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THE NET CHARGE PROCESS FOR INTERACTING, SIGNED DIFFUSIONS[1]

BY ROBERT J. ADLER

Technion–Israel Institute of Technology

We consider a system of N ($\to \infty$) interacting one-dimensional diffusions, in which each diffusion is assigned a random charge (± 1), and study the behavior of the net charge distribution through space and time.

The diffusion equations are a slight variation of those considered in the initial studies of "the propagation of chaos," but the interaction involves the signs of the diffusions and triplet rather than pairwise interactions. This has the effect of leading to a non-Gaussian fluctuation theory, which turns out to be close to the $P(:\Phi^4:)$ models of Euclidean quantum field theory.

The main tools of the proofs involve the Stroock–Varadhan martingale theory and a general theory of U-statistics.

1. Introduction. Our aim in this paper is twofold. On the one hand we wish to consider an interacting particle system in which the particles are of two types—positive and negative—and in which the interaction mechanism is of a fairly natural form. On the other hand, we wish to develop a system in which the "fluctuation theory" (defined below) is non-Gaussian, and yet still relatively easy to manage.

The motivation for our first aim is obvious—there are many physically interesting situations in which particles *are* of two kinds, and so it is natural to want to model them. The motivation of the second aim is not so clear, but comes from Euclidean quantum field theory (EQFT). The study of EQFT is essentially the study of Gaussian and related random fields, and it is really only the non-Gaussian ones that are physically interesting. (Physicists use the adjective "trivial" to describe Gaussian EQFT.) It is not generally an easy matter to construct non-Gaussian EQFTs, and certainly not with the tools readily available to probabilists. Thus we wanted to develop a model which was on the one hand probabilistically simple, and close in nature to models that probabilists are used to working with, as well as leading in a reasonably natural way to non-Gaussian random fields.

As a result of these aims, the model that we are about to introduce has a slightly artificial nature, although it is reasonably easy to understand the phenomenon that it describes. [Roughly speaking, it describes a movement of positive and negative particles in which clusters of positive particles tend to drift in one direction, clusters of negative particles tend to drift in the other, but since the clusters have trouble passing one another a mixture of positive

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and negative particles arises that, in simulations, looks like many of the standard models of interacting particle systems; cf. Liggett (1985).] For the moment, however, we ask the reader to bear with us and accept it as it stands. At the end of the paper it will be clear why we wrote down the model the way we did, and we shall discuss how changes to it would affect its $N \to \infty$ behavior.

We commence with $N \geq 1$ independent random signs $\sigma^1, \dots, \sigma^N$, such that

$$P\{\sigma^i = +1\} = P\{\sigma^i = -1\} = \tfrac{1}{2};$$

N standard, real-valued, Brownian motions $W^1, \dots, W^N$, independent among themselves and of the σ^i; a real $\beta > 0$ and a nice function $b(x, y, z)$ on $\mathbb{R}^3$, which together yield the N interacting diffusions $X^1, \dots, X^N$ given by the solution of the N stochastic differential equations

$$(1.1)\quad dX_t^i = dW_t^i + \sigma^i \beta N^{-3/2} \sum_{i \neq k,\, k \neq i,\, j \neq k} \sigma^j \sigma^k b(X_t^i, X_t^j, X_t^k)\, dt, \quad 1 \leq i \leq N.$$

(We shall specify the initial distribution of the X^i later.) Strictly speaking, the X_t^i form a true diffusion system only when the σ^i are nonrandom. We shall return to this point as well later. The parameter β should be thought of as an "interaction strength."

In order to study the temporal development of this system, we shall concentrate on two measure-valued stochastic processes. The first, $\mu_N(t, A)$ counts the total number of particles in the set $A \subset \mathbb{R}$ at time $t > 0$, while $\mu_N^\sigma(t, A)$, which is actually a *signed* measure, measures the "net charge" in A at time t. Formally,

$$(1.2)\quad \begin{aligned} \mu_N(t, A) &= \sum_{i=1}^N 1_A(X_t^i), \\ \mu_N^\sigma(t, A) &= \sum_{i=1}^N \sigma^i\, 1_A(X_t^i). \end{aligned}$$

In fact, because of the signed nature of μ_N^σ, it is more convenient to study function-indexed versions of these two processes, defined by

$$(1.3)\quad \begin{aligned} \mu_N(t, f) &= \int_{\mathbb{R}} f(x) \mu_N(t, dx) = \sum_{i=1}^N f(X_t^i), \\ \mu_N^\sigma(t, f) &= \int_{\mathbb{R}} f(x) \mu_N^\sigma(t, dx) = \sum_{i=1}^N \sigma^i f(X_t^i), \end{aligned}$$

where f belongs to some "nice" family of functions to be specified later. In particular, we shall generally take $f \in \mathscr{S}$, the space of Schwartz functions, so that both μ_N and μ_N^σ are $\mathscr{S}'$-valued processes, i.e., distribution-valued diffusions.

Our main interest is in the process μ_N^σ, and we shall study both its temporal development and its distribution for fixed t. Unfortunately, μ_N^σ is not a Markov process, but, fortunately, the pair $\langle \mu_N, \mu_N^\sigma \rangle$ is. Thus we shall study this pair as a $\mathscr{S}' \times \mathscr{S}'$-valued diffusion. In particular, we shall show that the $N \to \infty$ structure of the pair $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ is comparatively simple, and shall characterize its generator via a Stroock–Varadhan type martingale analysis. This is done in Section 4 of the paper, after we set up the martingale problem for the finite N situation in Section 2 and determine the structure of the finite-dimensional distributions of the limit process in Section 3.

Indeed, our most interesting result lies in characterizing the distribution of $\mu_N^\sigma(t, \cdot)$, for fixed t, as a $\mathscr{S}'$-valued random variable; i.e., as a random field. When the interaction parameter β is 0, then it is a straightforward consequence of the standard central limit theorem that the $N \to \infty$ distribution of $N^{-1/2}\mu_N^\sigma(t, \cdot)$ is that of a Gaussian random field. When $\beta > 0$, however, it turns out that this is no longer the case, and the corresponding distribution can be written in terms of its Radon-Nikodym derivative with respect to the $\beta = 0$ Gaussian field. In fact, the limiting distribution is somewhat akin to a $P(:\Phi_1^4:)$ quantum field theory, i.e., to a nontrivial EQFT. All of this is discussed in some detail in Section 5.

Before commencing in earnest it is probably worthwhile to explain where the model (1.1) actually came from. There is a large literature [e.g., McKean (1966, 1967), Tanaka (1982), Dawson (1983), Tanaka and Hitsuda (1981) and Sznitman (1982, 1984, 1986)] concerning the propagation of chaos and fluctuation theory for models of identical processes that, in general, look something like

$$(1.4)\quad dX_t^i = dW_t^i + g(X_t^i)\,dt + \beta N^{-1}\sum_{i \neq j} b(X_t^i, X_t^j)\,dt, \qquad 1 \le i \le N.$$

The fluctuation theory for such processes, i.e., the $N \to \infty$ distribution of

$$N^{-1/2}\{\mu_N(t,\cdot) - E[\mu_N(t,\cdot)]\},$$

is almost invariably Gaussian [the model of Dawson (1983) is a notable exception to this rule; however his non-Gaussian fluctuation theory is more a result of a delicate initial distribution than the interaction mechanism], and there is very little one can do to (1.4) to get a non-Gaussian fluctuation theory. For example, changing the interaction term to a triple sum as in (1.1) has some effect on the limiting distribution, but not enough to make it non-Gaussian. Introducing random signs, as in (1.1), also has a limited effect. Introducing *both* of these changes at once, however, provides just the right amount of impetus to yield something different. (In the process, of course, one should note that we have also shifted our attention from the particle density process μ_N to the net charge process μ_N^σ.) Precisely why this is what is required will become clear in Section 5, where we shall also discuss what other changes would arise from different perturbations of (1.4) and (1.1). [Previous results, similar in spirit to some of ours, but for deterministic ODEs, appeared in Geman and Hwang (1982).]

For the reader interested primarily in fluctuation theory, it is worth noting that Section 3 is essentially independent of everything in Sections 2 and 4, and so can be read quite independently of the remainder of the paper.

Finally, I should note that the impetus to introduce randomly signed particles in order to obtain a non-Gaussian fluctuation theory came from two (related) sources. Previous experience with somewhat simpler models [Adler (1989)] showed that this was likely to work, but, most of all, my wife the physicist was adamant in her long-term claim that "nothing interesting can ever happen if all your particles are identical"!

2. The structure of the finite N case. We start with some formalities, recalling, and somewhat refining, the setup of the previous section. For $i \geq 1$ let $\{\Omega^i, \mathscr{F}^i, (\mathscr{F}_t^i)_{0 \leq t \leq 1}, (W_t^i)_{0 \leq t \leq 1}, P\}$ be a probability space with a standard Brownian motion W^i, an $\mathscr{F}_0^i$-measurable random variable X_0^i with law u and a $\mathscr{F}_0^i$-measurable random Rademacher sign σ^i $[P(\sigma^i = +1) = P(\sigma^i = -1) = \frac{1}{2}]$ independent of X_0^i and W^i.

Let Ω_∞ denote the infinite product space, endowed with product probability P_∞ and the product filtration. For each $N \geq 1$ let P_N denote the distribution of $\langle W^1, \dots, W^N, X_0^1, \dots, X_0^N, \sigma^1, \dots, \sigma^N \rangle$ under P_∞, where $X_t^1, \dots, X_t^N$ start at $X_0^1, \dots, X_0^N$ and satisfy the following set of stochastic differential equations:

$$dX_t^i = dW_t^i + \beta N^{-3/2} \sigma^i \sum_{j \neq i,\, k \neq i,\, j \neq k} \sigma^j \sigma^k b(X_t^i, X_t^j, X_t^k)\, dt, \quad 1 \leq i \leq N. \tag{2.1}$$

Here $b\colon \mathbb{R}^3 \to \mathbb{R}$ is a real-valued, uniformly bounded, and uniformly Lipshitz function, so that there exists a finite $K > 0$ such that for all $x, x', y, y', z, z' \in \mathbb{R}$,

$$|b(x,y,z) - b(x',y',z')| \leq K\{|x - x'| + |y - y'| + |z - z'|\}. \tag{2.2}$$

It is easy to see that for a fixed sequence $\{\sigma^1, \dots, \sigma^N\}$ of random signs (2.1) determines a well-defined diffusion on $\mathbb{R}^N$. For $\beta = 0$ this is immediate, for then we have N i.i.d. Brownian motions with initial measure u, and for $\beta > 0$ existence and uniqueness follows from (2.1), (2.2) and a version, for fixed signs, of the Cameron–Martin–Girsanov formula given below.

There are a variety of ways to treat the randomness of the σ^i, perhaps the easiest being to introduce a sequence of constant processes σ_t^i, $t \geq 0$, given by $\sigma_t^i \equiv \sigma_0^i \equiv \sigma^i$, $1 \leq i \leq N$. Then for each $N \geq 1$ we obtain a true $2N$-dimensional diffusion $\langle X_t^1, \dots, X_t^N, \sigma_t^1, \dots, \sigma_t^N \rangle$, whose existence is no harder to establish.

In order to get some intuitive feeling for the diffusions described by (2.1), consider the special case $b \equiv 1$, and let χ_N denote the normalized random sum defined by

$$\chi_N = N^{-1} \sum_{j \neq i,\, k \neq i,\, j \neq k} \sigma^j \sigma^k.$$

(It is not hard to show that as $N \to \infty$, χ_N converges in distribution to a well-defined random variable, and so this is a reasonable variable to temporarily hold fixed.) With this notation, an approximate version of (2.1), for large N, is given by

$$dX_t^i = dW_t^i + \beta N^{-1/2} \sigma^i \chi_N\, dt, \qquad 1 \leq i \leq N.$$

In this format, the behavior of the X^i is transparent. If $\chi_N > 0$, then the X^i corresponding to positive σ^i behave as independent Brownian motions with positive drift, while those corresponding to negative σ^i propagate with a negative drift. If $\chi_N < 0$ then the situation is reversed.

There are essentially two different approaches to studying the diffusion (2.1). The first relies on determining explicit formulas for the finite-dimensional distributions of the X^i, and the second involves a Markov theory approach via the infinitesimal generator of the $2N$-dimensional diffusion. We shall require both approaches in what follows, and start the first by finding an explicit representation for the exponent in the Cameron–Martin–Girsanov formula relating the dependent case ($\beta > 0$) to the much easier independence case ($\beta = 0$).

For each $t \in [0, 1]$, set

$$
\begin{aligned}
J_N^\beta(t) = {} & \beta N^{-3/2} \sum_{i=1}^{N} \sum_{j \neq i,\, k \neq i,\, j \neq k} \sigma^i \sigma^j \sigma^k \int_0^t b(X_s^i, X_s^j, X_s^k)\, dW_s^i \\
& - \beta^2 N^{-3} \sum_{i=1}^{N} \sum_{\{j \neq i,\, k \neq i,\, j \neq k\}} \sum_{\{j' \neq i,\, k' \neq i,\, j' \neq k'\}} \sigma^j \sigma^{j'} \sigma^k \sigma^{k'} \\
& \times \int_0^t b(X_s^i, X_s^j, X_s^k) b(X_s^i, X_s^{j'}, X_s^{k'})\, ds.
\end{aligned} \tag{2.3}
$$

Then for all $\beta \geq 0$ the distribution of $\langle X_t^1, \dots, X_t^N, X_0^1, \dots, X_0^N, \sigma^1, \dots, \sigma^N \rangle$ is given by the probability measure

$$
Q_N^\beta(t, \cdot) := P_N(\cdot) \times e^{J_N^\beta(t)}. \tag{2.4}
$$

This now gives us a way of calculating finite-dimensional distributions for the X^i.

In order to facilitate the alternative approach, note that the infinitesimal generator of the $2N$-dimensional diffusion $\langle X_t^1, \dots, X_t^N, \sigma_t^1, \dots, \sigma_t^N \rangle$ is $\mathscr{A}_N$, where

$$
\begin{aligned}
\mathscr{A}_N f(x, \sigma) &= \mathscr{A}_N f(x_1, \dots, x_N, \sigma_1, \dots, \sigma_N) \\
&= \tfrac{1}{2} \sum_{i=1}^{N} \partial^2 f(x, \sigma)/\partial x_i^2 \\
&\quad + \beta N^{-3/2} \sum_{i=1}^{N} \sigma^i \sum_{j \neq i,\, j \neq k,\, k \neq i} \sigma^j \sigma^k b(x_i, x_j, x_k)\, \partial f(x, \sigma)/\partial x_i.
\end{aligned} \tag{2.5}
$$

The domain of $D(\mathscr{A}_N)$ of $\mathscr{A}_N$ contains $C_K^2(\mathbb{R}^{2N})$, the space of twice-differentiable functions with compact support on $\mathbb{R}^{2N}$.

Having set up the X_t^i of (2.1) formally, let us now turn to the measure-valued processes of central concern to us, as well as their distribution-valued versions. Define, for $t \geq 0$ and $A \in \mathbb{R}$,

$$
\mu_N^+(t, A) = \sum_{i=1}^{N} 1_{\{\sigma^i = +1\}} 1_{\{X_t^i \in A\}}, \qquad \mu_N^-(t, A) = \sum_{i=1}^{N} 1_{\{\sigma^i = -1\}} 1_{\{X_t^i \in A\}}, \tag{2.6}
$$

$$
\mu_N(t, A) = \sum_{i=1}^{N} 1_{\{X_t^i \in A\}} = \mu_N^+(t, A) + \mu_N^-(t, A), \tag{2.7}
$$

$$
\mu_N^\sigma(t, A) = \sum_{i=1}^{N} \sigma^i 1_{\{X_t^i \in A\}} = \mu_N^+(t, A) - \mu_N^-(t, A). \tag{2.8}
$$

The processes μ_N^+ and μ_N^- measure, respectively, the number of positive and negative particles in A at time t, while μ_N and μ_N^σ record the total number of particles and the "net charge" in A at time t. Their distribution-valued versions are given in the obvious way. For example,

$$\mu_N^\sigma(t,\phi) := \int_{\mathbb{R}} \phi(x)\mu_N^\sigma(t,dx) = \sum_{i=1}^{N} \sigma^i \phi(X_t^i),$$

where $\phi \in \mathscr{S}$, the Schwartz space of infinitely differentiable functions decaying at ∞ faster than any polynomial. Thus each of the above four measures can also be thought of as a $\mathscr{S}'$-valued process, where $\mathscr{S}'$ is the space of tempered distributions. We shall generally need to treat our processes in this form, but it will be more convenient to use a notation based on considering them as measures rather than distributions.

We now claim that the two pairs $\langle \mu_N, \mu_N^\sigma \rangle$ and $\langle \mu_N^+, \mu_N^- \rangle$ are actually $\mathscr{S}' \times \mathscr{S}'$-valued *Markov* processes. (It is *not* true, by the way, that any one of these four processes is, by itself, Markovian. *Any* pair, however, is.)

In fact, the Markovian nature of these pairs of processes is immediate from the Markovian nature of the X^i and σ^i processes. We shall, however, formalize this by setting up $\langle \mu_N, \mu_N^\sigma \rangle$ and $\langle \mu_N^+, \mu_N^- \rangle$ as solutions of two martingale problems [cf. Stroock and Varadhan (1979) and Ethier and Kurtz (1986) for details of this approach]. Recall, however, that the basic idea of the martingale formulation of Markov processes is as follows:

Let E be a locally compact, separable, metric space, $\mathscr{B}(E)$ the Borel σ-algebra of E, $B(E)$ the space of real, bounded functions on E and $\mathscr{P}(E)$ the space of Borel probability measures on $\mathscr{B}(E)$ with the topology of weak convergence. The set of continuous functions X: $[0,\infty) \to E$ with the topology of uniform convergence on bounded intervals is denoted by $C_E[0,\infty)$.

Given a linear operator $\mathscr{L}$ defined on a linear subspace $D = D(\mathscr{L}) \subset B(E)$, a solution of the $C_E[0,\infty)$ *martingale problem* for $(\mathscr{L}, D)$ is a stochastic process $\{X_t,\ t \geq 0\}$ with sample paths in $C_E[0,\infty)$ such that, for every $f \in D$, $f(X_t) - \int_0^t \mathscr{L} f(X_s)\,ds$ is a martingale with respect to $\sigma\{X_s\colon 0 \leq s \leq t\}$. The martingale problem is said to be *well posed* if, for each $u \in \mathscr{P}(E)$, there exists a solution of the $C_E[0,\infty)$ martingale problem for $(\mathscr{L}, D)$ under which X_0 has distribution u, and each such solution induces the same measure on $C_E[0,\infty)$. In this case the family of solutions (indexed by u) satisfies the Markov property, and $\mathscr{L}$ is the infinitesimal generator of the process.

To apply this to our case, we take $E = \mathscr{S}' \times \mathscr{S}'$ and consider the dense linear subspace D_0 of $B(\mathscr{S}' \times \mathscr{S}')$ defined by functions of the form

$$F_{f:\phi,\psi}(\eta,\nu) := f(\eta(\phi),\nu(\psi)), \tag{2.9}$$

where $\eta, \nu \in \mathscr{S}'$, $\phi, \psi \in \mathscr{S}$ and $f \in C_b^2(\mathbb{R}^2)$, the space of real-valued functions on $\mathbb{R}^2$ with bounded, continuous, second derivatives. For $n, m \in \{0,1,2\}$ and $f \in C_b^2(\mathbb{R}^2)$, write $f_{n,m}(x,y) = \partial^{n+m} f(x,y)/\partial x^n\,\partial y^m$.

To set up the appropriate operator for the martingale problem, we shall require something like a triple integral that ignores diagonal lines and planes.

Although we shall use integral notation for this, we must set it up in terms of distributions.

For $\eta_1, \eta_2, \eta_3 \in \mathscr{S}'(\mathbb{R})$ and $a \in \mathscr{S}(\mathbb{R}^3)$, set $\tilde{a}(x,y,z) = a(x,y,z)$ if all of x, y, z are distinct, and $a = 0$ otherwise. Then define the product $\eta_1 \otimes \eta_2 \otimes \eta_3 \in \mathscr{S}'(\mathbb{R}^3)$ as $(\eta_1 \otimes \eta_2 \otimes \eta_3)(a) = \eta_1(a^{(1)})$, where $a^{(1)}(x) := \eta_2(a^{(2)}(x, \cdot\,))$ and $a^{(2)}(x,y) := \eta_3(\tilde{a}(x,y,\cdot\,))$. We shall write all this in integral notation as follows:

$$\eta(\phi) = \langle \phi, \eta \rangle = \int_{\mathbb{R}} \phi(x)\eta(dx), \tag{2.10}$$

$$(\eta_1 \otimes \eta_2 \otimes \eta_3)(a) = \oint a(x,y,z)\eta_1(dx)\eta_2(dy)\eta_3(dz), \tag{2.11}$$

where we use the symbol $\oint$ precisely only via its definition through the triple $\otimes$ product. However, it can be thought of intuitively as a triple integral that does not charge the diagonal $\{x = y = z\}$ or the planes $\{x = y\}$, $\{x = z\}$, $\{y = z\}$, or as a regular triple integral minus three double integrals and one one-dimensional integral. (The latter gives one a better way of handling the dual process mentioned in Section 5.)

Finally, for $N \geq 1$, $\eta, \nu \in \mathscr{S}'$, $\phi, \psi \in \mathscr{S}$ and $f \in C_b^2(\mathbb{R}^2)$, define the operator $\mathscr{L}_N$ on D_0 as follows, writing $\xi = (\langle \phi, \eta \rangle, \langle \psi, \nu \rangle) \in \mathbb{R}^2$ to save on space:

$$\begin{aligned} \mathscr{L}_N F_{f:\phi,\psi}(\eta,\nu) = {} & \tfrac{1}{2} f_{10}(\xi)\langle \phi'', \eta\rangle + \tfrac{1}{2} f_{01}(\xi)\langle \psi'', \nu\rangle + \tfrac{1}{2} f_{02}(\xi)\langle (\psi')^2, \eta\rangle \\ & + \tfrac{1}{2} f_{20}(\xi)\langle (\phi')^2, \eta\rangle + f_{11}(\xi)\langle \phi'\psi', \nu\rangle \\ & + \beta N^{-3/2} f_{10}(\xi) \oint b(x,y,z)\phi'(x)\nu(dx)\nu(dy)\nu(dz) \\ & + \beta N^{-3/2} f_{01}(\xi) \oint b(x,y,z)\psi'(x)\eta(dx)\nu(dy)\nu(dz). \end{aligned} \tag{2.12}$$

We can now set up and prove our first result.

THEOREM 2.1. *Let b: $\mathbb{R}^3 \to \mathbb{R}$ be as described at the beginning of this section, and assume, furthermore, that $b \in \mathscr{S}(\mathbb{R}^3)$. Then the martingale problem associated with $(\mathscr{L}_N, D_0)$ is well posed, with solution given by the distribution of the $\mathscr{S}' \times \mathscr{S}'$-valued process $\langle \mu_N, \mu_N^\sigma \rangle$.*

REMARK. The additional restriction that $b \in \mathscr{S}(\mathbb{R}^3)$ is, in essence, unnecessary here, since both μ_N and μ_N^σ take values in the subspace of $\mathscr{S}' \times \mathscr{S}'$ made up of point measures. Furthermore, the care we have taken above in defining the triple integral $\oint$ is really superfluous, the integral being no more than a sum. Nevertheless, in preparation for the next section, in which we send $N \to \infty$, we introduce these restrictions at this point.

PROOF. The fact that the distribution of $\langle \mu_N, \mu_N^\sigma \rangle$ solves the martingale problem for $(\mathscr{L}_N, D_0)$ follows from straightforward computation using Itô's lemma and the following relationship between $\mathscr{L}_N$ and $\mathscr{A}_N$, the generator of $\langle X^1, \dots, X^N, \sigma^1, \dots, \sigma^N \rangle$:

Note that for each $F_{f:\,\phi,\psi} \in D_0$ there exists some $F^*: \mathbb{R}^{2N} \to \mathbb{R}$ satisfying

$$F_{f:\,\phi,\psi}(\mu_N, \mu_N^\sigma) = f\left(\sum_{i=1}^{N} \phi(X_t^i),\ \sum_{i=1}^{N} \sigma^i \psi(X_t^i)\right) = F^*(X_t^1, \dots, X_t^N, \sigma^1, \dots, \sigma^N). \tag{2.13}$$

Consider the effect of the operator $\mathscr{A}_N$ on F^*. Note

$$\begin{aligned} \frac{\partial F^*(X,\sigma)}{\partial X^i} &= f_{10}(\langle \phi, \mu_N \rangle, \langle \psi, \mu_N^\sigma \rangle)\phi'(X^i) \\ &\quad + f_{01}(\langle \phi, \mu_N \rangle, \langle \psi, \mu_N^\sigma \rangle)\sigma^i \psi'(X^i) \\ &= f_{10}(\xi_N)\phi'(X^i) + f_{01}(\xi_N)\sigma^i\psi'(X^i), \end{aligned} \tag{2.14}$$

on writing $\xi_N = (\langle \phi, \mu_N \rangle, \langle \psi, \mu_N^\sigma \rangle)$ to conserve space. Furthermore,

$$\begin{aligned} \frac{\partial^2 F^*(X,\sigma)}{\partial (X^i)^2} &= f_{20}(\xi_N)(\phi'(X^i))^2 + 2 f_{11}(\xi_N)\phi'(X^i)\psi'(X^i)\sigma^i \\ &\quad + f_{02}(\xi_N)(\psi'(X^i))^2 + f_{10}(\xi_N)\phi''(X^i) \\ &\quad + f_{01}(\xi_N)\sigma^i\psi''(X^i). \end{aligned} \tag{2.15}$$

Substitute (2.14) and (2.15) into (2.5), perform the summation over $i \in [1, N]$ and rewrite the result in the format of (2.12) to obtain

$$\mathscr{A}_N F^*(X,\sigma) \equiv \mathscr{L}_N F_{f:\,\phi,\psi}(\mu_N, \mu_N^\sigma). \tag{2.16}$$

The fact that $\langle \mu_N, \mu_N^\sigma \rangle$ solves the martingale problem for $(\mathscr{L}_N, D_0)$—i.e., that

$$F_{f:\,\phi,\psi}(\mu_N(t), \mu_N^\sigma(t)) - \int_0^t \mathscr{L}_N F_{f:\,\phi,\psi}(\mu_N(s), \mu_N^\sigma(s))\, ds \tag{2.17}$$

is a martingale—now follows from (2.16) and the martingale characterization of the system (2.1).

The uniqueness of the solution for $\beta = 0$ is standard, since the only true difficulty in the structure of $\mathscr{L}_N$ lies in the "multiple integral" terms. The uniqueness in the general case then follows either from a Cameron–Martin–Girsanov argument [cf. Dawson (1978)] substantially simplified by the discrete nature of the processes μ_N and μ_N^σ, or, alternatively, directly from the uniqueness of the martingale problem for $(\mathscr{A}_N, C_K^2(\mathbb{R}^{2N}))$. This completes the proof of the theorem. □

In much the same manner, the pair $\langle \mu_N^+, \mu_N^- \rangle$ can be associated with a related martingale problem. Again, take $F_{f:\,\phi,\psi} \in D_0$, and, with the same notation as in (2.12), define the operator $\mathscr{G}_N$ on D_0 by

$$\begin{aligned}\mathscr{G}_N F_{f:\phi,\psi}(\eta,\nu) &= \tfrac{1}{2}f_{10}(\xi)\langle\phi'',\eta\rangle + \tfrac{1}{2}f_{01}(\xi)\langle\psi'',\nu\rangle \\ &\quad + \tfrac{1}{2}f_{20}(\xi)\langle(\phi')^2,\eta\rangle + \tfrac{1}{2}f_{02}(\xi)\langle(\psi')^2,\nu\rangle \\ &\quad + \beta N^{-3/2}f_{10}(\xi)\oint b(x,y,z)\phi'(x)\eta(dx) \\ &\qquad \times(\eta-\nu)(dy)(\eta-\nu)(dz) \\ &\quad + \beta N^{-3/2}f_{01}(\xi)\oint b(x,y,z)\psi'(x)\nu(dx) \\ &\qquad \times(\eta-\nu)(dy)(\eta-\nu)(dz).\end{aligned} \tag{2.18}$$

Then we have the following theorem.

THEOREM 2.2. *Under the conditions of Theorem* 2.1 *the martingale problem associated with* $(\mathscr{G}_N, D_0)$ *is well posed, with solution given by the* $\mathscr{S}' \times \mathscr{S}'$*-valued process* $\langle \mu_N^+, \mu_N^- \rangle$.

PROOF. The proof requires only minor changes to that of the preceding theorem. Instead of (2.14), for example, we obtain

$$\begin{aligned}\frac{\partial F^*(X,\sigma)}{\partial X^i} &= f_{10}(\langle\phi,\mu_N^+\rangle,\langle\psi,\mu_N^-\rangle)1_{\{+1\}}(\sigma^i)\phi'(X^i) \\ &\quad + f_{01}(\langle\phi,\mu_N^+\rangle,\langle\psi,\mu_N^-\rangle)1_{\{-1\}}(\sigma^i)\psi'(X^i),\end{aligned} \tag{2.19}$$

with the indicator functions $1_{\{\pm 1\}}$ appearing also in the analog of (2.15). There we also obtain the product $1_{\{+1\}}1_{\{-1\}}$ ($\equiv 0$), which, on following the same steps as in the proof of Theorem 2.1, leads to the slightly simpler form of the generator $\mathscr{G}_N$, i.e., to (2.18). This proves the theorem. □

In the following two sections we shall study what happens to the processes μ_N, μ_N^+, μ_N^- and μ_N^σ as $N \to \infty$. Ideally, we would like to study this initially from the point of view of the associated martingale problems, but technical difficulties, which we shall discuss in some detail in Section 5, make a direct approach via this path somewhat difficult. Thus we commence with a more direct approach toward a weak convergence theory.

3. Fluctuation theory for μ_N^σ. Our aim in this section is to show that the $\mathscr{S}' \times \mathscr{S}'$-valued process $\langle \mu_N, \mu_N^\sigma \rangle$ has, after appropriate normalization, a weak limit as $N \to \infty$, and to obtain an explicit form for the corresponding univariate distributions of the limit process. With this end in mind, we shall from now on restrict our attention to the time interval $t \in [0, 1]$.

Our first project will be to handle these univariate distributions. The full weak convergence result, in the function space $C([0,1], \mathscr{S}' \times \mathscr{S}')$ (which naturally requires tightness arguments) we leave until the end of this section.

It turns out that the appropriate normalization is $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma\rangle$, and that $N^{-1}\mu_N$ tends to a constant, measure-valued process as $N \to \infty$. Thus primary interest settles on determining the univariate distribution of the normalized net charge process, i.e., of $N^{-1/2}\mu_N^\sigma(t)$. Since this is qualitatively the same for all $t > 0$, we shall soon concentrate on the case $t = 1$. Following standard usage, we call this problem "establishing the fluctuation theory" for μ_N^σ.

We require the limiting distribution of the random field $\langle N^{-1}\mu_N(t, \phi), N^{-1/2}\mu_N^\sigma(t, \psi)\rangle$, $\phi, \psi \in \mathscr{S}$. In view of the Cameron–Martin–Girsanov formula (2.4) and the linearity of μ_N and μ_N^σ as operators on $\mathscr{S}$, it is clearly sufficient to calculate the limiting characteristic function

$$\begin{aligned}(3.1)\quad &\lim_{N\to\infty}\int_{\Omega_\infty} \exp\{i\cdot[N^{-1}\mu_N(t,\phi) + N^{-1/2}\mu_N^\sigma(t,\psi)]\}\,dQ_N^\beta(t)\\ &= \lim_{N\to\infty}\int_{\Omega_\infty}\exp\left\{J_N^\beta(t) + i\cdot\left[N^{-1}\sum_{j=1}^N \phi(X_t^j) + N^{-1/2}\sum_{j=1}^N \sigma^j\psi(X_t^j)\right]\right\}dP_N\end{aligned}$$

where $J_N^\beta(t)$ is the rather complicated expression (2.3) appearing in the exponent of the change of measure formula (2.4).

We shall not attempt to evaluate (3.1) explicitly, but rather shall obtain a representation for it that will give us something recognizable. Furthermore, we shall now concentrate on the case $t = 1$, and shall discuss later how to adapt this to general t.

Unfortunately, we now have to make a substantial investment in notation, before we can even *state* the main result of this section. Some consolation, however, comes from the fact that the investment made in setting up appropriate notation pays dividends by generating relatively simple proofs.

To make our lives a little easier, we take the spaces Ω^i above to be $\{-1, +1\} \times \mathbb{R} \times C[0,1]$, so that we can take $\langle \sigma^i, X_0^i, W^i_{\{t:\ t\in[0,1]\}}\rangle$ canonical on Ω^i, and thus it makes sense to write X_t^i as $X_t(\omega^i)$. Note that $J_N^\beta(1)$ is then an element of $\mathscr{L}_0^2(\Omega, P)^{\otimes 4}$.

For an element of $\mathscr{L}_0^2(\Omega, P)^{\otimes k}$, $k \ge 1$, denote its symmetrized form in $\mathscr{L}_0^2(\Omega, P)^{\odot k}$ (symmetric tensor products) as symm F. For example, if

$$F(\omega^1,\omega^2,\omega^3) = \sigma^1\sigma^2\sigma^3\int_0^1 b(X_s(\omega^1), X_s(\omega^2), X_s(\omega^3))\,dW_s(\omega^1)$$

[so that $F \in \mathscr{L}_0^2(\Omega, P)^{\otimes 3}$], then

$$\begin{aligned}\text{symm } F(\omega^1,\omega^2,\omega^3) = \frac{\sigma^1\sigma^2\sigma^3}{6}\int_0^1 &b(X_s(\omega^1), X_s(\omega^2), X_s(\omega^3))\,dW_s(\omega^1)\\ &+ b(X_s(\omega^1), X_s(\omega^3), X_s(\omega^2))\,dW_s(\omega^1)\\ &+ b(X_s(\omega^2), X_s(\omega^1), X_s(\omega^3))\,dW_s(\omega^2)\\ &+ b(X_s(\omega^2), X_s(\omega^3), X_s(\omega^1))\,dW_s(\omega^2)\\ &+ b(X_s(\omega^3), X_s(\omega^1), X_s(\omega^2))\,dW_s(\omega^3)\\ &+ b(X_s(\omega^3), X_s(\omega^2), X_s(\omega^1))\,dW_s(\omega^3).\end{aligned}$$

Now define a collection of functions $F_1, \dots, F_5$, as follows:

$$F_1 \colon \Omega^3 \to \mathbb{R}, \qquad F_1 \in \mathscr{L}_0^2(\Omega, P)^{\odot 3},$$

$$\begin{aligned} F_1(\omega^1, \omega^2, \omega^3) \\ = \operatorname{symm}\left\{\sigma^1\sigma^2\sigma^3 \int_0^1 b\big(X_s(\omega^1), X_s(\omega^2), X_s(\omega^3)\big)\, dW_s(\omega^1)\right\}. \end{aligned} \tag{3.2}$$

$$F_2 \colon \Omega^4 \to \mathbb{R}, \qquad F_2 \in \mathscr{L}_0^2(\Omega, P)^{\odot 4},$$

$$\begin{aligned} F_2(\omega^1, \omega^2, \omega^3, \omega^4) \\ = \operatorname{symm}\left\{\sigma^1\sigma^2\sigma^3\sigma^4 \int_0^1 \bar{a}\big(X_s(\omega^1), X_s(\omega^2), X_s(\omega^3), X_s(\omega^4)\big)\, ds\right\}, \end{aligned} \tag{3.3}$$

where

$$\bar{a}(u, v, x, y) := \int_{\Omega_\infty} b(X_s(\omega), u, v) b(X_s(\omega), x, y)\, dP_1(\omega). \tag{3.4}$$

$$F_3 \colon \Omega^3 \to \mathbb{R}, \qquad F_3 \in \mathscr{L}_0^2(\Omega, P)^{\odot 3},$$

$$F_3(\omega^1, \omega^2, \omega^3) = \operatorname{symm}\left\{\sigma^1\sigma^2\sigma^3 \int_0^1 b^2\big(X_s(\omega^1), X_s(\omega^2), X_s(\omega^3)\big)\, ds\right\}. \tag{3.5}$$

$$\begin{aligned} &F_4 \colon \Omega^2 \to \mathbb{R}, \qquad F_4 \in \mathscr{L}_0^2(\Omega, P)^{\odot 2}, \\ &F_4(\omega^1, \omega^2) := \operatorname{symm}\left\{\sigma^1\sigma^2 \int_0^1 \bar{b}_s\big(X_s(\omega^1), X_s(\omega^2)\big)\, ds\right\}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \bar{b}_s(x, y) = \int_{\Omega_\infty} &b\big(X_s(\omega^1), X_s(\omega^2), x\big) \\ &\times b\big(X_s(\omega^1), X_s(\omega^2), y\big)\, dP(\omega^1)\, dP(\omega^2). \end{aligned} \tag{3.7}$$

$$\begin{aligned} &F_5 \colon \Omega^2 \to \mathbb{R}, \qquad F_5 \in \mathscr{L}_0^2(\Omega, P)^{\odot 2}, \\ &F_5(\omega^1, \omega^2) := \operatorname{symm}\left\{\sigma^1\sigma^2 \int_0^1 \bar{c}_s\big(X_s(\omega^1), X_s(\omega^2)\big)\, ds\right\}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &\bar{c}_s(x, y) \\ &= \int_{\Omega_\infty} b\big(X_s(\omega^1), x, X_s(\omega^2)\big) b\big(X_s(\omega^1), y, X_s(\omega^2)\big)\, dP(\omega^1)\, dP(\omega^2). \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\Phi(\phi) \colon \Omega \to \mathbb{R}, \qquad \Phi(\phi) \in \mathscr{L}_0^2(\Omega, P), \qquad \phi \in \mathscr{S}, \\ &\Phi(\phi)(\omega) := \sigma\phi(X(\omega)). \end{aligned} \tag{3.10}$$

With this notation, we can write $J_N^\beta(1)$ as a linear combination of the following expressions, in which we use $\mathscr{A}$ as shorthand for {all indices different} and we let $\mathscr{S}(i, j, k, j', k')$ denote the symmetrized version of the five-dimensional summand in the definition (2.3) of J_N^β. Set

$$S_1^N = N^{-3/2} \sum_{\mathscr{A}} F_1(\omega^i, \omega^j, \omega^k), \tag{3.11}$$

$$S_2^N = N^{-3} \sum_{\mathscr{A}} \{\mathscr{S}(i, j, k, j', k') - F_2(\omega^j, \omega^k, \omega^{j'}, \omega^{k'})\} + N^{-2} \sum_{\mathscr{A}} F_2(\omega^i, \omega^j, \omega^k, \omega^l), \tag{3.12}$$

$$S_3^N = N^{-3} \sum_{\mathscr{A}} F_3(\omega^i, \omega^j, \omega^k), \tag{3.13}$$

$$S_4^N = N^{-3} \sum_{i=1}^{N} \sum_{j \neq i} \sum \{\mathscr{S}(i, j, k, j', k') - F_4(\omega^k, \omega^{k'})\} + N^{-1} \sum_{i \neq j} F_4(\omega^i, \omega^j), \tag{3.14}$$

where the innermost sum is over $\{k \neq i \neq j,\ k' \neq i \neq j,\ k \neq k'\}$, and

$$S_5^N = N^{-3} \sum_{i=1}^{N} \sum_{k \neq i} \sum \{\mathscr{S}(i, j, k, j', k') - F_5(\omega^i, \omega^{j'})\} + N^{-1} \sum_{i \neq j} F_4(\omega^i, \omega^j), \tag{3.15}$$

where the innermost sum is now over $\{j \neq i \neq k,\ j' \neq i \neq k,\ j \neq j'\}$.

We thus have that

$$J_N^\beta(1) = \beta S_1^N - \beta^2\{S_2^N + S_3^N + S_4^N + S_5^N\}. \tag{3.16}$$

Despite the long formulas that we have generated, the $N \to \infty$ limit of J_N^β is now comparatively easy to ascertain, since each of the sums $S_1^N, \dots, S_5^N$ is now in a standard form to apply results from the general theory of U-statistics. Details of this can be found in Serfling (1980) and Dynkin and Mandelbaum (1983). Applications of the general theory, similar in spirit to that which we are about to make, can be found in Adler and Epstein (1987) and Adler (1989).

We are now near the end of the requisite notation, and are close to stating our first, and key, result. Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ be a probability space supporting a zero mean Gaussian random field $\Phi_u(f)$, $f \in \mathscr{L}_0^2(\tilde{\Omega}, \tilde{P})$ with covariance

$$E_{\tilde{P}}\{\Phi_u(f)\Phi_u(g)\} = \int_\Omega f(X_1(\omega))g(X_1(\omega))\, dP(\omega). \tag{3.17}$$

Note that the covariance function of Φ_u includes the initial distribution u of X_0 via P.

Denote by I: $\mathscr{L}_0^2(\Omega, P) \to \mathscr{L}_0^2(\tilde{\Omega}, \tilde{P})$ the canonical isometry $f \to \Phi_u(f)$, and by $I^{\odot k}$ the canonical isomorphism from $\mathscr{L}_0^2(\Omega, P)^{\odot k}$ into the kth chaos of the

Gaussian space $\mathscr{H} = I(\mathscr{L}_0^2(\tilde{\Omega}, \tilde{P}))$. [See, for example, Neveu (1968), Dynkin and Mandelbaum (1983) or Mandelbaum and Taqqu (1984).]

Then the general theory of U-statistics referred to above immediately yields the convergence in law of $J_N^\beta(1) + i \cdot N^{-1/2}\sum_{j=1}^N \sigma^j \phi(X_1^j)$ to

$$(3.18)\quad Z_\beta(\phi) := i\Phi_u(\phi) - \beta^2 I^{\odot 2}(F_4 + F_5) + \beta I^{\odot 3}(F_1) - \beta^2 I^{\odot 4}(F_2).$$

[Without going into details, this theory gives us that, under appropriate conditions, $N^{-k/2}$ times a symmetrized, centered, k-dimensional sum $\sum_{i_1=1}^N \cdots \sum_{i_k=1}^N F(Y_{i_1}, \dots, Y_{i_k})$ converges in distribution, as $N \to \infty$, to a random variable in the kth chaos of some Gaussian space. This fact gives us that the first terms in S_2^N, S_4^N and S_5^N, as well as all of S_3^N, go to their mean of 0 as $N \to \infty$, since the normalization is of too high an order. The remaining terms of interest converge to the appropriate terms of (3.18).]

We can now state the main result of the paper.

THEOREM 3.1. *Let the function b have the usual properties. Then, for $\beta \geq 0$ sufficiently small, the random distributions $N^{-1/2}\mu_N^\sigma(1, \psi)$, $\psi \in \mathscr{L}_0^2(\Omega, P)$, converge in law, under the interaction probabilities $Q_N^\beta(1)$, to a random $\mathscr{S}'$-valued distribution whose probability measure is absolutely continuous with respect to that of Φ_u, and the logarithm of whose Radon-Nikodym derivative with respect to this measure is given by*

$$(3.19)\quad - \beta^2 I^{\odot 2}(F_4 + F_5) + \beta I^{\odot 3}(F_1) - \beta^2 I^{\odot 4}(F_2).$$

Furthermore, under the same conditions, the random distributions $N^{-1}\mu_N(1, \phi)$, $\phi \in \mathscr{L}_0^2(\Omega, P)$, converge in probability to the deterministic distribution $\mu_\infty(1, \phi)$, where

$$(3.20)\quad \mu_\infty(t, \phi) = \langle \phi, \mu_\infty(t) \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} g_t(x, y)\phi(y) u(dx)\, dy,$$

$$(3.21)\quad g_t(\phi, y) = \int_{\mathbb{R}} g_t(x, y)\phi(x)\, dx, \qquad \phi \in \mathscr{S},$$

and g_t is the transition density of a standard Brownian motion, given by

$$(3.22)\quad g_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{(x-y)^2/2t}.$$

REMARKS. (a) We reemphasize that the conditions determining the initial points $X_0^1, X_0^2, \dots,$ of the Brownian motions appear in the above result, implicitly, via the parameter u determining the covariance function of Φ_u.

(b) Note that since one of the limiting distributions in Theorem 3.1 is degenerate, it involves no effort to actually consider the weak convergence there as *joint* weak convergence of the pair $\langle N^{-1}\mu_N(1, \phi), N^{-1/2}\mu_N^\sigma(1, \psi)\rangle$. This point of view is advantageous in the martingale problem formulation of the following section.

The proof of Theorem 3.1, as, indeed, the proofs of later results, relies on the following lemma, which forms the central technical result of this paper. We temporarily defer the proof.

LEMMA 3.2. *There is a universal constant $M < \infty$ such that for every $\alpha > \frac{1}{2}$ and for all sufficiently small $\beta \le \beta(\alpha)$*

$$\sup_N E_{P_N}\{\exp[\alpha J_N^\beta(1)]\} \le M. \tag{3.23}$$

PROOF OF THEOREM 3.1. We commence with the convergence of $N^{-1/2}\mu_N^\sigma(1, \psi)$, for which we have already proven almost everything. If we write $\tilde{Q}^\beta$ to denote the limiting probability referred to in the statement of the theorem, then the convergence in law of $N^{-1/2}\mu_N^\sigma(1, \cdot\,)$ under $Q_N^\beta(1)$ to a random distribution with law $\tilde{Q}^\beta$ follows directly from (3.1), the weak convergence result (3.18) and the comments surrounding it and the dominated convergence provided by Lemma 3.2.

As far as the term $N^{-1}\mu_N(1, \phi)$ is concerned, note that the result is true for $\beta = 0$ by the strong law of large numbers for *independent* random variables. The proof for $\beta \ge 0$, and sufficiently small, is then a trivial consequence of the Cameron–Martin–Girsanov formula and Lemma 3.2. □

We now turn to the proof of Lemma 3.2.

PROOF OF LEMMA 3.2. We commence by noting that

$$\begin{aligned}\alpha J_N^\beta &= \alpha\beta S_1^N - \alpha\beta^2\{S_2^N + S_3^N + S_4^N + S_5^N\}\\ &= \sqrt{\alpha/2}\,\beta S_1^N - \alpha\beta^2\{S_2^N + S_3^N + S_4^N + S_5^N\} + \beta\left(\alpha - \sqrt{\alpha/2}\,\right)S_1^N.\end{aligned}$$

By Hölder's inequality we thus have

$$\begin{aligned}E_{P_N}\{\exp[\alpha J_N^\beta]\} &\le \left\{E_{P_N}\left\{\exp\left[J_N^{\beta\sqrt{2\alpha}}\right]\right\}E_{P_N}\left\{\exp\left[2\beta\left(\alpha - \sqrt{\alpha/2}\,\right)S_1^N\right]\right\}\right\}^{1/2}\\ &= E_{P_N}^{1/2}\{\exp[\gamma S_1^N]\},\end{aligned}$$

where $\gamma = 2\beta(\alpha - \sqrt{\alpha/2}\,) > 0$ (since $\alpha > \frac{1}{2}$). Thus it suffices to show, for γ small enough, that

$$\sup_N E_{P_N}\{\exp[\gamma S_1^N]\} < M < \infty. \tag{3.24}$$

To show this, expand the exponential, interchange the order of summation and integration, and consider, for $n \ge 1$,

$$(n!)^{-1}E_{P_N}\{[\gamma S_1^N]^n\} = \frac{\gamma^n N^{-3n/2}}{n!}E_{P_N}\left\{\left[\sum_{\mathscr{A}} F_1(\omega^i, \omega^j, \omega^k)\right]^n\right\}. \tag{3.25}$$

We shall show that (3.25) tends to 0 fast enough as n, $N \to \infty$, for γ small enough, to justify the above steps and, at the same time, to establish (3.24) and the lemma.

Recall that the summation in (3.25) is given, in all its glory, by

$$\sum_{i,j,k \text{ distinct}} \operatorname{symm}\left\{\sigma^i\sigma^j\sigma^k \int_0^1 b\big(W_s(\omega^i), W_s(\omega^j), W_s(\omega^k)\big)\, dW_s(\omega^i)\right\}. \tag{3.26}$$

It is immediate, therefore, from the independence and the symmetry of the σ^i, that (3.25) is identically 0 for all odd n. Thus consider now $n = 2p$, even. Then standard bounds on stochastic integrals [e.g., Gihman and Skorohod (1979), page 75] yield that

$$\begin{aligned} &E_{P_N}\left\{\left|\int_0^1 b\big(W_s(\omega^i), W_s(\omega^j), W_s(\omega^k)\big)\, dW_s(\omega^i)\right|^{2p}\right\} \\ &\quad \le p(2p-1)^p \int_0^1 E_{P_N}\left\{\left|b\big(W_s(\omega^i), W_s(\omega^j), W_s(\omega^k)\big)\right|^{2p}\right\} ds \\ &\quad \le p(2p-1)^p B^p, \end{aligned} \tag{3.27}$$

where $B = \sup_{\{x,y,z\}} |b(x,y,z)|^2 < \infty$ by hypothesis. Now, however, simple counting will suffice to bound the right-hand side of (3.25).

Each term in (3.25) contributing to the total is of the form

$$E_{P_N}\{\sigma^{i_1} \cdots \sigma^{i_{k_n}} F_1(\omega^{i_1}, \omega^{j_1}, \omega^{k_1}) \cdots F_1(\omega^{i_n}, \omega^{j_n}, \omega^{k_n})\}.$$

The nonzero terms occur when the indices occur in pairs, quadruples, etc. Thus, if we write $\mathscr{P}(p)$ to denote all positive integer partitions of p (i.e., all sets of positive integers $\{r_1, \dots, r_k\}$, $1 \le k \le p$, such that $r_1 + \cdots + r_k = p$), then it is easy to check, via (3.27), that (3.25) is bounded by

$$\begin{aligned} &C \cdot (\gamma^2 B)^p \sum_{(\alpha_1, \dots, \alpha_k) \in \mathscr{P}(p)} N^{-3(k-p)} \alpha_1 \cdots \alpha_k \\ &\qquad \times (2\alpha_1 - 1)^{\alpha_1} \cdots (2\alpha_k - 1)^{\alpha_k} / (2p)!, \end{aligned} \tag{3.28}$$

where $n = 2p$ and C is a generic constant which we allow to change from line to line. But, for large p, there are $O((4p\sqrt{3})^{-1} \exp[\pi\sqrt{2p/3}])$ terms in this sum [e.g., Abramowitz and Stegun (1972), page 75] so that if we could show that for large enough p each summand was less than, say, $C \cdot 4^p$, we would have that (3.25) is bounded by a term of order $(4\gamma^2 B)^p \exp[C \cdot \sqrt{p}]$. By making γ small enough (by taking β small enough) this is bounded by η^p for some $\eta \in (0,1)$, so that (3.24), and thus the proof of the lemma, would follow.

To show that each term of (3.28) is, in fact, of the correct order, apply Stirling's formula to the factorial, so that noting that the constants C that appear below are N and p independent, we have

$$\begin{aligned}
&N^{3(k-p)}\alpha_1 \cdots \alpha_k(2\alpha_1 - 1)^{\alpha_1} \cdots (2\alpha_k - 1)^{\alpha_k}/(2p)! \\
&\quad \le C \cdot \alpha_1 \cdots \alpha_k(2\alpha_1 - 1)^{\alpha_1} \cdots (2\alpha_k - 1)^{\alpha_k}/\left[(2p)^{2p+1/2}e^{-2p}\right] \\
&\quad = C \cdot (2p)^{-1/2} \prod_{i=1}^{k} \left[\alpha_i^{1/\alpha_i}(2\alpha_i - 1)e^2/(2p)^2\right]^{\alpha_i} \\
&\quad \le C \cdot (2p)^{-1/2} \prod_{i=1}^{k} \left[4p^{1/\alpha_i - 1}\right]^{\alpha_i} \\
&\quad = C \cdot 4^p p^{k-p-1/2} \\
&\quad < C \cdot 4^p,
\end{aligned}$$

for all $1 \le k \le p$ and p large enough for the Stirling approximation. This completes the proof of Lemma 3.2. □

To complete the results of this section, and to prepare for the next, we now show that Theorem 3.1 can be raised in status to weak convergence on a space of distribution-valued processes. Unfortunately, however, specifying the finite-dimensional distributions of the limit process is a somewhat awkward proposition.

THEOREM 3.3. *Under the conditions of Theorem* 3.1, *as* $N \to \infty$ *the pair* $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma\rangle$ *converges, in the sense of weak convergence of measures on* $C([0,1], \mathscr{S}' \times \mathscr{S}')$, *to a pair* $\langle \mu_\infty, \mu_\infty^\sigma\rangle$.

The deterministic process $\mu_\infty(t, \phi)$ *is given by* (3.20). *The univariate distributions of* $\mu_\infty^\sigma(t, \psi)$ *can be described via Radon-Nikodym derivatives, as were those of* $\Phi_u(\psi) \equiv \mu_\infty^\sigma(1, \psi)$ *in Theorem* 3.1, *merely by replacing* 1 *by* t *throughout the definitions of the* F_i *and* Φ_u *of that result.*

REMARKS. (a) Since the notation is becoming somewhat cumbersome, we no longer note the explicit dependence of μ_∞ and μ_∞^σ on the initial distribution u, as we did for Φ_u. It should be remembered throughout, however, that this dependence is there.

(b) The difficulty that we are unable to explicitly write down finite-dimensional distributions for μ_∞^σ will be partially overcome in the martingale formulation of the following section.

PROOF. As usual, we have two things to prove: convergence of finite-dimensional distributions, and weak compactness. As regards the first, we shall merely claim that here the proof is exactly as in the proof of the weak convergence of Theorem 3.1, barring the irritation of much heavier notation. Since, in the current result, we have no desire to characterize the distribution of the limit process, there is no point in struggling through this.

Thus, what remains is to show that $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ is weakly compact in $C([0,1], \mathscr{S}\times \mathscr{S}')$. To show this, it suffices to establish the following two conditions:

(A) $$\sup_N \sup_{0\le t\le 1} E\{N^{-1}\mu_N(t,|x|) + N^{-1/2}\mu_N^\sigma(t,|x|)\} < \infty,$$

(B) for each $\langle \phi, \psi\rangle \in \mathscr{S}\times \mathscr{S}$, the bivariate process $\langle N^{-1}\mu_N(\cdot,\phi), N^{-1/2}\mu_N^\sigma(\cdot,\psi)\rangle$ is weakly compact in the topology of weak convergence of probability measures on $C([0,1]), \mathbb{R}^2)$.

We shall show how to prove (B). Condition (A) follows via a similar argument.

The key to the proof lies in noting that we really need only prove tightness, and this we attack by first noting that

$$\begin{aligned}
&\left\langle N^{-1}\mu_N(t,\phi), N^{-1/2}\mu_N^\sigma(t,\psi)\right\rangle - \left\langle N^{-1}\mu_N(s,\phi), N^{-1/2}\mu_N^\sigma(s,\psi)\right\rangle \\
&= \Bigg\langle N^{-1}\sum_{i=1}^N \phi(X_t^i) - N^{-1}\sum_{i=1}^N \phi(X_s^i), \\
&\qquad N^{-1/2}\sum_{i=1}^N \sigma^i\psi(X_t^i) - N^{-1/2}\sum_{i=1}^N \sigma^i\psi(X_s^i) \Bigg\rangle \\
&= \Bigg\langle N^{-1}\sum_{i=1}^N \left[\phi(X_t^i) - \phi(X_s^i)\right], N^{-1/2}\sum_{i=1}^N \sigma^i\left[\psi(X_t^i) - \psi(X_s^i)\right] \Bigg\rangle.
\end{aligned} \tag{3.29}$$

Tightness now follows from standard moment arguments, with a little help from Lemma 3.2. As an example, we consider the second term in the last bracket.

It suffices to show [easy extension of Ethier and Kurtz (1986), page 128] that for each $\psi \in \mathscr{S}$ there exists a finite $c > 0$ such that for that for every $\alpha > 0$ there exists a $\delta > 0$ for which

$$\sup_n Q_N^\beta\left\{\inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} N^{-1/2}\sum_{i=1}^N \sigma^i[\psi(X_t^i) - \psi(X_s^i)] > \alpha\right\} \le c\cdot\alpha, \tag{3.30}$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \cdots < t_n \le 1$ with $\min_{1\le i\le n}(t_i - t_{i-1}) > \delta$ and $n \ge 1$.

Note firstly that inspection of the proof of Lemma 3.2 immediately shows that (3.23) can be strengthened to

$$\sup_{0\le t\le 1} \sup_N E_{P_N}\{\exp[\alpha J_N^\beta(t)]\} \le M. \tag{3.31}$$

Let I_α denote the event in question at (3.30). Then it follows from Hölder's inequality that

$$Q_N^\beta\{I_\alpha\} \le P_N^{1/2}\{I_\alpha\}\cdot\left\{E_{P_N}\left[e^{2J_N^\beta(t)}\right]\right\}^{1/2}. \tag{3.32}$$

For sufficiently small β, the second term in this product is bounded uniformly in $t \in [0,1]$ by (3.31). As for the first, the fact that everything has been reduced to a calculation on independent Brownian motions makes the completion of (3.30) standard, and thus we leave it to the reader.

The remaining cases are treated similarly, combining the change of measure via $\exp[J_N^\beta]$ with simple calculations for Brownian motions. This completes the proof of Theorem 3.3. □

4. The infinite limit diffusions. In this section we shall consider the $N \to \infty$ behavior of normalized versions of the four distribution-valued processes μ_N, μ_N^σ, μ_N^+ and μ_N^- of Section 2, concentrating on their behavior as *processes* in t, rather than on their laws for fixed t. We shall start with the pair $\langle \mu_N, \mu_N^\sigma \rangle$, normalized, as in the previous section, to the pair $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$. Our approach in this section will be via martingale problems, whose structure will give us insight into the temporal development of the $N \to \infty$ limit of $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ that was lacking in the treatment of Section 3.

To develop the martingale problem associated with $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$, note that, in the notation of Section 2,

$$
\begin{aligned}
&F_{f:\,\phi,\psi}\big(N^{-1}\mu_N(t), N^{-1/2}\mu_N^\sigma(t)\big) \\
&\qquad = F_{f:\,N^{-1}\phi,\,N^{-1/2}\psi}\big(\mu_N(t), \mu_N^\sigma(t)\big), \qquad N \ge 1.
\end{aligned}
\tag{4.1}
$$

Applying Theorem 2.1 and recalling that the definition of the martingale problem relies on establishing a relationship of the form of (2.17), we immediately obtain that $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ solves the martingale problem for $(\mathscr{L}_N^*, D_0)$, where $\mathscr{L}_N^*$ is defined by

$$
\begin{aligned}
\mathscr{L}_N^* F_{f:\,\phi,\psi}(\eta,\nu) &= \tfrac{1}{2} f_{10}(\xi)\langle \phi'', \eta\rangle + \tfrac{1}{2} f_{01}(\xi)\langle \psi'', \nu\rangle + \tfrac{1}{2} f_{02}(\xi)\langle (\psi')^2, \eta\rangle \\
&\quad + N^{-1}\Big\{\tfrac{1}{2} f_{20}(\xi)\big\langle (\phi')^2, \eta\big\rangle + f_{11}(\xi)\langle \phi'\psi', \nu\rangle\Big\} \\
&\quad + N^{-1}\beta f_{10}(\xi) \oint b(x,y,z)\phi'(x)\nu(dx)\nu(dy)\nu(dz) \\
&\quad + \beta f_{01}(\xi) \oint b(x,y,z)\psi'(x)\eta(dx)\nu(dy)\nu(dz),
\end{aligned}
\tag{4.2}
$$

where, as before, $\xi = (\langle \phi, \eta\rangle, \langle \psi, \nu\rangle) \in \mathbb{R}^2$.

The limiting form of this generator, which we shall denote by $\mathscr{L}$, is defined as follows:

$$
\begin{aligned}
\mathscr{L} F_{f:\,\phi,\psi}(\eta,\nu) &= \tfrac{1}{2} f_{10}(\xi)\langle \phi'', \eta\rangle \\
&\quad + f_{01}(\xi)\Big\{\tfrac{1}{2}\langle \psi'', \nu\rangle + \beta \oint b(x,y,z)\psi'(x)\eta(dx)\nu(dy)\nu(dz)\Big\} \\
&\quad + \tfrac{1}{2} f_{02}(\xi)\big\langle (\psi')^2, \eta\big\rangle.
\end{aligned}
\tag{4.3}
$$

We can now state and prove the following result.

THEOREM 4.1. *Under the conditions of Theorem* 3.1, *the martingale problem associated with* $(\mathscr{L}, D_0)$ *has a solution as a* $\mathscr{S}' \times \mathscr{S}'$*-valued stochastic process. We denote the solution by* $\langle \mu_\infty, \mu_\infty^\sigma \rangle$. *Furthermore, as* $N \to \infty$, $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ *converges, in the sense of weak convergence of measures on* $C([0,1], \mathscr{S}' \times \mathscr{S}')$, *to* $\langle \mu_\infty, \mu_\infty^\sigma \rangle$.

PROOF. The fact that $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ has a weak limit in $C([0,1], \mathscr{S}' \times \mathscr{S}')$ is already known from Theorem 3.3. Call the limit process $\langle \mu_\infty, \mu_\infty^\sigma \rangle$. We need only show that this limit process solves the martingale problem for $(\mathscr{L}, D_0)$; that is, we must show that for all $F_{f:\,\phi,\psi} \in D_0$,

$$F_{f:\,\phi,\psi}(\mu_\infty(t), \mu_\infty^\sigma(t)) - \int_0^t \mathscr{L} F_{f:\,\phi,\psi}(\mu_\infty(s), \mu_\infty^\sigma(s))\,ds \tag{4.4}$$

is a martingale. To show this, it is sufficient to show that for every $k \geq 0$, $0 \leq t_1 < t_2 < \cdots < t_k \leq t < \tau \leq 1$ and $F, F_1, \dots, F_k \in D_0$,

$$\begin{aligned} 0 = E\Bigg[\bigg(&F(\langle \mu_\infty(\tau), \mu_\infty^\sigma(\tau)\rangle) - F(\langle \mu_\infty(t), \mu_\infty^\sigma(t)\rangle) \\ &- \int_t^\tau \mathscr{L}F(\langle \mu_\infty(s), \mu_\infty^\sigma(s)\rangle)\,ds\bigg) \prod_{i=1}^k F_i(\langle \mu_\infty(t_i), \mu_\infty^\sigma(t_i)\rangle)\Bigg]. \end{aligned} \tag{4.5}$$

The weak convergence result of Theorem 3.3, the restrictions inherent on F and the F_k as elements of D_0 and the structure of the operator $\mathscr{L}$, show that the right-hand side of (4.5) is equal to

$$\begin{aligned} \lim_{N\to\infty} E_{Q_N^\beta}\Bigg[\bigg(&F(\langle \mu_N(\tau), \mu_N^\sigma(\tau)\rangle) - F(\langle \mu_N(t), \mu_N^\sigma(t)\rangle) \\ &- \int_t^\tau \mathscr{L}F(\langle \mu_N(s), \mu_N^\sigma(s)\rangle)\,ds\bigg) \prod_{i=1}^k F_i(\langle \mu_N(t_i), \mu_N^\sigma(t_i)\rangle)\Bigg]. \end{aligned} \tag{4.6}$$

The structure of the operator $\mathscr{L}_N$, given by (4.2), shows that we can replace $\mathscr{L}$ by $\mathscr{L}_N$ in (4.6) without changing the limit.

If we now use the fact that $\langle N^{-1}\mu_N, N^{-1/2}\mu_N^\sigma \rangle$ satisfies the martingale problem for $(\mathscr{L}_N, D_0)$, we have that the expression within the limit of (4.6) (with $\mathscr{L}_N$ replacing $\mathscr{L}$) is identically 0 for all $N \geq 1$. This establishes (4.5) and so the theorem. □

An interesting consequence of the above two results is what they have to say about the "propagation of chaos" properties of the X_i processes. Note firstly that the process μ_∞ is not only deterministic, but its distribution in no way depends in the interaction strength parameter β. An immediate consequence of this [cf. Sznitman (1982) and Tanaka (1982)] is a propagation of chaos result for the processes $X^1, X^2, \dots,$ but only if one ignores the random signs attached to the individual processes. To make this comment somewhat more precise, write E_N^β to denote expectation with respect to the interacting system $\{X^1, \dots, X^N, \sigma^1, \dots, \sigma^N\}$ of (2.1), while, as usual, E denotes expectation with respect to the P defined at the beginning of Section 3 on the basic spaces Ω^i. Furthermore, let $\Phi^1, \dots, \Phi^k$, $k \le N$, be bounded, real-valued functions on $C([0,1])$. Then, for each $k \ge 1$,

$$E_N^\beta\left\{\prod_{i=1}^k \Phi^i(X^i)\right\} \to \prod_{i=1}^k E\{\Phi^i(W)\} \quad \text{as } N \to \infty. \tag{4.7}$$

A similar result, which incorporates the random signs, is not true. That is, if $\Psi^1, \dots, \Psi^k$, $k \le N$, are bounded, real-valued functions on $C([0,1] \times \{-1, +1\}$, then we generally have that

$$E_N^\beta\left\{\prod_{i=1}^k \Psi^i(X^i, \sigma^i)\right\} \nrightarrow \prod_{i=1}^k E\{\Psi^i(W, \sigma)\} \quad \text{as } N \to \infty. \tag{4.8}$$

Given the structure of the processes μ_N^σ and μ_∞^σ, this result, while negative, is not surprising. It is also strongly connected to the fluctuation theory for μ_N^σ of the preceding section.

Before concluding this section and turning to a general discussion, let us take a moment to consider the problem of attempting to conduct an asymptotic analysis for the pair $\langle \mu_N^+, \mu_N^- \rangle$ similar to that we have just completed for $\langle \mu_N, \mu_N^\sigma \rangle$. Arguing as at the beginning of this section, we note that

$$F_{f:\phi,\psi}\big(N^{-1}\mu_N^+(t), N^{-1}\mu_N^-(t)\big) = F_{f:N^{-1}\phi, N^{-1}\psi}\big(\mu_N^+(t), \mu_N^-(t)\big), \quad N \ge 1. \tag{4.9}$$

Then, via Theorem 2.2 and (2.17), it follows that $\langle N^{-1}\mu_N^+, N^{-1}\mu_N^- \rangle$ solves the martingale problem for $(\mathscr{G}_N^*, D_0)$, where $\mathscr{G}_N^*$ is defined by

$$\begin{aligned}
&\mathscr{G}_N^* F_{f:\phi,\psi}(\eta,\nu)\\
&\quad = \tfrac{1}{2} f_{10}(\xi)\langle \phi'', \eta\rangle + \tfrac{1}{2} f_{01}(\xi)\langle \psi'', \nu\rangle\\
&\qquad + \tfrac{1}{2}N^{-1}\Big\{ f_{20}(\xi)\big\langle (\phi')^2, \eta\big\rangle + \tfrac{1}{2} f_{02}(\xi)\big\langle (\psi')^2, \nu\big\rangle\Big\}\\
&\qquad + \beta N^{1/2} f_{10}(\xi)\oint b(x,y,z)\phi'(x)\eta(dx)(\eta-\nu)(dy)(\eta-\nu)(dz)\\
&\qquad + \beta N^{1/2} f_{01}(\xi)\oint b(x,y,z)\psi'(x)\nu(dx)(\eta-\nu)(dy)(\eta-\nu)(dz).
\end{aligned} \tag{4.10}$$

It is obvious that sending $N \to \infty$ to obtain a limiting diffusion will not work here. Two things, however, are worth noting, before we discard this problem completely.

The first is that if the normalization in the interaction term of the original diffusion equation (2.1) for $\langle X^1, \dots, X^N \rangle$ were changed from $N^{-3/2}$ to N^{-2}, then the problematic terms in the above definition of. $\mathscr{G}_N^*$ would disappear, leaving only an operator whose limiting form would be

$$\mathscr{G} F_{f:\phi,\psi}(\eta,\nu) = \tfrac{1}{2} f_{10}(\xi)\langle \phi'', \eta\rangle + \tfrac{1}{2} f_{01}(\xi)\langle \psi'', \nu\rangle. \tag{4.11}$$

This is a particularly easy generator to identify, since both components of this process correspond to deterministic processes of the kind given by (3.20). Thus a full propagation of chaos result does hold in this situation. The price that one has to pay for this weakening of the interaction mechanism is, however, substantial. In particular, it is easy to see from the calculations of the preceding section that the fluctuation theory is then Gaussian, and the entire aim of our project has been to produce a *non*-Gaussian fluctuation theory.

The second point is that if the values of (η, ν) are restricted to the diagonal in $\mathscr{S}' \times \mathscr{S}'$ then, even under the original normalization, we find that

$$\lim_{N \to \infty} \mathscr{G}_N^* F_{f:\phi,\psi}(\eta, \eta) = \mathscr{G} F_{f:\phi,\psi}(\eta, \eta). \tag{4.12}$$

The limit process is then clearly a pair of identical, deterministic distributions, which, starting together, develop together. The slightest deviation from equivalence, however, leads to immediate explosion. It is instructive to carry this argument backwards, to see what it tells us about the finite (but large) N-particle problem for $\langle \mu_N^+, \mu_N^- \rangle$, and, in particular, about the individual processes $\langle X^1, \dots, X^N \rangle$ themselves.

An $O(\varepsilon)$ difference between $N^{-1}\mu_N^+$ and $N^{-1}\mu_N^-$ translates to an $O(N\varepsilon) = O(N)$ difference between μ_N^+ and μ_N^-, and it easy to see by inspection of the diffusion mechanism (2.1) that this can easily raise the drift coefficient to $O(\sqrt{N})$ for some particles, thus leading to an explosion-type phenomenon for these particles. It is this phenomenon that corresponds to the above failure of $\langle N^{-1}\mu_N^+, N^{-1}\mu_N^- \rangle$ to converge to anything nondegenerate without changing the interaction coefficient.

5. Discussion. In this, final, section, we shall make a number of remarks concerning the general properties of the processes we have studied, and, in particular, consider what the effects of small changes in the model would be on our results.

(a) *Conditions on the function b.* Throughout the paper we placed a sequence of reasonably restrictive conditions on the interaction function b appearing in the drift coefficient of the diffusion system (2.1). There is no doubt that many of these could be relaxed in different parts of the argument.

For example, the analysis of the temporal development of the processes μ_N^σ and μ_∞^σ of Sections 2 and 4 undoubtedly holds under weaker boundedness conditions on b. In Section 3, boundedness was required at (3.27), in the proof of the crucial Lemma 3.2. Nevertheless, it is clear from the argument there that an appropriate bound on the moments of $b(W_s(\omega^1), W_s(\omega^2), W_s(\omega^3))$ is really all that is required.

In both cases, however, one can only weaken the conditions on b to a certain point, after which qualitatively quite different results arise. For example,it is natural to try to replace b with a delta function of the form $b(x, y, z)$"$=$"$\delta(x - y)\delta(y - z)$, so that the interaction mechanism is of a purely local nature; i.e., only particles actually touching one another interact.

The consequences of introducing a purely local interaction of this kind have been studied in models of unsigned particles, and lead not only to a much more involved analysis of the processes μ_N, but even the study of the diffusion system (2.1) itself becomes qualitatively more difficult [cf. Sznitman and

Varadhan (1986) and Sznitman (1986)]. Nevertheless, the introduction of an interaction of a more local nature is of interest, particularly in relation to obtaining fluctuation theories that correspond to true EQFTs.

(b) *Euclidean quantum field theories.* The fluctuation theory of Section 3 —i.e., the limit law of $N^{-1/2}\mu_N^\sigma$—was described via its Radon-Nikodym derivative with respect to a Gaussian random field. Since this derivative involves exponentials of random variables in chaoses up to and including the fourth, we earlier described the corresponding distribution as being akin to a $P(:\Phi_1^4:)$ EQFT.

It is important to note that "akin to" is not the same as "is." For a random field to be a true EQFT it must be Markovian in some reasonable sense, and this is not generally true of the random field described in Theorem 3.1. If the interaction function b is of compact support, and the diameter of its support given by some L, then we conjecture that the random field of Theorem 3.1 is "L-Markov" in the sense of Pitt (1975). For a more useful Markov structure, however, it is necessary that b be "local," in some sense. For example, were b a delta function of the form described above, then the corresponding fluctuation theory for μ_N^σ probably would be a true EQFT. However, in view of the difficulties already noted in (a), we have no proof of this.

(c) *Phase transition in β.* There are a number of interesting phenomena surrounding the interaction strength β.

Firstly, given the qualitatively different behavior of the fluctuation theory of μ_N^σ for $\beta = 0$ and $\beta > 0$, there is clearly a phase transition in the model at $\beta = 0$. It is worthwhile to note that this phenomenon is absent in the usual models of the type of (1.4), for which the fluctuation theory is invariably Gaussian, regardless of the size of β.

A more interesting question is what happens to the fluctuation theory as β increases further. All of our main results were proven only for "β small enough," this condition arising in the proof of Lemma 3.2. It is not clear to us, however, whether the need to restrict β is due to an insufficiently tight proof, or to an inherent restriction of the model. Since we have been unable to see how tightening the bounds leading to the proof of Lemma 3.2 could make a qualitative difference to our results, and since the final fluctuation theory is "akin" to that of a $P(:\Phi_1^4:)$ EQFT, we are inclined to conjecture the existence of a further phase transition; i.e., the existence of a $\beta_c > 0$ so that Theorem 3.1 fails to hold for $\beta > \beta_c$, in which case a different normalization may be required for μ_N^σ in order to obtain a nontrivial limit.

(d) *Higher-dimensional results.* It is reasonably easy to see that very little *qualitative* will change if the diffusions defined at (2.1) take values in $\mathbb{R}^d$, $d > 1$, rather than in $\mathbb{R}$. (This is not to say that such a generalization may not require a reasonable amount of hard work.) On the one hand, this is an encouraging indication of the robustness of the methodology and model. On the other, however, it indicates that the fluctuation theory is not quite as "akin" to a $P(:\Phi_1^4:)$ EQFT as one might hope, for such EQFTs behave very differently in different dimensions.

(e) *Higher-order interactions.* As noted in Section 1, it is possible to generalize our model to include higher-order interactions. For example, a fourth-order interaction might look something like

$$\beta_4 N^{-2}\sigma^i \sum_{\{\text{distinct indices}\}} \sigma^j\sigma^k\sigma^l b_4\big(X_t^i, X_t^j, X_t^k, X_t^l\big).$$

In such a case, under the appropriate conditions on b_4, the corresponding fluctuation theory would have a similar structure to that described in Theorem 3.1, but now the leading term in the exponent of the Radon-Nikodym derivative would come from the sixth, rather than the fourth, chaos of the corresponding Gaussian field.

This phenomenon—a fluctuation theory that changes qualitatively as the interaction order is changed—is not observed in the standard models in which the particles are all of the same kind. [For similar results in this direction, for related but different models, see Adler (1989).]

(f) *Uniqueness and duality.* A rather irritating loose end that remains in our results is that we have not shown that the solution $\langle \mu_\infty, \mu_\infty^\sigma \rangle$ to the martingale problem for $(\mathscr{L}, D)$, which forms the basis of Theorem 4.1, is unique.

One way that one could go about this would be to show that the dual problem [cf. Ethier and Kurtz (1986) for definitions and details] has a solution (not necessarily unique). While it is not in principle difficult to write down the generator for the dual problem, the result is somewhat complicated and does not represent a problem whose solution is obvious, or, as far as we can tell, at all known.

Nevertheless, in principle one can demonstrate the existence of the solution to the dual problem to $(\mathscr{L}, D)$ by obtaining it as the weak limit of the solutions for the dual problems to $(\mathscr{L}_N, D)$. However, given the amount of work involved in this task, for what seems like minimal reward, we have not carried out this program.

Finally, we note that there would be independent technical interest in establishing the uniqueness of the solution to the martingale problem for $(\mathscr{L}, D)$, for then the weak convergence result of Theorem 3.3 could be proven a little more neatly using martingale techniques. Nevertheless, these would still not yield the precise information on univariate distributions given by Theorem 3.1, which is, we feel, the most significant result of this paper.

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