 0,$$

which gives a contradiction. Hence $\Phi(x) > 0$ for all $x > 0$ and so $E\Phi(\|X_{m_1}\|) > 0$. Let

$$(4.16) \qquad k_j = \text{last } k \leq j\text{: } E\Phi(\|X_{m_k}\|) > \varepsilon_\alpha b_{k-1},$$

where $b_0 \equiv 0$. (Such a $k_j$ exists because the set of integers defining $k_j$ includes $k = 1$ and so is nonempty.) The idea behind the definition of $k_j$ is that $b_i$ is roughly constant for $k_j \leq i \leq j$. Hence if $j - k_j$ is sufficiently large, there will be so many $b_i$ comparable to $b_j$ that $2^{-3\alpha-2}(b_1 + \cdots + b_j)$ must exceed $\gamma_\alpha b_j$, which is turn exceeds $E\Phi_{j+1}$. Alternatively, if $j - k_j$ is relatively small, then $E\Phi(\|X_{m_{k_j}}\|)$ will be sizable compared to $b_{k_j}$ which in turn will be comparable to $b_j$ and hence to $E\Phi_{j+1}$.

We assert that

$$E\Phi\left(\|X_{m_{k_j}}\|\right) \leq 2E\Phi\left(\|X'_{m_{k_j}}\|\right). \tag{4.17}$$

PROOF OF (4.17). If $b_{k_j} \leq 2^{\alpha+1}b_{k_j-1}$, then by application of (4.16),

$$E\Phi\left(\|X_{m_{k_j}}\|\right) > 2^{-\alpha-1}\varepsilon_\alpha b_{k_j},$$

whence (4.17) holds [by (2.17)]. On the other hand, if $b_{k_j} > 2^{\alpha+1}b_{k_j-1}$, then since

$$\begin{aligned} b_{k_j} &\leq 2^\alpha E\Phi_{k_j} + 2^\alpha E\Phi\left(\|X_{m_{k_j}}\|\right) \\ &\leq 2^\alpha\gamma_\alpha b_{k_j-1} + 2^\alpha E\Phi\left(\|X_{m_{k_j}}\|\right) \\ &< 2^{-1}\gamma_\alpha b_{k_j} + 2^\alpha E\Phi\left(\|X_{m_{k_j}}\|\right), \end{aligned}$$

we also have

$$\begin{aligned} E\Phi\left(\|X_{m_{k_j}}\|\right) &> 2^{-\alpha}\left(1 - 2^{-1}\gamma_\alpha\right)b_{k_j} \\ &> 2^{-\alpha-1}\varepsilon_\alpha b_{k_j} \quad [\text{by } (4.14)], \end{aligned}$$

whence (4.17) again holds by (2.17). □

Next note that for $k_j < k \leq j$, $E\Phi(\|X_{m_k}\|) \leq \varepsilon_\alpha b_{k-1}$ and $E\Phi_k \leq \gamma_\alpha b_{k-1}$. Hence, invoking Lemma 4.1,

$$b_k \leq g_\alpha\left(\varepsilon_\alpha\gamma_\alpha^{-1}\right)\gamma_\alpha b_{k-1}. \tag{4.18}$$

Repeated use of (4.18) gives

$$b_j \leq \left(\gamma_\alpha g_\alpha\left(\varepsilon_\alpha\gamma_\alpha^{-1}\right)\right)^i b_{j-i} \quad \text{for all } k_j \leq j-i \leq j. \tag{4.19}$$

When $k_j \leq j - d_\alpha$,

$$\begin{aligned} E\Phi_{j+1} &< \gamma_\alpha b_j & &(\text{by construction}) \\ &\leq \gamma_\alpha\left(\gamma_\alpha g_\alpha\left(\varepsilon_\alpha\gamma_\alpha^{-1}\right)\right)^{d_\alpha} b_{j-d_\alpha} & &[\text{by } (4.19)] \\ &\leq \gamma_\alpha\left(\gamma_\alpha g_\alpha\left(\varepsilon_\alpha\gamma_\alpha^{-1}\right)\right)^{d_\alpha}(\gamma_\alpha - 1)\left(\gamma_\alpha^{d_\alpha+1} - 1\right)^{-1}(b_1 + \cdots + b_j) & & \\ & & &(\text{by Lemma 4.3}) \\ &= 2^{-3\alpha-2}(b_1 + \cdots + b_j) & &[\text{by } (4.13)]. \end{aligned}$$

Finally, when $k_j > j - d_\alpha$,

$$\begin{aligned} E\Phi_{j+1} &< \gamma_\alpha b_j && \text{(by construction)} \\ &\le \gamma_\alpha\big(\gamma_\alpha g_\alpha(\varepsilon_\alpha/\gamma_\alpha)\big)^{j-k_j} b_{k_j} && \text{[by (4.19)]} \\ &< \gamma_\alpha\big(\gamma_\alpha g_\alpha(\varepsilon_\alpha/\gamma_\alpha)\big)^{d_\alpha} b_{k_j}; \end{aligned}$$

and since

$$\begin{aligned} b_{k_j} &\le 2^\alpha E\Phi_{k_j} + 2^\alpha E\Phi\big(\|X_{m_{k_j}}\|\big) \\ &< 2^\alpha \gamma_\alpha b_{k_j - 1} + 2^\alpha E\Phi\big(\|X_{m_{k_j}}\|\big) \\ &< 2^\alpha(\gamma_\alpha/\varepsilon_\alpha + 1) E\Phi\big(\|X_{m_{k_j}}\|\big) && \text{[by (4.16)]} \\ &< 2^{\alpha+1}(1 + \gamma_\alpha/\varepsilon_\alpha) E\Phi\big(\|X'_{m_{k_j}}\|\big) && \text{[by (4.17)]} \\ &\le 2^{\alpha+1}(1 + \gamma_\alpha/\varepsilon_\alpha) \sum_{i=1}^{j} E\Phi\big(\|X'_{m_i}\|\big), \end{aligned}$$

the lemma holds with

$$c'_{\alpha,3} = 2^{4\alpha+3}\gamma_\alpha(1 + \gamma_\alpha/\varepsilon_\alpha)\big(\gamma_\alpha g_\alpha(\varepsilon_\alpha/\gamma_\alpha)\big)^{d_\alpha}. \qquad \square$$

**5. Main results.** Sections 3 and 4 have shown that inequalities (2.20), (2.21), (2.22), (2.25) and (2.28) [modulo (2.26)] hold. By virtue of the discussion presented in Section 2, our main theorem is valid. We restate it now for easy reference.

THEOREM 5.1. *Let $X_1, X_2, \dots$ be independent random elements taking values in a Banach space $(B, \|\cdot\|)$. Let $T$ be any possibly randomized stopping time with respect to $\{X_n\}$. Fix $\alpha > 0$ and let $\Phi(\cdot)$ be any nondecreasing function on $[0, \infty)$ with $\Phi(0) = 0$ such that*

$$\Phi(cx) \le c^\alpha \Phi(x) \quad \textit{for all } x > 0,\ c \ge 2. \tag{5.1}$$

*Let $S_n = X_1 + \cdots + X_n$ and*

$$a_n^* = E \max_{1 \le k \le n} \Phi(\|S_k\|). \tag{5.2}$$

*Then there exists a finite universal constant $c_\alpha^*$ depending only on $\alpha$ [and otherwise independent of $(B, \|\cdot\|)$, $\{X_n\}$, $T$ and $\Phi(\cdot)$] such that*

$$Ea_T^* \le c_\alpha^* E \max_{1 \le n \le T} \Phi(\|S_n\|). \tag{5.3}$$

As a consequence, we have the following corollary.

COROLLARY 5.2. *Let*

$$T_c = \begin{cases} \min\left\{n \geq 1: c \max_{1 \leq k \leq n} \Phi(\|S_k\|) < a_n^*\right\}, \\ \infty, \quad \text{if no such } n \text{ exists}. \end{cases} \tag{5.4}$$

*Suppose* $c \geq c_\alpha^*$ *and* $P(T_c < \infty) = 1$. *Then*

$$E \max_{1 \leq n \leq T_c} \Phi(\|S_n\|) = \infty. \tag{5.5}$$

PROOF. If $E \max_{1 \leq n \leq T_c} \Phi(\|S_n\|) < \infty$ then $cE \max_{1 \leq n \leq T_c} \Phi(\|S_n\|) < Ea_{T_c}^*$, which by (5.3) is at most $c_\alpha^* E \max_{1 \leq n \leq T_c} \Phi(\|S_n\|)$. Therefore $c < c_\alpha^*$, which contradicts our given assumptions. □

## APPENDIX

The following lemma strengthens and extends the Burkholder–Gundy Lemma 2.1. It seems significant that condition (2.1) can be weakened to (A.1) below without affecting the conclusion that $EU/EV$ is uniformly bounded above. We anticipate that there are problems for which this improvement is crucial. Nevertheless, we know of none at this time. We take the opportunity to issue a clarion call for would-be instances.

LEMMA A.1. *Let* $U$ *and* $V$ *be nonnegative random variables. Suppose there exist positive reals* $c, \beta, \delta, \gamma$ *such that* $c > 1$, $(1 - c^{-1})\beta^{-1} - \gamma > 0$ *and*

$$P(\beta y \leq U < c\beta y, V \leq \delta y) \leq \gamma P(U \geq y) \quad (\text{for all } y > 0). \tag{A.1}$$

*Then*

$$EU \leq \left((1 - c^{-1})\beta^{-1} - \gamma\right)^{-1} \delta^{-1} EV. \tag{A.2}$$

PROOF. First note that for any $c > 1$,

$$\begin{aligned} \int_0^\infty P(\beta y \leq U < c\beta y)\, dy &= \int_0^\infty P(\beta^{-1} U \geq y)\, dy - \int_0^\infty P\left((c\beta)^{-1} U \geq y\right) dy \\ &= E\beta^{-1} U - E(c\beta)^{-1} U \\ &= (1 - c^{-1})\beta^{-1} EU. \end{aligned}$$

Since for all $y > 0$,

$$\begin{aligned} P(\beta y \leq U < c\beta y) &= P(\beta y \leq U < c\beta y, V \leq \delta y) + P(\beta y \leq U < c\beta y, V > \delta y) \\ &\leq \gamma P(U \geq y) + P(V > \delta y), \end{aligned}$$

integrating with respect to $y > 0$,

$$(1 - c^{-1})\beta^{-1} EU \leq \gamma EU + \delta^{-1} EV.$$

Solving for $EU$ gives (A.2). □

The next lemma permits one to replace functions $\Phi \in \mathscr{F}_\alpha$ [see (1.1) for a definition of $\mathscr{F}_\alpha$] by functions $\Phi_a(\cdot)$ of comparable order of magnitude which satisfy (2.26) with $\alpha_a = 2^{2\alpha+1} - 1$ replacing $\alpha$.

LEMMA A.2. *Let $\Phi(\cdot) \in \mathscr{F}_\alpha$ and define*

$$\Phi_a(x) = \begin{cases} 0, & \text{if } x = 0, \\ x^{-1}\int_0^x \Phi(y)\,dy, & \text{if } x > 0. \end{cases} \tag{A.3}$$

*Let $\alpha_a = 2^{2\alpha+1} - 1$. Then $\Phi_a(x)$ is a nondecreasing continuous function with $\Phi_a(0) = 0$ such that*

$$\Phi_a(x) \le \Phi(x) \le 2^{\alpha+1}\Phi_a(x) \quad \text{for all } x \ge 0 \tag{A.4}$$

*and*

$$\Phi_a(cx) \le c^{\alpha_a}\Phi_a(x) \quad \text{for all } c \ge 1 \text{ and } x \ge 0. \tag{A.5}$$

PROOF. Clearly (for $x > 0$), $\Phi_a(x) \le x^{-1}\int_0^x \Phi(x)\,dy = \Phi(x)$. Since $\Phi_a(0) = 0 = \Phi(0)$, the left-hand side of (A.4) holds. As for the right-hand side (using $x > 0$),

$$\Phi_a(x) \ge x^{-1}\int_{2^{-1}x}^x \Phi(2^{-1}x)\,dy = 2^{-1}\Phi(2^{-1}x) \ge 2^{-1-\alpha}\Phi(x).$$

To prove (A.5), it suffices to consider $1 \le c \le 2$ and $x > 0$. For each such $c$ and $x$,

$$\begin{aligned} \Phi_a(xc) &= c^{-1}x^{-1}\int_0^{cx}\Phi(y)\,dy \\ &= c^{-1}\Phi_a(x) + c^{-1}x^{-1}\int_x^{cx}\Phi(y)\,dy \\ &\le c^{-1}\Phi_a(x) + c^{-1}(c-1)\Phi(2x) \\ &\le c^{-1}\Phi_a(x) + c^{-1}(c-1)2^\alpha\Phi(x) \\ &\le c^{-1}\left(1 + (c-1)2^{2\alpha+1}\right)\Phi_a(x) \quad [\text{by (A.4)}]. \end{aligned}$$

Using calculus, it is easily verified that $(1 + (c-1)y) \le c^y$ for all $y \ge 1$. Hence (A.5) holds. □

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