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ENTROPY ANALYSIS OF A NEAREST-NEIGHBOR ATTRACTIVE / REPULSIVE EXCLUSION PROCESS ON ONE-DIMENSIONAL LATTICES

By Hirotake Yaguchi

Hiroshima University

Stationary measures for an interactive exclusion process on $\mathbb{Z}$ are considered. The process is such that the jump rate of each particle to the empty neighboring site is $\alpha > 0$ (resp., $\beta > 0$) when another neighboring site is occupied (resp., unoccupied) by a particle, and that $\alpha \neq \beta$. According as $\alpha < \beta$ or $\alpha > \beta$ the process becomes nearest-neighbor attractive or repulsive, respectively. The method of relative entropy is used to determine the family $\mathscr{M}_{\beta/\alpha}$ of stationary measures. The member of $\mathscr{M}_\gamma$ is simply described as the probability measure having the regular clustering property which is a generalization of the exchangeable property of measures. It is shown that extremal points of $\mathscr{M}_\gamma$ are renewal measures. Thus the structure of stationary measures for the process is completely determined.

1. Introduction. Exclusion processes are continuous-time Markov processes on the state space $\mathscr{X} = \{0, 1\}^S$, $S = \mathbb{Z}^d$ or $\mathbb{R}^d$, which describe the stochastic movement of infinitely many indistinguishable particles on S such that more than one particle cannot occupy the same site at the same time. A simple exclusion process is the process such that $S = \mathbb{Z}^d$ and each particle executes an independent random walk with the same jump rate except that a jump to the occupied site is suppressed. Perhaps the simple exclusion process is at present the unique one for which the structure of stationary measures is completely known [see Liggett (1976) and the textbook by Liggett (1985) and also references given there]. The main tool that is usually used in the analysis of stationary measures for various types of simple exclusion processes is the method of a coupled Markov process. This method is used to introduce an order relation on the set of extremal points of stationary measures and is very effective if the number of sites at which particles jump at the same time on S is 1 and/or if the jump rate of particles is independent of the configuration of other particles. However, if the jump rate is influenced by the configuration of other particles (the exclusion process with speed change), we encounter a difficulty in applying the method of a coupled Markov process because the monotonicity concerning the Feller process [see Liggett (1985), Chapter 2, for the definition] is not assured and hence the usual argument of the coupled process does not hold: Even though one may make a coupled process, couples that are created during the time evolution of that process have chances of collapsing since each particle at the same coordinate may have a different jump rate due to the different configurations of the other particles.

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In this paper we treat an exclusion process on the one-dimensional lattices such that the jump rate of a particle is influenced by the nearest-neighbor sites of the particle, that is, the jump rate depends on whether or not a particle exists at the neighboring sites. More precisely, we consider a Markov process on $\mathscr{X} \equiv \{0, 1\}^{\mathbb{Z}}$ whose generator is of the form

$$(\Omega f)(\eta) = \sum_{x \in \mathbb{Z}} \{\alpha\chi_{11}(\eta_{x-1}\eta_x) + \beta\chi_{01}(\eta_{x-1}\eta_x) + \alpha\chi_{11}(\eta_{x+1}\eta_{x+2}) + \beta\chi_{10}(\eta_{x+1}\eta_{x+2})\}\left[f(\eta^{x,x+1}) - f(\eta)\right],$$

where α and β are fixed positive numbers (for details see Section 2). The exclusion process defined by the above generator is the process such that if $\alpha < \beta$ (resp., $\alpha > \beta$), then the existence of a particle at the neighboring sites has an effect which slows down (resp., speeds up) the jump rate of a particle. So we call our process a nearest-neighbor attractive/repulsive exclusion process.

The purpose of this paper is to investigate the structure of stationary measures for the above attractive/repulsive exclusion process. (Of course, if $\alpha = \beta$, our process becomes a usual simple exclusion process; and the stationary measures are equal to the totality of exchangeable measures). As a main tool of the analysis we employ the method of relative entropy, which was used in Holley (1971) and Holley and Stroock (1977) to prove the uniqueness of stationary measures for the stochastic Ising model [see Liggett (1985)]. However, the argument given there does not work directly since in an exclusion process the number of particles is preserved under the time evolution and hence for each ρ, $0 \le \rho \le 1$, there exists at least one stationary measure μ_ρ whose support is on the set of configurations such that the density of particles over $\mathbb{Z}$ is ρ, and as a result, the process has infinitely many stationary measures. We apply the relative entropy method to determine the condition for a probability measure on $\mathscr{X}$ to be stationary (Theorem 1). We will call that condition the regular clustering property with index β/α and write $(\mathrm{RCP})_{\beta/\alpha}$ for short. The proof of Theorem 1 is given in Section 4 and is fairly complicated because of the terms caused by the coming in and out of particles across the boundary of cylinder sets in the equilibrium equations. These terms do not exist in stochastic Ising models. The regular clustering property is a kind of generalization of the so-called exchangeable property of measures. The structure of measures having $(\mathrm{RCP})_\gamma$ is discussed in Section 3. It is shown that the extremal points of the set of measures having $(\mathrm{RCP})_\gamma$ are renewal measures (Theorem 2). Thus the structure of stationary measures for our nearest-neighbor attractive/repulsive exclusion process is completely determined. In Section 5 we state a result about a finite range interactive exclusion process on $\mathbb{Z}^1$ of the form

$$(\Omega f)(\eta) = \sum_{x<y} (x,y)c(x,y;\eta)\left[f(\eta^{x,y}) - f(\eta)\right].$$

An application of the entropy analysis to a discrete-time interactive exclusion process such that infinitely many particles can move at the same time is treated in Yaguchi (1989).

2. Definitions and results. Let $\mathbb{Z}$ be the set of all integers and $\mathscr{X}$ be the set $\{0,1\}^{\mathbb{Z}}$ equipped with the topology generated by the basic cylinder sets ${}_i[a_i \dots a_j]_j = \{\eta \equiv (\cdots \eta_{-1}\eta_0\eta_1 \cdots) \in \mathscr{X}\colon \eta_l = a_l,\ i \le l \le j\}$, $a_i \cdots a_j \in \{0,1\}^{j-i+1}$, $i \le j$, $i, j \in \mathbb{Z}$. We set $\mathscr{C}_{i,j} = \{{}_i[a_i \cdots a_j]_j\colon a_i \cdots a_j \in \{0,1\}^{j-i+1}\}$ and $\mathscr{C} = \{\varnothing\} \cup \bigcup_{i \le j} \mathscr{C}_{i,j}$. The Borel fields generated by $\mathscr{C}_{i,j}$ and $\mathscr{C}$ are denoted by $\mathscr{B}_{i,j}$ and $\mathscr{B}$, respectively. If we consider that there is a particle at the site i iff $\eta_i = 1$, then $\eta \in \mathscr{X}$ represents a configuration of finite/infinite particles on the one-dimensional lattices. By **0** (resp., **1**) we denote the configuration of particles on $\mathbb{Z}$ such that all sites are empty (resp., occupied) that is, $\mathbf{0} \equiv (\cdots 0000 \cdots)$ and $\mathbf{1} \equiv (\cdots 1111 \cdots)$.

We define a Markov process on $\mathscr{X}$, which we call a *nearest-neighbor attractive/repulsive exclusion process*, as follows. Let us fix α and β as positive real numbers. For each $\mathscr{B}_{i,j}$-measurable function f, define

$$(\Omega f)(\eta) = \sum_{x=i-1}^{j} \{\alpha\chi_{11}(\eta_{x-1}\eta_x) + \beta\chi_{01}(\eta_{x-1}\eta_x) + \alpha\chi_{11}(\eta_{x+1}\eta_{x+2}) + \beta\chi_{10}(\eta_{x+1}\eta_{x+2})\}\left[f(\eta^{x,x+1}) - f(\eta)\right],$$

where

$$\chi_{ab}(uv) = \begin{cases} 1 & \text{if } uv = ab, \\ 0 & \text{otherwise,} \end{cases}$$

and $\eta^{x,x+1}$ is the element of $\mathscr{X}$ obtained from $\eta = (\cdots \eta_{x-1}\eta_x\eta_{x+1}\eta_{x+2} \cdots) \in \mathscr{X}$ by exchanging the values at sites x and $x+1$, namely, $\eta^{x,x+1} = (\cdots \eta_{x-1}\eta_{x+1}\eta_x\eta_{x+2} \cdots)$. (Note that Ωf is a $\mathscr{B}_{i-2,j+2}$-measurable function on $\mathscr{X}$.) Then Ω defines a Markov generator $\overline{\Omega}$ on the set $C(\mathscr{X})$ of continuous functions on $\mathscr{X}$ [see Theorem 1.3.9 and page 27 of Liggett (1985)]. We can explain the Markov process defined by the generator $\overline{\Omega}$ as follows: Each particle scattered on $\mathbb{Z}$ has its own alarm clock which runs independently of the others; the clock rings the bell after the exponential holding time with parameter 2α or 2β according as the number of empty (nearest) neighbor sites is 1 or 2, respectively; when the bell rings, the clock clears its memory and the particle tries to move to its left- or right-neighbor site with probability $1/2$ and actually moves if the target site is empty. Note that if $\alpha < \beta$ (resp., $\alpha > \beta$) then a particle jumps more slowly (resp., rapidly) if there is a particle at one of the neighboring sites. This is the reason why we call our process the attractive/repulsive exclusion process (of course, if $\alpha = \beta$, then our process becomes a well-known simple exclusion process).

A probability measure ν on $\mathscr{X}$ is called a *stationary* (or invariant) *measure* for the Markov process defined by the generator $\overline{\Omega}$ if it satisfies $\int (S_t f)(\eta)\, d\nu(\eta) = \int f(\eta)\, d\nu(\eta)$ for all bounded measurable functions f on $\mathscr{X}$ for every $t \ge 0$. Here $\{S_t\}_{t \ge 0}$ is the Markov semigroup corresponding to the generator $\overline{\Omega}$. We denote by $\mathscr{S}$ the totality of stationary measures for our exclusion process.

We will say that a probability measure ν on $\mathscr{X}$ has the *regular clustering property with index* $\gamma(>0)$ and denote by $(\mathrm{RCP})_\gamma$ if it satisfies

$$\gamma^{\#_{01}({}_i[a_i \cdots a_j]_j)}\nu\left({}_i[a_i \cdots a_j]_j\right) = \gamma^{\#_{01}({}_i[b_i \cdots b_j]_j)}\nu\left({}_i[b_i \cdots b_j]_j\right),$$

for all ${}_i[a_i \cdots a_j]_j, {}_i[b_i \cdots b_j]_j \in \mathscr{C}_{i,j}$, $i \le j$, $i, j \in \mathbb{Z}$, with

$$a_i = b_i, \quad a_j = b_j \quad \text{and} \quad \sum_{k=i}^{j} a_k = \sum_{k=i}^{j} b_k,$$

where

$$\#_{uv}\big({}_i[a_i \cdots a_j]_j\big) = \#\{k: a_k a_{k+1} = uv, i \le k \le j-1\}.$$

The totality of probability measures having $(\mathrm{RCP})_\gamma$ is denoted by $\mathscr{M}_\gamma$. It is easy to check that $\mathscr{M}_1$ is equal to the set of exchangeable measures on $\mathscr{X}$.

A probability measure ν on $\mathscr{X}$ is said to be *translation* (or shift) *invariant* if it satisfies $\nu(A) = \nu(A + k)$ for every $k \in \mathbb{Z}$ and $A \in \mathscr{B}$, where

$$A + k = \{\xi \equiv (\cdots \xi_{-1}\xi_0\xi_1 \cdots) \in \mathscr{X}: \text{there exists } \eta = (\cdots \eta_{-1}\eta_0\eta_1 \cdots) \in A \text{ s.t. } \xi_i = \eta_{i+k} \text{ for all } i \in \mathbb{Z}\}.$$

Note that if ν is translation-invariant, then it is not necessary for us to specify the coordinates in the expression $\nu({}_i[a_i \cdots a_j]_j)$. (For example, we will use $\nu([00011])$ instead of $\nu({}_i[00011]_{i+4})$.)

We endow the set of probability measures on $\mathscr{X}$ with the topology of weak convergence as usual and denote by ext $\mathscr{A}$ the set of extremal points of a compact convex set $\mathscr{A}$ of probability measures on $\mathscr{X}$.

Now we can state our results as follows.

THEOREM 1. *A probability measure ν on $\mathscr{X}$ is a stationary measure for the Markov process with the generator $\overline{\Omega}$ if and only if ν has the regular clustering property with index β/α; that is, $\mathscr{S} = \mathscr{M}_{\beta/\alpha}$.*

THEOREM 2. Ext $\mathscr{M}_\gamma = \{\mu_\rho^{(\gamma)}\}_{0 \le \rho \le 1}$. *Here $\mu_0^{(\gamma)} = \delta_{\mathbf{0}}$ (Dirac measure concentrated at $\mathbf{0}$), $\mu_1^{(\gamma)} = \delta_{\mathbf{1}}$ and $\mu_\rho^{(\gamma)}$, $0 < \rho < 1$, is the translation-invariant probability measure on $\mathscr{X}$ defined by*

$$\begin{aligned} &\mu_\rho^{(\gamma)}([1]) = \rho, \qquad \mu_\rho^{(\gamma)}([0]) = 1 - \rho, \\ &\mu_\rho^{(\gamma)}\big([a_i \cdots a_j 00]\big) = q\mu_\rho^{(\gamma)}\big([a_i \cdots a_j 0]\big), \\ &\mu_\rho^{(\gamma)}\big([a_i \cdots a_j 11]\big) = q'\mu_\rho^{(\gamma)}\big([a_i \cdots a_j 1]\big), \end{aligned} \tag{2.1}$$

where q and q' is the unique pair of numbers in $(0,1)$ satisfying

$$qq'/[(1-q)(1-q')] = \gamma \quad \text{and} \quad (1-q')/(1-q) = (1-\rho)/\rho.$$

REMARK 2.1. We can interpret the measure $\mu_\rho^{(\gamma)}$, $0 < \rho < 1$, in various ways according to the context in which it appears. Below we give three examples for the convenience of readers.

1. $\mu_\rho^{(\gamma)}$ is the unique stationary distribution of the two-state Markov chain with transition matrix $\begin{pmatrix} q & 1-q \\ 1-q' & q' \end{pmatrix}$.

2. $\mu_\rho^{(\gamma)}$ is the renewal measure on $\mathscr{X}$ corresponding to the renewal process whose probability distribution function (p.d.f.) of interarrival time is given by

$$f_\rho(k) = \begin{cases} q' & \text{if } k = 1, \\ \gamma^{-1} q' q^{k-1} & \text{if } k > 1. \end{cases}$$

3. $\mu_\rho^{(\gamma)}$ is the Gibbs state with the nearest-neighbor interaction on $\mathbb{Z}$ such that the chemical potential J_0 and the interaction potential J_1 are given by

$$J_0 = kT\{2\log q - \log(1-q) - \log(1-q')\}$$

and

$$J_1 = -kT\{\log q + \log q' - \log(1-q) - \log(1-q')\} (= -kT \log \gamma),$$

respectively, where k is the Boltzmann constant and T is the absolute temperature. Here we say that a probability measure ν on $\mathscr{X}$ is the Gibbs state with the nearest-neighbor interaction on $\mathbb{Z}$ with the chemical potential J_0 and the interaction potential J_1 if its conditional probability $\nu\{{}_i[a_i \cdots a_j]_j | \mathscr{B}_{i,j}^c\}(\eta)$ of ${}_i[a_i \cdots a_j]_j \in \mathscr{C}_{i,j}$ given $\mathscr{B}_{i,j}^c$ ($\equiv$ the σ-field generated by $\mathscr{C}_{I,J}$, $I \le J < i$ and $j < I \le J$) is equal to

$$\Xi_{i,j}(\eta)^{-1} \exp\left[-(1/kT) \left\{ J_0 \sum_{k=i}^{j} a_k + J_1 \left(\eta_{i-1} a_i + a_j \eta_{j+1} + \sum_{k=i}^{j-1} a_k a_{k+1} \right) \right\} \right],$$

where $\Xi_{i,j}(\eta)$ is the normalizing factor which depends on i, j and $\eta \equiv (\eta_i)_{i \in \mathbb{Z}}$.

3. The structure of the set of regular clustering measures. In this section we will give the proof of Theorem 2. We postpone the proof of Theorem 1 to the next section since in that proof (more precisely, in the definition of relative entropy) we use the measure $\mu_\rho^{(\gamma)}$ whose explicit form is described in this section.

We first show that if ν has RCP then it is symmetric in the following sense.

LEMMA 3.1. *Suppose a probability measure ν on $\mathscr{X}$ has the regular clustering property. Then*

$$\nu\big({}_i[a_i a_{i+1} \cdots a_{j-1} a_j]_j\big) = \nu\big({}_i[a_j a_{j-1} \cdots a_{i+1} a_i]_j\big)$$

for every ${}_i[a_i a_{i+1} \cdots a_{j-1} a_j]_j \in \mathscr{C}$.

PROOF. Since the lemma is clear for $a_i = a_j$, we treat the case $a_i \neq a_j$. It is sufficient to prove that $\nu({}_i[00 \cdots 011 \cdots 1]_j) = \nu({}_i[11 \cdots 100 \cdots 0]_j)$ (the number of 1's in each cylinder is the same), since $\gamma^s \nu({}_i[0 a_{i+1} \cdots a_{j-1} 1]_j) = \nu({}_i[0 \cdots 01 \cdots 1]_j)$, $s = \#_{01}(a_i \cdots a_j) - 1$, if $1 + \sum_{k=i+1}^{j-1} a_k =$ the number of 1's in ${}_i[0 \cdots 01 \cdots 1]_j$, and so on. Let us show $\nu({}_i[0011]_{i+3}) = \nu({}_i[1100]_{i+3})$ to illustrate the way of proof.

For $c \in \{0,1\}$ let $\bar{c} = 1 - c$. Then for each $N \in \mathbb{N}$,

$$\begin{aligned}
&\nu({}_i[0011]_{i+3})\\
&\quad= \sum_{k=1}^{N} \sum_{c_1 \cdots c_N} \sum_{c'_1 \cdots c'_{k-1}} \nu\big({}_{i-N}[c_N \cdots c_{k+1} c_k c_{k-1} \cdots c_1 0011 c'_1 \\
&\qquad\qquad \cdots c'_{k-1} c_k \bar{c}_{k+1} \cdots \bar{c}_N]_{i+3+N}\big)\\
&\qquad + \sum_{c_1 \cdots c_N} \nu\big({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N}\big)\\
&\quad= \sum_{k=1}^{N} \sum_{c_1 \cdots c_N} \sum_{c'_1 \cdots c'_{k-1}} \nu\big({}_{i-N}[c_N \cdots c_{k+1} c_k c'_{k-1} \cdots c'_1 1100 c_1 \\
&\qquad\qquad \cdots c_{k-1} c_k \bar{c}_{k+1} \cdots \bar{c}_N]_{i+3+N}\big)\\
&\qquad + \sum_{c_1 \cdots c_N} \nu\big({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N}\big)\\
&\qquad\qquad \big[\text{since } \#_{01}(a a_i \cdots a_j a) = \#_{01}(a a_j \cdots a_i a)\big]\\
&\quad= \sum_{k=1}^{N} \sum_{c'_1 \cdots c'_{k-1} c_k \cdots c_N} \sum_{c_1 \cdots c_k} \nu\big({}_{i-N}[c_N \cdots c_{k+1} c_k c'_{k-1} \cdots c'_1 1100 c_1 \\
&\qquad\qquad \cdots c_{k-1} c_k \bar{c}_{k+1} \cdots \bar{c}_N]_{i+3+N}\big)\\
&\qquad + \sum_{c_1 \cdots c_N} \nu\big({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N}\big)
\end{aligned}$$

and

$$\begin{aligned}
&\nu({}_i[1100]_{i+3})\\
&\quad= \sum_{k=1}^{N} \sum_{c_1 \cdots c_N} \sum_{c'_1 \cdots c'_{k-1}} \nu\big({}_{i-N}[c_N \cdots c_{k+1} c_k c_{k-1} \cdots c_1 1100 c'_1 \\
&\qquad\qquad \cdots c'_{k-1} c_k \bar{c}_{k+1} \cdots \bar{c}_N]_{i+3+N}\big)\\
&\qquad + \sum_{c_1 \cdots c_N} \nu\big({}_{i-N}[c_N c_{N-1} \cdots c_1 1100 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N}\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\nu({}_i[0011]_{i+3}) - \nu({}_i[1100]_{i+3})\\
&\quad= \sum_{c_1 \cdots c_N} \big\{\nu\big({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N}\big)\\
&\qquad\qquad - \nu\big({}_{i-N}[c_N c_{N-1} \cdots c_1 1100 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N}\big)\big\},
\end{aligned}$$

for every $N \in \mathbb{N}$.

If we set

$$A_{c_1 \cdots c_{N-1}} = \left\{ c_1' \cdots c_{N-1}' \in \{0,1\}^{N-1} : \sum_{k=1}^{N-1} c_k' = \sum_{k=1}^{N-1} \bar{c}_k, \right.$$
$$\left. \left| \sum_{k=2}^{N-1} |c_k' - c_{k-1}'| - \sum_{k=2}^{N-1} |c_k - c_{k-1}| \right| \le 2 \right\},$$

for each $c_1 \cdots c_{N-1} \in \{0,1\}^{N-1}$, then for every $c_1' \cdots c_{N-1}' \in A_{c_1 \cdots c_{N-1}}$

$$\nu({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_{n-1} \bar{c}_N]_{i+3+N})$$
$$\le \theta^2 \nu({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 c_1' c_2' \cdots c_{n-1}' \bar{c}_N]_{i+3+N})$$

by RCP, where $\theta = \gamma + \gamma^{-1}$. Hence we have

$$\# A_{c_1 \cdots c_{N-1}} \nu({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_{N-1} \bar{c}_N]_{i+3+N})$$
$$\le \theta^2 \sum_{c_1' \cdots c_{N-1}' \in A_{c_1 \cdots c_{N-1}}} \nu({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 c_1' c_2' \cdots c_{N-1}' \bar{c}_N]_{i+3+N})$$
$$\le \theta^2 \nu({}_{i-N}[c_N \cdots c_1]_{i-1}).$$

Thus

$$\nu({}_{i-N}[c_N c_{N-1} \cdots c_1 0011 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N})$$
$$\le \theta^2 (\# A_{c_1 \cdots c_{N-1}})^{-1} \nu({}_{i-N}[c_N \cdots c_1]_{i-1}),$$

for every $c_1 \cdots c_N \in \{0,1\}^N$, $N \in \mathbb{N}$. The same estimate holds for $\nu({}_{i-N}[c_N c_{N-1} \cdots c_1 1100 \bar{c}_1 \bar{c}_2 \cdots \bar{c}_N]_{i+3+N})$. Since $\# A_{c_1 \cdots c_{N-1}} \ge (N-1)/16$ provided that $\sum_{k=2}^{N-1} |c_k - c_{k-1}| > 0$, we have

$$|\nu({}_i[0011]_{i+3}) - \nu({}_i[1100]_{i+3})| \le 32\theta^2 (N-1)^{-1}$$
$$+ \nu({}_{i-N+1}[11 \cdots 100110 \cdots 00]_{i+N+2})$$
$$+ \nu({}_{i-N+1}[00 \cdots 000111 \cdots 11]_{i+N+2})$$
$$+ \nu({}_{i-N+1}[11 \cdots 111000 \cdots 00]_{i+N+2})$$
$$+ \nu({}_{i-N+1}[00 \cdots 011001 \cdots 11]_{i+N+2}).$$

Letting $N \to \infty$ and using the fact that $\nu(\cdots 111100110000 \cdots) = 0$ and so on, which follow from RCP, we have $\nu({}_i[0011]_{i+3}) = \nu({}_i[1100]_{i+3})$. □

PROPOSITION 3.1. *If a probability measure ν on $\mathscr{X}$ has the regular clustering property, then ν is translation-invariant.*

PROOF. Let us show first that if $a_i = a_j$ then $\nu({}_i[a_i a_{i+1} \cdots a_j]_j) = \nu({}_{i-1}[a_i a_{i+1} \cdots a_j]_{j-1})$. By RCP it suffices to check the case when

$\#_{01}(a_i \cdots a_j) \le 1$. If $a_i \cdots a_j = 00110$, for example,

$$\begin{aligned}\nu({}_i[00110]_{i+4}) &= \nu({}_{i-1}[000110]_{i+4}) + \nu({}_{i-1}[100110]_{i+4})\\ &= \nu({}_{i-1}[001100]_{i+4}) + \nu({}_{i-1}[001101]_{i+4})\\ &\qquad\qquad\text{(by RCP and Lemma 3.1)}\\ &= \nu({}_{i-1}[00110]_{i+3}).\end{aligned}$$

Thus we have $\nu({}_i[a_i \cdots a_j]_j) = \nu({}_{i-n}[a_i \cdots a_j]_{j-n})$ for all $n \in \mathbb{Z}$ provided $a_i = a_j$.

Now suppose $a_i \ne a_j$. If $a_i \cdots a_j = 0011$, for example, then

$$\begin{aligned}\nu({}_i[0011]_{i+3}) &= \sum_{k=0}^{\infty} \nu\Big({}_i[0011\underbrace{1\cdots 1}_{k}0]_{i+4+k}\Big)\\ &\quad + \nu(\{\eta \in \mathscr{X}\colon \eta_i\eta_{i+1} = 00,\ \eta_n = 1, \forall\ n \ge i+2\}),\end{aligned}$$

and the last term is 0 by RCP. Since $\nu({}_i[00111 \cdots 10]_{i+4+k}) = \nu({}_{i-1}[00111 \cdots 10]_{i+3+k})$ from the above result, and since $\nu(\{\eta \in \mathscr{X}\colon \eta_{i-1}\eta_i = 00, \eta_n = 1, \forall\ n \ge i+1\})$ is 0 by RCP, we have $\nu({}_i[0011]_{i+3}) = \nu({}_{i-1}[0011]_{i+2})$. Thus $\nu({}_i[a_i \cdots a_j]_j) = \nu({}_{i-1}[a_i \cdots a_j]_{j-1})$ even if $a_i \ne a_j$, which completes the proof. □

By Proposition 3.1 it is known that in the repulsive case, that is, in the case $\alpha > \beta$, $\mathscr{M}_{\beta/\alpha}$ is equal to the set $\mathscr{S}$ which appeared in Section 3 of Yaguchi (1986) [of course, $1-\alpha$ in that paper should be replaced by the present β/α; and by the way present q and q' are related to α, β and γ of that paper by $q = (1-\beta)(\beta\gamma/\alpha)$ and $q' = (1-\beta\gamma)(\beta/\alpha)$, respectively]. However, the proof given there does not cover the attractive case $\alpha < \beta$ directly, and so we give a proof which is rather simple and valid for the both cases.

LEMMA 3.2. *Suppose $\nu \in \operatorname{ext} \mathscr{M}_\gamma$ and $\nu \ne \delta_0, \delta_1$. Then there exist constants q and q' with $0 < q, q' < 1$ such that*

$$\nu([00a_i \cdots a_j])/\nu([0a_i \cdots a_j]) \equiv q = \nu([00])/\nu([0]),$$
$$\nu([a_i \cdots a_j11])/\nu([a_i \cdots a_j1]) \equiv q' = \nu([11])/\nu([1]),$$

for all $a_i \cdots a_j$.

(Note that ν is translation-invariant by Proposition 3.1, and hence it is not necessary for us to specify the coordinates of cylinders in the statement of the lemma.)

PROOF. (i) We have $\nu({}_i[a_i \cdots a_j]_j) > 0$ for all ${}_i[a_i \cdots a_j]_j$. In fact let

$$M_1 = \left\{\eta \in \mathscr{X}\colon \sum_{-1}^{-\infty} \eta_x = \sum_{-1}^{-\infty} (1-\eta_x) = \sum_{1}^{\infty} \eta_x = \sum_{1}^{\infty} (1-\eta_x) = \infty\right\},$$
$$M_2 = \mathscr{X} \setminus M_1.$$

Then $\nu(M_1) = 1$ since $\nu(M_2) = 0$ by the translation invariance of ν and the assumption that $\nu \neq \delta_0, \delta_1$. It is easy to see that the family of basic cylinders $\zeta \equiv {}_I[\zeta_I \cdots \zeta_J]_J$ satisfying $I < i < j < J$ and

$$\sum_{k=I+1}^{J-1} \zeta_k \geq \sum_{k=i}^{j} a_k \quad \text{and} \quad \sum_{k=I+1}^{J-1} (1 - \zeta_k) \geq \sum_{k=i}^{j} (1 - a_k)$$

is countable and covers M_1. Hence we can take a $\zeta' \equiv {}_{I'}[\zeta_{I'} \cdots \zeta_{J'}]_{J'}$ such that $\nu(\zeta') > 0$ among them. Since the configuration $a_i \cdots a_j$ on the sites $\{i, \ldots, j\}$ can be obtained from ζ' by rearranging the configuration $\zeta_{I'+1} \cdots \zeta_{J'-1}$ on the sites $\{I' + 1, \ldots, J' - 1\}$, it follows from RCP that $\nu({}_i[a_i \cdots a_j]_j) > 0$.

(ii) Now define a nonnegative translation-invariant function $\tilde{\lambda}$ on $\mathscr{C}$ as follows:

$$\tilde{\lambda}(\varnothing) = 0, \qquad \tilde{\lambda}([1]) = \nu([001]) + \gamma\nu([101]),$$

$$\tilde{\lambda}\big([0a_i \cdots a_j]\big) = \nu\big([00a_i \cdots a_j]\big),$$

$$\tilde{\lambda}\big([1 \cdots 10a_i \cdots a_j]\big) = \nu\big([1 \cdots 100a_i \cdots a_j]\big),$$

$$\tilde{\lambda}([\overbrace{11 \cdots 11}^{n}]) = \gamma\nu([10\overbrace{1 \cdots 11}^{n-1}]), \qquad n > 1.$$

It is easy to check that $\tilde{\lambda}$ satisfies the consistency condition of measures and $(\mathrm{RCP})_\gamma$. Hence we can extend $\tilde{\lambda}$ to a finite $(\mathrm{RCP})_\gamma$ measure λ on $\mathscr{X}$ uniquely. Further we have

$$\lambda\big([a_i \cdots a_j]\big) < \nu\big([a_i \cdots a_j]\big) \quad \text{for all } \varnothing \neq [a_i \cdots a_j] \in \mathscr{C}.$$

Indeed if $\min\{a_i, a_j\} = 0$ it is obvious from (i). If $a_i = a_j = 1$, we have, for example,

$$\begin{aligned}
\lambda([11 \cdots 11]) &= \gamma\nu([101 \cdots 11]) \\
&= \gamma\{\nu([0101 \ldots 11]) + \nu([01101 \cdots 11]) \\
&\qquad\qquad + \nu([011101 \cdots 11]) + \cdots\} \quad [\text{by } \nu(M_2) = 0] \\
&= \nu([0011 \cdots 11]) + \nu([00111 \cdots 11]) \\
&\qquad + \nu([001111 \cdots 11]) + \cdots \quad [\text{by } (\mathrm{RCP})_\gamma] \\
&< \nu([011 \cdots 11]) + \nu([0111 \cdots 11]) + \nu([01111 \cdots 11]) + \cdots \\
&\qquad [\text{by (i)}] \\
&= \nu([11 \cdots 11]) \quad [\text{by } \nu(M_2) = 0].
\end{aligned}$$

Therefore ν is represented by two $(\mathrm{RCP})_\gamma$ measures $\kappa_1 = (1 - \lambda(\mathscr{X}))^{-1}(\nu - \lambda)$ and $\kappa_2 = \lambda(\mathscr{X})^{-1}\lambda$ such as $\nu = (1 - \lambda(\mathscr{X}))\kappa_1 + \lambda(\mathscr{X})\kappa_2$ with $0 < \lambda(\mathscr{X}) < 1$, which implies $\kappa_1 = \kappa_2$ since $\nu \in \text{ext}\,\mathscr{M}_\gamma$. Then a direct computation gives us $\lambda = (\lambda(\mathscr{X})/\nu(\mathscr{X}))\nu$. Thus $q = \lambda(\mathscr{X})/\nu(\mathscr{X}) = \nu([00])/\nu([0])$. □

Now we can complete the proof of Theorem 2 as follows.

Proof of Theorem 2. It is clear that $\delta_0, \delta_1 \in \text{ext}\,\mathscr{M}_\gamma$. Suppose $\nu \in \text{ext}\,\mathscr{M}_\gamma$ and $\nu \neq \delta_0, \delta_1$; and let q and q' be those of Lemma 3.2. (Note that ν is translation-invariant by Proposition 3.1.) Then we have

$$1 = \nu([00]) + \nu([01]) + \nu([10]) + \nu([11])$$

$$= q\nu([0]) + 2(1-q')\nu([1]) + q'\nu([1]) \quad \text{(by Lemmas 3.1, and 3.2)},$$

and hence

$$(1-\rho)(1-q) = \rho(1-q'), \tag{3.1}$$

where $\rho = \nu([1])$. It follows from the translation invariance of ν that $0 < \rho < 1$. On the other hand,

$$qq'\nu([01]) = \nu([0011]) = \gamma\nu([0101])$$

$$= \gamma\{\nu([01]) - \nu([011]) - \nu([0100])\}$$

$$= \gamma\{\nu([01]) - q'\nu([01]) - [q\nu([01]) - qq'\nu([01])]\},$$

and hence

$$qq' = \gamma(1-q)(1-q'). \tag{3.2}$$

Here we have used the fact that $\nu([01]) > 0$ which follows from the assumption $\nu \neq \delta_0, \delta_1$ and the translation invariance of ν.

Let us see that (3.1) and (3.2) determine q and q' in (0. 1) uniquely. Set $x = q/(1-q)$ and $y = q'/(1-q')$. Then the equations become $xy = \gamma$ and $(1+x)/(1+y) = (1-\rho)/\rho$; and the problem is reduced to finding a positive solution x and y. If $(1-\rho)/\rho > 1$, we can easily find a unique positive y by eliminating x, which determines x automatically. To treat the case $(1-\rho)/\rho < 1$, we have only to exchange the role of x and y. Thus a pair q and q' is uniquely determined in (0, 1). Since (2.1) defines a translation-invariant measure $\mu_\rho^{(\gamma)}$ on $\mathscr{X}$ uniquely, we have $\text{ext}\,\mathscr{M}_\gamma \subset \{\mu_\rho^{(\gamma)}\}_{0 \le \rho \le 1}$.

It is easy to check that $\{\mu_\rho^{(\gamma)}\} \subset \mathscr{M}_\gamma$. Then to complete the proof, it is sufficient to show that $\{\mu_\rho^{(\gamma)}\}_{0<\rho<1}$ are mutually singular. For $\eta \equiv (\cdots \eta_{-1}\eta_0\eta_1 \cdots) \in \mathscr{X}$ define $X_0(\eta) = \min\{k \ge 0\colon \eta_k = 1\}$ and $X_n(\eta) = \min\{k > X_{n-1}(\eta)\colon \eta_k = 1\}$, $n \in \mathbb{N}$. Then X_n's are random variables on $(\mathscr{X}, \mu_\rho^{(\gamma)})$ such that $\{X_n - X_{n-1}\}_{n\in\mathbb{N}}$ are i.i.d. of which distribution $f_\rho(k) \equiv \mu_\rho^{(\gamma)}(\{\eta\colon X_n(\eta) - X_{n-1}(\eta) = k\})$, $k \in \mathbb{N}$, is given by

$$f_\rho(k) = \begin{cases} q' & \text{if } k = 1, \\ \gamma^{-1}q'q^{k-1} & \text{if } k > 1. \end{cases}$$

Since $\sum_{k=1}^{\infty} k f_\rho(k) = \rho^{-1}$, we have

$$\mu_\rho^{(\gamma)}\left\{\eta\colon \lim_{n\to\infty} \frac{1}{n}\min\left\{I\colon \sum_{i=0}^{I} \eta_i = n\right\} = \rho^{-1}\right\} = 1$$

by the law of large numbers, which implies

$$\mu_\rho^{(\gamma)}\left\{\eta\colon \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \eta_i = \rho\right\} = 1.$$

In the same way we have

$$\mu_\rho^{(\gamma)}\left\{\eta\colon \lim_{n\to\infty} \frac{1}{n}\sum_{i=-1}^{-n} \eta_i = \rho\right\} = 1.$$

These show us that $\{\mu_\rho^{(\gamma)}\}$ are mutually singular. □

4. The entropy analysis of exclusion process. In this section we will give the proof of Theorem 1. Before giving the proof let us prepare some notation. For $\eta = (\cdots \eta_{-1}\eta_0\eta_1 \cdots) \in \mathscr{X}$ define

$$\chi_{abcd}(\eta_{x-1}\eta_x\eta_{x+1}\eta_{x+2}) = \begin{cases} 1 & \text{if } \eta_{x-1}\eta_x\eta_{x-1}\eta_{x+2} = abcd, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$N(\eta_{x-1}\eta_x\eta_{x+1}\eta_{x+2}) = \big[\alpha(\chi_{0011} + \chi_{1011} + \chi_{1100} + \chi_{1101}) \\ +\beta(\chi_{0010} + \chi_{1010} + \chi_{0100} + \chi_{0101})\big](\eta_{x-1}\eta_x\eta_{x+1}\eta_{x+2}).$$

We sometimes write ${}_i[\zeta]_j$ for $\zeta \in \mathscr{C}_{i,j}$ to stress i and j and denote by $\zeta^{x,x+1}$ or ${}_i[\zeta]_j^{x,x+1}$, $i \le x \le j-1$, the set of configurations obtained from the elements of ${}_i[\zeta]_j$ by exchanging the values at the sites x and $x+1$. For $\zeta \equiv {}_i[\zeta]_j$ we denote by ζ_x, $i \le x \le j$, the value of ζ at the site x, and by ${}_{i-1}[\eta_{i-1}\zeta]_j$ the element of $\mathscr{C}_{i-1,j}$ such that the configuration from $i-1$ to j is $\eta_{i-1}\zeta_i \cdots \zeta_j$, and so on. We remark that if we set

$$\Gamma(x;\zeta) = \begin{cases} N(\zeta_{x-1}\zeta_x\zeta_{x+1}\zeta_{x+2})\nu(\zeta) & \text{if } i+1 \le x \le j-2, \\ 0 & \text{otherwise,} \end{cases}$$

for $\zeta \in \mathscr{C}_{i,j}$, then the regular clustering property is equivalent to the condition

$$\Gamma(x;\zeta) = \Gamma(x;\zeta^{x,x+1}) \quad \text{for all } x \in \mathbb{Z} \text{ and } \zeta \in \mathscr{C}. \tag{4.1}$$

Proof of the sufficiency part of Theorem 1. Since $\overline{\Omega}$ is a Markov generator, we have only to show $\int \Omega f(\eta)\, d\nu(\eta) = 0$ for every $\mathscr{B}_{i,j}$-measurable function f [see Liggett (1985), page 51]. We will show that $\int \Omega f_\zeta(\eta)\, d\nu(\eta) = 0$ for every $\zeta \equiv {}_i[\zeta_i \cdots \zeta_j]_j \in \mathscr{C}$, where f_ζ is the indicator function of ζ. Since

$$(\Omega f_\zeta)(\eta) = \sum_{x=i-1}^{j} N(\eta_{x-1}\eta_x\eta_{x+1}\eta_{x+2})\{f_\zeta(\eta^{x,x+1}) - f_\zeta(\eta)\},$$

we have

$$
\begin{aligned}
&\int (\Omega f_\zeta)(\eta)\, d\nu(\eta) \\
&\quad = \sum_{x=i+1}^{j-2} \{\Gamma(x; \zeta^{x,x+1}) - \Gamma(x;\zeta)\} \\
&\qquad + \sum_{\eta_{i-1}=0,1} \Big\{\Gamma\Big(i; {}_{i-1}[\eta_{i-1}\zeta]_j^{i,i+1}\Big) - \Gamma\big(i; {}_{i-1}[\eta_{i-1}\zeta]_j\big)\Big\} \\
&\qquad + \sum_{\eta_{j+1}=0,1} \Big\{\Gamma\Big(j-1; {}_i\big[\zeta\eta_{j+1}\big]_{j+1}^{j-1,j}\Big) - \Gamma\Big(j-1; {}_i\big[\zeta\eta_{j+1}\big]_{j+1}\Big)\Big\} \\
(4.2)\qquad &\qquad + \sum_{\eta_{i-2}=0,1} \sum_{\eta_{i-1}=0,1} \Big\{\Gamma\Big(i-1; {}_{i-2}[\eta_{i-2}\eta_{i-1}\zeta]_j^{i-1,i}\Big) \\
&\qquad\qquad\qquad - \Gamma\big(i-1; {}_{i-2}[\eta_{i-2}\eta_{i-1}\zeta]_j\big)\Big\} \\
&\qquad + \sum_{\eta_{j+1}=0,1} \sum_{\eta_{j+2}=0,1} \Big\{\Gamma\Big(j; {}_i\big[\zeta\eta_{j+1}\eta_{j+2}\big]_{j+2}^{j,j+1}\Big) - \Gamma\Big(j; {}_i\big[\zeta\eta_{j+1}\eta_{j+2}\big]_{j+2}\Big)\Big\} \\
&\quad \equiv S_1(i,j) + S_2(i,j) + S_3(i,j) + S_4(i,j) + S_5(i,j) \\
&\quad = 0 \quad [\text{by } (4.1)],
\end{aligned}
$$

which was to be shown. □

The following proposition shows that if the set $\{\mathbf{0}, \mathbf{1}\}$ does not have full measure for $\nu \in \mathscr{S}$ then $\nu(\zeta) > 0$ for every basic cylinder ζ.

PROPOSITION 4.1. *Suppose $\nu \in \mathscr{S}$ and $\nu(\{\mathbf{0}, \mathbf{1}\}) < 1$. Then $\nu(\zeta) > 0$ for all nonempty $\zeta \in \mathscr{C}$.*

We divide the proof into two steps. Given ${}_i[\zeta_i \cdots \zeta_j]_j$ let us write ${}_I[\zeta_I' \cdots \zeta_J']_J \rhd {}_i[\zeta_i \cdots \zeta_j]_j$ if $I \le i \le j \le J$ and if for some x with $i-1 \le x \le j$ the restriction of ${}_I[\zeta_I' \cdots \zeta_J']_J^{x,x+1}$ to the sites $\{i, \dots, j\}$ is $\zeta_i \cdots \zeta_j$. The meaning is that the configuration $\zeta_i \cdots \zeta_j$ on $\{i, \dots, j\}$ is obtained from $\zeta_I' \cdots \zeta_J'$ by the one-step transition of a particle. We will write $\zeta' \rhd\rhd \zeta$ if there is a $\rhd$-chain from ζ' to ζ: $\zeta' \rhd \zeta^{(1)} \rhd \cdots \rhd \zeta^{(k)} \rhd \zeta$.

LEMMA 4.1. *If $\nu \in \mathscr{S}$ and $\nu(\zeta) = 0$ for some $\zeta \equiv {}_i[\zeta_i \cdots \zeta_j]_j$, then $\nu(A_\zeta) = 1$, where*

$$
A_\zeta = \left\{\eta \in \mathscr{X}: \sum_{k=-\infty}^{\infty} \eta_k < \sum_{k=i}^{j} \zeta_k \quad or \quad \sum_{k=-\infty}^{\infty} (1-\eta_k) < \sum_{k=i}^{j} (1-\zeta_k)\right\}.
$$

PROOF. Since $\nu \in \mathscr{S}$, we have $\int (\Omega f_\zeta)(\eta)\, d\nu(\eta) = 0$. Then by (4.2) and by the assumption that $\nu(\zeta) = 0$ it holds that $\nu(\zeta') = 0$ for all ζ' satisfying

$\zeta' \rhd \rhd \zeta$ as well as ζ' satisfying $\zeta' \rhd \zeta$. Since

$$A_\zeta^c = \left\{\eta \in \mathscr{X}: \sum_{k=-\infty}^{\infty} \eta_k \geq \sum_{k=i}^{j} \zeta_k \quad \text{and} \quad \sum_{k=-\infty}^{\infty} (1-\eta_k) \geq \sum_{k=i}^{j} (1-\zeta_k)\right\}$$

$$\subset \bigcup_{n=0}^{\infty} \left\{\eta \in \mathscr{X}: \sum_{k=i-n}^{j+n} \eta_k \geq \sum_{k=i}^{j} \zeta_k \quad \text{and} \quad \sum_{k=i-n}^{j+n} (1-\eta_k) \geq \sum_{k=i}^{j} (1-\zeta_k)\right\}$$

$$\equiv \bigcup_{n=0}^{\infty} F_n(\zeta)$$

and since $\zeta' \rhd \rhd \zeta$ for every basic cylinder ζ' in $F_n(\zeta)$, the lemma follows. □

Proof of Proposition 4.1. It is obvious that δ_0 and δ_1 are elements of $\mathscr{S}$. Hence without loss of generality we can assume that $\nu(\{\mathbf{0}, \mathbf{1}\}) = 0$ (consider the measure $[\nu(\cdot) - \nu(\{\mathbf{0}, \mathbf{1}\})]/[\nu(\mathscr{X}) - \nu(\{\mathbf{0}, \mathbf{1}\})]$ instead of ν). Then suppose that the conclusion is false, that is, $\nu(\zeta) = 0$ for some $\zeta \equiv {}_i[\zeta_i \cdots \zeta_j]_j \neq \varnothing$. Let A_ζ be the set introduced in the above lemma. It is clear that A_ζ is an infinite countable set (indeed every η' with $\eta' \rhd \eta$ for some $\eta \in A_\zeta$ is an element of A_ζ), which does not contain $\mathbf{0}$ and $\mathbf{1}$ by the assumption. As ν is a probability measure there exists $\max\{(\beta/\alpha)^{\#_{01}(\eta)}\nu(\eta)\colon \eta \in A_\zeta\}$, which is positive and is attained by the finite elements of A_ζ, say, $\{\eta^{(1)}, \dots, \eta^{(l)}\} \subset A_\zeta$. Since the proof is the same, we assume $l = 1$ and will write η^* instead of $\eta^{(1)}$ for simplicity. We will treat only the case that $\eta^*(\in A_\zeta)$ satisfies $\sum_{k=-\infty}^{\infty} \eta_k^* < \sum_{k=i}^{j} \zeta_k$.

Let $B_n = {}_{-n}[\eta_{-n}^* \eta_{-n+1}^* \cdots \eta_n^*]_n$. Since $\nu \in \mathscr{S}$, it follows that $\int (\Omega f_{B_n})(\eta)\, d\nu(\eta) = 0$ for all $n \in \mathbb{N}$, and hence we have

$$0 = \sum_{x=-\infty}^{\infty} \left\{N(\eta_{x-1}^* \eta_{x+1}^* \eta_x^* \eta_{x+2}^*)\nu\left((\eta^*)^{x,x+1}\right) \right.$$
$$\left. - N(\eta_{x-1}^* \eta_x^* \eta_{x+1}^* \eta_{x+2}^*)\nu((\eta^*))\right\}$$

by (4.2). However, the r.h.s. of the above equation, which is a finite summation over x essentially, is negative because the equation

$$N(\eta_{x-1}^* \eta_{x+1}^* \eta_x^* \eta_{x+2}^*)(\alpha/\beta)^{\#_{01}((\eta^*)^{x,x+1})}$$
$$= N(\eta_{x-1}^* \eta_x^* \eta_{x+1}^* \eta_{x+2}^*)(\alpha/\beta)^{\#_{01}(\eta^*)}$$

holds for all $x \in \mathbb{N}$, and η^* is by assumption the unique element which attains the maximum of $(\beta/\alpha)^{\#_{01}(\eta)}\nu(\eta)$, $\eta \in A_\zeta$. This is a contradiction. □

Now let us begin the proof of the necessity part of Theorem 1.

Suppose ν and μ are probability measures on $\mathscr{X}$ and that $\mu(\zeta) > 0$ for all nonempty $\zeta \in \mathscr{C}$. *The entropy of* ν *on* $T_n \equiv \{-n, -n+1, \dots, n\}$ *relative to* μ

is defined as

$$\begin{aligned}H_{T_n}(\nu) &= \sum_{\zeta\in\mathscr{C}_{-n,n}} \nu(\zeta)\log(\nu(\zeta)/\mu(\zeta))\\ &= \sum_{\zeta\in\mathscr{C}_{-n,n}} \nu(\zeta)\log\nu(\zeta) - \sum_{\zeta\in\mathscr{C}_{-n,n}} \nu(\zeta)\log\mu(\zeta).\end{aligned}$$

Let ν_t, $t > 0$, be the probability measure on $\mathscr{X}$ defined by

$$\nu_t(f) = \int d\nu(\eta)(S_t f)(\eta), \qquad f\in C(\mathscr{X}),$$

where S_t is the semigroup corresponding to $\overline{\Omega}$ (of course, $\nu_0 = \nu$). What we want to do in the following is to investigate the "monotonicity" of $H_{T_n}(\nu_t)$ as a function of t when μ is taken from ext $\mathscr{M}_{\beta/\alpha}$ the structure of which is completely known by Theorem 2. Hence we fix here one $\mu_\rho(\neq \delta_0, \delta_1)$ from ext $\mathscr{M}_{\beta/\alpha}$ and denote it by μ instead of μ_ρ for short. It is easy from Theorem 2 to find a positive constant K_μ such that for every ${}_i[\zeta_i \cdots \zeta_j]_j \in \mathscr{C}$, $i<j$,

$$\begin{aligned}&\left|\log\{\mu(\zeta^{x,x+1})/\mu(\zeta)\}\right| < K_\mu, \qquad x\in\{i, i+1,\dots,j-1\},\\ (4.3)\qquad &\left|\log\left\{\mu\left({}_i\left[\bar{\zeta}_i\zeta_{i+1}\cdots\zeta_j\right]_j\right)/\mu\left({}_i\left[\zeta_i\zeta_{i+1}\cdots\zeta_j\right]_j\right)\right\}\right| < K_\mu,\end{aligned}$$

where $\bar{\zeta}_i = 1-\zeta_i$.

In order to investigate the relative entropy of ν_t we differentiate $H_{T_n}(\nu_t)$ at time $t=0$:

$$\begin{aligned}\frac{d}{dt}\left(\sum_{\zeta\in\mathscr{C}_{-n,n}} \nu_t(\zeta)\log\nu_t(\zeta)\right)_{t=0} &= \sum_\zeta (1+\log\nu(\zeta))\int(\Omega f_\zeta)(\eta)\,d\nu(\eta)\\ &= \sum_\zeta (\log\nu(\zeta))[S_1(-n,n)+S_2(-n,n)+S_3(-n,n)\\ &\qquad +S_4(-n,n)+S_5(-n,n)],\end{aligned}$$

where $S_k(i,j)$, $k=1,\dots,5$, are those which appeared in (4.2); and

$$\begin{aligned}\frac{d}{dt}\left(\sum_{\zeta\in\mathscr{C}_{-n,n}} \nu_t(\zeta)\log\mu(\zeta)\right)_{t=0} &= \sum_\zeta (\log\mu(\zeta))\int(\Omega f_\zeta)(\eta)\,d\nu(\eta)\\ &= \sum_\zeta (\log\mu(\zeta))[S_1(-n,n)+S_2(-n,n)+S_3(-n,n)\\ &\qquad +S_4(-n,n)+S_5(-n,n)].\end{aligned}$$

For simplicity we set

$$S_k = \sum_{\zeta} (\log \nu(\zeta)) S_k(-n,n) \quad \text{and} \quad T_k = \sum_{\zeta} (\log \mu(\zeta)) S_k(-n,n),$$

$$k = 1, \dots, 5.$$

We will evaluate these S_k and T_k in the following four lemmas.

LEMMA 4.2. *Suppose $\nu(\zeta) > 0$ for every nonempty $\zeta \in \mathscr{C}$. Then*

$$S_1 - T_1 = -\frac{1}{2} \sum_{x=-n+1}^{n-2} \sum_{\zeta \in \mathscr{C}_{-n,n}} \phi\big(\Gamma(x;\zeta^{x,x+1}), \Gamma(x;\zeta)\big) \le 0,$$

where

$$\phi(u,v) = (u-v)\log(u/v), \qquad u,v > 0.$$

PROOF. We have

$$\begin{aligned} S_1 &= \frac{1}{2} \sum_{x=-n+1}^{n-2} \sum_{\zeta} (\log \nu(\zeta))\big(\Gamma(x;\zeta^{x,x+1}) - \Gamma(x;\zeta)\big) \\ &\quad + \frac{1}{2} \sum_{x=-n+1}^{n-2} \sum_{\zeta} (\log \nu(\zeta^{x,x+1}))\big(\Gamma(x;\zeta) - \Gamma(x;\zeta^{x,x+1})\big) \\ &= -\frac{1}{2} \sum_{x=-n+1}^{n-2} \sum_{\zeta} \left(\log \frac{\nu(\zeta^{x,x+1})}{\nu(\zeta)}\right)\big(\Gamma(x;\zeta^{x,x+1}) - \Gamma(x;\zeta)\big) \end{aligned}$$

and similarly

$$T_1 = -\frac{1}{2} \sum_{x=-n+1}^{n-2} \sum_{\zeta} \left(\log \frac{\mu(\zeta^{x,x+1})}{\mu(\zeta)}\right)\big(\Gamma(x;\zeta^{x,x+1}) - \Gamma(x;\zeta)\big).$$

Then the lemma is immediate from the equation

$$N(\zeta_{x-1}\zeta_{x+1}\zeta_x\zeta_{x+2})\mu(\zeta^{x,x+1}) = N(\zeta_{x-1}\zeta_x\zeta_{x+1}\zeta_{x+2})\mu(\zeta),$$

which follows from the RCP of μ [cf. (4.1)]. □

For $n \in \mathbb{N}$ let

$$\begin{aligned} U^-(n) &= \sum_{\xi \in \mathscr{C}_{-n-1,n}} \big|\Gamma(-n;\xi^{-n,-n+1}) - \Gamma(-n;\xi)\big|, \\ U^+(n) &= \sum_{\xi \in \mathscr{C}_{-n,n+1}} \big|\Gamma(n-1;\xi^{n-1,n}) - \Gamma(n-1;\xi)\big|, \\ V^-(n) &= \sum_{\xi \in \mathscr{C}_{-n-2,n}} \big|\Gamma(-n-1;\xi^{-n-1,-n}) - \Gamma(-n-1;\xi)\big|, \\ V^+(n) &= \sum_{\xi \in \mathscr{C}_{-n,n+2}} \big|\Gamma(n;\xi^{n,n+1}) - \Gamma(n;\xi)\big|. \end{aligned}$$

LEMMA 4.3. *$|T_2| \le 2^{-1}K_\mu U^-(n)$, $|T_3| \le 2^{-1}K_\mu U^+(n)$, $|T_4| \le 2^{-1}K_\mu V^-(n)$ and $|T_5| \le 2^{-1}K_\mu V^+(n)$.*

PROOF. Just as in the proof of the above lemma we have

$$T_2 = -\frac{1}{2}\sum_{\eta_{-n-1}}\sum_{\zeta\in\mathscr{C}_{-n,n}}\left(\log\frac{\mu(\zeta^{-n,-n+1})}{\mu(\zeta)}\right) \times\left\{\Gamma\left(-n;\, {}_{-n-1}[\eta_{-n-1}\zeta]_n^{-n,-n+1}\right) - \Gamma\left(-n;\, {}_{-n-1}[\eta_{-n-1}\zeta]_n\right)\right\}.$$

Then the first half of the lemma is immediate from (4.3). Since

$$\begin{aligned} T_4 &= \sum_{\eta_{-n-2}\eta_{-n-1}}\sum_{\zeta_{-n}}\sum_{\zeta'\in\mathscr{C}_{-n+1,n}}\left(\log\mu({}_{-n}[\zeta_{-n}\zeta']_n)\right) \\ &\qquad\times\left\{\Gamma\left(-n-1;\, {}_{-n-2}[\eta_{-n-2}\eta_{-n-1}\zeta_{-n}\zeta']_n^{-n-1,-n}\right)\right. \\ &\qquad\qquad\left.-\Gamma\left(-n-1;\, {}_{-n-2}[\eta_{-n-2}\eta_{-n-1}\zeta_{-n}\zeta']_n\right)\right\} \\ &= -\frac{1}{2}\sum_{\eta_{-n-2}}\sum_{\zeta_{-n}}\sum_{\zeta'\in\mathscr{C}_{-n+1,n}}\left(\log\frac{\mu\left({}_{-n}[\bar\zeta_{-n}\zeta']_n\right)}{\mu\left({}_{-n}[\zeta_{-n}\zeta']_n\right)}\right) \\ &\qquad\times\left\{\Gamma\left(-n-1;\, {}_{-n-2}[\eta_{-n-2}\zeta_{-n}\bar\zeta_{-n}\zeta']_n\right)\right. \\ &\qquad\qquad\left.-\Gamma\left(-n-1;\, {}_{-n-2}[\eta_{-n-2}\bar\zeta_{-n}\zeta_{-n}\zeta']_n\right)\right\}, \end{aligned}$$

it follows from (4.3) that $|T_4| \le 2^{-1}K_\mu V^-(n)$. The same estimates hold for T_3 and T_5, respectively. □

LEMMA 4.4. *If $\nu(\zeta) > 0$ for every nonempty $\zeta \in \mathscr{C}$, then*

$$S_2 \le 2^{-1}MU^-(n) + 2(\alpha+\beta)(M+1)e^{-(M-1)},$$

$$S_3 \le 2^{-1}MU^+(n) + 2(\alpha+\beta)(M+1)e^{-(M-1)}$$

for every $M \in \mathbb{N}$.

PROOF. Just as in the proof of the above lemma we have

$$\begin{aligned} S_2 &= -\frac{1}{2}\sum_{\eta_{-n-1}}\sum_{\zeta\in\mathscr{C}_{-n,n}}\left(\log\frac{\nu(\zeta^{-n,-n+1})}{\nu(\zeta)}\right) \\ &\qquad\times\left\{\Gamma\left(-n;\, {}_{-n-1}[\eta_{-n-1}\zeta]_n^{-n,-n+1}\right) - \Gamma\left(-n;\, {}_{-n-1}[\eta_{-n-1}\zeta]_n\right)\right\} \\ &= -\frac{1}{2}\sum_{\eta_{-n-1}}\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M)} - \frac{1}{2}\sum_{\eta_{-n-1}}\sum_{k=0}^{\infty}\sum_{\varepsilon=+,-}\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,\varepsilon}, \end{aligned}$$

where $M \in \mathbb{N}$ and $\Sigma^{(M)}_{\zeta\in\mathscr{C}_{-n,n}}$, $\Sigma^{(M),k,+}_{\zeta\in\mathscr{C}_{-n,n}}$ and $\Sigma^{(M),k,-}_{\zeta\in\mathscr{C}_{-n,n}}$ are summations over ζ

satisfying

$$
(4.4)\qquad
\begin{aligned}
e^{-M} &< \nu(\zeta^{-n,n+1})/\nu(\zeta) < e^{M},\\
e^{M+k} &\le \nu(\zeta^{-n,n+1})/\nu(\zeta) < e^{M+k+1},\\
e^{-(M+k+1)} &< \nu(\zeta^{-n,-n+1})/\nu(\zeta) \le e^{-(M+k)},
\end{aligned}
$$

respectively. By $\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,++}$ we denote the partial sum of $\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,+}$ over ζ satisfying

$$
\Gamma\big(-n;\,_{-n-1}[\eta_{-n-1}\zeta]_n^{-n,-n+1}\big) < \Gamma(-n;\,_{-n-1}[\eta_{-n-1}\zeta]_n).
$$

Then

$$
\begin{aligned}
&-\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,+}\left(\log\frac{\nu(\zeta^{-n,-n+1})}{\nu(\zeta)}\right)\\
&\qquad\times\left\{\Gamma\big(-n;\,_{-n-1}[\eta_{-n-1}\zeta]_n^{-n,-n+1}\big) - \Gamma(-n;\,_{-n-1}[\eta_{-n-1}\zeta]_n)\right\}\\
&\quad\le \sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,++}(M+k+1)\\
&\qquad\times\left\{\Gamma(-n;\,_{-n-1}[\eta_{-n-1}\zeta]_n) - \Gamma\big(-n;\,_{-n-1}[\eta_{-n-1}\zeta]_n^{-n,-n+1}\big)\right\}\\
&\quad\le \sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,++}(M+k+1)(\alpha+\beta)\nu(_{-n-1}[\zeta_{-n-1}\cdots\zeta_n]_n)\\
&\quad\le (\alpha+\beta)(M+k+1)e^{-(M+k)}\sum_{\zeta\in\mathscr{C}_{-n,n}}\nu(\zeta^{-n,-n+1}) \quad [\text{by (4.4)}]\\
&\quad= (\alpha+\beta)(M+k+1)e^{-(M+k)},
\end{aligned}
$$

which implies that

$$
-\frac{1}{2}\sum_{\eta_{-n-1}}\sum_{k=0}^{\infty}\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,+} \le (\alpha+\beta)(M+1)e^{-(M-1)}.
$$

Here we have used $\sum_{k=0}^{\infty}(M+k+1)e^{-(M+k)} \le (M+1)e^{-(M-1)}$. Since the same estimate holds for $\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M),k,-}$ and since

$$
\begin{aligned}
\left|\sum_{\eta_{-n-1}}\sum_{\zeta\in\mathscr{C}_{-n,n}}^{(M)}\right| &\le M\sum_{\xi\in\mathscr{C}_{-n-1,n}}\left|\Gamma(-n;\xi^{-n,-n+1}) - \Gamma(-n;\xi)\right|\\
&\equiv MU^{-}(n),
\end{aligned}
$$

we obtain the first half of the lemma. □

Lemma 4.5. *If* $\nu(\zeta) > 0$ *for every nonempty* $\zeta\in\mathscr{C}$, *then*

$$
\begin{aligned}
S_4 &\le 2^{-1}MV^{-}(n) + 4(\alpha+\beta)(M+1)e^{-(M-1)},\\
S_5 &\le 2^{-1}MV^{+}(n) + 4(\alpha+\beta)(M+1)e^{-(M-1)},
\end{aligned}
$$

for every $M\in\mathbb{N}$.

PROOF. Since the proof is the same as that of the above lemma except for

$$S_4 = -\frac{1}{2}\sum_{\eta_{-n-2}}\sum_{\eta_{-n-1}}\sum_{\zeta\in\mathscr{C}_{-n,n}}\left(\log\frac{\nu({}_{-n}[\eta_{-n-1}\zeta_{-n+1}\cdots\zeta_n]_n)}{\nu({}_{-n}[\zeta_{-n}\zeta_{-n+1}\cdots\zeta_n]_n)}\right)$$
$$\times\{\Gamma(-n-1;{}_{-n-2}[\eta_{-n-2}\eta_{-n-1}\zeta]_n^{-n-1,-n}) - \Gamma(-n-1;{}_{-n-2}[\eta_{-n-2}\eta_{-n-1}\zeta]_n)\},$$

which is obtained in the same way as for T_4 in Lemma 4.3, we omit details. □

For integers i, j and x with $i+1\le x\le j-2$, define

$$E_{i,j}(x) = \sum_{\zeta\in\mathscr{C}_{i,j}}\phi(\Gamma(x;\zeta^{x,x+1}),\Gamma(x;\zeta)),$$

where $\phi(\cdot,\cdot)$ is the function defined in Lemma 4.2. We remark that

(4.5) $\quad$ if $I\le i$ and $J\ge j$ then $E_{I,J}(x)\ge E_{i,j}(x)(\ge 0)$,

which follows from the subadditivity of $\phi(u,v)$, $u,v>0$, that is,

(4.6) $$\phi(u_1,v_1)+\phi(u_2,v_2)\ge\phi(u_1+u_2,v_1+v_2),$$

and the fact that

$$\sum_{\eta_I\cdots\eta_{i-1}}\sum_{\eta_{j+1}\cdots\eta_J}\Gamma\left(x;{}_I[\eta_I\cdots\eta_{i-1}\zeta\eta_{j+1}\cdots\eta_J]_J^{x,x+1}\right)=\Gamma(x;\zeta^{x,x+1}),$$

and so on.

LEMMA 4.6. *Suppose $\nu\in\mathscr{S}$ and $\nu(\zeta)>0$ for every nonempty $\zeta\in\mathscr{C}$. Then*

(4.7)
$$\begin{aligned}0&\le 2^{-1}\sum_{x=-n+1}^{n-2}E_{-n,n}(x)\\&\le 2^{-1}(K_\mu+M)(U^-(n)+U^+(n)+V^-(n)+V^+(n))\\&\quad+12(\alpha+\beta)(M+1)e^{-(M-1)},\end{aligned}$$

for every $M\in\mathbb{N}$. In particular, $\lim_{n\to\infty}\sum_{x=-n+1}^{n-2}E_{-n,n}(x)$ exists, and

(4.8)
$$\begin{aligned}\lim_{x\to\infty}E_{-x-1,x}(-x)&=\lim_{x\to\infty}E_{-x,x+1}(x-1)=0,\\ \lim_{x\to\infty}E_{-x-2,x}(-x-1)&=\lim_{x\to\infty}E_{-x,x+2}(x)=0.\end{aligned}$$

PROOF. Since $\nu\in\mathscr{S}$, $((d/dt)H_{T_n}(\nu_t))_{t=0}$ must be 0, that is,

$$-(S_1-T_1)=S_2+S_3+S_4+S_5-(T_2+T_3+T_4+T_5).$$

Then combining Lemmas 4.2 to 4.5 yields (4.7). If M is taken to be 1 in (4.7), we have

$$0 \le \sum_{x=-n+1}^{n-2} E_{-n,n}(x) < (8K_\mu + 56)(\alpha + \beta),$$

since $|U^{\pm}(n)| \le 2(\alpha + \beta)$ and $|V^{\pm}(n)| \le 2(\alpha + \beta)$. Hence the existence of $\lim_{n\to\infty}\sum_{x=-n+1}^{n-2} E_{-n,n}(x)$ is assured by (4.5). The latter half of the lemma is then clear from

$$\sum_{x=-n+1}^{n-2} E_{-n,n}(x) \ge \sum_{x=1}^{n-1} \{E_{-x-1,x}(-x) + E_{-x,x+1}(x-1)\} \ge 0,$$

$$\sum_{x=-n+1}^{n-2} E_{-n,n}(x) \ge \sum_{x=1}^{n-2} \{E_{-x-2,x}(-x-1) + E_{-x,x+2}(x)\} \ge 0. \qquad \square$$

PROOF OF THE NECESSITY PART OF THEOREM 1. Since δ_0 and δ_1 are elements of $\mathscr{S}$ and have RCP, it is sufficient for the proof of necessity to show that every $\nu \in \mathscr{S}$ with $\nu(\{\mathbf{0}, \mathbf{1}\}) = 0$ satisfies (4.1).

By Proposition 4.1 we have $\nu(\zeta) > 0$ for every nonempty $\zeta \in \mathscr{C}$. For integers i, j and x with $i < x < j - 1$ set

$$R \equiv R_{i,j}(x) = \sum_{\zeta \in \mathscr{C}_{i,j}} \max\{\Gamma(x;\zeta^{x,x+1}), \Gamma(x,\zeta)\},$$

$$r \equiv r_{i,j}(x) = \sum_{\zeta \in \mathscr{C}_{i,j}} \min\{\Gamma(x;\zeta^{x,x+1}), \Gamma(x,\zeta)\},$$

and note that

$$R - r = \sum_{\zeta \in \mathscr{C}_{i,j}} \left|\Gamma(x;\zeta^{x,x+1}) - \Gamma(x;\zeta)\right|.$$

Then by the equality $\phi(u, v) = \phi(v, u)$ and (4.6)

$$E_{i,j}(x) \ge \phi(R, r) \ge (R - r)^2/R \ge (R - r)^2/(\alpha + \beta),$$

which is the inequality similar to that of Lemma 4.5.8 in Liggett (1985); in particular,

$$E_{-x-1,x}(-x) \ge U^-(x)^2/(\alpha + \beta), \qquad E_{-x,x+1}(x-1) \ge U^+(x)^2/(\alpha + \beta),$$

$$E_{-x-2,x}(-x-1) \ge V^-(x)^2/(\alpha + \beta), \qquad E_{-x,x+2}(x) \ge V^+(x)^2/(\alpha + \beta),$$

for $x = 1, 2, \ldots$. Then by (4.8)

$$\lim_{x\to\infty} U^{\pm}(x) = \lim_{x\to\infty} V^{\pm}(x) = 0.$$

Therefore letting $n \to \infty$ in (4.7) yields

$$0 \le \lim_{n\to\infty} \lim_{x=-n+1}^{n-2} E_{-n,n}(x) \le 24(\alpha + \beta)(M + 1)e^{-(M-1)}.$$

As $M \in \mathbb{N}$ is arbitrary it follows that

$$\lim_{n\to\infty} \sum_{x=-n+1}^{n-2} E_{-n,n}(x) = 0,$$

and hence

$$E_{-n,n}(x) = 0 \quad \text{for every } x \in \{-n+1, -n+2, \dots, n-2\}, n \in \mathbb{N},$$

by (4.5). Therefore for every $\zeta \in \mathscr{C}_{-n,n}$, $n \ge 2$,

$$\Gamma(x; \zeta^{x,x+1}) = \Gamma(x, \zeta), \qquad x \in \{-n+1, \dots, n-2\},$$

which establishes (4.1). □

5. An extension to finite range interactive exclusion process. The proofs given in Section 4 can be applied to the analysis of finite range interactive exclusion processes (speed change model) on $\mathbb{Z}^1$ after a minor change.

Let L be a positive integer and $p(x,y)$ be transition probabilities for a discrete-time irreducible Markov chain on $\mathbb{Z}^1$ with the property

$$(5.1)\quad p(x,y) = p(y,x) = p(0,|x-y|) \quad \text{and} \quad C_p \equiv \sum_{k=1}^{\infty} kp(0,k) < \infty.$$

For a function $c(x,y;\eta)$: $\mathbb{Z}^1 \times \mathbb{Z}^1 \times \mathscr{X} \to (0,\infty)$ satisfying

$$0 < c(x,y;\eta) \le c_0 \quad \text{for all } x, y \text{ and } \eta,$$

$$c(x,y;\cdot) \text{ is } \mathscr{B}_{x-L,x+L} \vee \mathscr{B}_{y-L,y+L}\text{-measurable},$$

consider an exclusion process on $\mathscr{X}$ defined by

$$(\Omega f)(\eta) = \sum_{\substack{x<y \\ \{x,y\}\cap[i,j]\neq\varnothing}} p(x,y)c(x,y;\eta)[f(\eta^{x,y}) - f(\eta)],$$

for $\mathscr{B}_{i,j}$-measurable functions f. Then we can prove the following result (a sketch of the proof is given in the Appendix).

PROPOSITION 5.1. *Suppose there exists a stationary measure μ for the above process such that $\mu(\zeta) > 0$ for every nonempty $\zeta \in \mathscr{C}$ and*

(i) *there is a constant $K_\mu > 0$ such that for all $\zeta \in \mathscr{C}_{i,j}$, $i \le j$,*

$$\left|\log\frac{\mu(\zeta^{x,y})}{\mu(\zeta)}\right| \le K_\mu, \qquad i+L \le x < y \le j-L,$$

$$\left|\log\frac{\mu\left({}_i[\zeta_i \cdots \zeta_{x-1}\bar{\zeta}_x\zeta_{x+1} \cdots \zeta_j]_j\right)}{\mu\left({}_i[\zeta_i \cdots \zeta_{x-1}\zeta_x\zeta_{x+1} \cdots \zeta_j]_j\right)}\right| \le K_\mu, \qquad i \le x \le j;$$

(ii) *for all $\zeta \in \mathscr{C}_{i,j}$, $i \le j$,*

$$\mu(\zeta^{x,y})c(x,y;\zeta^{x,y}) = \mu(\zeta)c(x,y;\zeta),$$

$$i+L \le x < y < j-L \qquad (\textit{reversibility condition}).$$

Then a probability measure ν on $\mathscr{X}$ is stationary for the process if and only if it satisfies (ii) *replacing μ by ν.*

As an example of c and μ having the above property we can give an exclusion process associated with a Gibbs state relative to a finite range pair potential $\{J_0, J_1, \dots, J_L\}$ (we put $J_{L+k} = 0$, $k \in \mathbb{N}$, for convenience):

$$c(x,y;\eta) = \exp\left[(1/kT)\left\{\eta_x \sum_u J_{|x-u|}\eta_u + \eta_y \sum_u J_{|y-u|}\eta_u\right\}\right],$$

μ = a probability measure on $\mathscr{X}$ whose conditional probability$\mu\{_i[a_i \cdots a_j]_j | \mathscr{B}_{i,j}^c\}(\eta)$ is equal to

$$\Xi_{i,j}(\eta)^{-1}\exp\left[-(1/kT)\left\{\sum_{i\le k\le k'\le j} J_{k'-k}a_k a_{k'} + \sum_{u<i\le k\le j} J_{k-u}\eta_u a_k + \sum_{i\le k\le j<u} J_{u-k}a_k\eta_u\right\}\right],$$

where $\Xi_{i,j}(\eta)$ is a normalizing constant.

However, we do not know at present the way of determining the structure of a corresponding family of measures satisfying condition (ii).

The above proposition shows us that if the coefficient $c(x,y;\eta)$ is the same, then so is the totality of stationary measures regardless of the arbitrariness of $p(x,y)$ having the property (5.1). And so just as exchangeable measures for simple exclusion processes our $\mathscr{M}_\gamma$ can become a family of stationary measures for various nearest-neighbor interactive exclusion processes. For example, we have the following.

Proposition 5.2. *Let*

$$c(x,y;\eta) = \gamma^{-\{\chi_{11}(\eta_{x-1}\eta_x)+\chi_{11}(\eta_x\eta_{x+1})+\chi_{11}(\eta_{y-1}\eta_y)+\chi_{11}(\eta_y\eta_{y+1})\}}$$

in the above. Then the set of stationary measures for the corresponding exclusion process is equal to $\mathscr{M}_\gamma$ for every $p(x,y)$ with (5.1).

Remark 5.1. Our argument essentially relies upon the derivation of the form

$$-\frac{1}{2}\sum_{-n+L\le x<y\le n-L} p(x,y)\sum_{\zeta\in\mathscr{C}_{-n,n}} \phi(\Gamma(x,y;\zeta^{x,y}),\Gamma(x,y;\zeta)) \le 0$$

(cf. Lemma 4.2), where

$$\phi(u,v) = (u-v)\log(u/v) \quad \text{and} \quad \Gamma(x,y;\zeta) = c(x,y;\zeta)\nu(\zeta),$$

and does not go through if the symmetricity of $p(x,y)$ in (5.1) does not hold [if $p(x,y) \ne p(y,x)$ for some x and y]. However, we think that the entropy analysis is still effective for that asymmetric case.

APPENDIX

In this Appendix we will sketch the proof of Proposition 5.1. Since the proof is almost the same as that of Theorem 1 except for the existence of additional probabilities $p(x, y)$, we will in the following mainly be concerned with the crucial part of the proof.

It is easy to check that if a probability measure ν on $\mathscr{X}$ satisfies condition (ii), then it is stationary for the process. Indeed if we set

$$\Gamma(x,y;\zeta) = \begin{cases} c(x,y;\zeta)\nu(\zeta) & \text{if } i+L \le x \neq y \le j-L, \\ 0 & \text{otherwise,} \end{cases}$$

condition (ii) is equivalent to

$$\Gamma(x,y;\zeta) = \Gamma(x,y;\zeta^{x,y}) \quad \text{for all } x, y \text{ and } \zeta.$$

Using the dominated convergence theorem, we have for $\zeta \in \mathscr{C}_{i,j}$,

$$\begin{aligned}
&\int (\Omega f_\zeta)(\eta)\, d\nu(\eta) \\
&= \sum_{x<y} p(x,y) \int c(x,y;\eta)\big[f_\zeta(\eta^{x,y}) - f_\zeta(\eta)\big]\, d\nu(\eta) \\
&= \sum_{i+L\le x<y\le j-L} p(x,y)\{\Gamma(x,y;\zeta^{x,y}) - \Gamma(x,y;\zeta)\} \\
&\quad + \sum_{\substack{x<y\le j-L \\ x<i+L,\, i\le y}} p(x,y) \sum_{\xi\in\mathscr{C}_{x-L,i-1}} \big\{\Gamma\big(x,y;(\xi\zeta)^{x,y}\big) - \Gamma(x,y;\xi\zeta)\big\} \\
&\quad + \sum_{\substack{i+L\le x<y \\ x\le j,\, j-L<y}} p(x,y) \sum_{\xi'\in\mathscr{C}_{j+1,j+L}} \big\{\Gamma\big(x,y;(\zeta\xi')^{x,y}\big) - \Gamma(x,y;\zeta\xi')\big\} \\
&\quad + \sum_{\substack{x<i+L,\, j-L<y \\ \{x,y\}\cap[i,j]\neq\varnothing}} p(x,y) \\
&\qquad\qquad \times \sum_{\xi\in\mathscr{C}_{x-L,i-1}} \sum_{\xi'\in\mathscr{C}_{j+1,j+L}} \big\{\Gamma\big(x,y;(\xi\zeta\xi')^{x,y}\big) - \Gamma(x,y;\xi\zeta\xi')\big\} \\
&\equiv G_1(i,j) + G_2(i,j) + G_3(i,j) + G_4(i,j) \\
&= 0,
\end{aligned}$$

where we have denoted $\xi \cap \zeta$ by $\xi\zeta$ and so on.

Next for a probability measure ν on $\mathscr{X}$ define H_{T_n} and ν_t as in Section 4. Then we have

$$\frac{d}{dt}\big(H_{T_n}(\nu_t)\big)_{t=0} = \sum_{k=1}^{4} S_k^{\sim} - \sum_{k=1}^{4} T_k^{\sim},$$

where

$$S_k^{\sim} = \sum_{\zeta\in\mathscr{C}_{-n,n}} (\log \nu(\zeta)) G_k(i,j) \quad \text{and} \quad T_k^{\sim} = \sum_{\zeta\in\mathscr{C}_{-n,n}} (\log \mu(\zeta)) G_k(i,j).$$

For $n \in \mathbb{N}$, $n > L$, let

$$\Lambda_n^- = \{(x,y): x < y,\ x < -n + L,\ -n \le y \le n - L\},$$
$$\Lambda_n^+ = \{(x,y): x < y,\ -n + L \le x \le n,\ n - L < y\},$$
$$\Lambda_n^\pm = \{(x,y): x < -n + L,\ n - L < y,\ \{x,y\} \cap [-n,n] \ne \varnothing\},$$
$$\Lambda_n^* = \{(x,y): -n + L \le x < y \le n - L\},$$

and set

$$W^-(n) = \sum_{\Lambda_n^-} p(x,y) D_{x-L,n}(x,y),$$
$$W^+(n) = \sum_{\Lambda_n^+} p(x,y) D_{-n,y+L}(x,y),$$
$$W^\pm(n) = \sum_{\Lambda_n^\pm} p(x,y) D_{x-L,y+L}(x,y),$$

and

$$Z^*(n) = \sum_{\Lambda_n^*} p(x,y) E_{-n,n}(x,y),$$
$$Z^-(n) = \sum_{\Lambda_n^-} p(x,y) E_{x-L,n}(x,y),$$
$$Z^+(n) = \sum_{\Lambda_n^+} p(x,y) E_{-n,y+L}(x,y),$$
$$Z^\pm(n) = \sum_{\Lambda_n^\pm} p(x,y) E_{x-L,y+L}(x,y),$$

where

$$D_{i,j}(x,y) = \sum_{\xi \in \mathscr{C}_{i,j}} |\Gamma(x,y;\xi^{x,y}) - \Gamma(x,y;\xi)|,$$
$$E_{i,j}(x,y) = \sum_{\xi \in \mathscr{C}_{i,j}} \phi(\Gamma(x,y;\xi^{x,y}), \Gamma(x,y;\xi)),$$

for i, j and x, y with $i + L \le x < y \le j - L$. We remark that W^-, W^+ and $W^\pm$ are bounded with respect to n. In fact we have $D_{x-L,n}(x,y) \le 2c_0$ by (5.1) and hence

$$W^-(n) \le 2c_0 \sum_{\substack{x<y \\ x<L,\ 0\le y}} p(x,y) \le 2c_0 L \sum_{x<0\le y} p(x,y) = 2c_0 L C_p,$$

and so on. Then just like Lemmas 4.2 to 4.5 we can prove

$$S_1^\sim - T_1^\sim = -2^{-1} Z^*(n),$$
$$|T_2^\sim| \le 2^{-1} K_\mu W^-(n), \qquad |T_3^\sim| \le 2^{-1} K_\mu W^+(n), \qquad |T_4^\sim| \le 2^{-1} K_\mu W^\pm(n),$$

and

$$S_2^{\sim} \leq 2^{-1}MW^-(n) + 2^{3L+1}(M+1)e^{-(M-1)}c_0 LC_p,$$
$$S_3^{\sim} \leq 2^{-1}MW^+(n) + 2^{3L+1}(M+1)e^{-(M-1)}c_0 LC_p,$$
$$S_4^{\sim} \leq 2^{-1}MW^{\pm}(n) + 2^{3L+1}(M+1)e^{-(M-1)}c_0 C_p,$$

for every $M \in \mathbb{N}$. Therefore if ν is stationary for the process, it holds that

$$\begin{aligned} 0 &\leq 2^{-1}Z^*(n) \\ &\leq 2^{-1}(K_\mu + M)\big(W^-(n) + W^+(n) + W^{\pm}(n)\big) \\ &\quad + 3 \cdot 2^{3L+1}(M+1)e^{-(M-1)}c_0 LC_p \end{aligned}$$

[cf. (4.7)]. In particular, we have $\lim_{n\to\infty} Z^*(n) < \infty$, which implies

$$\lim_{n\to\infty} Z^-(n) = \lim_{n\to\infty} Z^+(n) = \lim_{n\to\infty} Z^{\pm}(n) = 0.$$

Indeed we have only to use the fact $\lim_{m\to\infty}(Z^*(m+n) - Z^*(n)) \geq Z^-(n)$.

To complete the proof of Proposition 5.1, it is sufficient to show that the fact $\lim_{n\to\infty} Z^-(n) = 0$ implies $\lim_{n\to\infty} W^-(n) = 0$ since the rest of the proof is almost the same as that of the necessity part of Theorem 1. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{l_k\}_{k\in\mathbb{N}}$ be sequences of positive numbers satisfying

$$\varepsilon_k \downarrow 0, \qquad l_k \uparrow \infty \quad \text{and} \quad \sum_{l > l_k} lp(0,l) < \varepsilon_k.$$

Using the inequality

$$E_{x-L,n}(x,y) \geq D_{x-L,n}(x,y)^2 \Bigg\{ \sum_{\xi \in \mathscr{C}_{x-L,n}} \big|\Gamma(x,y;\xi^{x,y}) + \Gamma(x,y;\xi)\big| \Bigg\}^{-1},$$

it follows from $Z^-(n) \to 0$ that

$$\lim_{n\to\infty} \sum_{\Lambda_n^-} p(x,y) D_{x-L,n}(x,y)^2 = 0,$$

and hence

$$\lim_{n\to\infty} \sum_{\Lambda_n^- \cap \{y-x\leq l_k\}} p(x,y) D_{x-L,n}(x,y) = 0,$$

for every fixed l_k since $\#\{(x,y) \in \Lambda_n^-: y - x \leq l_k\}$ is bounded with respect to n. Dividing the summation in $W^-(n)$ into $\Lambda_n^- \cap \{y - x \leq l_k\}$ and $\Lambda_n^- \cap \{y - x > l_k\}$, we have

$$0 \leq \limsup_{n\to\infty} W^-(n) \leq 2c_0 \sum_{\substack{x<y,\, x<L,\, 0\leq y \\ y-x>l_k}} p(x,y) \leq 2c_0 L\varepsilon_k,$$

and consequently $\lim_{n\to\infty} W^-(n) = 0$.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HIROSHIMA UNIVERSITY
HIROSHIMA 730
JAPAN