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EXTREME VALUES AND HIGH BOUNDARY CROSSINGS OF LOCALLY STATIONARY GAUSSIAN PROCESSES

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We consider the large values of a locally stationary Gaussian process which satisfies Berman's condition on the long range dependence. The paper presents some limit results on the exceedances of the process above a certain general smooth high boundary. This allows deriving the limiting distribution of the maximum up to time T, for example, in the case of a standardized process with a constant boundary or in the case of a nonstandardized process with a smooth trend.

1. Introduction. Let $\{X(t),\ t \geq 0\}$ be a real Gaussian process with mean 0, variance 1 and continuous sample functions. In the following we deal with the maximum of X up to time T: $M_T = \max\{X(t),\ 0 \leq t \leq T\}$ and its asymptotic distribution as $T \to \infty$. Usually $M_T \to \infty$ as $T \to \infty$, which implies introducing some normalization. Since the Gaussian process is standardized, we allow the normalization to depend on t also; for, any Gaussian process with trend and nonconstant variance function can be transformed into a standardized one, where the standardization implies the moving boundary. This general boundary we denote by $\{u_T(t),\ t \leq T\}$, being a continuous real function for every T with $u_{T,\min} = \min\{u_T(t),\ t \leq T\} \to \infty$ as $T \to \infty$.

In the stationary case, the limiting distribution of M_T is known to be of Gumbel type [Pickands (1969), Berman (1971), Qualls and Watanabe (1972), Lindgren, Maré and Rootzén (1975) and Leadbetter, Lindgren and Rootzén (1983)], assuming that the correlation function satisfies the condition

$$r(\tau) = 1 - C|\tau|^\alpha + o(|\tau|^\alpha) \quad \text{as } \tau \to 0, \tag{1}$$

with $0 < \alpha \leq 2$, and Berman's condition

$$\sup_{t \geq \tau} r(t) \log \tau \to 0 \quad \text{as } \tau \to \infty. \tag{2}$$

The important quantity in this derivation is the local behaviour of M_h for small h. We have for fixed $h > 0$,

$$P\{M_h > u\} \sim C^{1/\alpha} H_\alpha \phi(u) u^{2/\alpha - 1} h \quad \text{as } u \to \infty, \tag{3}$$

where $\phi(u)$ denotes the density of a unit normal law and H_α is a positive constant depending only on α [cf. Leadbetter, Lindgren and Rootzén (1983)]. We denote

$$\psi(u) = \phi(u) u^{2/\alpha - 1}.$$

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For the convergence of the distribution of M_T one has to find $u_T = a_T x + b_T$ such that

$$C^{1/\alpha} H_\alpha T \psi(u_T) \to \tau = \tau(x) < \infty \quad \text{as } T \to \infty.$$

This, in the stationary case, is the same as

$$\sum_{j=1}^{n} P\{M(I_j) > u_T\} \to \tau, \tag{4}$$

with $I_j = ((j-1)h, jh]$, $j \le n = [T/h]$ and $M(I_j) = M(I_j, X) = \sup\{X(t), t \in I_j\}$.

Our aim in this paper is to extend these results to the case with a moving boundary and certain nonstationary Gaussian processes. The case of a moving barrier for Gaussian processes is discussed, e.g., in Berman (1974) and Cuzick (1981). Berman deals with the limiting probability of M_T above a boundary which tends to ∞ but with a fixed T. Cuzick treats stationary Gaussian processes and very high boundaries where T may be fixed or may tend to ∞. Our case differs by using rather smooth boundaries with respect to nonstationary Gaussian processes; therefore, our results are similar to the statements in the stationary case with a fixed level. We restrict the class of nonstationary processes by dealing with locally stationary Gaussian processes, which were introduced by Berman (1974).

The Gaussian process is called *locally stationary* if there exist a continuous function $C(t)$, $t \ge 0$, with $0 < \min\{C(t), t \ge 0\} \le \sup\{C(t), t \ge 0\} < \infty$ and a continuous monotone function $K(s)$ with $K(0) = 0$, $K(s) > 0$ $(s > 0)$ such that

$$\lim_{s \to 0} \frac{E(X(t+s) - X(t))^2}{2K(|s|)} = C(t) \quad \text{uniformly in } t \ge 0. \tag{5}$$

Following Berman (1974), we assume that $K(s)$ is regularly varying with index α, $0 < \alpha \le 2$, as $s \to 0$. For simplicity, we assume that

$$K(s) = s^\alpha + o(s^\alpha) \quad \text{as } s \to 0. \tag{6}$$

[Note that a possible constant C with the term s^α can be moved to the function $C(t)$.] This implies that $r(t, t+s) = 1 - C(t)|s|^\alpha + o(|s|^\alpha)$, uniformly in t, and that for small h and $I_t = [t, t+h]$,

$$P\{M(I_t) > u\} \sim C^{1/\alpha}(t) H_\alpha \psi(u) h$$

(see Section 2). Instead of (4) we have now to consider sums of the form

$$\sum_{j=1}^{n} P\{M(I_j) > u_{T,j}\} \sim \sum_{j=1}^{n} C^{1/\alpha}(jh) H_\alpha \psi(u_{T,j}) h,$$

with $u_{T,j}$ depending on $u_T(t)$ on I_j for small h and $T \to \infty$. If h tends to 0, the sum will be approximated by a Riemann integral, which should converge as $T \to \infty$:

$$(7) \qquad J(T) = H_\alpha \int_0^T C^{1/\alpha}(t)\psi(u_T(t))\,dt \to \tau < \infty \quad \text{as } t \to \infty.$$

The approximation of the integral by upper and lower sums has to be sufficiently accurate which restricts the behaviour of $u_T(t)$. Obviously, the Riemann integral (7) can be approximated for every T by using $h = h(T)$, such that

$$\begin{aligned}\Delta(T,h) &= J^-(T) - J^+(T)\\ &= \sum_{j=1}^{n} \left(C_-^{1/\alpha}(jh)\psi(u_T^-(jh)) - C_+^{1/\alpha}(jh)\psi(u_T^+(jh))\right)h \le \Delta\end{aligned}$$

for some fixed $\Delta > 0$, with

$$u_T^-(jh) = \inf\left(u_T(t), t \in I_j\right) \le \sup\left(u_T(t), t \in I_j\right) = u_T^+(jh)$$

and

$$C_+(jh) = \inf\left(C(t), t \in I_j\right) \le \sup\left(C(t), t \in I_j\right) = C_-(jh).$$

As $T \to \infty$, $h(T)$ may tend fast to 0 for a given Δ. We have to restrict $h(T)$ such that $h(T)$ tends slowly enough to 0. We assume that there exists $h(T)$ such that as $T \to \infty$,

$$(8) \qquad h(T)\left(u_{T,\min}^2/\log u_{T,\min}\right)^{1/\alpha} \to \infty$$

and

$$(9) \qquad \Delta(T, h(T)) \to 0.$$

Because of the nonstationarity we have to rewrite condition (2). We denote $\delta(\tau) = \sup\{r(t,s), |t-s| \ge \tau\}$. Berman's condition becomes now

$$(10) \qquad \delta(\tau)\log\tau \to 0 \quad \text{as } \tau \to \infty, \text{ with } r(t,s) = 1 \text{ iff } t = s.$$

Under these assumptions we deal with the limiting behaviour of $P\{X(t) \le u_T(t), t \le T\}$ tending to $\exp(-\tau)$, which is equivalent to

$$P\{X(t) \le u_T(t), t \le T\} - \exp(-J(T)) \to 0 \quad \text{as } T \to \infty.$$

The limiting distribution of the maximum M_T depends then on the suitable choice of the normalization $u_T(t) = u_{T,x}(t)$ for $\tau = \tau(x)$. This idea is used in Section 4 to find the asymptotic distribution of M_T with respect to a locally stationary Gaussian process, not only for the case with mean 0, but also for the case of a nonstandardized process. It shows the influence of the function $C(t)$ in the normalization for M_T with respect to nonstationary processes. The role of α in the normalization is the same as in the stationary case, discussed by Pickands (1969).

2. The maximum on small intervals. Because of the local structure of X with a varying constant $C(t)$, we have to select $h = h(T) \to 0$ in the limiting procedure. Therefore, we deal first with the question of the maximum in small intervals of length h. If h is very small with respect to u, then $P\{\max(X(t),\ t \in [s, s+h]) > u\} \sim 1 - \Phi(u)$ with Φ being the unit normal law. Note that we assume in (7) that h is not too small, which excludes this limiting behaviour. As mentioned, for $h > 0$ fixed, the approximation (3) holds in the stationary case, as well as in the locally stationary case with an adjustment of the constant [cf. Berman (1972)]. By checking the proof of the stationary result in Leadbetter, Lindgren and Rootzén (1983), Section 12.2, we find that the following statement is true in the stationary case, if $h \to 0$.

THEOREM 2.1. *Let X be a standardized stationary Gaussian process. If $r(t)$ satisfies* (1) *with* $0 < \alpha \le 2$ *and* $h = h(u) \to 0$ *satisfies* $hu^{2/\alpha} \to \infty$ $(u \to \infty)$, *then*

$$\lim_{u \to \infty} P\{M(h) > u\}/h\psi(u) = C^{1/\alpha} H_\alpha.$$

The proof follows the lines of several lemmas of the mentioned textbook, which we do not repeat in this paper. More important is the extension of this result for locally stationary Gaussian processes.

THEOREM 2.2. *Let X be a standardized locally stationary Gaussian process, satisfying* (5) *and* (6) *with* $0 < \alpha \le 2$. *If* $h = h(u)$ *satisfies* $hu^{2/\alpha} \to \infty$ $(u \to \infty)$, *then*

$$\lim_{u \to \infty} P\{\sup(X(t),\ s \le t \le s+h) > u\}/h\psi(u) = C^{1/\alpha}(s) H_\alpha$$

uniformly in s.

PROOF. $r_1(t) = 1/(1 + C_1|t|^\alpha)$ is a correlation function [cf. Lukacs (1970)] with $r_1(t) = 1 - C_1|t|^\alpha + o(|t|^\alpha)$, as $t \to 0$. Let $\varepsilon > 0$ be small and C_1 such that $r(t, t') \le r_1(|t - t'|)$ for every $t, t' \in [s, s+\varepsilon]$. In the same way we get a lower bound of $r(t, t')$ by using a suitable constant C_2 instead of C_1, where $C_2 \ge C_1 > 0$. We denote by Y_1 and Y_2 the corresponding stationary standardized Gaussian processes on $[s, s+\varepsilon]$. By Slepian's lemma [see Leadbetter, Lindgren and Rootzén (1983)] we have for any h, u, with $h \le \varepsilon$ (hence for all u sufficiently large),

$$P\{\max(Y_1(t),\ s \le t \le s+h) > u\} \le P\{\max(X(t),\ s \le t \le s+h) > u\}$$

$$\le P\{\max(Y_2(t),\ s \le t \le s+h) > u\}.$$

By using Theorem 2.1 we get

$$C_1^{1/\alpha} H_\alpha \le \liminf_{u \to \infty} P\{\max(X(t), s \le t \le s+h) > u\}/h\psi(u)$$
$$\le \limsup_{u \to \infty} P\{\max(X(t), s \le t \le s+h) > u\}/h\psi(u) \le C_2^{1/\alpha} H_\alpha.$$

Since ε is arbitrary, $C_1 \to C(s)$ and $C_2 \to C(s)$ uniformly in s as $\varepsilon \to 0$. This implies our statement. □

3. Some lemmas. For the proof of the results in the next section we use the following comparison lemmas. They are based on the ideas which have been introduced in the stationary continuous time case [see, e.g., Leadbetter, Lindgren and Rootzén (1983)] and in the nonstationary discrete time case [Hüsler (1983), (1986)]. We assume in this section that X is a locally stationary standardized Gaussian process with some α, $0 < \alpha \le 2$, and that $\limsup_{T \to \infty} J(T) < \infty$ holds with respect to a boundary $u_T(\cdot)$ with $u_{T,\min} \to \infty$. It means that

$$\limsup_{n \to \infty} \sum_{j=1}^{n} \psi(u_T^-(jh))h < \infty. \tag{11}$$

Note that we may replace $u_T^-(\cdot)$ in (11) by $u_T^+(\cdot)$ and that (7) obviously implies (11). Most of the following proofs are based on this assumption and do not depend on the particular choice of $u_T^-(\cdot)$. Instead of thusly writing some of the statements for $u_T^+(\cdot)$ and for $u_T^-(\cdot)$, we formulate a particular statement with $u_T^*(\cdot)$ and mean that it holds for both particular boundaries [or even any piecewise constant boundary satisfying (11)]. We use also the notation $u_T^*(t) = u_T^*(jh)$ for $t \in I_j$, as well as $u_T^+(t) = \sup(u_T(s),\ s \in (nh, T])$, $u_T^-(t) = \inf(u_T(s),\ s \in (nh, T])$ and $u_T^*(t) = u_T^*(T)$ for $t \in (nh, T]$.

LEMMA 3.1. *Assume that* (11) *holds with respect to some boundary* $\{u_T(\cdot)\}$. *Let* $I_j = ((j-1)h, jh]$ *be subintervals of* $[0, T]$, $j \le n = [T/h]$, *of length* $h = h(T) \to 0$, *satisfying* (8) *and* (9). *Let* $\varepsilon = \varepsilon(T) < h(T)$ *and* $I_j^* = I_j \setminus (jh - \varepsilon, jh]$. *Then as* $T \to \infty$,

(i)

$$0 \le P\{X(t) \le u_T^+(t), t \le T\} - P\{X(t) \le u_T(t), t \le T\} \to 0 \text{ and}$$
$$0 \le P\{X(t) \le u_T(t), t \le T\} - P\{X(t) \le u_T^-(t), t \le T\} \to 0. \tag{12}$$

(ii) *For any boundary* $u_T^*(t)$ *satisfying* (11)

$$0 \le P\left\{X(t) \le u_T^*(t), t \in \bigcup_{j=1}^{n} I_j^*\right\} - P\{X(t) \le u_T^*(t), t \le T\} \to 0 \tag{13}$$

as $T \to \infty$, *if* $\varepsilon(T)/h(T) \to 0$ *and* $\varepsilon(T)(\min(u_T^*(t), t \le T))^{2/\alpha} \to \infty$.

PROOF. (i) Obviously, the differences of the two probabilities in (12) are nonnegative. The first difference is bounded above by

$$\sum_{j=1}^{n} P\Big\{X(t) \le u_T^+(t), t \in I_j, \sup_{s \in I_j} (X(s) - u_T(s)) > 0\Big\}$$

$$+P\Big\{X(t) \le u_T^+(t), t \in \big(nh, T\big], \sup_{s \in (nh,T]} (X(s) - u_T(s)) > 0\Big\}$$

$$\le \sum_{j=1}^{n} P\{u_T^-(jh) \le M(I_j) \le u_T^+(jh)\}$$

$$+ P\{u_T^-(T) \le M((nh,T]) \le u_T^+(T)\}.$$

Note that by (8), $hu_T^{*2/\alpha}(jh) \to \infty$ for all $j \ge 1$. Using Theorem 2.2 we get for this upper bound

$$\sum_{j=1}^{n} h\big(C_-^{1/\alpha}(jh) H_\alpha \psi(u_T^-(jh))(1 + o(1))$$

$$-C_+^{1/\alpha}(jh) H_\alpha \psi(u_T^+(jh))(1 + o(1))\big)$$

$$+ h\big(C_-^{1/\alpha}(T) H_\alpha \psi(u_T^-(T))(1 + o(1))$$

$$-C_+^{1/\alpha}(T) H_\alpha \psi(u_T^+(T))(1 + o(1))\big),$$

with $o(1)$ uniformly in $j \le n$.

Hence the sum is bounded by

$$H_\alpha \sum_{j=1}^{n} h\big(C_-^{1/\alpha}(jh)\psi(u_T^-(jh)) - C_+^{1/\alpha}(jh)\psi(u_T^+(jh))\big) + o(1),$$

which tends to 0 by (9). In the same way, the second difference in (12) tends to 0.

(ii) Also the difference of the probabilities in (13) is nonnegative and bounded by

$$\sum_{j=1}^{n} P\{\sup(X(t), t \in (jh - \varepsilon, jh]) > u_T^*(jh)\}$$

$$+ P\{\sup(X(t), t \in (nh,T]) > u_T^*(T)\}$$

$$\le O(1)(\varepsilon/h) \sum_{j=1}^{n} h\psi(u_T^*(jh)) + o(1) \to 0$$

as $T \to \infty$ by the assumptions. □

Note that we did not use an assumption on the correlation function besides (5) and (6). Hence the statement (13) is also true in the version

$$\prod_{j=1}^{n} P\{X(t) \le u_T^*(t), t \in I_j^*\} - \prod_{j=1}^{n} P\{X(t) \le u_T^*(t), t \in I_j\} \to 0.$$

The next lemma relates the continuous time case to the discrete time case.

LEMMA 3.2. *With the same notations and assumptions as in Lemma* 3.1, *let*

$$q_j = q_0 / u_T^{*2/\alpha}(jh) \tag{14}$$

for every $j \le n$. *Then*

$$0 \le \limsup_{T \to \infty} \big(P\{X(iq_j) \le u_T^*(jh), iq_j \in I_j, j \le n\}$$
$$-P\{X(t) \le u_T^*(jh), t \in I_j, j \le n\}\big) \to 0$$

as $q_0 \to 0$, *where* $h = h(T)$ *satisfies* (8).

PROOF. We use now the stationary Gaussian processes Y_1 and Y_2 of Section 2 to locally approximate the process X, where we adapt the constants $C_1 = C_{1,j}$ and $C_2 = C_{2,j}$ for every interval in an obvious way. Note that as in Lemma 3.1, the difference of probabilities in the statement of the lemma is nonnegative and is bounded above in the same way by

$$\sum_{j=1}^{n} \big(P\{X(iq_j) \le u_T^*(jh), iq_j \in I_j\} - P\{M(I_j) \le u_T^*(jh)\}\big) =: S^*.$$

By Slepian's lemma we again get for any $u = u_T^*(jh)$,

$$P\{X(iq_j) \le u, iq_j \in I_j\} - P\{M(I_j, X) \le u\}$$
$$\le P\{Y_1(iq_j) \le u, iq_j \in I_j\} - P\{M(I_j, Y_2) \le u\}$$
$$\le \big(P\{Y_1(iq_j) \le u, iq_j \in I_j\} - P\{M(I_j, Y_1) \le u\}\big)$$
$$+ \big(P\{M(I_j, Y_1) \le u\} - P\{M(I_j, Y_2) \le u\}\big).$$

The first term is bounded by using Lemma 12.2.11 of Leadbetter, Lindgren and Rootzén (1983) (modified for $h \to 0$) by

$$hH_\alpha C_{1,j}^{1/\alpha} \psi(u) \rho(q_0) + \phi(u)/u,$$

with $\rho(q_0) \to 0$ as $q_0 \to 0$, not depending on $C_{1,j}$, since again

$$hu^{2/\alpha} = h(T) u_T^{*2/\alpha}(jh) \ge h(T) u_{T,\min}^{*2/\alpha} \to \infty$$

and $u^{*2/\alpha} q_j = q_0$. Observe that $\phi(u)/u = o(\psi(u)h)$, by the same reasoning.

The second term is bounded by using Theorem 2.1:

$$o(1)h\psi(u) + \left(C_{2,j}^{1/\alpha} - C_{1,j}^{1/\alpha}\right)hH_\alpha\psi(u)$$

uniformly in j, where $o(1) \to 0$ as $T \to \infty$.

Taking the sum, we observe that S^* is bounded by a constant times

$$(\rho(q_0) + o(1))\sum_{j=1}^{n} h\psi(u_T^*(jh)) + \sum_{j=1}^{n}\left(C_{2,j}^{1/\alpha} - C_{1,j}^{1/\alpha}\right)H_\alpha\psi(u_T^*(jh))h.$$

Letting $T \to \infty$, the lim sup of S is bounded by $O(\rho(q_0))$, using (9) and (11). Hence the statement follows as $q_0 \to 0$. □

Note that Lemma 3.2 also implies that

$$\limsup_{T\to\infty}\left(\prod_{j=1}^{n} P\{X(iq_j) \le u_T^*(jh),\, iq_j \in I_j\} - \prod_{j=1}^{n} P\{X(t) \le u_T^*(t),\, t \in I_j\}\right) \to 0$$

as $q_0 \to 0$. Both statements also hold if the I_j's are replaced by the I_j^*'s.

The next step is to show the asymptotic independence of exceedances above the boundary which occur in different time intervals I_j^*. This is true if we assume Berman's condition (10). But first we show that we may also delete certain intervals I_j from our consideration. This is important for the use of Berman's comparison lemma. It means that we delete intervals I_j which have a negligible influence in (7), but which get too much weight in the double sum of the comparison lemma (see Lemma 3.4). We use the technique which was introduced in Hüsler (1983), by adapting it to the new situation.

For given T and $h(T)$, define

$$a_1 = \min\{u_T^*(jh),\, j \le n\} \quad\text{and}\quad J_1 = \{j\colon a_1 \le u_T^*(jh) \le 2a_1\},$$

$$a_2 = \min\{u_T^*(jh) > a_1\} \quad\text{and}\quad J_2 = \{j\colon a_2 \le u_T^*(jh) \le 2a_2\}$$

and so on: a_m, J_m with $m \le H \le n$.

Let

$$f_m = \sum_{j\in J_m} \psi(u_T^*(jh))h/q_j$$

with q_j given in Lemma 3.2 and

$$G = \left\{m \le H\colon f_m \ge \exp\left(-a_m^2/4\right)\right\}.$$

Let $J_0 = \bigcup_{m\in G} J_m$. Note that for every $m \le H$,

$$f_m \ge \psi(a_m)ha_m^{2/\alpha}/q_0, \tag{15}$$

since there exists at least one $j \in J_m$ with $u_T^*(jh) = a_m$.

LEMMA 3.3. *Assume that* (11) *holds. Then*

$$\text{(i)}\quad 0 \le P\{X(iq_j) \le u_T^*(jh),\, iq_j \in I_j,\, j \in J_0\} - P\{X(iq_j) \le u_T^*(jh),\, iq_j \in I_j,\, j \le n\} \to 0 \quad \text{as } T \to \infty,$$

and also

$$\text{(ii)}\quad 0 \le \prod_{j \in J_0} P\{X(iq_j) \le u_T^*(jh),\, iq_j \in I_j\} - \prod_{j=1}^{n} P\{X(iq_j) \le u_T^*(jh),\, iq_j \in I_j\} \to 0.$$

Both statements hold also with I_j^* instead of I_j.

PROOF. Both differences are bounded by

$$\sum_{m \notin G} P\{X(iq_j) > u_T^*(jh) \text{ for some } iq_j \in I_j,\, j \in J_m\}$$
$$\le \sum_{m \notin G} \sum_{j \in J_m} \sum_{iq_j \in I_j} \phi(u_T^*(jh))/u_T^*(jh) \le \sum_{m \notin G} \sum_{j \in J_m} h\psi(u_T^*(jh))/q_0$$
$$\le \sum_{m \notin G} f_m/a_m^{2/\alpha}.$$

But

$$f_m \le \exp(-a_m^2/4) \le \exp(-a_{m-1}^2) \le f_{m-1} \exp(-a_{m-1}^2/2) a_{m-1}^{1-4/\alpha} c q_0/h$$

with $c = (2\pi)^{1/2}$ using (15), for $m > 1$, $m \notin G$. If $1 \notin G$, then $f_1/a_1^{2/\alpha} \le \exp(-a_1^2/4)a_1^{2/\alpha} \to 0$ as $T \to \infty$. Hence

$$\sum_{m \notin G} f_m/a_m^{2/\alpha}$$
$$\le o(1) + (cq_0/h) \sum_{m>1,\, m \notin G} a_{m-1}^{1-4/\alpha} \exp(-a_{m-1}^2/2) f_{m-1}/a_{m-1}^{2/\alpha} \tag{16}$$
$$\le o(1) + (cq_0/h) a_1^{1-2/\alpha} \exp(-a_1^2/2) \sum_{m=1}^{H} f_m/a_m^{2/\alpha}.$$

Now observe that $\Sigma_m f_m/a_m^{2/\alpha} = O(\Sigma_{j=1}^n h\psi(u_T^*(jh)))$, which is bounded by assumption. By (8) the term (16) converges to 0 as $T \to \infty$, for every $q_0 > 0$. □

Since we may restrict our considerations on the time subset J_0 we can now use Berman's comparison lemma [Berman (1964); cf. various versions in Leadbetter, Lindgren and Rootzén (1983)].

LEMMA 3.4. *If* (5), (6), (8), (10) *and* (11) *hold, we have that*

$$S = \sum_{iq_j \in I_j^*} \sum_{i'q_{j'} \in I_{j'}^*,\, j \neq j' \in J_0} \left| r(iq_j, i'q_{j'}) \right| \exp\Big(-\tfrac{1}{2}\big(u_T^{*2}(jh) + u_T^{*2}(j'h) \big) \Big/ \big(1 + r(iq_j, i'q_{j'})\big)\Big)$$

$$= \sum d(i, i') \to 0 \quad \text{as } T \to \infty$$

for every $q_0 > 0$.

PROOF. Note that all points $|iq_j - i'q_{j'}| \geq \varepsilon$ for all $j \neq j'$, while $\varepsilon = \varepsilon(T) \to 0$ as $T \to \infty$. First we consider the points i, i' with

$$(17) \qquad \varepsilon \leq |iq_j - i'q_{j'}| \leq \exp\big((1 - \delta) a_1^2 / (1 + \delta) 4\big) = \lambda = \lambda(T),$$

with $\delta = \delta(\varepsilon) < 1$ for fixed T; but $\delta(\varepsilon(T)) \to 1$ as $T \to \infty$. Observe that by (5) and (6), $(1 - \delta(\varepsilon(T)))/(1 + \delta(\varepsilon(T))) \geq c\varepsilon^\alpha(T)$ for all large T where the constant c is positive because $\inf C(t) > 0$. (17) implies that $\lambda > \exp(c\varepsilon^\alpha(T) a_1^2)$. Hence $\lambda \to \infty$, if we select ε such that $\varepsilon a_1^{2/\alpha}/(\log a_1)^{1/\alpha} \to \infty$ and $\varepsilon(T)/h(T) \to 0$ (because of Lemma 3.1). For example, $\varepsilon(T) = h(T)/y(T)$ with $y^2(T) = h(T) a_1^{2/\alpha}/(\log a_1)^{1/\alpha}$ satisfies both conditions. Our first partial sum of S is denoted by $S_1 = \sum d(i, i')$ with i, i' such that (17) holds.

(i) Split the sum S_1 into partial sums $S_{1,m}$ which add the terms $d(i, i')$ with $iq_j \in I_j^*$, $j \in J_m$, $i'q_{j'} \in I_{j'}^*$, $j' \in J_{m'}$, $m' \geq m$ and (17) holds.

If now $m < m'$, then for all $iq_j \in I_j^*$ with $j \in J_m$, there are at most λ/h intervals $I_{j'}^*$ $(j \neq j')$ with such points $i'q_{j'}$. In each of these intervals $I_{j'}^*$ there are at most $h/q_{j'}$ points $i'q_{j'}$ with the required property. Hence the sum on these points $i'q_{j'} \in I_{j'}^*$, $j' \in J_{m'}$, $iq_j \in I_j^*$ fixed, is bounded by a constant times

$$\exp\big(-\tfrac{1}{2} u_T^{*2}(j'h)/(1 + \delta) \big) h/q_{j'} \leq \exp\big(-\tfrac{1}{2} a_{m'}^2/(1 + \delta) \big) h a_{m'}^{2/\alpha}/q_0$$
$$\leq \exp\big(-\tfrac{1}{2} a_{m+1}^2/(1 + \delta) \big) h a_{m+1}^{2/\alpha}/q_0,$$

since $u_T^*(j'h) \geq a_{m'} \geq a_{m+1}$ for $m' > m$. If $m = m'$, then by using

$$\exp(-x - y) \leq \exp(-2x) + \exp(-2y)$$

and the same argument again, we get that the sum $S_{1,m}$ is bounded by

$$\sum_{iq_j \in I_j^*,\, j \in J_m} \exp\big(-\tfrac{1}{2} u_T^{*2}(jh)/(1 + \delta) \big) \exp\big(-\tfrac{1}{2} a_{m+1}^2/(1 + \delta) \big) h a_{m+1}^{2/\alpha} \lambda / h q_0$$
$$+ \big(\lambda h a_m^{2/\alpha}/h q_0 \big) \sum_{iq_j \in I_j^*,\, j \in J_m} \exp\big(-u_T^{*2}(jh)/(1 + \delta) \big)$$
$$+ \big(\lambda h a_m^{2/\alpha}/h q_0 \big) \sum_{i'q_{j'} \in I_{j'}^*,\, j' \in J_m} \exp\big(-u_T^{*2}(j'h)/(1 + \delta) \big)$$
$$\leq c a_m \Big(\big(\lambda a_{m+1}^{2/\alpha}/q_0 \big) \exp\big(-\tfrac{1}{2} a_{m+1}^2 (1 - \delta)/(1 + \delta) \big)$$
$$+ 2 \big(\lambda a_m^{2/\alpha}/q_0 \big) \exp\big(-\tfrac{1}{2} a_m^2 (1 - \delta)/(1 + \delta) \big) \Big) \sum_{j \in J_m} h \psi(u_T^*(jh))/q_0$$
$$\leq \big(c \lambda a_m^{1 + 2/\alpha} \exp\big(-\tfrac{1}{2} a_m^2 (1 - \delta)/(1 + \delta) \big)/q_0^2 \big) \sum_{j \in J_m} h \psi(u_T^*(jh))$$

with some constant $c > 0$.

Taking the sum of $S_{1,m}$ on all m, gives the upper bound

$$c\left[\lambda a_1^{1+2/\alpha}\exp\left(-\tfrac{1}{2}a_1^2(1-\delta)/(1+\delta)\right)/q_0^2\right]\sum_{j=1}^{n} h\psi(u_T^*(jh)).$$

Since the remaining terms with $m' < m$ are bounded in the same way, by interchanging m and m', it remains to prove that the term in the brackets tends to 0 as $T \to \infty$. By definition of $\lambda = \lambda(T)$, this is true for any $q_0 > 0$ since by (17),

$$\begin{aligned}a_1^{1+2/\alpha}\exp\left(-\tfrac{1}{4}a_1^2(1-\delta)/(1+\delta)\right) &\le \exp\left(-ca_1^2\varepsilon^\alpha + (1+2/\alpha)\log a_1\right)\\ &\le \exp\left(-(c+o(1))a_1^2\varepsilon^\alpha\right) \to 0 \quad \text{as } T \to \infty.\end{aligned}$$

(ii) Now we consider the sum on i, i' points such that

$$\lambda(T) \le |iq_j - i'q_{j'}| \le \gamma_{m,m'} \quad \text{and} \quad |iq_j - i'q_{j'}| \ge \gamma_{m,m'}$$

with $j \in J_m$, $j' \in J_{m'}$, where $\gamma_{m,m'} = \exp(\max(a_m^2, a_{m'}^2)/8)$. Let us denote these partial sums for given m, m' by $S'_{m,m'}$ and $S''_{m,m'}$, respectively. We also use $\delta' = \delta(\lambda(T))$ and $\delta'' = \delta(\gamma_{m,m'})$.

(a) We approximate first the term $S'_{m,m'}$. As in (i), we get for $m < m'$ that the exponential term in $d(i,i')$ is bounded by

$$\exp\left(-\tfrac{1}{2}u_T^{*2}(jh) - \tfrac{1}{2}u_T^{*2}(j'h)(1-2\delta')\right) \le \exp\left(-\tfrac{1}{2}u_T^{*2}(jh) - \tfrac{1}{2}a_{m'}^2(1-2\delta')\right),$$

using $u_T^*(jh) < u_T^*(j'h)$ and T sufficiently large ($\delta' < \frac{1}{2}$). If $m = m'$, we use again the bound

$$\begin{aligned}&\exp\left(-\tfrac{1}{2}u_T^{*2}(jh)(1-\delta') - \tfrac{1}{2}u_T^{*2}(j'h)(1-\delta')\right)\\ &\quad\le \exp\left(-\tfrac{1}{2}a_m^2(1-\delta') - \tfrac{1}{2}u_T^{*2}(j'h) + 2a_m^2\delta'\right)\\ &\quad\le \exp\left(-\tfrac{1}{2}u_T^{*2}(jh) - \tfrac{1}{2}a_m^2(1-5\delta')\right).\end{aligned}$$

Hence, together for all $m \le m'$, we use the factor $5\delta'$. For a fixed i there are at most $\gamma_{m,m'}/h$ intervals $I_{j'}^*$ with at most $h/q_{j'}$ points. Thus

$$\begin{aligned}S'_{m,m'} &\le c\delta'(\gamma_{m,m'}/h)\left(ha_{m'}^{2/\alpha}/q_0\right)\sum_{j\in J_m}\exp\left(-\tfrac{1}{2}u_T^{*2}(jh)\right)hu_T^{*2/\alpha}(jh)\\ &\qquad\times\exp\left(-\tfrac{1}{2}a_{m'}^2(1-5\delta')\right)/q_0\\ &\le c\delta'\left(\gamma_{m,m'}/q_0^2\right)a_{m'}^{2/\alpha}\exp\left(-\tfrac{1}{2}a_{m'}^2(1-5\delta')\right)\sum_{j\in J_m} h\psi(u_T^*(jh))2a_m.\end{aligned}$$

Now we derive for $m \in G$,

$$\begin{aligned}2^{2/\alpha}\sum_{j\in J_m} h\psi(u_T^*(jh)) &\ge \sum_{j\in J_m} h\psi(u_T^*(jh))q_0/\left(q_j a_m^{2/\alpha}\right)\\ &\ge f_m q_0/a_m^{2/\alpha} \ge q_0\exp\left(-\tfrac{1}{4}a_m^2\right)/a_m^{2/\alpha}.\end{aligned}$$

Inserting this inequality for the term $\exp(-\frac{1}{4}a_{m'}^2)$, it follows that

$$\begin{aligned} S'_{m,m'} &\le c\delta'(\gamma_{m,m'}/q_0^3)a_{m'}^{4/\alpha}a_m\exp\left(-\tfrac{1}{2}a_{m'}^2(\tfrac{1}{2}-5\delta')\right)\\ &\quad\times\sum_{j\in J_m}h\psi(u_T^*(jh))\sum_{j'\in J_{m'}}h\psi(u_T^*(j'h))\\ &\le c\delta'(a_1^{1+4/\alpha}/q_0^3)\exp\left(-\tfrac{1}{2}a_1^2(\tfrac{1}{4}-5\delta')\right)\\ &\quad\times\sum_{j\in J_m}h\psi(u_T^*(jh))\sum_{j'\in J_{m'}}h\psi(u_T^*(j'h)) \end{aligned}$$

for T sufficiently large ($5\delta' < \frac{1}{4}$).

The same holds by interchanging m with m'. Thus taking the sum on all m and m', we find that $\sum_{m,m'\in G}S'_{m,m'}$ is bounded by

$$c\delta'(a_1^{1+4/\alpha}/q_0^3)\exp\left(-\tfrac{1}{2}a_1^2(\tfrac{1}{4}-5\delta')\right)\left(\sum_{m\in G}\sum_{j\in J_m}h\psi(u_T^*(jh))\right)^2,$$

which tends to 0 as $T\to\infty$, for all $q_0>0$, since the last sum is again bounded.

(b) Finally, we deal with $S''_{m,m'}$. For the exponential term in the sum $S''_{m,m'}$, we use the bound

$$\begin{aligned} &\exp\left(-\tfrac{1}{2}u_T^{*2}(jh)(1-\delta'')-\tfrac{1}{2}u_T^{*2}(j'h)(1-\delta'')\right)\\ &\qquad\le\exp\left(-\tfrac{1}{2}u_T^{*2}(jh)+2\delta''a_m^2-\tfrac{1}{2}u_T^{*2}(j'h)+2\delta''a_{m'}^2\right). \end{aligned}$$

Hence for $m\le m'$, the sum $S''_{m,m'}$ is bounded by

$$\begin{aligned} &\delta''\exp(2\delta''a_m^2)\sum_{j\in J_m}\psi(u_T^*(jh))2a_m(h/q_0)\exp(2\delta''a_{m'}^2)\\ &\quad\times\sum_{j'\in J_{m'}}\psi(u_T^*(j'h))2a_{m'}(h/q_0)\\ &\qquad\le 4\delta''a_{m'}^2q_0^{-2}\exp(4\delta''a_{m'}^2)\sum_{j\in J_m}h\psi(u_T^*(jh))\sum_{j'\in J_{m'}}h\psi(u_T^*(j'h))\\ &\qquad\le 4\Delta_Tq_0^{-2}\exp(4\Delta_T)\sum_{j\in J_m}h\psi(u_T^*(jh))\sum_{j'\in J_{m'}}h\psi(u_T^*(j'h)), \end{aligned}$$

with $\Delta_T=\Delta(\gamma_{1,1})\ge\Delta(\gamma_{m,m'})$, where $\Delta(x)=\sup(8\delta(y)\log y;\ y\ge x)\to 0$ as $x\to\infty$ by (10). Note that $\gamma_{1,1}=\exp(a_1^2/8)\to\infty$ as $T\to\infty$, hence $\Delta_T\to 0$.

The same inequality holds for $S''_{m,m'}$ with $m'<m$. Summing these $S''_{m,m'}$, we get as in (a) an upper bound which tends to 0.

Combining all these partial results implies the statement of the lemma. □

Lemma 3.3 and 3.4 imply now with Berman's lemma the asymptotic independence of exceedances in the intervals I_j^* and I_j.

COROLLARY 3.5. *Assume that* (5), (6), (8), (10) *and* (11) *hold. Then*

$$P\{X(iq_j) \le u_T^*(jh), iq_j \in I_j^*, j \le n\}$$

$$- \prod_{j=1}^{n} P\{X(iq_j) \le u_T^*(jh), iq_j \in I_j^*\} \to 0$$

and

$$P\{X(iq_j) \le u_T^*(jh), iq_j \in I_j, j \le n\}$$

$$- \prod_{j=1}^{n} P\{X(iq_j) \le u_T^*(jh), iq_j \in I_j\} \to 0,$$

as $T \to \infty$, *for every* $q_0 > 0$.

Obviously, the second statement of the corollary follows by the first one and Lemma 3.1(ii).

4. Results for increasing T. With the lemmas of the previous section we can now derive the asymptotic behaviour of the probabilities of no exceedances of the local stationary Gaussian process X above a moving barrier. We suppose again that X is standardized.

THEOREM 4.1. *Assume that the standardized Gaussian process X satisfies* (5), (6) *and* (10). *Let* $\{u_T(t), t \le T\}$ *be a real continuous function for every positive T such that* (8) *and* (9) *hold. Then*

(i) $$P\{X(t) \le u_T(t), t \le T\} - \exp(-J(T)) \to 0$$

as $T \to \infty$, *if* $\limsup J(T) < \infty$.

(ii) $P\{X(t) \le u_T(t), t \le T\} \to \exp(-\tau)$ *iff* $J(T) \to \tau < \infty$ *as* $T \to \infty$.

PROOF. (ii) follows obviously from (i). We approximate $P\{X(t) \le u_T(t), t \le T\}$ by $P\{X(t) \le u_T^+(t), t \le T\}$ and $P\{X(t) \le u_T^-(t), t \le T\}$ with the use of $h = h(T)$ in (8). By Lemma 3.1(i) and the assumptions (8) and (9), we have to show that

$$P\{X(t) \le u_T^+(t), t \le T\} - \exp(-J^+(T)) \to 0$$

and

$$P\{X(t) \le u_T^-(t), t \le T\} - \exp(-J^-(T)) \to 0$$

as $T \to \infty$. For every difference the proof is the same. By Lemma 3.1(ii) it remains for the first limit to verify that with $\varepsilon = \varepsilon(T)$ defined in Lemma 3.4 and with q_j given in (14),

$$P\left\{X(t) \le u_T^+(t), t \in \bigcup_{j=1}^n I_j^*\right\} - \exp(-J^+(T))$$

$$= \left(P\left\{X(t) \le u_T^+(t), t \in \bigcup_{j=1}^n I_j^*\right\} - P\{X(iq_j) \le u_T^+(jh), iq_j \in I_j^*, j \le n\}\right)$$

$$+ \left(P\{X(iq_j) \le u_T^+(jh), iq_j \in I_j^*, j \le n\}\right.$$

$$\left. - \prod_{j=1}^n P\{X(iq_j) \le u_T^+(jh), iq_j \in I_j^*\}\right)$$

$$+ \left(\prod_{j=1}^n P\{X(iq_j) \le u_T^+(jh), iq_j \in I_j^*\} - \prod_{j=1}^n P\{X(t) \le u_T^+(t), t \in I_j^*\}\right)$$

$$+ \left(\prod_{j=1}^n P\{X(t) \le u_T^+(t), t \in I_j^*\} - \prod_{j=1}^n P\{X(t) \le u_T^+(t), t \in I_j\}\right)$$

$$+ \left(\prod_{j=1}^n P\{X(t) \le u_T^+(t), t \in I_j\} - \exp(-J^+(T))\right).$$

By the results of Section 3, the $\limsup_{T \to \infty}$ of each difference is either $O(q_0)$ (for the first and third difference) or else 0. The convergence of the last difference follows simply by using Theorem 2.2, (8), (9) and the fact that $h\psi(u_T^+(jh)) \to 0$ uniformly in $j \le n$. Hence the statement follows by letting $q_0 \to 0$. □

We now apply this general result with a moving barrier for some particular boundaries. We begin with a fixed level $u_T(t) = u_T$, to derive the limiting distribution of M_T of a locally stationary, standardized Gaussian process.

THEOREM 4.2. *Assume that X is a locally stationary Gaussian process with mean 0 and variance 1, satisfying (5), (6) and (10). Let $\{u_T, T > 0\}$ be such that (7) holds with a $\tau < \infty$, i.e.,*

$$H_\alpha \psi(u_T) \int_0^T C^{1/\alpha}(t)\, dt \to \tau.$$

Then

(i) $P\{M_T \le u_T\} \to \exp(-\tau)$ *as* $T \to \infty$.

(ii) *Let* $C_T = \int_0^T C^{1/\alpha}(t)\,dt$ *and*

$$u_{T,x} = x(2\log C_T)^{-1/2} + (2\log C_T)^{1/2} + (2\log C_T)^{-1/2}\Big(((2-\alpha)/2\alpha)\log\log C_T + \log\big(H_\alpha 2^{(2-\alpha)/2\alpha}(2\pi)^{-1/2}\big)\Big).$$

Then for every $x \in R$,

$$P\{M_T \le u_{T,x}\} \to \exp(-\exp(-x)) \quad \text{as } T \to \infty.$$

Note that $u_{T,x}$ is a linear normalization, giving

$$P\{M_T \le a_T x + b_T\} \to \exp(-\exp(-x)),$$

and that C_T plays the role of the process specific time. If $C(t) = C$ as in the stationary case, then $C_T = TC^{1/\alpha}$ and $u_{T,x}$ is the known linear normalization of the stationary case. The proof of Theorem 4.2 is straightforward, since we have mainly to verify that

$$J(T) = H_\alpha \psi(u_{T,x})C_T \to \exp(-x) \quad \text{as } T \to \infty,$$

which is an easy calculation. Also the verification of (8) and (9) is obvious.

Since $C(t)$ is bounded we also might use the normalization $u_{T,x}$ of the stationary case with T instead of C_T and an additional varying term C_T^*:

$$u_{T,x} = x(2\log T)^{-1/2} + (2\log T)^{1/2} + (2\log T)^{-1/2} \times\Big(((2-\alpha)/2\alpha)\log\log T + \log\big(C_T^* H_\alpha 2^{(2-\alpha)/2\alpha}(2\pi)^{1/2}\big)\Big) \tag{18}$$

with bounded

$$C_T^* = (1/T)\int_0^T C^{1/\alpha}(t)\,dt.$$

This shows explicitly the influence of local stationarity by the function $C(t)$ in the limit distribution of M_T. Assume now that X is locally stationary with index α, $0 < \alpha \le 2$, with mean $-td(T)$ and variance 1. Assume that $0 < d(T) \to 0$ as $T \to \infty$ and that $C(s) \to C_\infty > 0$ as $s \to \infty$ to simplify. Then we have to consider $u_{T,x}(t) = a_T x + b_T + td(T) = u_{T,x} + td(T)$ as boundary with

$$a_T = (-2\log d(T))^{-1/2}$$

and

$$b_T = a_T^{-1} + a_T\Big(\log\big(C^* H_\alpha 2^{1/\alpha - 1}(2\pi)^{-1/2}\big) + (1/\alpha - 1)\log\log 1/d(T)\Big) \tag{19}$$

with C^* appropriately chosen.

Assume that

$$Td(T) \to d_1 < \infty \quad \text{and} \quad Td(T)u_T \to d_2 \in (0,\infty] \tag{20}$$

as $T \to \infty$. Then let $C^* = C_\infty^{1/\alpha}(1 - \exp(-d_2))$. Note that if $d_2 = 0$, then the trend has asymptotically no influence on the extreme behaviour and we find the same result as in Theorem 4.2. If $d_2 < \infty$, then $d_1 = 0$; if $d_1 > 0$, then $d_2 = \infty$. If $d_2 < \infty$, we may simplify the normalization by using $u_{T,x}$ defined in (18) with $C_\infty^{1/\alpha}(1 - e^{-d_2})/d_2$ instead of C_T^*.

THEOREM 4.3. *Let X be a locally stationary Gaussian process with index* α, $0 < \alpha \le 2$, *and with trend* $-td(T)$ *and variance* 1, *such that* (5), (6) *and* (10) *hold. With the normalization* (19) *and conditions* (20), *we have*

$$P\{X(t) \le u_{T,x}, t \le T\} \to \exp(-\exp(-x)) \quad \text{as } T \to \infty.$$

PROOF. We verify that $J(T) \to \exp(-x)$ for every x. With $u_T = u_{T,x}$, $J(T)$ is asymptotically equal to

$$H_\alpha \psi(u_T) \int_0^{Tu_T d(T)} C^{1/\alpha}(y/u_T d(T)) \exp(-y - y^2/2u_T^2)\, dy/u_T d(T)$$

$$\sim H_\alpha \psi(u_T) C_\infty^{1/\alpha} \int_0^{d_2} \exp(-y)\, dy/u_T d(T),$$

which tends to $\exp(-x)$ by a similar calculation as in Theorem 4.2. It is rather straightforward to verify the conditions (8) and (9). □

Note that the important role is played by d_2 not by d_1 in this case with $d_1 < \infty$. The theorem deals with a particular case with a bounded trend, since $Td(T) \to d_1 < \infty$. This can be generalized in the following way, similar to the case of random sequences [Leadbetter, Lindgren and Rootzén (1983)]. Let

$$\gamma_T = \sup_{t \le T} m(t) = o\big((\log T)^{1/2}\big), \tag{21}$$

where $m(t)$ denotes the continuous trend function of X. The extension is possible if we find m_T^* such that

$$J^*(T) = 1/T \int_0^T C^{1/\alpha}(t) \exp\{a_T^*(m(t) - m_T^*) - (m(t) - m_T^*)^2/2\}\, dt \to 1 \tag{22}$$

as $T \to \infty$, where $a_T^* = (2\log T)^{1/2} - ((2 - \alpha)/2\alpha)(\log\log T)/(2\log T)^{1/2}$. We assume that the integral $J^*(T)$ in (22) can be approximated in the same way as $J(T)$ in conditions (8) and (9).

THEOREM 4.4. *Let X be a locally stationary Gaussian process with index* α, $0 < \alpha \le 2$, *with continuous trend* $m(t)$ *and variance* 1, *satisfying* (5), (6) *and*

(10). *Assume that* (21) *and* (22) *hold with a suitable* m_T^* *and that* $J^*(T)$ *can be approximated as* $J(T)$ *in* (8) *and* (9). *Then*

$$P\{X(t) \le u_{T,x} + m_T^*, t \le T\} \to \exp(-\exp(-x)) \quad \text{as } T \to \infty,$$

where $u_{T,x}$ *is given in* (18) *with* $C_T^* = 1$.

PROOF. Note that by (21) the boundary values $u_{T,x} - m(t) + m_T^*$ tend uniformly to ∞. Hence one verifies along the same lines that $J(T)$ converges to $\exp(-x)$. (8) and (9) follow directly by the assumption on the approximation of (22), since $J(T) \sim \exp(-x)J^*(T)$. □

5. Final remarks. We used an approach to the problem which is strongly related to the methodology in the stationary case. This implies that our considered processes are behaving in a certain sense like stationary processes. This is included in the condition of the local stationarity. Also the index α, describing the local behaviour of the correlation function, is assumed to be the same for all time points. Obviously, we might try to extend our results to processes where the index α may change also. If $\alpha = \alpha(t)$ is piecewise constant, our results can easily be adjusted. But for more general cases the problem becomes tedious to deal with.

Another interesting question arises in cases where the conditions (8) and (9) are not satisfied. These conditions restrict the boundary to be rather smooth in relation to the local behaviour of the sample functions. It seems that another approach might be used, which is more related to Berman's paper (1985).

The assumption that $0 < \inf C(s) \le \sup C(s) < \infty$ is sometimes too restrictive also. For instance, in some applications we have that $C(s) \to 0$ or ∞ as $s \to 0$ or ∞. We believe that some similar results can be derived, depending on the limit behaviour of $C(s)$.

These results are applied to the particular statistical problem of the first zero R_n of the empirical characteristic function. Heathcote and Hüsler (1990) derive some limit distributions of R_n, depending on the assumptions of the underlying statistical sample of size n [see also the review and some extensions in Hüsler (1989)].

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