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INDUCED DIRICHLET FORMS AND CAPACITARY INEQUALITIES

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A Dirichlet form on a large (complicated or multidimensional) space may be carried over onto a small (simple or one-dimensional) space. Here conditions are given ensuring the induced form is regular. A capacitary inequality between the two forms allows one to estimate the probability of a large deviation on the large space by that on the small space. Also asymptotically sharp results are derived in a one-dimensional setting.

0. Introduction. Let $x(\cdot)$ be a reversible Markov process on a state space X, with respect to a stationary measure m; a more precise description of the setting is given below [consult Fukushima (1980) as a general reference]. In this article among other things we consider the problem of estimating $P_m(\sup_{t \in [0,T]} F(x(t)) > l)$, where $F\colon X \to \mathbb{R}$ is some given function. Note that in general $F(x(\cdot))$ is not itself Markovian. For the sake of clarity we will work in the framework of general Dirichlet forms.

We consider a Dirichlet form $\mathscr{E}$ on $\mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \subset L^2(X; m) \times L^2(X; m)$, where X is a locally compact, separable Hausdorff space and m is a positive Radon measure on $\mathscr{B}$, the Borel σ-algebra on X, with $\operatorname{supp}[m] = X$ [see Fukushima (1980) for definitions]. Define

$$(0.1)\qquad \mathscr{E}_\theta(u,v) := \mathscr{E}(u,v) + \theta(u,v), \qquad u,v \in \mathscr{D}(\mathscr{E}),$$

where $(\cdot,\cdot) \equiv (\cdot,\cdot)_m$ denotes the inner product on $L^2(X; m)$, and for A open

$$(0.2)\qquad \operatorname{Cap}_\theta(A) := \begin{cases} \inf\{\mathscr{E}_\theta(u,u) \colon u \in D_A\} & \text{if } D_A \neq \varnothing, \\ \infty & \text{if } D_A = \varnothing, \end{cases}$$

where

$$(0.3)\qquad D_A = \{u \in \mathscr{D}(\mathscr{E}) \colon u \geq 1 \ m\text{-a.e. on } A\}$$

($\operatorname{Cap}_\theta$ is called the θ-capacity).

As is shown in Lemma 3.1.1 in Fukushima (1980) the infimum in (0.2) is attained by a unique element $e_A \equiv e_{A;\theta} \in \mathscr{D}(\mathscr{E})$; and moreover

$$(0.4)\qquad \operatorname{Cap}_\theta(A) = \mathscr{E}_\theta(e_A, 1).$$

If $\mathscr{E}$ is regular the form may be associated with a Hunt process $x(\cdot)$ on X [see Theorems (6.2.1) and (4.3.6) of Fukushima (1980)]. In this case, assuming

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that m is a probability measure and that there is no killing,

$$\text{Cap}_\theta(A) = \theta E_m e^{-\theta\tau} \quad \text{and} \quad e_{A;\theta}(x) = E_x e^{-\theta\tau} \quad m\text{-a.e.}, \tag{0.5}$$

where $\tau \doteq \inf\{t \geq 0: x(t) \in A\}$.

In order to estimate θ-capacities we will consider a proper map F: $X \to X^*$ onto a (smaller) locally compact, separable, Hausdorff space X^* [see Dugundji (1966), Chapter 11, Section 5]. Let $\mathscr{B}^*$ denote the Borel σ-algebra of X^* and let m^* denote the image of m under F; evidently $\text{supp}[m^*] = X^*$ and m^* is Radon. In general anything associated with X^* will carry an asterisk-superscript.

We may construct a form $\mathscr{E}^*$ on $\mathscr{D}(\mathscr{E}^*) \times \mathscr{D}(\mathscr{E}^*) \subset L^2(X^*; m^*) \times L^2(X^*; m^*)$ by

$$\mathscr{E}^*(u, v) := \mathscr{E}(u \circ F, v \circ F), \qquad u, v \in \mathscr{D}(\mathscr{E}^*), \tag{0.6}$$

where

$$\mathscr{D}(\mathscr{E}^*) = \{u \in L^2(X^*; m^*) | u \circ F \in \mathscr{D}(\mathscr{E})\}. \tag{0.7}$$

It will be shown in Proposition (1.4) under a mild condition on the conditional expectation operator $E_m[\cdot|F]$ that $\mathscr{E}^*$ is a Dirichlet form. If we suppose in addition that it is regular then we can associate with it a certain X^*-valued Markov process, say $x^*(\cdot)$. If $F(x(\cdot))$ happens to be Markovian then $\mathscr{E}^*$ is its associated Dirichlet form [see Rogers and Pitman (1981) for conditions implying $F(x(\cdot))$ is Markov]. Note if A^* is open in X^* and $A = F^{-1}(A^*)$ then we can consider the two first hitting times $\tau \equiv \tau_A$ and $\tau^* = \tau_{A^*}$. As will be shown in Proposition (1.12), x^* is conservative if x is conservative, which we are presently assuming. We then have the following capacitary inequality.

METATHEOREM (0.8). *Under the previous assumptions, if the Dirichlet form* $\mathscr{E}^*$ *defined at* (0.6) *and* (0.7) *is regular, then*

$$\text{Cap}_\theta(A) = \theta E_m e^{-\theta\tau} \leq \theta E_{m^*} e^{-\theta\tau^*} = \text{Cap}_\theta^*(A^*). \tag{0.9}$$

PROOF. Let $u(r) = E_r e^{-\theta\tau^*}$, $u \in D_{A^*}$, $u = 1$, m^*-a.e. $r \in A^*$, which implies $u \circ F \in D_A$, $u \circ F = 1$, m-a.e. $x \in A$, i.e., $u \circ F \in D_A$.

$$\begin{aligned}\theta E_{m^*} e^{-\theta\tau^*} &= \text{Cap}_\theta^*(A^*) \\ &= \mathscr{E}^*(u, u) + \theta(u, u)_{m^*} = \mathscr{E}(u \circ F, u \circ F) + \theta(u \circ F, u \circ F)_m \\ &\geq \min\{\mathscr{E}_\theta(v, v) | v \in D_A\} \\ &= \text{Cap}_\theta(A) = \theta E_m e^{-\theta\tau}.\end{aligned}$$

□

We refer to (0.8) as a "metatheorem" because it seems difficult to find a practical condition to impose on F which would yield the regularity of $\mathscr{E}^*$. This difficulty is circumvented in Section 1 by imposing some extra mild conditions on the conditional expectation operator $E_m[\cdot|F]$ and then restricting the domain of $\mathscr{E}^*$. In effect, the maximal domain of $\mathscr{E}^*$, as described at

(0.7), may simply be too large for $\mathscr{E}^*$ to be regular. Note that the outer inequality in (0.9) does not require regularity of $\mathscr{E}^*$ when A^* is open.

Returning to (0.5), we can then estimate, in the case where $A = F^{-1}(A^*)$,

$$\begin{aligned} P_m(\tau \le 1) &\le e^{\theta} E_m e^{-\theta\tau} \\ &= \theta^{-1} e^{\theta} \operatorname{Cap}_\theta(A) \\ &\le \theta^{-1} e^{\theta} \operatorname{Cap}^*_\theta(A^*) \\ &\le \theta^{-1} e^{\theta} \min\{\mathscr{E}^*_\theta(u,u) | u \in D_{A^*}\} \\ &\le \theta^{-1} e^{\theta} \mathscr{E}^*_\theta(v,v), \end{aligned} \tag{0.10}$$

for any suitable choice of $v \in D_{A^*}$. In the case where $X^* \subseteq \mathbb{R}$ we have transferred our original problem to one for a one-dimensional process. In particular when $A^* = (l, \infty)$ we are addressing the problem stated in the opening paragraph.

The layout of the remainder of this article is as follows. In Section 1 we drop the assumption that there is no killing and by introducing the notion of a *core* map F we study a suitably restricted form $\mathscr{E}^*$. In particular we show that if $x(\cdot)$ is an m-symmetric diffusion (resp. jump process) then so is $x^*(\cdot)$; and if $x(\cdot)$ is conservative then so is $x^*(\cdot)$. A (practical) extension of Metatheorem (0.8) is then given which includes some killing terms. As an example, a more concrete expression is derived for the form $\mathscr{E}^*$ in the case of a general conservative m-symmetric diffusion $x(\cdot)$ and proper map F: $X \to I$, I a subinterval of $\mathbb{R}$. The derivation relies quite heavily on the stochastic calculus of such diffusions.

In Section 2 we return to the problem of estimating the right-hand side of (0.10) for $x(\cdot) \in \Xi$, a certain class of classical reversible diffusions on $X = \mathbb{R}^n$ with $F = |\cdot|$ and $A^* = (l, \infty)$, i.e., $A = B^c(0; l) \equiv \{x \in \mathbb{R}^n | |x| > l\}$. A suitable choice of $v \in D_{A^*}$ is made, based on a generalization of some calculations in Newell (1962) and results in Section 3. Upper estimates are given for $\operatorname{Cap}_\theta(A)$ in this case [cf. Corollaries (2.19) and (2.21)].

In Section 3 we estimate $\operatorname{Cap}^*_\theta((l,\infty))$ both above and below and show that $v^* = v$, given by (3.4) in Proposition (3.2), is an asymptotically optimal test function. Under certain conditions [see Proposition (3.19)] we show $\lim_{l\to\infty} \nu(l)\operatorname{Cap}^*_\theta((l,\infty)) = 1$, where ν is defined in Proposition (3.2). It follows that

$$\lim_{l\to\infty} \nu(l) P_m(\tau^* \le T) = T.$$

This generalizes and makes precise (by identifying the dominant term) the asymptotics as $l \to \infty$ of (3.23a) in Newell (1962).

1. Induced Dirichlet forms. Let X, X^* be locally compact, separable, Hausdorff spaces, m a positive Radon measure on $\mathscr{B}$, the Borel σ-algebra on X, with $\operatorname{supp}[m] = X$, and F: $X \to X^*$ a proper map. We denote by $\mathscr{B}^*$ the Borel σ-algebra of X^*, and the image of m under F by m^*; evidently supp$[m^*] = X^*$. In general anything associated with X^* will carry an asterisk-superscript in this section.

Denote the σ-algebra generated by F by $\sigma(F)$, and the projection operator from $L^2(X, \mathscr{B}, m)$ to $L^2(X, \sigma(F), m)$ by $\mathscr{F}$; $\mathscr{F} \equiv E_m[\cdot|F]$ in case m is a probability measure. A straightforward localization argument yields the following lemma which is well known in case m is a probability measure. We omit the proof.

LEMMA (1.1).

(i) $\int_B \mathscr{F}u\,dm = \int_B u\,dm$, $\forall\, B \in \sigma(F)$, $u \in L^2(X, \mathscr{B}, m)_+$.

(ii) $(\mathscr{F}u)^2 \le \mathscr{F}u^2$, *m-a.e.*, $\forall\, u \in L^2(X, \mathscr{B}, m)$.

(iii) $u, v \in L^2(X, \mathscr{B}, m)$ *and* $u \le v$, *m-a.e.* $\Rightarrow \mathscr{F}u \le \mathscr{F}v$, *m-a.e.*

(iv) *If* $u \in L^2(X, \mathscr{B}, m)$ *then* $\mathscr{F}u = u^* \circ F$ *for some Borel-measurable* u^*.

(v) $u \in L^2(X, \mathscr{B}, m)$ *and* $v \in L^\infty(X^*, \mathscr{B}^*, m^*) \Rightarrow \mathscr{F}(u \cdot [v \circ F]) = [v \circ F] \cdot \mathscr{F}u$, *m-a.e.*

Let $\mathscr{E}$ *be a Dirichlet form in* $L^2(X; m)$. F *induces a form* $\mathscr{E}_0^*$ *in* $L^2(X^*; m^*)$ *by*

$$\mathscr{E}_0^*(u^*, v^*) = \mathscr{E}(u^* \circ F, v^* \circ F), \qquad u^*, v^* \in \mathscr{D}(\mathscr{E}_0^*), \tag{1.2}$$

where

$$\mathscr{D}(\mathscr{E}_0^*) = \{u^* \in L^2(X^*, m^*) | u^* \circ F \in \mathscr{D}(\mathscr{E})\}. \tag{1.3}$$

PROPOSITION (1.4). *If* $\mathscr{F}(\mathscr{D}) \subset \mathscr{D}(\mathscr{E})$ *where* $\mathscr{D}$ *is some* L^2*-dense subset of* $\mathscr{D}(\mathscr{E})$, *then* $\mathscr{E}_0^*$ *is a Dirichlet form which is local if* $\mathscr{E}$ *is.*

PROOF. That $\mathscr{E}_0^*$ is a positive semidefinite, symmetric, Markovian bilinear form on $\mathscr{D}(\mathscr{E}_0^*) \times \mathscr{D}(\mathscr{E}_0^*)$ is evident. We show that $\mathscr{D}(\mathscr{E}_0^*)$ is dense in $L^2(X^*; m^*)$ and that $\mathscr{E}_0^*$ is closed.

Let $\varepsilon > 0$ and $u^* \in L^2(X^*; m^*)$ be given; so that $u^* \circ F \in L^2(X; m)$. Choose $u \in \mathscr{D}$ such that $(u - u^* \circ F, u - u^* \circ F)_m < \varepsilon$, and set $\mathscr{F}u = u_1^* \circ F$ as in Lemma (1.1)(iv). Then by Lemma (1.1)(i), (ii),

$$\begin{aligned}(u_1^* - u^*, u_1^* - u^*)_{m^*} &= \int_X [u_1^* \circ F - u^* \circ F]^2\,dm \\ &= \int_X (\mathscr{F}[u - u^* \circ F])^2\,dm \\ &\le \int_X \mathscr{F}\big([u - u^* \circ F]^2\big)\,dm \\ &= \int_X [u - u^* \circ F]^2\,dm \\ &< \varepsilon.\end{aligned} \tag{1.5}$$

Also if $(u_n^*)_{n \in \mathbb{N}}$ is $(\mathscr{E}_0^*)_1$-Cauchy then $(u_n^* \circ F)_{n \in \mathbb{N}}$ is $\mathscr{E}_1$-Cauchy, hence convergent to some $u \in \mathscr{D}(\mathscr{E})$. Clearly u is $\sigma(F)$-measurable, so $u =$

$u^* \circ F$ for some $u^* \in L^2(X^*; m^*)$. By definition $u^* \in \mathscr{D}(\mathscr{E}_0^*)$ and $\|u_n^* - u^*\|_1 = \|u_n \circ F - u\|_1 \to 0$, as $n \to \infty$. Thus $\mathscr{E}_0^*$ is closed.

Finally if $\mathscr{E}$ is local and $u^*, v^* \in \mathscr{D}(\mathscr{E}_0^*)$ with disjoint compact supports then $u^* \circ F, v^* \circ F \in \mathscr{D}(\mathscr{E})$ with disjoint compact supports since $\operatorname{supp}[u \circ F] \subset F^{-1}(\operatorname{supp}[u])$. Therefore $\mathscr{E}_0^*(u^*, v^*) = \mathscr{E}(u^* \circ F, v^* \circ F) = 0$. □

Suppose in addition $\mathscr{E}$ is regular. It seems difficult to find a practical condition to impose on F which would guarantee that $\mathscr{E}_0^*$, as defined at (1.2) and (1.3), is also regular. To circumvent this problem, we will restrict the original domain $\mathscr{D}(\mathscr{E}_0^*)$ and impose the following condition on F.

DEFINITION (1.6). Let $C^* \subset C_0(X^*)$ be a dense subspace and C_1 a core for $\mathscr{E}$. Set $C_1^* = \{u^* \in C^*: u^* \circ F \in \mathscr{D}(\mathscr{E})\}$. We say that $F: X \to X^*$ is a *core* map if for each $u \in C_1$,

(i) there exists a continuous version of $\mathscr{F}u$, still denoted by $\mathscr{F}u$, with $\mathscr{F}u \in \mathscr{D}(\mathscr{E})$;

(ii) $\mathscr{F}u = u^* \circ F$ with $u^* \in C^*$; u^* is denoted by $\mathscr{F}^*u$.

Thus a core map F induces two maps $\mathscr{F}: C_1 \to C(X) \cap \mathscr{D}(\mathscr{E})$ and $\mathscr{F}^*: C_1 \to C_1^*$. In light of Lemma (1.1)(iv), a measurable version of u^* certainly exists so that $\mathscr{F}u = u^* \circ F$, m-a.e. The content of (1.6)(ii) is that a version $u^* \in C^* \subset C_0(X^*)$ exists. In some cases when X is compact then (1.6)(ii) follows from (1.6)(i) as we now show.

LEMMA (1.7). *Let F: $X \to X^*$ be a proper map such that* (1.6)(i) *holds. Then for $u \in C_1$, $\mathscr{F}u = u^* \circ F$ with $u^* \in C(X^*)$. Hence if X is compact and $C^* = C(X^*)$ then* (1.6)(ii) *holds.*

PROOF. If $\mathscr{F}u = u^* \circ F$, with u^* Borel, and $E \subset \mathbb{R}$ is a closed subset, then $(u^*)^{-1}(E) = F((\mathscr{F}u)^{-1}(E))$ is closed since $u^{-1}(E)$ is closed and F is a closed map. □

Let $\mathscr{E}$ be a regular Dirichlet form, C_1, C^* and C_1^* as in Definition (1.6), and F a proper core map [relative to (C_1, C^*)]. Define

$$\mathscr{E}_p^*(u^*, v^*) = \mathscr{E}(u^* \circ F, v^* \circ F), \qquad u^*, v^* \in \mathscr{D}(\mathscr{E}_p^*) = C_1^*.$$

THEOREM (1.8). *The form $\mathscr{E}_p^*$ is closable and its closure, $\mathscr{E}^*$, is a regular Dirichlet form with C_1^* serving as a core. Moreover $\mathscr{D}(\mathscr{E}^*) \subset \mathscr{D}(\mathscr{E}_0^*) = \{u^* \in L^2(X^*; m^*): u^* \circ F \in \mathscr{D}(\mathscr{E})\}$. If $A^* \subset X^*$ is open and $A := F^{-1}(A^*)$ then*

$$\operatorname{Cap}_\theta(A) \le \operatorname{Cap}_\theta^*(A^*). \tag{1.9}$$

Furthermore, if $\mathscr{E}$ has the local property then so does $\mathscr{E}^*$.

PROOF. That $\mathscr{E}_p^*$ is a positive semidefinite, symmetric, Markovian bilinear form is evident. The proof that $\mathscr{D}(\mathscr{E}_p^*) \equiv C_1^*$ is dense in $L^2(X^*; m^*)$ follows as in Proposition (1.4) with u now chosen from C_1 which implies, by our hypothesis, that $u_1^* \equiv \mathscr{F}^* u \in C_1^*$.

Next we show that C_1^* is $\|\cdot\|_\infty$-dense in $C_0(X^*)$. Let $u^* \in C_0(X^*)$; then $u^* \circ F \in C_0(X)$ as F is proper. Since C_1 is a core for $\mathscr{E}$, given $\varepsilon > 0$ there exists a $u_1 \in C_1$ such that $\|u_1 - u^* \circ F\|_\infty < \varepsilon$. Set $u_1^* = \mathscr{F}^* u_1 \in C_1^*$. Then

$$\begin{aligned}\|u_1^* - u^*\|_\infty &= \sup_{x \in X} |u_1^* \circ F(x) - u^* \circ F(x)| \\ &= \operatorname*{ess\,sup}_{x \in X} |\mathscr{F}(u_1 - u^* \circ F)(x)| \\ &\leq \operatorname*{ess\,sup}_{x \in X} \mathscr{F}(|u_1(x) - u^* \circ F(x)|) \\ &\leq \operatorname*{ess\,sup}_{x \in X} \mathscr{F}(\varepsilon) = \varepsilon.\end{aligned}$$

That $\mathscr{E}_p^*$ is closable follows easily since $\mathscr{E}$ is closed; we omit the straightforward notational details. Let $\mathscr{E}^*$ denote the closure of $\mathscr{E}_p^*$.

By construction C_1^* is $\mathscr{E}_1^*$-dense in $\mathscr{D}(\mathscr{E}^*)$, the $(\mathscr{E}_p^*)_1$-completion of C_1^* in $L^2(X^*; m^*)$, and we have already seen that C_1^* is $\|\cdot\|_\infty$-dense in $C_0(X^*)$. Thus $\mathscr{E}^*$ is regular with C_1^* serving as a core. The preservation of the Markovian property, in passing from a closable Markovian symmetric form (here $\mathscr{E}_p^*$) to its closure is a result of a general nature—see Theorem 2.1.1 of Fukushima (1980). Also $\mathscr{D}(\mathscr{E}^*) \subset \mathscr{D}(\mathscr{E}_0)$ because $\mathscr{E}_0^*$ is a closed extension of $\mathscr{E}_p^*$, by Proposition (1.4), and $\mathscr{E}^*$ is the smallest closed extension of $\mathscr{E}_p^*$. The local property of $\mathscr{E}^*$ then follows as in Proposition (1.4).

Finally, the capacitary inequality follows as in the proof of Metatheorem (0.8). □

The Beurling–Deny representation for a regular Dirichlet form $\mathscr{E}$ [see Theorem 2.2.1 and Lemma 4.5.4 in Fukushima (1980)] is the decomposition

$$\begin{aligned}\mathscr{E}(u,v) &= \mathscr{E}^{(c)}(u,v) + \int_{X \times X \setminus d} [u(x) - u(y)][v(x) - v(y)] J(dx, dy) \\ &\quad + \int_X u(x) v(x) k(dx), \qquad u, v \in \mathscr{D}(\mathscr{E}) \cap C_0(X).\end{aligned} \tag{1.10}$$

Here $\mathscr{E}^{(c)}$ is a symmetric form with domain $\mathscr{D}(\mathscr{E}^{(c)}) = \mathscr{D}(\mathscr{E})$ which satisfies the property

$$\mathscr{E}^{(c)}(u,v) = 0 \quad \text{if } u, v \in \mathscr{D}(\mathscr{E}) \text{ have compact support and } v \text{ is constant on a neighbourhood of } \operatorname{supp}[u]. \tag{1.11}$$

J is a symmetric positive Radon measure on $X \times X \setminus d$, d being the diagonal; and k is a positive Radon measure on X. The form $\mathscr{E}^{(c)}$ and measures J and k are uniquely determined by $\mathscr{E}$; $\mathscr{E}^{(c)}$ is called the diffusion part of $\mathscr{E}$, and J and k are called the *jump* measure and the *killing* measure, respectively, associated with $\mathscr{E}$. The representation at (1.10) is also valid for general $u, v \in \mathscr{D}(\mathscr{E})$ if one replaces u and v with quasicontinuous modifications $\tilde{u}$ and $\tilde{v}$, respectively, in the integral terms.

COROLLARY (1.12). *The form $\mathscr{E}^*$ of Theorem* (1.8) *has Beurling–Deny representation*

$$\begin{aligned}\mathscr{E}^*(u^*, v^*) &= \mathscr{E}^{*(c)}(u^*, v^*)\\ &\quad+\int_{X^*\times X^*\setminus d^*}[u^*(r) - u^*(s)][v^*(r) - v^*(s)]J^*(dr, ds) \qquad (1.13)\\ &\quad+\int_{X^*} u^*(r)v^*(r)k^*(dr), \qquad u^*, v^* \cap \mathscr{D}(\mathscr{E}^*) \cap C_0(X^*),\end{aligned}$$

where $\mathscr{E}^{*(c)}(u^*, v^*) = \mathscr{E}^{(c)}(u^* \circ F, v^* \circ F)$, $J^* = \bar{J} \circ \hat{F}^{-1}$ restricted to $X^* \times X^* \setminus d^*$, where $\hat{F}(x, y) = (F(x), F(y)) \in X^* \times X^*$ for $x, y \in X$ and $\bar{J}$ is the zero extension of J to $X \times X$, and $k^* = k \circ F^{-1}$. Also the representation at (1.13) *remains valid for $u^*, v^* \in \mathscr{D}(\mathscr{E}^*)$ upon the replacement of u^* and v^* in the integral terms by quasicontinuous modifications $\tilde{u}^*$ and $\tilde{v}^*$, respectively.*

PROOF. The representation of (1.13) is evident since $u^* \in \mathscr{D}(\mathscr{E}^*) \cap C_0(X^*)$ [resp. $\mathscr{D}(\mathscr{E}^*)$] implies that $u^* \circ F \in \mathscr{D}(\mathscr{E}) \cap C_0(X)$ [resp. $\mathscr{D}(\mathscr{E})$] by Theorem (1.8), the assumption that F is proper, and the usual change of variables formula for integrals. To conclude that (1.13) is the Beurling–Deny representation, it suffices to note that k^* and J^* are Radon measures. The former is clear. For the latter, if K^* is a compact subset of the (open) set $X^* \times X^* \setminus d^*$ then $K \equiv \hat{F}^{-1}(K^*)$ is compact [K is a closed subset of the compact $F^{-1}(\pi_2 K) \times F^{-1}(\pi_1 K)$, where π_1, π_2 denote the continuous coordinate projections from $X \times X$ onto its first and second factors]. If $(x, x) \in K \cap d$ then $(F(x), F(x)) \in K^* \cap d^* = \varnothing$, which is absurd. Thus K is compact and $K \cap d = \varnothing$, so $J^*(K^*) = J(K) < +\infty$. □

COROLLARY (1.14). *If $u^* \in \mathscr{D}(\mathscr{E}^*)$ and $\tilde{u}^*$ is a quasicontinuous modification of u^* then $\tilde{u}^* \circ F$ is a quasicontinuous modification of $u^* \circ F \in \mathscr{D}(\mathscr{E})$.*

PROOF. The existence of $\tilde{u}^*$ is guaranteed by Theorem 3.1.3 of Fukushima (1980). Also we have $m(\tilde{u}^* \circ F \neq u^* \circ F) = m^*(\tilde{u}^* \neq u^*) = 0$. Finally, if $\varepsilon > 0$ is given and a corresponding open $G^* \subset X^*$ is chosen with $\mathrm{Cap}^*(G^*) < \varepsilon$ and $\tilde{u}^*|_{X^* \setminus G^*}$ continuous, then $G := F^{-1}(G^*)$ is open and by (1.9) $\mathrm{Cap}(G) \leq$

$\mathrm{Cap}^*(G^*) < \varepsilon$. By construction $F(X \setminus G) = X^* \setminus G^*$ and $\tilde{u}^* \circ F|_{X \setminus G} = \tilde{u}^*|_{X^* \setminus G^*} \circ F|_{X \setminus G}$ is evidently continuous. □

We now take a closer look at the form of $\mathscr{E}^{*(c)}$ in a special case. Specifically, we assume that:

$(1.15)_1$ the Hunt process $x(\cdot)$ associated with the regular Dirichlet form $\mathscr{E}$ is a diffusion on X;
$(1.15)_2$ $X^* = I$, an interval in $\mathbb{R}^1$;
$(1.15)_3$ the subspace $C^* \subset C_0(I)$ [appearing in Definition (1.6) of the (eventual) core C_1^* of $\mathscr{E}^*$] is contained in $C_0^1(I)$;
$(1.15)_4$ denoting the infinitesimal generator of $x(\cdot)$ by $\mathscr{L}$, assume $F, F^2 \in \mathscr{D}(\mathscr{L})$ and $F \cdot \mathscr{L}F \in L^2(X; m)$.

THEOREM (1.16). *Under assumptions* $(1.15)_1$–$(1.15)_4$, *the Hunt process* $x^*(\cdot)$ *associated with* $\mathscr{E}^*$ *of Theorem* (1.8) *is a diffusion and*

$$\mathscr{E}^{*(c)}(u, u) = \int_I a^*(r) u'(r)^2 m^*(dr), \qquad u \in C_1^*, \tag{1.17}$$

where $a^* = \mathscr{F}^* \mathscr{D}$ *and* $\mathscr{D} := \frac{1}{2}\mathscr{L}F^2 - F \cdot \mathscr{L}F$. [*Note that by the result of Hamza* (1975) m^* *has a density.*]

PROOF. By Theorem 5.4.3 of Fukushima (1980)

$$\mathscr{E}^{*(c)}(u, u) \equiv \mathscr{E}^{(c)}(u \circ F, u \circ F) = \tfrac{1}{2}\int_X [u' \circ F]^2 \, d\mu_{\langle F \rangle},$$

where $\mu_{\langle F \rangle}$ is the energy measure of the martingale part, $M^{[F]}$, of the additive functional $A_t^{[F]} = F(x(t)) - F(x(0))$. Since $u' \circ F \in C_0(X)$ and $\mathscr{E}$ is regular there exists a sequence $u_n \in \mathscr{D}(\mathscr{E}) \cap C_0(X)$ such that $\|u_n\|_\infty \le \|u' \circ F\|_\infty$, $\lim_{n \to \infty} \|u_n - u' \circ F\|_\infty = 0$, and moreover $\mathrm{supp}[u_n] \subset \mathrm{supp}[u' \circ F]$ for all n [see Theorem 1.4.2(iii) and Lemma 1.4.2 of Fukushima (1980)]. Therefore by the bounded convergence theorem (because $\mu_{\langle F \rangle}$ is Radon) and Lemma 5.4.6 and relations (5.4.24) and (5.4.25) of Fukushima (1980)

$$\mathscr{E}^{*(c)}(u, u) = \tfrac{1}{2}\lim_{n \to \infty} \int_X u_n \, d\mu_{\langle F \rangle} \equiv \tfrac{1}{2}\lim_{n \to \infty} \int_X u_n \, d\mu_{\langle \bar{F} \rangle}, \tag{1.18}$$

where $\bar{F} := (F \wedge K) \vee (-K) \in \mathscr{D}(\mathscr{E})_b$ again by Theorem 1.4.2(iii) of Fukushima (1980), where K is chosen so that $\mathrm{supp}[u' \circ F] \subset \{-(K-1) < F < K-1\}$. Since $u_n \in \mathscr{D}(\mathscr{E})_b \cap C_0(X)$ and $\bar{F} \in \mathscr{D}(\mathscr{E})_b$ we can apply Theorem 5.2.3 of Fukushima (1980) to the extreme right-hand side of (1.18):

$$\begin{aligned}\mathscr{E}^{*(c)}(u, u) &= \tfrac{1}{2}\lim_{n \to \infty} \{2\mathscr{E}^{(c)}(\bar{F} \cdot u_n, \bar{F}) - \mathscr{E}^{(c)}(\bar{F}^2, u_n)\} \\ &\equiv \tfrac{1}{2}\lim_{n \to \infty} \{2\mathscr{E}^{(c)}(F \cdot u_n, F) - \mathscr{E}^{(c)}(F^2, u_n)\}\end{aligned}$$

by Theorem 4.5.3(i) of Fukushima (1980), noting that $\bar{F} = F$ on an open neighbourhood of $\operatorname{supp}[u_n]$. Therefore since $\mathscr{E}^{(c)}(f, g) = -(\mathscr{L}f, g)_m$ whenever $f \in \mathscr{D}(\mathscr{L})$ and $g \in \mathscr{D}(\mathscr{E})$:

$$\begin{aligned}\mathscr{E}^{*(c)}(u, u) &= \lim_{n \to \infty} \left\{ \left(\tfrac{1}{2}\mathscr{L}F^2, u_n \right)_m - (F \cdot u_n, \mathscr{L}F)_m \right\} \\ &\equiv \lim_{n \to \infty} (\mathscr{Q}, u_n)_m \\ &= \left(\mathscr{Q}, [u' \circ F]^2 \right)_m,\end{aligned}$$

the last equality following from the dominated convergence theorem since $|u_n| \le \|u' \circ F\|_\infty \cdot \chi_{\operatorname{supp}[u' \circ F]} \in L^2(X; m)$ ($\operatorname{supp}[u' \circ F]$ is compact and m is Radon), and $\mathscr{Q} \in L^2(X; m)$.

Finally, since $\mathscr{Q} \in L^2(X; m)$, $a^* := \mathscr{F}\mathscr{Q}$: $I \to \mathbb{R}$ is defined and

$$\begin{aligned}\mathscr{E}^{*(c)}(u, u) &= \int_X \mathscr{Q}[u' \circ F]^2 \, dm \\ &= \int_X \mathscr{F}\{\mathscr{Q}[u' \circ F]^2\} \, dm \\ &= \int_X [(\mathscr{F}^*\mathscr{Q}) \circ F][u' \circ F]^2 \, dm \\ &\qquad [\text{by Lemma } (1.1)(\text{v}) \text{ and Definition } (1.6)(\text{ii})] \\ &= \int_I \mathscr{F}^*\mathscr{Q}[u']^2 \, dm^* \\ &\equiv \int_I a^*(r) u'(r)^2 m^*(dr).\end{aligned}$$

That $x^*(\cdot)$ is a diffusion follows from the general result of Theorem 6.2.2 (and Theorem 4.3.6) of Fukushima (1980) since $\mathscr{E}^*$ is regular and possesses the local property [see Theorem (1.8)]. □

For the remainder of this section we shall assume that $m(X) < \infty$, $1 \in \mathscr{D}(\mathscr{E})$ *and* $k(X) < \infty$, *where the killing measure* k *was described in the Beurling–Deny representation* (1.10).

As such the constant functions belong to $\mathscr{D}(\mathscr{E})$. An equivalent condition, in terms of the semigroup $(T_t)_{t \ge 0}$ associated with $\mathscr{E}$, is that $\lim_{t \to 0^+} t^{-1} \int_X (1 - T_t 1) \, dm < \infty$.

We now translate the capacitary inequality (1.9) into semiprobabilistic terms for the Hunt processes $x(\cdot)$ and $x^*(\cdot)$ associated with $\mathscr{E}$ and $\mathscr{E}^*$ [see Theorems 6.2.1 and 4.3.6 of Fukushima (1980)]. We begin with a lemma.

LEMMA (1.19). *If* $m(X) < \infty$, $k(X) < \infty$, $1 \in \mathscr{D}(\mathscr{E})$ *and* $u \in \mathscr{D}(\mathscr{E})$ *then* $\mathscr{E}(u, 1) = \int_X \tilde{u} \, dk$, *where* $\tilde{u}$ *denotes a quasicontinuous version of* u.

PROOF. Polarizing (4.5.22) of Fukushima (1980) gives

$$\mathscr{E}^{\mathrm{res}}(u,v) = \lim_{t\to 0}\frac{1}{2t}E_m[u(X_t)-u(X_0)][v(X_t)-v(X_0)]$$

for any $u, v \in \mathscr{D}(\mathscr{E})$, where

$$\mathscr{E}^{\mathrm{res}}(u,v) := \mathscr{E}(u,v) - \int_X \tilde{u}(x)\tilde{v}(x)k(dx).$$

Substituting $v = 1$ yields the lemma. □

PROPOSITION (1.20). *If $m(X) < \infty$, $k(X) < \infty$, $1 \in \mathscr{D}(\mathscr{E})$ and $A^* \subset X^*$ is open then*

$$\int_X E_x e^{-\theta\tau}(\theta m + k)(dx) \le \int_{X^*} E_r e^{-\theta\tau^*}(\theta m^* + k^*)(dr), \tag{1.21}$$

where τ^* is the first hitting time of A^* by $x^*(\cdot)$ and τ is the first hitting time of $A := F^{-1}(A^*)$ by $x(\cdot)$. In particular if $k = 0$, then

$$E_m e^{-\theta\tau} \le E_m e^{-\theta\tau^*}. \tag{1.22}$$

PROOF. In view of (1.9) and Proposition (1.12) one need only make the general identification

$$\mathrm{Cap}_\theta(A) = \int_X E_x e^{-\theta\tau}(\theta m + k)(dx) \tag{1.23}$$

under the hypotheses: $m(X) < \infty$, $1 \in \mathscr{D}(\mathscr{E})$ and $k(X) < \infty$. This follows immediately from Lemma 3.1.1(iv) and Theorem 4.3.5 of Fukushima (1980), and Lemma (1.19) with $e_A(x) := E_x e^{-\theta\tau}$:

$$\mathrm{Cap}_\theta(A) = \mathscr{E}(e_A, 1) + \theta(e_A, 1)_m \equiv \int_X e_A\,dk + \theta\int_X e_A\,dm. \qquad \square$$

We close this section with an upper estimate on $P_m(\tau \le T)$ which is a generalization of the one announced at (0.10).

PROPOSITION (1.24). *Let $m(X) < \infty$, $k(X) < \infty$, $1 \in \mathscr{D}(\mathscr{E})$ and $A^* \subset X^*$ be an open subset of finite* Cap**-capacity; set $A = F^{-1}(A^*)$, $\tau = \tau_A$ and $\tau^* = \tau_{A^*}$. Then for all $\theta > 0$,*

$$P_m(\tau \le T) \le e^\theta\left\{E_{m^*}e^{-\theta\tau^*/T} + T\theta^{-1}\int_X E_r e^{-\theta\tau^*/T}k^*(dr)\right\}. \tag{1.25}$$

Also

$$\begin{aligned} P_m(\tau \le T) \le \theta^{-1}e^\theta \min\{&T\mathscr{E}^*(u^*,u^*) + \theta(u^*,u^*)_m | u^* \in \mathscr{D}(\mathscr{E}^*),\\ &u^* \ge 1,\ m^*\text{-}a.e.\ on\ A^*\}. \end{aligned} \tag{1.26}$$

PROOF. Denoting θ/T by θ_1,

$$P_m(\tau \le T) = P_m(\theta_1 \tau \le \theta)$$

$$\le e^{\theta} E_m e^{-\theta_1 \tau}$$

$$(1.27) \qquad \le e^{\theta}\theta_1^{-1}\left\{\int_{X^*} E_r e^{-\theta_1 \tau^*}(\theta_1 m^* + k^*)(dr)\right\} \quad [\text{by } (1.21)]$$

$$= e^{\theta}\theta_1^{-1}\,\mathrm{Cap}^*_{\theta_1}(A^*) \quad [\text{by } (1.23)]$$

$$(1.28) \qquad \equiv e^{\theta}\theta_1^{-1}\min\{\mathscr{E}^*(u^*,u^*) + \theta_1(u^*,u^*)_{m^*} | u^* \in D_{A^*}\},$$

the last identification being the result of Lemma 3.1.1(i) of Fukushima (1980), where $D_{A^*} = \{u^* \in \mathscr{D}(\mathscr{E}^*) | u^* \ge 1,\ m^* - \text{a.e. on } A^*\}$ as in (0.4). Evidently (1.27) becomes (1.25), and (1.28) becomes (1.26) upon replacing θ_1 by θ/T. □

2. Reversible diffusions on $\mathbb{R}^n$. In this section we specialize and exemplify the results of the preceding section for the case $X = \mathbb{R}^n$, $X^* = [0,\infty) \subset \mathbb{R}^1$ and $F = |\cdot|$, the Euclidean norm, which is clearly a proper map. The forms considered will correspond to a subclass of uniformly elliptic, classical reversible diffusions. Specifically, we will assume that the Dirichlet form $\mathscr{E}$ is the closure of $\mathscr{E}'$, where

$$(2.1)_1 \qquad \mathscr{E}'(u,v) = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij}(y) \frac{\partial}{\partial y_i} u(y) \frac{\partial}{\partial y_j} v(y) m(y)\, dy,$$

$$u, v \in \mathscr{D}(\mathscr{E}') = C_0^{\infty}(\mathbb{R}^n);$$

$$(2.1)_2 \qquad m(\cdot) \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n; dy) \text{ and } m(y) > 0 \text{ for all } y \in \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} m(y)\, dy = 1;$$

$$(2.1)_3 \qquad \forall\, i,j\colon a_{ij}(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^n; m(dy)) \quad \text{where } m(dy) \equiv m(y)\, dy,$$

$$\text{and } a_{ij}(\cdot) = a_{ji}(\cdot);$$

$$(2.1)_4 \qquad \exists\, \delta > 0 \text{ such that } \forall y \in \mathbb{R}^n\colon \sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \ge \delta|\xi|^2\ \forall\, \xi \in \mathbb{R}^n.$$

For some results we will also assume in addition to $(2.1)_1$–$(2.1)_4$,

$$(2.1)_5 \qquad \forall\, i,j\colon a_{ij}(\cdot) \in L^1(\mathbb{R}^n; m).$$

In Lemma (2.2), we will show that $(2.1)_1$–$(2.1)_4$ indeed ensure that $\mathscr{E}'$ is closable in $L^2(\mathbb{R}^n; m)$; $C_1 = C_0^{\infty}(\mathbb{R}^n)$ then serves as a core for $\mathscr{E}$. Theorems 2.1.1 and 2.1.2 of Fukushima (1980) then yield that $\mathscr{E}$ is a regular, local Dirichlet form. We will denote the corresponding diffusion by $x(\cdot)$ and the class of all m-symmetric diffusions in $\mathbb{R}^n$, whose Dirichlet forms arise as outlined above, by Ξ.

LEMMA (2.2). *Under assumptions* $(2.1)_1$–$(2.1)_4$, $\mathscr{E}'$ *is closable in* $L^2(\mathbb{R}^n; m)$.

PROOF. Lemma (2.2) can be shown to follow from Lemma 4.3 and Corollary 3.5 of Röckner and Wielens (1985) but for the reader's convenience we will carry through the type of argument given in Case (1°) of Section 2.1 of Fukushima (1980). [Note that our assumptions (2.1) do not quite fit into that case.]

Let $(u_l)_{l\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ with $\mathscr{E}'(u_l - u_k, u_l - u_k) + (u_l, u_l)_m \to 0$ as $l, k \to \infty$. We must show that $\mathscr{E}'(u_l, u_l) \to 0$ as $l \to \infty$. By $(2.1)_4$

$$D(u_l - u_k, u_l - u_k) \le \delta^{-1}\mathscr{E}'(u_l - u_k, u_l - u_k) \to 0 \quad \text{as } l, k \to \infty,$$

where

$$D(u,v) := \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial u}{\partial y_i}(y) \frac{\partial v}{\partial y_i}(y) m(y)\, dy, \qquad u, v \in C_0^\infty(\mathbb{R}^n).$$

Now D is closable on $L^2(\mathbb{R}^n; m)$ by (1.1.3) of Fukushima (1980). Indeed if $(v_j)_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ with $(v_j, v_j)_m \to 0$ as $j \to \infty$, then for each $v \in C_0^\infty(\mathbb{R}^n)$,

$$D(v_j, v) = -\int_K v_j(y)(Sv)(y)\, dy,$$

where $K = \operatorname{supp}[v]$ and

$$(Sv)(y) := \sum_{i=1}^n \frac{\partial}{\partial y_i}\left[m(y)\frac{\partial v}{\partial y_i}(y)\right].$$

By Cauchy–Schwarz

$$|D(v_j, v)| \le \|[(Sv)/m] \cdot \chi_K\|_\infty \sqrt{\operatorname{vol}(K)} \cdot (v_j, v_j)_m^{1/2} \to 0 \quad \text{as } j \to \infty.$$

Thus $D(u_k, u_k) \to 0$ as $k \to \infty$. By restricting attention to a subsequence if necessary, we may assume that $\partial u_k/\partial y_i \to 0$ m – a.e. for each $i = 1, 2, \ldots, n$. Then by Fatou's lemma

$$\mathscr{E}'(u_l, u_l) \le \liminf_{k\to\infty} \mathscr{E}'(u_l - u_k, u_l - u_k) \to 0 \quad \text{as } l \to \infty. \qquad \square$$

LEMMA (2.3). *Under assumptions* $(2.1)_1$–$(2.1)_4$, $\mathscr{D}(\mathscr{E}) \supset C_0^1(\mathbb{R}^n)$. *If also* $(2.1)_5$ *holds, then* $\mathscr{D}(\mathscr{E}) \supset C_b^1(\mathbb{R}^n)$, *the set of bounded continuously differentiable functions having bounded first partial derivatives. The expression* $(2.1)_1$ *is valid for* $u, v \in C_0^1(\mathbb{R}^n)$ [*resp.* $C_b^1(\mathbb{R}^n)$, *if* $(2.1)_5$ *holds*].

PROOF. Fix a nonnegative $w \in C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} w(y)\, dy = 1$, and set u_ε equal to the convolution $u_\varepsilon = u * w_\varepsilon$, where for $\varepsilon > 0$: $w_\varepsilon(y) = \varepsilon^{-n} w(\varepsilon^{-1} y)$, $y \in \mathbb{R}^n$. Then if $u \in C_0^1(\mathbb{R}^n)$, $u_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to u$, $\partial u_\varepsilon/\partial y_i = (\partial u/\partial y_i)_\varepsilon \equiv (\partial u/\partial y_i) * w_\varepsilon \to \partial u/\partial y_i$ for each i pointwise boundedly on $\mathbb{R}^n$ as $\varepsilon \to 0^+$; $\|u_\varepsilon\|_\infty \le \|u\|_\infty$ and $\|\partial u_\varepsilon/\partial y_i\|_\infty \le \|\partial u/\partial y_i\|_\infty$ for each i. Then by the bounded convergence theorem $(u_\varepsilon - u, u_\varepsilon - u)_m \to 0$ as $\varepsilon \to 0^+$ and by the dominated convergence theorem, as $a_{ij}(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^n; m)$ and $\operatorname{supp}[u_\varepsilon] \subset \operatorname{supp}[u] + \operatorname{supp}[w]$ for $\varepsilon < 1$,

$$\begin{aligned}\mathscr{E}(u_\varepsilon - u_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) &\leq 2 \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij}(y)[(\partial u/\partial y_i)_\varepsilon(y) - \partial u/\partial y_i(y)] \\ &\quad \times \left[(\partial u/\partial y_j)_\varepsilon(y) - \partial u/\partial y_j(y)\right] m(dy) \\ &\quad + 2 \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij}(y)[(\partial u/\partial y_i)_{\varepsilon'}(y) - \partial u/\partial y_i(y)] \\ &\qquad \times \left[(\partial u/\partial y_j)_{\varepsilon'}(y) - \partial u/\partial y_j(y)\right] m(dy) \\ &\to 0 \quad \text{as } \varepsilon, \varepsilon' \to 0^+.\end{aligned}$$

Since $\mathscr{E}$ is closed, $u \in \mathscr{D}(\mathscr{E})$.

Finally, if $u, v \in C_0^1(\mathbb{R}^n)$ then

$$\mathscr{E}(u,v) = \lim_{\varepsilon \to 0^+} \mathscr{E}(u_\varepsilon, v_\varepsilon) = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial u}{\partial y_i}(y) \frac{\partial v}{\partial y_i}(y) m(y)\, dy,$$

by another application of the dominated convergence theorem.

In case $u \in C_b^1(\mathbb{R}^n)$ only, but each $a_{ij}(\cdot) \in L^1(\mathbb{R}^n; m)$, then $u_n := u \cdot v_n \in C_0^1(\mathbb{R}^n) \subset \mathscr{D}(\mathscr{E})$, where $v_n \in C_0^\infty(\mathbb{R}^n)$ with $v_n \equiv 1$ on $\{|y| < n\}$ and $v_n \equiv 0$ on $\{|y| > 2n\}$ and $\|\partial v_n/\partial y_i\|_\infty \leq 1$ for $i = 1, 2, \ldots, n$. Then using the product rule, one can show that $(u_n)_{n \in \mathbb{N}}$ is $\mathscr{E}_1$-Cauchy and $(u - u_n, u - u_n)_m \to 0$ as $n \to \infty$, using the dominated convergence theorem; we omit the straightforward details. The expression for $\mathscr{E}(u,v)$ when $u, v \in C_b^1(\mathbb{R}^n)$ follows as in the previous paragraph. □

In order to apply the results of Section 1 we must verify that $F \equiv |\cdot|$ is a (proper) core map [relative to some pair (C_1, C^*); see Definition (1.6)]; and then calculate the resulting form $\mathscr{E}^*$. This we now do.

LEMMA (2.4). *Under assumptions* $(2.1)_1$–$(2.1)_4$, $F \equiv |\cdot|$ *is a (proper) core map relative to* $C_1 = C_0^\infty(\mathbb{R}^n)$ *and* $C^* = C_0^1([0, \infty))$.

PROOF. Since $m(\mathbb{R}^n) = 1$, if $u \in C_1$, then $\mathscr{F}u$ and $\mathscr{F}^*u$ will just be versions of $E_m[u|F]$ and $E_m[u|F = \cdot]$, respectively. We proceed to verify the conditions (1.6)(i), (ii) of Definition (1.6). Denote by σ the usual hypersurface measure of $S \equiv \{y \in \mathbb{R}^n: |y| = 1\}$; so that $\int_{\mathbb{R}^n} u(y)\, dy = \int_0^\infty \int_S u(r\theta)\, d\sigma(\theta) r^{n-1}\, dr$ for all $u \in L^1(\mathbb{R}^n; dy)$.

For any $u \in L^2(\mathbb{R}^m; m) \cap C(\mathbb{R}^n)$ (in particular for $u \in C_1$) and $v \circ F \in L^2(\mathbb{R}^n, \sigma(F), m)$, ($v$: $\mathbb{R}_+ \to \mathbb{R}$ and is Borel measurable)

$$\begin{aligned} E_m[(v \circ F)u] &= \int_0^\infty v(r) \int_S u(r\theta) m(r\theta)\, d\sigma(\theta) r^{n-1}\, dr \\ &= \int_0^\infty v(r) \mathscr{F}^* u(r) \int_S m(r\theta)\, d\sigma(\theta) r^{n-1}\, dr \\ &= E_m[(v \circ F)\mathscr{F} u], \end{aligned}$$

where

$$\mathscr{F}^* u(r) = \int_S u(r\theta) m(r\theta)\, d\sigma(\theta) \Big/ \int_S m(r\theta)\, d\sigma(\theta) \tag{2.5}$$

and

$$\mathscr{F} u(y) := \mathscr{F}^* u(|y|) \equiv (\mathscr{F}^* u) \circ F(y). \tag{2.6}$$

If $u \in C_1$, then clearly $\mathscr{F} u$ and $\mathscr{F}^* u$ are continuous versions of $E_m[u \mid F]$ and $E_m[m \mid F = \cdot]$, respectively, and $\mathscr{F}^* u \in C^* \equiv C_0^1([0, \infty))$. It only remains to check that $\mathscr{F} u \in \mathscr{D}(\mathscr{E})$ if $u \in C_1 \equiv C_0^\infty(\mathbb{R}^n)$.

For $\varepsilon > 0$ consider

$$u_\varepsilon(y) := \mathscr{F}^* u\left(\left[\sum_{i=1}^n y_i^2 + \varepsilon\right]^{1/2}\right), \qquad u_\varepsilon \in C_0^1(\mathbb{R}^n) \subset \mathscr{D}(\mathscr{E}),$$

by Lemma (2.3), and moreover $\operatorname{supp}[u_\varepsilon] \subset \operatorname{supp}[\mathscr{F}^* u] \subset [0, R]$, say. Set $|y|_\varepsilon := [\sum_{i=1}^n y_i^2 + \varepsilon]^{1/2}$; $|y|_\varepsilon \searrow |y|$ as $\varepsilon \searrow 0$. By the bounded convergence theorem $(u_\varepsilon - \mathscr{F} u, u_\varepsilon - \mathscr{F} u)_m \to 0$, as $\varepsilon \to 0^+$. Also, by the chain rule

$$\mathscr{E}(u_\varepsilon - u_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) = \int_{\{|y| \le R\}} \left\{ \sum_{i,j=1}^n a_{ij}(y) y_i y_j \left[\frac{(\mathscr{F}^* u)'(|y|_\varepsilon)}{|y|_\varepsilon} - \frac{(\mathscr{F}^* u)'(|y|_{\varepsilon'})}{|y|_{\varepsilon'}} \right]^2 \right\} m(dy), \tag{2.7}$$

where $(\mathscr{F}^* u)' \equiv (d/dr)\mathscr{F}^* u$, since the integrand at (2.7) is identically zero for $|y| \ge R$. Continuing,

$$\begin{aligned} &\mathscr{E}(u_\varepsilon - u_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) \\ &\le 2 \int_{\{|y| \le R\}} \sum_{i,j=1}^n a_{ij}(y) y_i y_j \left(\left[\frac{(\mathscr{F}^* u)'(|y|_\varepsilon)}{|y|_\varepsilon} - \frac{(\mathscr{F}^* u)'(|y|)}{|y|} \right]^2 \right. \\ &\qquad \left. + \left[\frac{(\mathscr{F}^* u)'(|y|_{\varepsilon'})}{|y|_{\varepsilon'}} - \frac{\mathscr{F}^* u'(|y|)}{|y|} \right]^2 \right) m(dy). \end{aligned} \tag{2.8}$$

Now the integrand at (2.8) is majorized (since $|y_i y_j| / |y|_\varepsilon^2 \le 1$) by

$$8 \|(\mathscr{F}^* u)'\|_\infty \sum_{i,j=1}^n |a_{ij}(\cdot)| \in L^1_{\text{loc}}(\mathbb{R}^n; m).$$

By the dominated convergence theorem, $\mathscr{E}(u_\varepsilon - u_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) \to 0$ as $\varepsilon, \varepsilon' \to 0$. Since $\mathscr{E}$ is closed, $\mathscr{F}u = \lim_{\varepsilon\to 0^+} u_\varepsilon \in \mathscr{D}(\mathscr{E})$. □

As an immediate consequence we obtain the following proposition.

PROPOSITION (2.9). *Under assumptions* $(2.1)_1$–$(1.2)_4$, *when* $F = |\cdot|$, *the induced regular Dirichlet form* $\mathscr{E}^*$ *is determined on the core* $C_1^* = \{u \in C_0^1([0,\infty)): u \circ F \in \mathscr{D}(\mathscr{E})\}$ *by*

$$\mathscr{E}^*(u,v) = \int_0^\infty f(r)u'(r)v'(r)\,dr, \qquad u,v \in C_1^*, \tag{2.10}$$

where

$$f(r) = \int_S \sum_{i,j=1}^n a_{ij}(r\theta)\theta_i\theta_j m(r\theta)\,d\sigma(\theta)r^{n-1}, \quad a.e.\ r \ge 0.$$

PROOF. The existence and regularity of $\mathscr{E}^*$ follows from Theorem (1.8) since $F \equiv |\cdot|$ is a proper core map. Finally, by the proof of Lemma (2.4) [see the paragraph containing (2.7) and (2.8)], with $u_\varepsilon(y) := u(|y|_\varepsilon)$, since $\mathscr{F}^*(u \circ F) = u$,

$$\begin{aligned}
\mathscr{E}^*(u,v) &= \mathscr{E}(u \circ F, v \circ F) \\
&= \lim_{\varepsilon\to 0^+} \mathscr{E}(u_\varepsilon, v_\varepsilon) \\
&= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(y)y_iy_ju'(|y|_\varepsilon)v'(|y|_\varepsilon)/|y|_\varepsilon^2 m(y)\,dy \\
&= \int_0^\infty f(r)u'(r)v'(r)\,dr
\end{aligned}$$

upon writing the integral in spherical coordinates. The interchange of limit and integral is justified by the dominated convergence theorem. □

REMARK (2.11). With some additional smoothness and integrability assumptions on the coefficients $a_{ij}(\cdot)$, one can use the arguments given in Lemmas (2.3) and (2.4) to verify that $F \equiv |\cdot|$ satisfies the hypotheses of Theorem (1.16). In particular,

$$\mathscr{L}u(y) = \frac{1}{m(y)} \sum_{i,j=1}^n \frac{\partial}{\partial y_j}\left[a_{ij}(y)m(y)\frac{\partial}{\partial y_i}u(y)\right], \qquad u \in C_0^\infty(\mathbb{R}^n)\ (\text{at least}).$$

Also one calculates that $\mathscr{Q} = \frac{1}{2}\mathscr{L}F^2 - F \cdot \mathscr{L}F$ is given by

$$\mathscr{Q}(y) = \left[\sum_{i,j=1}^n a_{ij}(y)y_iy_j\right]\Big/|y|^2.$$

[Note that $\mathscr{Q} \in L^1(\mathbb{R}^n; m)$ if each $a_{ij}(\cdot) \in L^1(\mathbb{R}^n; m)$; no smoothness assumptions are required for this.] The coefficient a^* and density of the measure m^*

appearing in the formula (1.17) would be given by [as at (2.5)]

$$a^*(r) = E_m[\mathscr{D}|F = r] = \int_S \sum_{i,j=1}^{n} a_{ij}(r\theta)\theta_i\theta_j m(r\theta)\, d\sigma(\theta) \div \int_S m(r\theta)\, d\sigma(\theta), \quad \text{a.e. } r \geq 0, \tag{2.12}$$

$$m^*(r) = \left[\frac{d}{d\lambda}(m \circ F^{-1})\right](r) = \int_S m(r\theta)\, d\sigma(\theta) \cdot r^{n-1}, \tag{2.13}$$

where λ is one-dimensional Lebesgue measure and $d/d\lambda$ stands for the Radon-Nikodym derivative with respect to λ. Thus the coefficient f of (2.10) satisfies

$$f(r) = a^*(r) \cdot m^*(r), \tag{2.14}$$

since the two approaches must yield the same representation for $\mathscr{E}^*$. We shall not pursue a rigorous derivation of Proposition (2.9) from Theorem (1.16) since the proof given for the former was quite simple and direct and required no assumptions additional to $(2.1)_1$–$(2.1)_4$.

REMARK (2.15). If $n \geq 2$ and each $a_{ij}(\cdot)$ is bounded near the origin then the diffusion coefficient f, of (2.10), is degenerate, i.e., $f(r) = \mathscr{O}(r^{n-1})$ as $r \to 0^+$. In the terminology of Feller's classification of one-dimensional diffusions [see Itô and McKean (1965)] the boundary point $r = 0$ of $X^* = [0, \infty)$ is an "entrance boundary" for the diffusion $x^*(\cdot)$ associated with $\mathscr{E}^*$. If $n = 1$ then 0 is a reflecting boundary.

As a preliminary step towards estimating the capacity $\mathrm{Cap}_\theta^*((x, \infty))$ for $x > 0$, we must make more precise the domain, $\mathscr{D}(\mathscr{E}^*)$, of $\mathscr{E}^*$. The description is given in the next corollary which is a consequence of more general results developed in Section 3.

COROLLARY (2.16). *Under assumptions* $(2.1)_1$–$(2.1)_5$,

$$\mathscr{D}(\mathscr{E}^*) := \{u \in L^2(\mathbb{R}_+; m^*) | u \in AC((0, \infty)),\, u' \in L^2(\mathbb{R}_+; f(r)\, dr)\}.$$

PROOF. Due to $(2.1)_4$: $f(r) \geq \delta\sigma(S) r^{n-1}$ [see (2.10)] which implies that $1/f \in L^1_{\mathrm{loc}}((0, \infty); dr)$. By $(2.1)_5$, $f \in L^1(\mathbb{R}_+; dr)$ so $f^{-1} \notin L^1([1, \infty); dr)$ by Cauchy–Schwarz. Clearly $m^* \in L^1(\mathbb{R}_+; dr)$, and $(m^*)^{-1} \in L^1_{\mathrm{loc}}((0, \infty); dr)$ by $(2.1)_2$. Moreover $\mathscr{D}(\mathscr{E}^*)$ contains $C_0^1(\mathbb{R}_+)$ (densely). Indeed, given $u^* \in C_0^1(\mathbb{R}_+)$, we define

$$u_\varepsilon^*(r) := u^*\left([r^2 + \varepsilon]^{1/2}\right)$$

for each $\varepsilon > 0$. Then as in Lemma (2.4) [see the paragraph after (2.6)] $u_\varepsilon^* \in C_0^1(\mathbb{R}_+)$, $u_\varepsilon^*(|\cdot|) \in C_0^1(\mathbb{R}^n) \subset \mathscr{D}(\mathscr{E})$ [by Lemma (2.3)], and $u_\varepsilon^*(|\cdot|)$ converges in $\mathscr{E}_1$-norm to $u^*(|\cdot|)$ as $\varepsilon \to 0^+$. Hence u_ε^* converges to u^* in $\mathscr{E}_1^*$-norm, as $\varepsilon \to 0^+$. Since $\mathscr{E}$ [from Proposition (3.1)] and $\mathscr{E}^*$ are closed and agree [identifying $m(r)$ with $m^*(r)$] on a common core, $C_0^1(\mathbb{R}_+)$, $\mathscr{E} = \mathscr{E}^*$ and the corollary is a particular instance of Proposition (3.1). □

We are now in a position to give an upper and lower estimate of $\mathrm{Cap}_\theta^*((x,\infty))$ for $x > 0$.

COROLLARY (2.17). *Under assumptions* $(2.1)_1$–$(2.1)_5$

$$\begin{aligned} &\frac{\theta}{1+\theta\nu(x)}[1+\theta(1-M(x))\nu(x)] \\ (2.18) \qquad &\le \mathrm{Cap}_\theta^*((x,\infty)) \\ &\le \frac{1}{\nu(x)} + \theta\int_0^x v(r;x)^2 m^*(r)\,dr + \theta\int_x^\infty m^*(r)\,dr, \end{aligned}$$

where $v(r;x) = \nu(r)/\nu(x)$, $\nu(r) = \int_0^r M(s)/f(s)]\,ds$, $M(r) = \int_0^r m^*(s)\,ds$ *and* f *is given in Proposition* (2.9).

PROOF. As verified in the proof of Corollary (2.16), f and m^* satisfy the hypotheses of Proposition (3.1) with $\mathscr{E} \equiv \mathscr{E}^*$; moreover $\int_0^\infty m^*(r)\,dr = 1$ by $(2.1)_2$. To apply Proposition (3.2), it only remains to verify that $M/f \in L^1_{\mathrm{loc}}(\mathbb{R}_+; dr)$. By $(2.1)_4$ and the description of f in Proposition (2.9), $f(r) \ge \delta\sigma(S)r^{n-1}$, and by (2.13), $M(r) = \mathscr{O}(r^n)$ as $r \to 0^+$; so $M(r)/f(r) = \mathscr{O}(r)$ as $r \to 0^+$. Moreover $M(r) \le 1$. Therefore $M/f \in L^1_{\mathrm{loc}}(\mathbb{R}_+; dr)$. □

COROLLARY (2.19). *In* (3.15) *take m to be* m^* *above and assume* $\delta(x)$ *tends to* 0 *as* $x \to \infty$. *Under assumptions* $(2.1)_1$–$(2.1)_4$ *and assuming there exists a constant* $\Delta > 0$ *such that*

$$(2.20) \qquad \sum_{i,j=1}^n a_{ij}(y)\xi_i\xi_j \le \Delta|\xi|^2,$$

then

$$\lim_{x\to\infty} \nu(x)\mathrm{Cap}_\theta^*((x,\infty)) = 1.$$

PROOF. Simply note that $\delta \le a^*(r) \le \Delta$, where a^* is defined in (2.12). The result follows from Proposition (3.19). □

COROLLARY (2.21). *Under the assumptions of Corollary* (2.19),

$$\limsup_{x\to\infty} \nu(x)\cdot \mathrm{Cap}_\theta\big(B(0;x)^c\big) \le 1.$$

PROOF. This follows from Theorem (1.8) and Corollary (2.19). □

COROLLARY (2.22). *If* $x(\cdot) \in \Xi$, *the class of m-symmetric diffusions in* $\mathbb{R}^n$ *whose Dirichlet forms* $\mathscr{E}$ *satisfy* $(2.1)_1$–$(2.1)_5$, $x > 0$ *and* τ *is the first hitting* *time of* $B_x^c = \{y \in \mathbb{R}^n : |y| > x\}$, *then for all* $\theta, T > 0$,

$$\begin{aligned} &P_m(\tau \le T) \\ (2.23)\qquad &\le \theta^{-1} e^{\theta} \left\{ T[\nu(x)]^{-1} + \theta \int_{B_x} v(|y|; x) m(y)\, dy + \theta \int_{B_x^c} m(y)\, dy \right\} \to 0 \end{aligned}$$

as $x \to \infty$, *where* ν *and* v *are as described in Corollary* (2.17).

PROOF. This is an immediate consequence of (1.26) with $A^* = (x, \infty)$ and the replacement of u^* by v; $\mathscr{E}^*(v, v)$ is estimated at (3.5) in the proof of Proposition (3.2) □

In Iscoe and McDonald (1989a) the asymptotics of the estimate given at (2.23) are worked out for an n-dimensional Ornstein–Uhlenbeck process. Subsequently, a lower bound of the same order was derived in Iscoe and McDonald (1989b).

3. Estimates and asymptotics for one-dimensional diffusions.

PROPOSITION (3.1). *Let* f *and* m *be nonnegative Borel functions on* $\mathbb{R}_+ = [0, \infty)$ *such that*

$$m, f \in L^1_{\mathrm{loc}}(\mathbb{R}_+; dr); \qquad m^{-1}, f^{-1} \in L^1_{\mathrm{loc}}((0,\infty); dr); \qquad f^{-1} \notin L^1([1,\infty); dr).$$

Define the form $\mathscr{E}$ *in* $L^2(\mathbb{R}_+; m(r)\, dr)$ *by*

$$\mathscr{E}(u, v) = \int_0^\infty f(r) u'(r) v'(r)\, dr, \qquad u, v \in \mathscr{D}(\mathscr{E}),$$

$$\mathscr{D}(\mathscr{E}) = \{u \in L^2(\mathbb{R}_+; m(r)\, dr) | u \text{ is AC on } (0, \infty) \text{ and } u' \in L^2(\mathbb{R}_+; f(r)\, dr)\}.$$

Then $\mathscr{E}$ *is a regular Dirichlet form; and* $C_0^1(\mathbb{R}_+)$ *is a core for* $\mathscr{E}$.

PROOF. Theorem 3.2 in Röckner and Wielens (1985) shows that $\mathscr{E}$ is a Dirichlet form [see also Rullkötter and Spönemann (1983) for the specific one-dimensional case.] Concerning the regularity of $\mathscr{E}$, it suffices to verify that $C_0^1(\mathbb{R}_+)$ is $\mathscr{E}_1$-dense among the bounded functions in $\mathscr{D}(\mathscr{E})$ since the latter are already dense in $\mathscr{D}(\mathscr{E})$ [see Theorem 1.4.2(iii) of Fukushima (1980)]. Next, it suffices to treat the case where u is, in addition, compactly supported in $\mathbb{R}_+$. Indeed if $u \in \mathscr{D}(\mathscr{E})$, $u \ne 0$ and is bounded, then given $\varepsilon > 0$, we choose $R_1 > 0$ such that

$$\int_{R_1}^\infty \left[u'(r)^2 f(r) + u(r)^2 m(r) \right] dr < \varepsilon/4$$

and then $R_2 > R_1$ such that

$$\left(\int_{R_1}^{R_2} f(r)^{-1}\,dr\right)^{-1} < \varepsilon/\left(4\|u\|_\infty^2\right).$$

Then defining

$$v(r) = \begin{cases} 1, & 0 \le r \le R_1, \\ 1 - V(r)/V(R_2), & R_1 < r < R_2, \\ 0, & R_2 \le r, \end{cases}$$

where

$$V(r) = \int_{R_1}^{r} f(s)^{-1}\,ds,$$

it is easy to see that $uv \in \mathscr{D}(\mathscr{E})$, uv is bounded and has compact support in $\mathbb{R}_+$ and $\mathscr{E}_1(u - uv, u - uv) < \varepsilon$. Finally, if $u \in \mathscr{D}(\mathscr{E})$ is of compact support, we may use the hypotheses m, $f \in L^1_{\text{loc}}(\mathbb{R}_+; dr)$ and regularization to construct a sequence $u_\delta \in C_0^1(\mathbb{R}_+)$ such that $\mathscr{E}_1(u - u_\delta, u - u_\delta) \to 0$. □

PROPOSITION (3.2). *Let $\mathscr{E}$, f and m be as in Proposition* (3.1) *with moreover $m \in L^1(\mathbb{R}_+; dr)$ and $\int_0^\infty m(r)\,dr = 1$. If further $M/f \in L^1_{\text{loc}}(\mathbb{R}_+; dr)$, where $M(r) := \int_0^r m(s)\,ds$, then relative to $\mathscr{E}$*

$$\begin{aligned} &\frac{\theta}{1 + \theta\nu(x)}\left[1 + \theta(1 - M(x))\nu(x)\right] \\ &\le \operatorname{Cap}_\theta((x,\infty)) \\ &\le \frac{1}{\nu(x)} + \theta\int_0^x v(r;x)^2 m(r)\,dr + \theta\int_x^\infty m(r)\,dr, \qquad \theta > 0,\ x > 0, \\ &\to 0 \text{ as } x \to \infty, \end{aligned} \tag{3.3}$$

where $\nu(r) := \int_0^r [M(s)/f(s)]\,ds$ and $v(r;x) := \nu(r)/\nu(x)$ for $0 \le r \le x$.

PROOF. Since $\int_x^\infty m(r)\,dr < \infty$, (x,∞) has finite capacity; moreover $\operatorname{Cap}_\theta((x,\infty)) \le \mathscr{E}_\theta(v,v)$, where $v(r) \equiv v(r;x)$ is given by

$$v(r;x) := \begin{cases} \nu(r)/\nu(x), & 0 \le r \le x, \\ 1, & x < r. \end{cases} \tag{3.4}$$

Note that $v \in D_{(x,\infty)}$ [see (0.4)] by Proposition (3.1) since $M/f \in L^1_{\text{loc}}(\mathbb{R}_+; dr)$ implies that ν, and hence v is well defined and absolutely continuous on $\mathbb{R}_+$;

also $v \in L^{\infty}(\mathbb{R}_+; dr) \subset L^2(\mathbb{R}_+; m(r)\,dr)$ and $v' \in L^2(\mathbb{R}_+; f(r)\,dr)$:

$$
\begin{aligned}
\int_0^{\infty} v'(r)^2 f(r)\,dr &= \nu(x)^{-2}\int_0^x \left[M(r)^2/f(r)\right] dr \\
&\le \nu(x)^{-2} M(x)\int_0^x [M(r)/f(r)]\,dr \qquad (3.5)\\
&\equiv M(x)/\nu(x) \\
&\le 1/\nu(x) < \infty.
\end{aligned}
$$

The upper bound at (3.3) follows immediately from (3.5), which is an upper bound for $\mathscr{E}(v,v)$ and the definition: $\mathscr{E}_\theta(v,v) = \mathscr{E}(v,v) + \theta\int_0^\infty v(r)^2 m(r)\,dr$.

Concerning the lower bound in (3.3): By Lemma 3.1.1(i), (iii) of Fukushima (1980) and Proposition (3.1), $\mathrm{Cap}_\theta((x,\infty)) = \mathscr{E}_\theta(u,u)$ for some $u \in \mathscr{D}(\mathscr{E}) \subset AC((0,\infty))$, such that $u \equiv 1$ on (x,∞). Since $\mathscr{E}_\theta(u,u)$ is the minimum value of $\mathscr{E}_\theta(v_1,v_1)$ as v_1 ranges over $D_{(x,\infty)}$, it follows that the quadratic polynomial (in t) $\mathscr{E}_\theta(u+th, u+th)$ has its minimum value at $t=0$ for any $h \in \mathscr{D}(\mathscr{E})$ such that $h \equiv 0$ on $[x,\infty)$. Developing the quadratic polynomial and minimality property yields

$$
\begin{aligned}
0 &= \theta\int_0^x u(r)h(r)m(r)\,dr + \int_0^x u'(r)h'(r)f(r)\,dr \\
&= \int_0^x h'(r)\left\{f(r)u'(r) - \theta\int_0^r u(s)m(s)\,ds\right\} dr, \qquad (3.6)
\end{aligned}
$$

after an integration by parts. We now choose for h the absolutely continuous function: $h(r) = -\int_r^x M(s)/f(s)\,ds \equiv \nu(r) - \nu(x)$ for $0 \le r \le x$ and $h \equiv 0$ for $r > x$. Then $h'(r) = M(r)/f(r) \equiv \nu'(r)$ for a.e. $r \in (0,x)$; so $h' \in L^2(\mathbb{R}_+; f(r)\,dr)$ [see (3.5)]. Thus $h \in \mathscr{D}(\mathscr{E})$ by Proposition (3.1) and so is a legitimate choice. Continuing from (3.6), we obtain, by another integration by parts and an application of Fubini's theorem

$$
\begin{aligned}
0 &= \int_0^x M(r)u'(r)\,dr - \theta\int_0^x\int_0^r [M(r)/f(r)]u(s)m(s)\,ds\,dr \\
&= M(x) - \int_0^x u(r)m(r)\,dr - \theta\left\{\int_0^x u(s)m(s)\int_s^x [M(r)/f(r)]\,dr\,ds\right\} \\
&\qquad\qquad [M(0) = 0,\ u(x) = 1] \\
&= 1 - \int_0^\infty u(r)m(r)\,dr - \theta\left\{\left[\int_0^x u(s)m(s)\,ds\right]\nu(x) + \int_0^x u(s)\nu(s)m(s)\,ds\right\} \\
&\ge 1 - \left[\int_0^\infty u(r)m(r)\,dr\right][1+\theta\nu(x)] + \theta[1-M(x)]\nu(x).
\end{aligned}
$$

In the last equality we used the facts: $u \equiv 1$ on $[x,\infty)$ and $M(\infty) = \int_0^\infty m(r)\,dr = 1$; and to obtain the inequality we discarded the positive integral $\int_0^x u(s)\nu(s)m(s)\,ds$.

Thus we obtain, upon rearranging the last inequality,

$$\int_0^\infty u(r)m(r)\,dr \ge 1/[1+\theta\nu(x)] + \theta[1-M(x)]\nu(x)/[1+\theta\nu(x)].$$

It only remains to recall that $\mathrm{Cap}_\theta((x,\infty)) = \mathscr{E}_\theta(u,1) \equiv \theta\int_0^\infty u(r)m(r)\,dr$ [see Lemma 3.1.1(iv) of Fukushima (1980)]. Finally, $\int_1^\infty f^{-1}(r)\,dr = \infty$ implies that $\lim_{x\to\infty}\nu(x) = \infty$ by the Cauchy–Schwarz inequality since $\lim_{x\to\infty} M(x) = 1$. □

The distributional limiting behaviour of $\tau \equiv \tau_x := x$-level crossing time, as $x \to \infty$, of the diffusion associated with $\mathscr{E}$, can be easily deduced from the previous proposition [cf. also Mandl (1968)].

COROLLARY (3.7). *Let f and m satisfy the hypotheses of Proposition* (3.2). *Then, under P_m, $\tau_x/\nu(x)$ converges in distribution as $x \to \infty$ to an exponential random variable* (*with mean* 1).

PROOF. Define

$$w(r;x) := \begin{cases} [1+\theta\nu(r)]/[1+\theta\nu(x)], & 0 \le r \le x, \\ 1, & x < r. \end{cases}$$

As in Proposition (3.2), writing simply τ for τ_x,

$$\begin{aligned} \theta E_m e^{-\theta\tau} &= \mathrm{Cap}_\theta((x,\infty)) \le \mathscr{E}_\theta(w,w) \\ (*) \qquad &\le \theta^2\nu(x)/(1+\theta\nu(x))^2 + \left[\theta/(1+\theta\nu(x))^2\right] \\ &\quad \times \int_0^x [1+\theta\nu(r)]^2 m(r)\,dr + \theta(1-M(x)). \end{aligned}$$

Also, by (3.3)

$$(**) \qquad \frac{\theta}{1+\theta\nu(x)}[1+\theta\nu(x)(1-M(x))] \le \mathrm{Cap}_\theta((x,\infty)) = \theta E_m e^{-\theta\tau}.$$

Cancelling θ from the inequalities $(*)$ and $(**)$, and then replacing θ with $\theta/\nu(x)$ yields

$$\begin{aligned} &\frac{1}{1+\theta} + \frac{\theta}{1+\theta}[1-M(x)] \\ (***) \qquad &\le E_m e^{-\theta\tau/\nu(x)} \\ &\le \frac{\theta}{(1+\theta)^2} + \frac{1}{(1+\theta)^2}\int_0^x \left[1+\theta\frac{\nu(r)}{\nu(x)}\right]^2 m(r)\,dr + (1-M(x)). \end{aligned}$$

By the dominated convergence theorem

$$\lim_{x\to\infty}\int_0^x\left[1+\theta\frac{\nu(r)}{\nu(x)}\right]^2 m(r)\,dr = 1.$$

Letting $x \to \infty$ in $(***)$ yields that $\lim_{x\to\infty} E_m e^{-\theta\tau/\nu(x)} = 1/(1+\theta)$. □

We now give conditions on f and m which ensure that the term $\nu(x)^{-1}$, in the estimate at (3.3) is the dominant one.

LEMMA (3.8). *Let f and m be as in Proposition* (3.2). *Then the following are equivalent*:

$$\frac{1}{f(x)} = \frac{1}{M(x)}\left[\int_x^\infty \frac{m(s)}{\varepsilon(s)}\,ds\right]^{-2}\frac{m(x)}{\varepsilon(x)} \quad \text{for a.e. } x > x_0 \tag{3.9}$$

where $\varepsilon(x)$ is positive and measurable and $\nu(x_0) = [\int_{x_0}^\infty (m(s)/\varepsilon(s))\,ds]^{-1}$.

$$\varepsilon(x) = \frac{f(x)m(x)}{M(x)}\nu(x)^2 < \infty \quad \text{for a.e. } x > x_0 \tag{3.10}$$

for some $x_0 > 0$.

PROOF. Assuming (3.9)

$$\frac{M(x)}{f(x)} = \frac{d}{dx}\left[\int_x^\infty \frac{m(s)}{\varepsilon(s)}\,ds\right]^{-1} \quad \text{a.e.}$$

Integrating from x_0 to x,

$$\nu(x) - \nu(x_0) = \left[\int_x^\infty \frac{m(s)}{\varepsilon(s)}\,ds\right]^{-1} - \left[\int_{x_0}^\infty \frac{m(s)}{\varepsilon(s)}\,ds\right]^{-1}$$

so $\int_x^\infty (m(s)/\varepsilon(s))\,ds = \nu(x)^{-1}$; hence

$$\varepsilon(x) = \frac{f(x)m(x)}{M(x)}\nu(x)^2 \quad \text{a.e. } x.$$

On the other hand if (3.10) holds, then

$$\int_x^\infty \frac{m(s)}{\varepsilon(s)}\,ds = \int_x^\infty \frac{M(s)}{f(s)}\nu(s)^{-2}\,ds = \nu(x)^{-1} < \infty, \tag{3.11}$$

since $\lim_{x\to\infty}\nu(x) = \infty$; so $\nu(x_0) = [\int_{x_0}^\infty (m(s)/\varepsilon(s))\,ds]^{-1} < \infty$. Also

$$\frac{1}{M(x)}\left[\int_x^\infty \frac{m(s)}{\varepsilon(s)}\,ds\right]^{-2}\frac{m(x)}{\varepsilon(x)} = \frac{1}{f(x)} \quad \text{a.e.}$$

□

PROPOSITION (3.12). *Let f and m be as in Lemma* (3.8). *Suppose $\varepsilon(x)$ in representation* (3.10) *is such that $\lim_{x\to\infty}\varepsilon(x) = 0$, then*

$$\lim_{x\to\infty} \nu(x)\int_0^x v(r;x)^2 m(r)\,dr = 0$$

and

$$\lim_{x\to\infty} \nu(x)(1 - M(x)) = 0.$$

PROOF. Let $\eta > x_0$ [cf. (3.10)]. Then

$$\begin{aligned}
&\limsup_{x\to\infty} \nu(x)\int_0^x v(r;x)^2 m(r)\,dr \\
&\quad= \limsup_{x\to\infty}\left[\int_0^x \varepsilon(r)\frac{M(r)}{f(r)}\right]dr\bigg/\nu(x) \\
&\quad\le \limsup_{x\to\infty}\left[\int_0^\eta \varepsilon(r)\frac{M(r)}{f(r)}\,dr\bigg/\nu(x) + \sup\{\varepsilon(r)\colon r\ge\eta\}\int_0^x \frac{M(r)}{f(r)}\,dr\bigg/\nu(x)\right] \\
&\quad\le \sup\{\varepsilon(r)\colon r\ge\eta\}.
\end{aligned}$$

The first result follows since η can be chosen arbitrarily large and $\lim_{x\to\infty}\varepsilon(x) = 0$. Next, using (3.11),

$$\begin{aligned}
\limsup_{x\to\infty}\nu(x)(1 - M(x)) &= \limsup_{x\to\infty}\int_x^\infty m(r)\,dr\bigg/\int_x^\infty \frac{m(r)}{\varepsilon(r)}\,dr \\
&\le \limsup_{x\to\infty}\{\varepsilon(r)\colon r\ge x\} = 0. \qquad \square
\end{aligned}$$

LEMMA (3.13). *Let m, δ be nonnegative Borel functions on $\mathbb{R}_+$ such that $m^{-1}\in L^1_{\mathrm{loc}}((x_0,\infty);dr)$ for some $x_0\ge 0$. Then the following are equivalent:*

$$m(x) = \delta(x)\exp\left\{-\int_{x_0}^x \frac{dr}{\delta(r)}\right\} < \infty \quad a.e.\ x > x_0, \tag{3.14}$$

$$\delta(x) = m(x)\left(1 + \int_{x_0}^x \frac{dr}{m(r)}\right) < \infty \quad a.e.\ x > x_0. \tag{3.15}$$

PROOF. If (3.15) holds then integrating $1/\delta(x)$ gives

$$\int_{x_0}^x \frac{1}{\delta(r)}\,dr = \int_{x_0}^x \frac{d}{dr}\log\left(1 + \int_{x_0}^r \frac{ds}{m(s)}\right)dr = \log\left(1 + \int_{x_0}^x \frac{dr}{m(r)}\right).$$

Hence

$$\exp\left\{-\int_{x_0}^{x}\frac{dr}{\delta(r)}\right\} = \frac{1}{1+\int_{x_0}^{x}m^{-1}(r)\,dr} = \frac{m(x)}{\delta(x)} \quad \text{a.e. by (3.15)}$$

so (3.14) follows.

Conversely if (3.14) holds, then for a.e. x,

$$\begin{aligned} m(x)\left(1+\int_{x_0}^{x}\frac{dr}{m(r)}\right) &= \delta(x)\exp\left\{-\int_{x_0}^{x}\frac{dr}{\delta(r)}\right\}\left(1+\int_{x_0}^{x}\frac{1}{\delta(r)}\exp\left\{\int_{x_0}^{r}\frac{ds}{\delta(s)}\right\}dr\right) \\ &= \delta(x)\exp\left\{-\int_{x_0}^{x}\frac{dr}{\delta(r)}\right\}\exp\left\{\int_{x_0}^{x}\frac{dr}{\delta(r)}\right\} \\ &= \delta(x). \end{aligned}$$

Hence (3.15) holds. □

PROPOSITION (3.16). *Suppose $f(r) = a(r)m(r)$ where f and m are as in Proposition* (3.2) *and where a is bounded above and away from* 0 *for large r. If $m(x)$ is represented as in* (3.14), *then $\delta(x) \to 0$ as $x \to \infty$ is a necessary and sufficient condition for $\varepsilon(x) \to 0$ as $x \to \infty$, where $\varepsilon(x)$ is given by* (3.10).

PROOF. Suppose $\delta(x) \to 0$ as $x \to \infty$ in representation (3.14). Then by (3.10) $\varepsilon(x) = a(x)m(x)^2\nu(x)^2/M(x)$, so with $\bar{a} := \limsup a(x)$ and $\underline{a} := \liminf a(x)$,

$$\begin{aligned} \limsup_{x\to\infty}\varepsilon(x) &\le \bar{a}\left(\limsup_{x\to\infty} m(x)\nu(x)\right)^2 \\ &= \bar{a}\left(\limsup_{x\to\infty} m(x)\int_0^x \frac{M(r)}{a(r)m(r)}\,dr\right)^2 \\ &\le \bar{a}\left(\limsup_{x\to\infty} m(x)\left[(\underline{a})^{-1}\int_{x_0}^{x}\frac{dr}{m(r)} + \int_0^{x_0}\frac{M(r)}{a(r)m(r)}\,dr\right]\right)^2 \\ &= 0, \end{aligned}$$

by (3.15) since $\lim_{x\to\infty}\delta(x) = 0$.

The converse follows similarly. □

EXAMPLE (3.17). Consider the form

$$\mathscr{E}(u,v) = \int_0^\infty u'(r)v'(r)be^{-(b/a)r}\,dr \quad \text{in } L^2(\mathbb{R}_+; m(r)\,dr),$$

where $m(r) = (b/a)e^{-(b/a)r}$ which corresponds to the diffusion $dx(t) = -b\,dt + \sqrt{2a}\,dB(t)$ with reflection at 0. Then clearly $f(r) = be^{-(b/a)r} = am(r)$. Moreover by the representation (3.15) for $\delta(x)$ we see $\lim_{x\to\infty}\delta(x) \ne 0$ so the criterion of Proposition (3.16) fails (barely). We note that

$$1 - M(x) = e^{-(b/a)x},$$

$$\nu(x) = \int_0^x \frac{M(r)}{f(r)}\,dr = \int_0^x \frac{(1 - e^{-(b/a)r})}{b} e^{(b/a)r}\,dr = \frac{a}{b^2}e^{(b/a)x} - \frac{a}{b^2} - \frac{x}{b}$$

and $\nu(x)/\int_0^x v(r)^2 m(r)\,dr \to a/b^2$ as $x \to \infty$ by l'Hôpital's rule. Hence $(1 - M(x))$, $\nu(x)^{-1}$ and $\int_0^x v(r)^2 m(r)\,dr$ in (3.3) are all of the same order.

EXAMPLE (3.18). Consider the Ornstein–Uhlenbeck form

$$\mathscr{E}(u,v) = \int_0^\infty au'(r)v'(r)m(r)\,dr \quad \text{in } L^2(\mathbb{R}_+; m(r)\,dr),$$

where

$$m(r) = \sqrt{2\lambda/\pi a}\exp\left\{-\frac{\lambda}{2a}r^2\right\}.$$

One easily checks that the criterion of Proposition (3.16) holds via (3.15). $\mathscr{E}$ corresponds to the diffusion $dx(t) = -\lambda x(t)\,dt + \sqrt{2a}\,dB(t)$ with reflection at 0.

PROPOSITION (3.19). *Let $x(\cdot)$ be a reversible diffusion associated with the form*

$$\mathscr{E}(u,v) = \int_0^\infty f(r)u'(r)v'(r)\,dr \quad in\ L^2(\mathbb{R}_+; m(r)\,dr),$$

where f and m are nonnegative Borel functions on $\mathbb{R}_+$ such that m^{-1} and $f^{-1} \in L^1_{\text{loc}}((0,\infty); dr)$; $\int_0^\infty m(r)\,dr = 1$; $f \in L^1_{\text{loc}}(\mathbb{R}_+; dr)$; $\int_1^\infty f^{-1}(r)\,dr = \infty$; and $M/f \in L^1_{\text{loc}}(\mathbb{R}_+; dr)$, where $M(x) = \int_0^x m(r)\,dr$. Let $\nu(x) = \int_0^x M(r)/f(r)\,dr$ and $\varepsilon(x) = f(x)m(x)\nu(x)^2/M(x)$. Then if

$$\lim_{x\to\infty} \varepsilon(x) = 0$$

we have

$$\lim_{x\to\infty} \nu(x)\mathrm{Cap}_\theta((x,\infty)) = 1 \tag{3.20}$$

and

$$\lim_{x\to\infty} \nu(x)P\Big(\sup_{t\in[0,T]} x(t) > x\Big) = T. \tag{3.21}$$

In particular if $a(r) := f(r)/m(r)$ is also bounded from above and away from 0 for large r then $\lim_{x\to\infty}\varepsilon(x) = 0$ if and only if $\lim_{x\to\infty}\delta(x) = 0$, where m is represented in subexponential form (3.14) with δ given in (3.15).

PROOF. Using (3.3) along with Proposition (3.12), we get (3.20). The last statement of the proposition is Proposition (3.16). Now $\mathrm{Cap}_\theta((x,\infty)) = \theta E_m e^{-\theta\tau_x}$; so upon dividing through the inequalities (3.3) by θ and replacing θ

by $\theta/\nu(x)$ we obtain, by Proposition (3.12)

$$\frac{1}{1+\theta} \le E_m e^{-\theta\tau_x/\nu(x)} \le \frac{1}{\theta} + o\big(\nu(x)^{-1}\big), \tag{3.22}$$

where the o-term is uniform with respect to θ as $x \to \infty$.

Set

$$l_x(\theta) \equiv E_m e^{-\theta\tau/\nu(x)} = \int_0^\infty e^{-\theta t} P_m(\tau_x/\nu(x) \in dt) \equiv \int_0^\infty e^{-\theta t}\, dU_x(t)$$

and

$$L_x(\theta) \equiv l_x(\theta\nu(x))/l_x(\nu(x)) = \int_0^\infty e^{-\theta t}\, d\big[U_x(t)/l_x(\nu(x))\big].$$

By (3.22)

$$\begin{aligned}(1+\theta\nu(x))^{-1}/\big(\nu(x)^{-1} + o\big(\nu(x)^{-1}\big)\big)&\\ \le L_x(\theta) \le \big([\theta\nu(x)]^{-1} + o\big(\nu(x)^{-1}\big)\big)/(1+\nu(x))^{-1}&.\end{aligned}$$

Therefore $\lim_{x\to\infty} L_x(\theta) = \theta^{-1} = \int_0^\infty e^{-\theta t}\, dt$.

By the extended continuity theorem for Laplace transforms [cf. Feller (1971), page 433],

$$\lim_{x\to\infty} U_x(t)/l_x(\nu(x)) = t.$$

Substituting $\theta = \nu(x)$ into (3.22) yields that $\lim_{x\to\infty}\nu(x) l_x(\nu(x)) = 1$, and so

$$\lim_{x\to\infty} \nu(x) P_m(\tau_x \le T) = \lim_{x\to\infty} \frac{U_x(T)}{l_x(\nu(x))} = T. \qquad \square$$

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