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PARTIALLY OBSERVED CONTROL OF MARKOV PROCESSES. III[1]

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Let v denote the value function of a partially observed control problem. If v is once differentiable in a certain direction $\hat{B}$, then optimal controls are characterized by a feedback involving the directional derivative $\hat{B}v$. It is also shown that v satisfies the corresponding Bellman equation, an infinite-dimensional PDE on the space of measures, in the viscosity sense of Crandall and Lions.

0. Introduction. Let x be a controlled Markov process evolving on some state space X and let observations be taken according to

$$y(t) = \int_0^t c(x(s))\, ds + \eta(t), \qquad t \geq 0, \tag{0.1}$$

where η is standard Brownian motion. Consider the cost criterion

$$v^u(m) = E\left(\int_0^\infty e^{-t} L(x(t), u(t))\, dt\right) \tag{0.2}$$

corresponding to a control u and initial probability distribution m of $x(0)$ on X. Let

$$v(m) = \inf_u v^u(m)$$

denote the *value function* of the problem; here the infimum is taken over all controls that depend only on the past of the corresponding observations y.

In this paper we are interested in characterizing optimal controls, i.e., controls u satisfying $v^u(m) = v(m)$.

This variational problem, motivated by engineering considerations, is one formulation of the general problem of stabilizing a controlled dynamical system in the presence of noisy or partial information concerning the state.

From a purely mathematical point of view, this problem involves the analysis of a solution of a nonlinear PDE (0.5) in infinite dimensions.

Since information about x is available only through y, it turns out that the conditional distribution $\mu_m^u(t)$ of $x(t)$ given $y(s)$, $0 \leq s \leq t$, plays a crucial role in the analysis. Let $M(X)$ denote the set of probability measures on X, endowed with the weak topology. Specifically then, it turns out optimal

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controls satisfy a feedback of the form

$$u(t) = F(\mu_m^u(t)), \qquad t \geq 0,$$

for a specific F: $M(X) \to \mathbf{R}$ involving the value function v and the Lagrangian L. It is precisely such a fixed-point formula for u that is most useful in applications.

To describe the feedback F more explicitly we will need to assume that the infinitesimal generator A^u of the Markov process x is *affine* in the control, $A^u = A - Bu$, where we assume that controls take values in a compact convex subset U of euclidean space, taken to be one dimensional for now. Here we have inserted a minus sign in A^u in anticipation of the *restoring* nature of the optimal control. We will also need to assume that both $+B$ and $-B$ generate Markov semigroups on $C(X)$ (see Section 1 for precise assumptions). Let T_t^B, $-\infty < t < \infty$, denote the corresponding Markov *group*. The flow T_t^B then naturally induces a flow $\mu \to \mu T_t^B$ on $M(X)$. For Φ: $M(X) \to \mathbf{R}$, set

$$(0.3) \qquad (\hat{B}\Phi)(\mu) = \frac{d}{dt}\bigg|_{t=0} \Phi(\mu T_t^B)$$

whenever the (two-sided) limit exists.

Heuristic considerations (Appendix B) then indicate that the feedback F is the Legendre transform of the derivative of the value function v in the direction of the vector field $\hat{B}$,

$$F(\mu) = \hat{H}_p(\mu, \hat{B}v(\mu)),$$

where $\hat{H}$ is the convex conjugate of $\hat{L}(\mu, u) = \int_X L(x, u)\, d\mu(x)$,

$$(0.4) \qquad \hat{H}(\mu, p) = \sup_{u \in U} (p \cdot u - \hat{L}(\mu, u))$$

and $\hat{H}_p$ denotes the partial derivative of $\hat{H}$ with respect to p.

The first result established here states that the existence of $\hat{B}v(\mu_m^u(t))$ for all $t \geq 0$ implies the validity of the above stated feedback.

The plan of the paper is as follows. In Section 1 we state the above result precisely while Section 2 is the proof. In Section 3 we derive the second result: v satisfies the Bellman equation corresponding to the above variational problem,

$$(0.5) \qquad -\hat{A}v + \hat{H}(\mu, \hat{B}v) + v = 0, \qquad \mu \in M(X),$$

in the viscosity sense of Crandall and Lions (1983).

The heuristic considerations referred to above rely heavily on the assumption that v satisfy (0.5) in some strong sense, e.g., v is in the domain of the generator of the semigroup generated by $\hat{A}$. Since $\hat{A}$ in (0.5) is a second-order diffusion generator on $M(X)$ (see Section 1), this is tantamount to assuming some sort of C^2 smoothness. If $X = \{0, \dots, N\}$, then (0.5) reduces to a finite-dimensional degenerate elliptic PDE on the domain $M(X) \subset \mathbf{R}^N$. Because of the degeneracy of $\hat{A}$, the fact that v solves (0.5) in a weak sense does not allow us to conclude that v solves (0.5) in a strong or classical sense. *A fortiori*, we certainly cannot conclude this in the general case, since a regularity theory is not always available in infinite dimensions; see Lions (1988). An important aspect of our main result is then that the optimal feedback holds without having v satisfy (0.5) in any classical sense. This is because the optimal feedback depends only on $\hat{B}v$, a first-order derivative of v.

In Appendix A we derive the necessary results concerning $\hat{L}$ and its conjugate $\hat{H}$. In Appendix B we provide the above-mentioned nonrigorous derivation of the main result using the Bellman equation. In Appendix C we establish a PDE-theoretic estimate that is necessary in the proof of the main result.

The main idea behind the proof is the fact that v, being the infimum of affine functions, is concave on $M(X)$. This is put to good use via the well-known observation that convergence of a sequence of differentiable functions to a differentiable function *does* imply the corresponding convergence of the first derivatives, provided all the functions are concave. Given this, the proof proceeds via an approximation argument: In effect X is approximated by a finite set X_n of n points, the operators A, B are approximated by $n \times n$ matrices A_n, B_n and so on. Then, for each n, $M(X_n)$ is a bounded domain in $\mathbf{R}^{n-1}$ and the analogue $(0.5)_n$ of (0.5) is a finite-dimensional degenerate elliptic PDE involving analogous operators $\hat{A}_n, \hat{B}_n$ (see Section 2). Let v_n denote the value function of the approximating variational problem. Since $(0.5)_n$ is degenerate, v_n is not necessarily smooth. To remedy this we add n observation equations to (0.2) in such a way that the information in each equation decreases to zero as $n \uparrow \infty$ and $\hat{A}_n$ is replaced by $\hat{A}_n + \varepsilon_n^2 \hat{\Delta}_n$, where $\hat{\Delta}_n$ is an explicit second-order differential operator that is *nondegenerate* on the interior of $M(X_n)$. Because $(0.5)_n$ is now nondegenerate, v_n is smooth enough for the application of the Itô rule and so the heuristic derivation presented in Appendix B is fully rigorous and we obtain the correct feedback $u_n(t) = F_n(\mu^u_{n,m}(t))$, $t \geq 0$, $F_n = (\hat{H}_n)_p(\cdot, \hat{B}_n v_n)$, for the optimal control of the approximating problem. Since this involves a first-order derivative of v_n, we let $n \uparrow \infty$ and appeal to the concavity of v_n, v to complete the proof. A crucial estimate (2.13), independent of the dimension n, is required to establish the uniformity of the concavity of v_n in n.

For the above to work, it is crucial to establish the convergence $v_n \to v$. This is the main result in Hijab (1989b); its proof relies on the technique of generalized controls introduced by Fleming and Pardoux (1982). To establish the estimate (2.13), certain results from pseudo-differential operator theory are utilized. It may be possible with additional work to derive (2.13) without ΨDO techniques, but we do not address this issue here. In simple cases (e.g., $A, B_1, \dots, B_d$ with constant coefficients on the torus) one can get by with more elementary arguments. We emphasize that the ΨDO technique we use is very natural (see Appendix C).

A flaw in Theorem B is the assumption of existence of $\hat{B}v$ at each point of the trajectory $\mu^u_m(t)$, $t \geq 0$. Nevertheless the proof of Theorem B shows that such a condition is extremely natural. This leads us to ask whether optimality actually *implies* the existence of the directional derivative. If this were so, this would in some sense completely characterize optimal controls.

It turns out that in certain situations the result described in the last paragraph is true. The classical case arises in the calculus of variations [Fleming (1969)]; there it turns out that v is differentiable at each point of a minimizing geodesic, except perhaps the initial point. This is a finite-dimensional result involving solutions of a first-order Hamilton–Jacobi equation. The corresponding finite-dimensional result for controlled degenerate diffusions, although never established, is also true; this work will appear elsewhere. The result alluded to in the previous paragraph is a corresponding result for a controlled diffusion in infinite dimensions; since the techniques are probabilistic, this result is expected to hold. For these results, the technique of proof is somewhat different from the techniques of the present paper and involves the Euler–Lagrange equations for this variational problem.

1. Statement of results. Any undefined notation is as in Ethier and Kurtz (1986) or Stroock and Varadhan (1979). The Hijab (1989a, b) papers are referred to as Part I and Part II, respectively. We begin by describing the setting natural to the problem. Then we state our somewhat stricter assumptions. Let X be a Polish space and U a closed convex subset of $\mathbf{R}^d$. Let $\mathscr{D} \subset C_b(X)$ be a linear subspace containing 1 such that the closure of $\mathscr{D}$ under bounded pointwise convergence is all of $B(X)$. Let $M(X)$ denote the space of (Borel) probability measures on X equipped with the *weak* topology. Then $M(X)$ is a Polish space. Let L^2 denote the space $L^2([0,\infty), U; e^{-t}\,dt)$ of all maps u: $[0,\infty) \to U$ that are square integrable against $e^{-t}\,dt$ over $[0,\infty)$, equipped with the *weak* topology. When U is compact, as we shall assume below, L^2 is Polish and the Borel σ-field of L^2 agrees with the σ-field generated by the norm topology.

Let $A, B_1, \dots, B_d$: $\mathscr{D} \to C_b(X)$ be linear operators and set $A^u = A - B_1 u_1 - \cdots - B_d u_d$ for $u = (u_1, \dots, u_d) \in \mathbf{R}^d$. Let c: $X \to \mathbf{R}$ be a function and set

$$G^u = A^u + \frac{1}{2}\frac{\partial^2}{\partial y^2} + c(x)\frac{\partial}{\partial y}. \tag{1.1}$$

Then G^u acts on the space $\mathscr{D} \otimes C_0^\infty(\mathbf{R})$ of all sums of productlsmaps,s $\phi\psi$ where $\phi \in \mathscr{D}$, $\psi \in C_0^\infty(\mathbf{R})$. Let $\Omega = D([0,\infty); X) \times C([0,\infty); \mathbf{R})$, $x(t)$: $\Omega \to X$, $y(t)$: $\Omega \to \mathbf{R}$, $t \geq 0$, the canonical evaluation maps, $\mathscr{X}_t = \sigma(x(s), 0 \leq s \leq t)$, $\mathscr{Y}_t = \sigma(y(s), 0 \leq s \leq t)$, $\mathscr{F}_t = \mathscr{X}_t \times \mathscr{Y}_t$, $\mathscr{X} = \sigma(\mathscr{X}_t, t \geq 0)$, $\mathscr{Y} = \sigma(\mathscr{Y}_t, t \geq 0)$, $\mathscr{F} = \mathscr{X} \times \mathscr{Y} = \sigma(\mathscr{F}_t, t \geq 0)$. Then $\mathscr{F}$ is the Borel σ-field of the Polish space Ω.

Let W denote Wiener measure on $\mathscr{Y}$. A *control* is a $\mathscr{Y}_t$ progressively measurable map u: $[0,\infty) \times \Omega \to U$ such that for W-almost all ω, $u(\cdot, \omega)$ is in L^2. Then the joint state-observations dynamics is specified by solutions to the martingale problem for G^u on $(\Omega, \mathscr{F})$. In particular for any such solution P the process η defined by (0.1) is a Brownian motion under P.

The following is our first set of assumptions.

1. c is in $C_b(X)$.
2. L is in $C^{0,1}(X \times \mathbf{R})$, $L \geq 0$, for each $x \in X$, $L(x, \cdot\,)$ is strictly convex on $\mathbf{R}^d$ and

$$\inf_{x \in X} \frac{L(x,u)}{|u|} \to +\infty \quad \text{as } |u| \to \infty.$$

3. For each $u \in L^2$, the martingale problem for A^u on $D([0,\infty); X)$ is well-posed.
4. The martingale problems for $\pm B_1, \dots, \pm B_d$ on $D([0,\infty); X)$ are well-posed.

In particular these assumptions imply continuity of the map $X \times L^2 \to M(D([0,\infty); X))$ that takes (x, u) to the solution of the martingale problem for A^u starting from x. This implies the well-posedness of the martingale problem for G^u on Ω (Part I, Lemma 4.3) for any *control* u. Let u be a control, let $m \in M(X)$ and let P_x^u denote the solution to the martingale problem for G^u starting from x. It follows then that the solution P_m^u of the martingale problem for G^u and starting from m in the sense $P_m^u(x(0) \in B, y(0) = 0) = m(B)$ satisfies $P_m^u = \int_X P_x^u \, dm(x)$. Let $E = E_m^u$ denote expectation against P_m^u. Then we define $v^u(m)$ by (0.2) and the value function $v(m)$ as the infimum of the quantities $v^u(m)$ as u varies over the set of controls.

Let $\hat{L}(\mu, u) = \int_X L(\cdot, u)\, d\mu$ and for $p \in \mathbf{R}^d$, let $\hat{H}(\mu, p)$ be given by (0.4). Then $\hat{H} \in C^{0,1}(M(X) \times \mathbf{R}^d)$ (Appendix A). Define $\hat{B}_i\Phi$, $i = 1, \dots, d$, as in (0.3) and set $\hat{B}\Phi = (\hat{B}_1\Phi, \dots, \hat{B}_d\Phi)$.

In Part I it is shown that for each control u and $m \in M(X)$ there is a right-continuous P_m^u-a.s. continuous $\mathscr{Y}_t$ progressively measurable map μ_m^u: $[0,\infty) \times \Omega \to M(X)$ satisfying

$$\mu_m^u(t)(\phi) = E_m^u(\phi(x(t)) | \mathscr{Y}_t)$$

a.s.-P_m^u for all $t \geq 0$, $\phi \in B(X)$. Here and elsewhere $\mu(\phi)$ denotes the integral of ϕ against the measure μ over X. In particular $\mu_m^u(0) = m$. The process μ_m^u is the *conditional distribution process*.

Our second set of assumptions is:

1. X is compact.
2. U is compact.

Most of the proof of the main result holds true under the above sets of assumptions. The one exception is the estimate (2.13) whose proof is in Appendix C. For this we need our third and final set of assumptions:

1. X is a compact C^∞ manifold and $\mathscr{D} = C^\infty(X)$.
2. U is a compact convex subset of $\mathbf{R}^d$.
3. $c \in C^\infty(X)$ is smooth.
4. $L \in C^\infty(X \times \mathbf{R}^d)$, $L \geq 0$ and $L_{uu} > 0$.
5. A is a *nondegenerate* C^∞ Markov generator, in the sense that $A = A_0 + A_1$, where A_0 is a second-order nondegenerate elliptic differential operator

with C^∞ coefficients and A_1 is a jump process generator

$$A_1\phi(x) = \int_X (\phi(z) - \phi(x))\lambda(x; dz), \tag{1.2}$$

where λ is a transition probability kernel with C^∞ density relative to Lebesgue measure on X.

6. $B_1, \dots, B_d$ are C^∞ vector fields.

These assumptions imply the well-posedness of the martingale problems stated above as well as the compactness of $M(X)$ and L^2. A control u is *optimal at* m if $v^u(m) = v(m)$. We can now state the first result.

THEOREM A. *Fix an arbitrary* $m \in M(X)$. *Let* u *be a control such that the derivatives* $\hat{B}_1 v(\mu^u_m(t)), \dots, \hat{B}_d v(\mu^u_m(t))$, $t \ge 0$, *exist a.s.-*$(dt \times P^u_m)$. *Then* u *is optimal at* m *iff*

$$u(t) = \hat{H}_p\big(\mu^u_m(t), \hat{B}v(\mu^u_m(t))\big), \qquad t \ge 0, \tag{1.3}$$

a.s.-$(dt \times P^u_m)$.

As yet the existence of controls optimal at m has not been established. Because of this Fleming and Pardoux (1982) introduced a notion of generalized control and established the existence of optimal generalized controls. We therefore derive Theorem A as a consequence of the analogous result for generalized controls, Theorem B below. We begin by recalling the notion of generalized control [Fleming and Pardoux (1982), Part II]. To this end we need to think of L^2 as a path space.

Recall that elements of L^2 are not functions but equivalence classes of functions equal almost everywhere. Let $\mathscr{U}$ denote the Borel σ-field of L^2. To define martingale problems on L^2, we need an evaluation map. An evaluation map on L^2 is a Borel map $u: [0, \infty) \times L^2 \to U$ such that for all $\omega \in L^2$, $u(\cdot, \omega)$ is in the equivalence class ω. Let $L^2_t = L^2([0, t]; U)$ and let $\mathscr{U}_t$ denote the σ-field on L^2 that is the inverse image of the Borel σ-field of L^2_t under the natural projection $L^2 \to L^2_t$. Then $\mathscr{U}_t$, $t \ge 0$, is a filtration on L^2 generating $\mathscr{U}$.

Using the martingale convergence theorem on L^2, one can show that an evaluation map exists. Moreover an evaluation map can be chosen $\mathscr{U}_{t+}$ progressively measurable. Thus, for f in $B([0, \infty) \times U)$, the process $\int_0^t f(s, u(s))\, ds$, $t \ge 0$, is $\mathscr{U}_t$ progressively measurable and is independent of the choice of evaluation map.

To define generalized controls we extend our basic probability space $(\Omega, \mathscr{F})$ by setting $\Omega' = \Omega \times L^2$, $\mathscr{F}_t' = \mathscr{F}_t \times \mathscr{U}_t$, $\mathscr{Y}_t' = \mathscr{Y}_t \times \mathscr{U}_t$, $\mathscr{F}' = \mathscr{F} \times \mathscr{U}$, $\mathscr{Y}' = \mathscr{Y} \times \mathscr{U}$. Then Ω' is a Polish space with Borel σ-field $\mathscr{F}'$. *Generalized controls* are certain probability measures Q on $\mathscr{F}'$ solving the martingale problem for $G^{\ddot{u}}$ in the sense that for $\phi\psi \in \mathscr{D} \otimes C_0^\infty(\mathbf{R})$,

$$\phi\psi(x(t), y(t)) - \int_0^t G^{u(s)}(\phi\psi)(x(s), y(s))\, ds, \qquad t \ge 0, \tag{1.4}$$

is a $(\Omega', \mathscr{F}_t', Q)$ martingale. Note that in (1.4), u denotes an evaluation map on L^2 and not a control as defined earlier. In particular every control u induces a natural map $(x, y) \to (x, y, u)$ from $\Omega \to \Omega'$. Then the image of P^u under this map is a generalized control. The exact class of generalized controls is described in Part II. A generalized control starts at m if $Q(x(0) \in B, y(0) = 0) = m(B)$. Given a generalized control Q set

$$v(Q) = E^Q\left(\int_0^\infty e^{-t} L(x(t), u(t))\, dt\right),$$

$$v'(m) = \inf\{v(Q) | Q \text{ starts at } m\}.$$

We refer to v' as the *generalized* value function. By the above remarks, $v(m) \geq v'(m)$ for all m. In Part II it is shown that under our assumptions $v = v'$.

In this context the conditional distribution process can be defined (Part II) as follows: To each $m \in M(X)$, there is a right-continuous $\mathscr{Y}_t'$ progressively measurable map μ_m: $[0, \infty) \times \Omega' \to M(X)$ such that for all generalized controls Q starting at m, $\phi \in B(X)$ and $t \geq 0$,

$$\mu_m(t)(\phi) = E^Q(\phi(x(t)) | \mathscr{Y}_t')$$

a.s.-Q. A generalized control Q is *optimal* at m if Q starts at m and $v(Q) = v(m)$. We can now state the following.

THEOREM B. *Fix an arbitrary $m \in M(X)$. Let Q be a generalized control, starting at m, such that the derivatives $\hat{B}_1 v(\mu_m(t)), \dots, \hat{B}_d v(\mu_m(t))$, $t \geq 0$, exist a.s.-$(dt \times Q)$. Then Q is optimal at m iff*

$$u(t) = \hat{H}_p\big(\mu_m(t), \hat{B}v(\mu_m(t))\big), \qquad t \geq 0, \tag{1.5}$$

a.s.-$(dt \times Q)$.

The proof is in Section 2. Clearly Theorem A follows from Theorem B.

To describe the operators $\hat{A}$, $\hat{B}$ appearing in (0.5), let $\hat{\mathscr{D}}$ be the set of all $\Phi \in C(M(X))$ of the form

$$\Phi(\mu) = F(\mu(\phi_1), \dots, \mu(\phi_n))$$

for some $n \geq 0$, $\phi_1, \dots, \phi_n \in \mathscr{D}$, $F \in C^\infty(\mathbf{R}^n)$. Here as before $\mu(\phi)$ denotes the integral of ϕ against the measure μ over X. Then $\hat{\mathscr{D}}$ is uniformly dense in $C(M(X))$ by the Stone-Weierstrass theorem. For $\Phi \in \hat{\mathscr{D}}$ set

$$\begin{aligned}\hat{A}(\Phi)(\mu) &= \sum_{i=1}^n \partial_i F(\mu(\phi_1), \dots, \mu(\phi_n))\mu(A\phi_i) \\ &\quad + \tfrac{1}{2} \sum_{i,j=1}^n \partial_i \partial_j F(\mu(\phi_1), \dots, \mu(\phi_n))\langle c, \phi_i\rangle_\mu \langle c, \phi_j\rangle_\mu,\end{aligned} \tag{1.6}$$

where $\langle \phi, \phi' \rangle_\mu = \mu(\phi\phi') - \mu(\phi)\mu(\phi')$ denotes the covariance of ϕ, ϕ' under μ.

Then for $\Phi \in \hat{\mathscr{D}}$ and $\hat{B}$ defined as in (0.3),

$$\hat{B}\Phi(\mu) = \sum_{i=1}^{n} \partial_i F(\mu(\phi_1), \dots, \mu(\phi_n))\mu(B\phi_i).$$

Now for $\Phi \in \hat{\mathscr{D}}$ define $\hat{A}^u$ by setting $\hat{A}^u\Phi = \hat{A}\Phi - \hat{B}_1\Phi u_1 - \cdots - \hat{B}_d\Phi u_d$. The second result of the paper is the following.

THEOREM C. *The value function v is in $C(M(X))$ and is a viscosity solution of* (0.5) *relative to the test function space $\hat{\mathscr{D}}$.*

The definition of viscosity solution and the proof are in Section 3. In Lions (1988) the test function space considered is significantly larger; there existence and uniqueness are established for the analogous PDE on $L^2(X)$, rather than $M(X)$.

2. Proof of Theorem B. The key to establishing Theorem B is to establish the following: Let $m \in M(X)$ and let Q be a generalized control starting at m. Suppose Q is such that $\hat{B}v(\mu_m(t))$ exists a.s.-$(dt \times Q)$. Then, as (2.23) shows,

$$v(Q) = v(m) + E^Q\biggl(\int_0^\infty e^{-t}\bigl(\hat{L}(\mu_m(t), u(t)) - u(t)\hat{B}v(\mu_m(t)) + \hat{H}\bigl(\mu_m(t), \hat{B}v(\mu_m(t))\bigr)\bigr)\, dt\biggr).$$

If v were in the domain of the generator of the semigroup generated by $\hat{A}$ and (0.5) held in an operator sense, then (2.23) would follow immediately. Because this is not so, the bulk of the proof is the approximation process outlined in Section 0.

Once (2.23) is established, Theorem B follows by noting (Appendix A)

$$\hat{L}(\mu, u) - pu + \hat{H}(\mu, p) \geq 0,$$

for all u, μ, p with equality iff $u = \hat{H}_p(\mu, p)$. (2.23) is established in Corollary 2.9.

To begin, by translating $U \to U + a$, $a \in \mathbf{R}^d$, if necessary, we can assume $-U \subset \mathbf{R}^d_+$, i.e., $u_i \leq 0$, $i = 1, \dots, d$, for $u = (u_1, \dots, u_d) \in U$. For simplicity of notation we also take $d = 1$, i.e., $A^u = A - Bu$. The integral of ϕ against μ is denoted $\mu(\phi)$ and $\langle \phi, \psi \rangle_m$ denotes the *covariance* of ϕ, ψ under m, $m(\phi\psi) - m(\phi)m(\psi)$. For T: $\mathscr{D} \to C(X)$ linear and $\mu \in M(X)$, let μT: $\mathscr{D} \to \mathbf{R}$ denote the linear functional $(\mu T)(\phi) = \mu(T\phi)$. Given L: $\mathscr{D} \to \mathbf{R}$ linear recall that the *variational norm* of L is the quantity

$$|L|_{\text{var}} = \sup_{\substack{\phi \in \mathscr{D} \\ |\phi| \leq 1}} |L\phi|.$$

Here $|\phi|$ denotes the sup norm of ϕ. Then $|L|_{\text{var}} < \infty$ implies the existence of a signed measure μ such that $L(\phi) = \mu(\phi)$, with $|L|_{\text{var}} = |\mu|(X)$.

Since A_0 is nondegenerate (notation of Section 1) there is a unique choice of Riemannian metric on X and a C^∞ vector field B_0 such that $A_0 = \Delta + B_0$, where Δ is the Laplacian corresponding to the metric. We fix this choice of metric throughout. We use the notation $|\phi|_{C^r}$ for the sum of the sup norms of all partial derivatives of ϕ of order less than or equal to r, taken relative to a fixed collection of charts. We say $G \subset X$ is *geodesically small* if any two points of G can be joined by a unique minimizing geodesic. $L^2(X)$ denotes the Hilbert space of all square integrable functions relative to the volume element induced by the Riemannian metric. Let $\langle \phi, \psi \rangle_{L^2}$ denote the corresponding inner product.

A *partition* $\mathscr{P}$ of X is a collection of subsets $\{G_1, \dots, G_N\}$ such that for $i = 1, \dots, N$, the sets G_i are disjoint, geodesically small, have nonempty interior and their union is all of X.

The *size* of the partition $\mathscr{P}$ is the integer N and the *mesh* is the maximum of the diameters of G_i, $i = 1, \dots, N$, where the diameters are measured relative to the induced metric. It is easy to see that partitions of arbitrarily small mesh exist. Throughout ϕ denotes an element of $\mathscr{D}$, i.e., a C^∞ function.

Let $x(t)$, $0 \le t \le 1$, be the minimizing geodesic joining x and x'. Then

$$\phi(x) - \phi(x') = \int_0^1 \frac{d}{dt}\phi(x(t))\, dt = \int_0^1 D\phi(x(t))\dot{x}(t)\, dt$$

and so

$$|\phi(x) - \phi(x')| \le |\phi|_{C^1} d(x, x'). \tag{2.1}$$

Let P denote the orthogonal projection of $L^2(X)$ onto the Hilbert subspace $L^2(X, \mathscr{P})$ of all functions piecewise constant over $\mathscr{P}$. Then $|P\phi| \le |\phi|$, P is self-adjoint on L^2 and (2.1) implies

$$|P\phi - \phi| \le |\phi|_{C^1} \operatorname{mesh}(\mathscr{P}). \tag{2.2}$$

Although we are interested only in the case of a single observation equation (0.1) (the vector-valued case involves trivial modifications) for the proof below we will actually need observations valued in $\mathbf{R}^\infty$, i.e., $y = (y_1, y_2, \dots)$. This is because (see below) for each n we construct approximations to c that take values in $\mathbf{R}^{N_n}$, where $N_n \to \infty$ as $n \to \infty$ is the size of the nth approximation partition. To this end we redefine Ω to be $D([0,\infty); X) \times C([0,\infty); \mathbf{R}^\infty)$ and for c: $X \to \mathbf{R}^\infty$ redefine G^u to be $A^u + \frac{1}{2}\Delta_y + c(x) \cdot \nabla_y$ where ∇_y, Δ_y denote the gradient and Laplacian in $\mathbf{R}^\infty$. In Part II it is shown that the value functions v, v' are well-defined in this more general setting and in particular if $c = (c, 0, 0, \dots)$ we recover the value functions as defined in Section 1. In fact we shall only deal with c that have only finitely many nonzero components. Nevertheless we still need to work with $\mathbf{R}^\infty$ as we need to consider sequences of observations c_n: $X \to \mathbf{R}^{N+1}$ where the number $N = N_n$ of (nonzero) observations goes to infinity with n.

Given $c: X \to \mathbf{R}^\infty$ with finitely many nonzero components, $c = (c_1, c_2, \dots)$, $y = (y_1, y_2, \dots)$, set

$$\|c\|^2 = |c_1|^2 + |c_2|^2 + \cdots,$$
$$cy = c_1 y_1 + c_2 y_2 + \cdots,$$
$$c^2 = c_1^2 + c_2^2 + \cdots, \quad \text{etc.}$$

A measure $m \in M(X)$ is *smooth* if in each coordinate chart m is absolutely continuous relative to Lebesgue measure with $dm/dleb$ positive and C^∞. In particular the support of a smooth measure is all of X.

Although we do not need it, we begin by establishing that $m \in M(X)$ smooth implies the smoothness of $\mu_m(t)$, $t \geq 0$, a.s.-Q. We do this as motivation for the estimate (2.13) below. Let $c = (c, 0, 0, \dots)$. Then $cy = cy_1$ for $y \in \mathbf{R}^\infty$. Fix $m \in M(X)$ and let p_0 denote the corresponding density. Since A, B have C^∞ coefficients there are *adjoint* operators A^*, B^*,

$$\langle A\phi, \psi \rangle_{L^2} = \langle \phi, A^*\psi \rangle_{L^2}, \langle B\phi, \psi \rangle_{L^2} = \langle \phi, B^*\psi \rangle_{L^2}.$$

Then A^*, B^* are operators with C^∞ coefficients whose zeroth order terms may be nonzero. Nevertheless $A^* - A^*1$ is a Markov generator (as in Section 1) and $B^* - B^*1$ is a vector field. Set $A^{*u} = A^* - B^*u$. For $u \in U$, $y \in \mathbf{R}^\infty$, set

$$B^{u,y} = e^{cy} A^u e^{-cy} - \tfrac{1}{2}c^2,$$
$$B^{*u,y} = e^{-cy} A^{*u} e^{cy} - \tfrac{1}{2}c^2,$$
$$B_0^{u,y} = B^{u,y} - B^{u,y}1.$$

Then $(A^u)^* = A^{*u}$, $(B^{u,y})^* = B^{*u,y}$. These are operators with C^∞ coefficients. For $u \in L^2$ and $y \in C([0,\infty); \mathbf{R}^\infty)$, let B_t, B_t^*, $B_{0,t}$ denote the time-dependent operators obtained from $B^{u,y}$, $B^{*u,y}$, $B_0^{u,y}$ by plugging in $u(t), y(t)$. Let $T_{t,s}$ denote the time-dependent semigroup generated by B_t and let $p = p(t,x)$ be the solution of the initial value problem

$$\frac{\partial p}{\partial t} = B_t^* p, \qquad t > 0,\ p(0) = p_0. \tag{2.3}$$

Then by the Feynman–Kac formula, $p(t,x)$ is continuous and positive. Now suppose $\pi \in C([0,\infty); M(X))$ solves the initial value problem (recall $\langle \cdot , \cdot \rangle_\pi$ denotes the covariance)

$$\frac{d}{dt}\pi(\phi) = \pi(B_{0,t}\phi) + \langle B_t 1, \phi \rangle_\pi, \qquad t > 0,\ \phi \in \mathscr{D},\ \pi(0) = m. \tag{2.4}$$

Let $R(t)$, $t \geq 0$, be the solution of $\dot{R} = R\pi(B_t 1)$, $R(0) = 1$. Set $\rho(t)(\phi) = R(t)\pi(t)(\phi)$. Then by differentiating $\rho(s)(T_{t,s}\phi)$, $0 < s < t$, it follows that $\pi(t)(\phi) = \langle p_0, T_{t,0}\phi \rangle_{L^2} / \langle p_0, T_{t,0}1 \rangle_{L^2}$ is the unique solution of (2.4). Now set $\tilde{\pi}(t)(\phi) = \langle p(t), \phi \rangle_{L^2} / \langle p(t), 1 \rangle_{L^2}$; then by (2.3) π satisfies (2.4). Moreover since the coefficients of B_t are C^∞ and bounded in u, standard techniques from PDE theory (see Appendix C) yield the following result: For each $r \geq 0$

there is a progressively measurable continuous map k_r: $[0,\infty) \times C([0,\infty); \mathbf{R}) \to \mathbf{R}_+$ such that

$$\sup_{0 \le t \le T} |p(t)|_{C^r} \le k_r(T, y_1). \tag{2.5}$$

Thus the solution of (2.4) exists, is unique and the density $p(t)$ satisfies (2.5). In particular p is in $C^{0,\infty}([0,\infty) \times X)$.

Now let $\mu(t) = \mu_m(t)$ denote the conditional distribution of $x(t)$ given $\mathscr{Y}_t'$, $t \ge 0$. Then (Part II) for all $\phi \in \mathscr{D}$ and all generalized controls Q starting at m,

$$\begin{aligned} \mu(t)(\phi) &= m(\phi) + \int_0^t \mu(s)(A^{u(s)}\phi)\,ds \\ &\quad + \int_0^t \langle c, \phi \rangle_{\mu(s)}(dy(s) - \mu(s)(c)\,ds), \qquad t \ge 0, \end{aligned} \tag{2.6}$$

a.s.-Q.

Moreover using the Itô differential rule (2.6) implies that for $\Phi \in \hat{\mathscr{D}}$,

$$\Phi(\mu_m(t)) - \int_0^t \hat{A}^{u(s)}\Phi(\mu_m(s))\,ds, \qquad t \ge 0, \tag{2.7}$$

is a $(\Omega', \mathscr{Y}_t', Q)$-martingale. In fact for $u \in U$ constant, the martingale problem for $\hat{A}^u$ is well-posed on $C([0,\infty); M(X))$ (Part I), although we do not use this here. With reference to the discussion in Section 0, we note the well-posedness of the martingale problem is the same as saying that $\hat{A}^u$ (in particular $\hat{A}$) extends uniquely to a (Hille–Yosida) generator for a Markov semigroup on $C(M(X))$.

Recall that c denotes the function $X \to \mathbf{R}^\infty$ given by $(c, 0, 0, \dots)$ and y denotes the canonical process in $\mathbf{R}^\infty$. Then (2.6) suggests that μ_m can be chosen $\sigma(u(s), y_1(s), 0 \le s \le t)$ progressively measurable where y_1 denotes the first component of y. That this is indeed the case is established in Part I, Lemma 5.1. Set

$$\pi^{u,y}(t)(\phi) = \mu_m(t)(e^{-cy(t)}\phi)/\mu_m(t)(e^{-cy(t)}).$$

Then a standard application of the Itô differential rule establishes that $\pi^{u,y}$ solves (2.4) a.s.-Q. Since

$$\mu_m(t)(\phi) = \pi^{u,y}(t)(e^{cy(t)}\phi)\mu_m(t)(e^{-cy(t)}),$$

it follows from the above that $\mu_m(t)$ is smooth for $t \ge 0$, a.s.-Q, and in particular satisfies

$$\begin{aligned} \sup_u \sup_{0 \le t \le T} |\mu_m(t) B|_{\text{var}} &\le k(T, y_1), \\ \sup_u \sup_{0 \le t \le T} |\mu_m(t) B^2|_{\text{var}} &\le k(T, y_1) \end{aligned} \tag{2.8}$$

for some progressively measurable W-a.s.-continuous map k: $[0,\infty)\times C([0,\infty);\mathbf{R})\to\mathbf{R}$ when m is smooth. It also follows that the solution of (2.6) is unique.

The second step begins by choosing, for each $n\ge 1$, a partition $\mathscr{P}_n$ with $\delta_n=\mathrm{mesh}(\mathscr{P}_n)$ satisfying $n^3\,\delta_n\le 1$, for all n. Let P_n denote the corresponding projection operator onto $L^2(X,\mathscr{P}_n)$ and let $N=N_n$ denote the size of $\mathscr{P}_n$.

With B_0 as defined above, let B_0', B' be the unique first-order differential operators such that B_0', B' are skew-adjoint on $L^2(X)$ and differ from B_0, B by multiplication operators. Then $e^{t\Delta}$, $t\ge 0$, are self-adjoint and $e^{tB_0'}, e^{tB'}$, $t\ge 0$, are unitary, both on $L^2(X)$. The operators $e^{tB_0'}, e^{tB'}$ can be written down explicitly; for example,

$$(e^{tB'}\phi)(x)=\phi(\beta_t(x))\exp\left(\int_0^t(B'1)(\beta_s(x))\,ds\right),$$

where β_t: $X\to X$ is the flow corresponding to B.

For $n\ge 1$, $u\in U$, set (recall $A=A_0+A_1=\Delta+B_0+A_1$)

$$\begin{aligned}
A_n&=n\left(P_ne^{\Delta/n}P_n-I\right)+n\left(P_ne^{B_0'/n}P_n-P_ne^{B_0'/n}1\right)+A_{1,n},\\
B_n&=n\left(P_ne^{B'/n}P_n-P_ne^{B'/n}1\right),\\
A_n^u&=A_n-B_nu,\\
c_n&=\left(P_nc,\varepsilon_n1_{G_1},\dots,\varepsilon_n1_{G_N},0,0,\dots\right),\ \varepsilon_n=\delta_n/N_n^3,\\
L_n(\cdot,u)&=P_nL(\cdot,u).
\end{aligned}$$

Here $A_{1,n}$ is the jump process generator whose transition probability kernel is the piecewise constant projection of that of A_1 onto $\mathscr{P}_n$. Then A_n^u satisfies the maximum principle (recall $-u\ge 0$), $A_n^u1=0$, c_n: $X\to\mathbf{R}^\infty$ and we have the estimates

(2.9) $$|A_n^u\phi|=O(n)|\phi|,$$

(2.10) $$|A_n^u\phi-A^u\phi|=O\left(n\,\delta_n+\frac{1}{n}\right)|\phi|_{C^4}=O\left(\frac{1}{n}\right)|\phi|_{C^4},$$

(2.11) $$\|c_n-(c,0,0,\dots)\|=O(\delta_n),$$

(2.12) $$\sup_u|L_n(\cdot,u)-L(\cdot,u)|=O(\delta_n).$$

By construction for each $n\ge 1$ the operator A_n^u is of the form

$$A_n^u\phi(x)=\lambda_n^u\int_X(\phi(x')-\phi(x))\mu_n^u(x,dx')$$

for some number $\lambda_n^u>0$ and measurable transition kernel $\mu_n^u(x,\cdot)$ whose explicit forms are not needed. Consequently [Ethier and Kurtz (1986), page 162], for each $u \in L^2$ the martingale problem for A_n^u is well-posed on $D([0,\infty); X)$. This implies that to each $m \in M(X)$ and $u \in L^2$ corresponds a unique P solving the martingale problem for A_n^u on $D([0,\infty); X)$. It is easy to see that P depends measurably on $u \in L^2$. This implies (Part I, Corollary 4.3) that the martingale problem for $G_n^u = A_n^u + c_n(x) \cdot \nabla_y + \frac{1}{2}\Delta_y$ is well-posed on $D([0,\infty); X) \times C([0,\infty); \mathbf{R}^\infty)$. Let ${}_nP_m^u$ be the unique solution to this martingale problem. With E denoting expectation against ${}_nP_m^u$, $v_n^u(m)$ is defined as in (0.2) and the corresponding value function v_n: $M(X) \to \mathbf{R}$ is defined as before. With the above definitions, (2.10), (2.11) and (2.12) imply that v_n converges uniformly to v on $M(X)$ as $n \uparrow \infty$ (Part II).

Let $\mu_{n,m}(t)$ denote the conditional distribution of $x(t)$ given $\mathscr{Y}_t'$ corresponding to the approximating system (A_n, B_n, c_n, L_n). Then $\mu_{n,m}$ is a progressively measurable function of u and the first $N_n + 1$ components of y.

Let $m \in M(X)$ be smooth. Below we shall need the estimates

$$(2.13)\qquad \begin{aligned} &\sup_u \sup_{0\le t\le T} |\mu_{n,m}(t) B_n|_{\text{var}} \le k(T, y_1, y_2, \dots), \quad \text{a.s.-}W,\\ &\sup_u \sup_{0\le t\le T} |\mu_{n,m}(t) B_n^2|_{\text{var}} \le k(T, y_1, y_2, \dots), \quad \text{a.s.-}W, \end{aligned}$$

the point being that k: $[0,\infty) \times C([0,\infty); \mathbf{R}^\infty) \to \mathbf{R}$ does not depend on n; W is Wiener measure on $C([0,\infty); \mathbf{R}^\infty)$. Since for n large, B_n approximates B and $\mu_{n,m}$ approximates μ_m, the fact that (2.8) holds certainly makes (2.13) plausible. Nevertheless the proof, in Appendix C, is somewhat involved. In what follows we let $\hat{L}_n$, $\hat{H}_n$ be the quantities corresponding to L_n as in Section 0.

The third step is the analysis of v_n and the corresponding control problem for each n fixed. Let $\overline{X}_n$ denote the finite set $\{1, \dots, N_n\}$ and let π_n: $X \to \overline{X}_n$ be the piecewise constant map whose value on G_i is i. Then there are $N_n \times N_n$ matrices $\overline{A}_n$, $\overline{B}_n$, an N_n-vector $\bar{c}_n$: $\overline{X}_n \to \mathbf{R}^\infty$ and an N_n-vector of functions $\overline{L}_n$: $\overline{X}_n \times U \to \mathbf{R}$ such that with $\overline{A}_n^u = \overline{A}_n - \overline{B}_n u$,

$$(\overline{A}_n^u \overline{\phi}) \circ \pi_n = A_n^u(\overline{\phi} \circ \pi_n),$$

$$\bar{c}_n \circ \pi_n = c_n,$$

$$\overline{L}_n \circ \pi_n = L_n,$$

where A_n^u, c_n, L_n are already defined. In other words, the map π_n *intertwines* the systems $(A_n, B_n, c_n, L_n), (\overline{A}_n, \overline{B}_n, \bar{c}_n, \overline{L}_n)$. Since the matrices $\overline{A}_n^u$ satisfy the maximum principle, $\overline{A}_n^u 1 = 0$ and $(\overline{A}_n, \overline{B}_n, \bar{c}_n, \overline{L}_n)$ is smooth ($\overline{X}_n$ is a finite set), the corresponding variational problem and value function $\bar{v}_n$: $M(\overline{X}_n) \to \mathbf{R}$ are well-defined. Let $\hat{\overline{A}}_n, \hat{\overline{B}}_n, \hat{\overline{L}}_n, \hat{\overline{H}}_n, \bar{\mu}_{n,m}^u$ be the corresponding quantities. Since the map π_n induces maps π_{n*}: $M(X) \to M(\overline{X}_n)$, $D([0,\infty); X) \to D([0,\infty); \overline{X}_n)$ and $M(D([0,\infty); X)) \to M(D([0,\infty); \overline{X}_n))$, chasing through the definitions yields Lemma 2.1.

LEMMA 2.1. *Let $\mathscr{P}_n = \{G_1, \ldots, G_N\}$. Then*

$$\begin{aligned}
v_n^u(m) &= \bar{v}_n^u(\pi_{n*}m),\\
v_n(m) &= \bar{v}_n(\pi_{n*}m) = \bar{v}_n(m(G_1), \ldots, m(G_N)),\\
\bar{\mu}_{n,\bar{m}}^u(t) &= \pi_{n*}\mu_{n,m}^u(t), \qquad t \geq 0, \bar{m} = \pi_{n*}m,\\
\pi_{n*}(\mu T_t^{B_n}) &= (\pi_{n*}\mu)T_t^{\bar{B}_n},\\
\hat{B}_n v_n(\mu) &= \hat{\bar{B}}_n \bar{v}_n(\pi_{n*}\mu),\\
\hat{L}_n(\mu, u) &= \hat{\bar{L}}_n(\pi_{n*}\mu, u),\\
\hat{H}_n(\mu, p) &= \hat{\bar{H}}_n(\pi_{n*}\mu, p).
\end{aligned}$$

Moreover if the support of m is X, then $\bar{m} = \pi_{n*}m$ lies in the interior of $M(\bar{X}_n)$.

For $\bar{m} \in M(\bar{X}_n)$ and control u, let $\bar{\mu}_{n,\bar{m}}^u$ denote the conditional distribution process corresponding to $(\bar{A}_n, \bar{B}_n, \bar{c}_n, \bar{L}_n)$. Then $\bar{\mu} = \bar{\mu}_{n,\bar{m}}^u$ satisfies (W-a.s.)

$$\begin{aligned}
\bar{\mu}(t)(\bar{\phi}) &= \bar{m}(\bar{\phi}) + \int_0^t \bar{\mu}(s)\big(\bar{A}_n^{u(s)}\bar{\phi}\big)\, ds\\
&\quad + \int_0^t \langle \bar{c}_n, \bar{\phi}\rangle_{\bar{\mu}(s)}(dy(s) - \bar{\mu}(s)(\bar{c}_n)\, ds), \qquad t \geq 0, \bar{\phi} \in B(\bar{X}_n).
\end{aligned} \tag{2.14}$$

Since $\bar{X}_n$ is a finite set (2.14) is a finite-dimensional system of coupled stochastic differential equations with polynomial coefficients. Moreover our definition of c_n guarantees the following.

LEMMA 2.2. *For $\bar{m} \in M(\bar{X}_n)$, $\bar{\mu}_{n,\bar{m}}^u$ is a C^∞ diffusion valued in $M(\bar{X}_n) \subset \mathbf{R}^{N_n}$ with generator $\hat{\bar{A}}_n^u$ that is nondegenerate on the interior of $M(\bar{X}_n)$.*

To describe $\hat{\bar{A}}_n^u$ more explicitly, we specialize Lemma 2.2 to two cases: First we choose $A = 0$, $B = 0$, $c = 0$, and $\varepsilon_n = 1$ in the definition of c_n. Then $\hat{\bar{A}}_n^u$ does not depend on u and is a specific second-order *nondegenerate* differential operator $\hat{\Delta}_n$ on $M(\bar{X}_n) \subset \mathbf{R}^{N_n}$. Second we choose (A, B, c) arbitrary but choose $\varepsilon_n = 0$ in the definition of c_n. Denote the generator given in Lemma 2.2 in this case by $\hat{\bar{A}}_{0n}^u$. Given these generators, $\hat{\bar{A}}_n^u$ in the general case turns out to be

$$\hat{\bar{A}}_n^u = \hat{\bar{A}}_{0n}^u + \varepsilon_n^2\, \hat{\Delta}_n.$$

Now note that

$$\bar{v}_n^u(\bar{m}) = \bar{E}_{n,\bar{m}}^u\left(\int_0^\infty e^{-t}\bar{L}_n(\bar{x}(t), u(t))\, dt\right) \tag{2.15}$$

$$= \bar{E}_{n,\bar{m}}^u\left(\int_0^\infty e^{-t}\hat{\bar{L}}_n(\bar{\mu}_{n,m}^u(t), u(t))\, dt\right). \tag{2.16}$$

This and Lemma 2.2 imply that $\bar{v}_n$ is a viscosity solution [Crandall and Lions (1983)] of the Bellman equation

$$-\hat{\bar{A}}_n\bar{v}_n + \hat{\bar{H}}_n\left(\bar{\mu}, \hat{\bar{B}}_n\bar{v}_n\right) + \bar{v}_n = 0 \tag{2.17}$$

on $M(\bar{X}_n)$.

LEMMA 2.3. *$\bar{v}_n$ is C^2 on the interior of $M(\bar{X}_n)$.*

PROOF. Since (2.17) is nondegenerate, it follows [Lions (1981)] that $\bar{v}_n$ is $C^{1,1}$ on the interior of $M(\bar{X}_n)$. Since in our case (2.17) is in addition quasilinear (as opposed to fully nonlinear), it follows by Schauder that $\bar{v}_n$ is $C^{2,\alpha}$. In fact one can show that $\hat{\bar{H}}_n(\bar{\mu}, p)$ is $C^{1,1}$ and obtain $\bar{v}_n$ in $C^{3,\alpha}$. □

Note also that (2.15) shows that $\bar{v}_n^u(\bar{m})$ is affine in $\bar{m}$. An immediate consequence is the following *a priori* inequality.

LEMMA 2.4. *For all $n \geq 1$, we have*

$$\max_{\bar{\mu}} \max_{1\leq i\leq N_n} |\partial_i \bar{v}_n(\bar{\mu})| \leq \sup_u \max_{\bar{\mu}} \max_{1\leq i\leq N_n} |\partial_i \bar{v}_n^u(\bar{\mu})| \leq |\bar{L}_n| \leq |L|$$

and $\bar{v}_n$ is concave on $M(\bar{X}_n)$.

Let $\bar{Q}_n$ be a generalized control starting at $\bar{m}_n$ and let $\bar{\mu}_n(t)$ denote the conditional distribution process, both for the system $(\bar{A}_n, \bar{B}_n, \bar{c}_n, \bar{L}_n)$. Then (2.17) and Lemma 2.3 imply

$$e^{-t}\bar{v}_n(\bar{\mu}_n(t)) - \int_0^t e^{-s}\Big(-u(s)\hat{\bar{B}}_n\bar{v}_n(\bar{\mu}_n(s)) + \hat{\bar{H}}_n\Big(\bar{\mu}_n(s), \hat{\bar{B}}_n\bar{v}_n(\bar{\mu}_n(s))\Big)\Big)\, ds, \qquad t \geq 0,$$

is a $(\Omega', \mathscr{Y}_t', \bar{Q}_n)$-martingale, provided $\bar{m}_n$ lies in the interior of $M(\bar{X}_n)$. We conclude the following. Below $M(X)^0$ denotes the measures whose support is all of X.

LEMMA 2.5. *For each $n \geq 1$, $\hat{B}_n v_n$ exists on all of $M(X)^0$. Let $m \in M(X)^0$ and let Q_n be a generalized control starting at m. Let τ be a $\mathscr{Y}_t'$-stopping time. Then*

$$E^{Q_n}(e^{-\tau}v_n(\mu_n(\tau))) = v_n(m) + E^{Q_n}\left(\int_0^\tau e^{-t}\big(-u(t)\hat{B}_n v_n(\mu_n(t)) + \hat{H}_n(\mu_n(t), \hat{B}_n v_n(\mu_n(t)))\big)\, dt\right). \tag{2.18}$$

Here μ_n is the conditional distribution process (given $\mathscr{Y}_t'$) for the system (A_n, B_n, c_n, L_n).

Lemma 2.5 completes the analysis of the approximating value function v_n and step 3. The fourth step involves passing to the limit $n \uparrow \infty$.

LEMMA 2.6. *Let $\mu_n, \mu \in M(X)^0$ satisfy $\mu_n \to \mu$. If there exists a constant $C > 0$ such that $|\mu_n B_n^2|_{\text{var}} \le C$, then*

$$v_n\left(\mu_n T_t^{B_n}\right) - |L|Ct^2 \tag{2.19}$$

is a concave function of $t \in \mathbf{R}$ for each n. If moreover $\hat{B}v(\mu)$ exists, then $\hat{B}_n v_n(\mu_n) \to \hat{B}v(\mu)$.

PROOF. By Lemma 2.1,

$$v_n\left(\mu_n T_t^{B_n}\right) = \bar{v}_n\left(\mu_n\left(T_t^{B_n} 1_{G_1}\right), \dots, \mu_n\left(T_t^{B_n} 1_{G_N}\right)\right).$$

Differentiating twice and using Lemma 2.4 yields

$$\begin{aligned}
\frac{d^2}{dt^2} v_n\left(\mu_n T_t^{B_n}\right) &= \sum_{i,j=1}^{N} (\partial_i \partial_j \bar{v}_n) \mu_n\left(B_n T_t^{B_n} 1_{G_i}\right) \mu_n\left(B_n T_t^{B_n} 1_{G_j}\right) \\
&\quad + \sum_{i=1}^{N} (\partial_i \bar{v}_n) \mu_n\left(B_n^2 T_t^{B_n} 1_{G_i}\right) \\
&\le |L| \sum_{i=1}^{N} \left|\mu_n\left(B_n^2 T_t^{B_n} 1_{G_i}\right)\right| \\
&\le 2|L| \left|\sum{}' \mu_n\left(B_n^2 T_t^{B_n} 1_{G_i}\right)\right| \\
&\le 2|L| |\mu_n B_n^2|_{\text{var}},
\end{aligned}$$

where Σ' denotes the sum over a subset of $\{1, \dots, N_n\}$. This establishes the concavity of (2.19). Now this implies

$$\hat{B}_n v_n(\mu_n) \ge \frac{v_n\left(\mu_n T_t^{B_n}\right) - v_n(\mu_n)}{t} - |L|Ct$$

for $t > 0$, $n \ge 1$. Letting $n \uparrow \infty$, $t \downarrow 0$, in that order, yields

$$\liminf_{n \to \infty} \hat{B}_n v_n(\mu_n) \ge \hat{B}v(\mu). \tag{2.20}$$

Replacing B_n by $-B_n$ in (2.20) the result follows. □

Let

$$R(T) = \exp\left(\int_0^T c(x(t))\, dy(t) - \tfrac{1}{2}\int_0^T |c(x(t))|^2\, dt\right),$$

$$\hat{R}(T) = \exp\left(\int_0^T \mu_m(t)(c)\, dy(t) - \tfrac{1}{2}\int_0^T |\mu_m(t)(c)|^2\, dt\right).$$

Let Q be a generalized control for (A, B, c, L) and define $Q^\#$ by setting $dQ/dQ^\# = R(T)$ on $\mathcal{F}_T'$. Then $Q^\#$ is a generalized control for $(A, B, 0, L)$. Let

U denote its marginal on $\mathscr{Y}'$. Then the marginal of U on $\mathscr{Y}$ is W and the defining property of $\mu_m(t)$ implies

$$\begin{aligned} v(Q) &= E^Q\left(\int_0^\infty e^{-t} L(x(t), u(t))\, dt\right) \\ &= E^Q\left(\int_0^\infty e^{-t} \hat{L}(\mu_m(t), u(t))\, dt\right). \end{aligned} \tag{2.21}$$

Let $Q_\omega^\#$ denote a regular conditional distribution of $Q^\#$ given $\mathscr{Y}'$. Then (Part II) $Q_\omega^\#$ solves the martingale problem for A^u on $D([0,\infty); X)$, where $\omega = (y, u)$. Thus letting P_m^u denote the unique solution to the martingale problem for A^u on $D([0,\infty); X)$, we have $Q_\omega^\# = P_m^u$. It follows that (Part II)

$$\mu_m(t, \omega)(\phi) = E_m^u(\phi(x(t))R(t))/E_m^u(R(t)), \qquad \text{a.s.-}U. \tag{2.22}$$

Now let $\mu_{n,m}$ denote the conditional distribution process for the system (A_n, B_n, c_n, L_n). Using the formula analogous to (2.22), one can verify the following.

LEMMA 2.7. *$\mu_{m,n}(t)(\phi) \to \mu_m(t)(\phi)$ in $L^1(U)$ for all $\phi \in C(X)$.*

PROPOSITION 2.8. *Let $m \in M(X)$ be smooth. Let Q be such that the derivatives $\hat{B}v(\mu_m(t))$, $t \geq 0$, exist a.s.-$(dt \times Q)$ [$=$ a.s.-$(dt \times U)$]. Then the following holds*:

$$\begin{aligned} v(Q) = v(m) + E^Q\Big(\int_0^\infty e^{-t}\big(&\hat{L}(\mu_m(t), u(t)) - u(t)\hat{B}v(\mu_m(t)) \\ &+ \hat{H}(\mu_m(t), \hat{B}v(\mu_m(t)))\big)\, dt\Big). \end{aligned} \tag{2.23}$$

PROOF. Since m is smooth, m has full support and so (see above) $\overline{m}_n = \pi_{n*}m$ lies in the interior of $M(\overline{X}_n)$. Thus (2.18) can be applied.

Now by passing to a subsequence we have $\mu_{m,n}(t) \to \mu_m(t)$ in $M(X)$ a.s.-U. Note that $|\hat{B}_n v_n(\mu_n)| \leq |L||\mu_n B_n|_{\text{var}}$. Let τ_C be the first time t that $|\mu_{mn}(t)B_n|_{\text{var}}, |\mu_{m,n}(t)B_n^2|_{\text{var}}$ exceeds C for some $n \geq 1$. Then by (2.13) it follows that $\tau_C \to \infty$ as $C \to \infty$. For each n let R_n, $\hat{R}_n$ be defined as above but with $\mu_{m,n}, c_n$ replacing μ_m, c. Now let $Q_n^\#$ be the unique generalized control for $(A_n, B_n, 0, L_n)$ whose marginal on $\mathscr{Y}'$ is U and set $dQ_n/dQ_n^\# = R_n(T)$ on $\mathscr{F}_T'$. Then Q_n is a generalized control for (A_n, B_n, c_n, L_n) satisfying $dQ_n/dU = \hat{R}_n(T)$ on $\mathscr{Y}_T'$ (the proof is analogous to that of Lemma 5.3 in Part I). Using this, one checks that as $n \uparrow \infty$, $dQ_n/dU \to dQ/dU$ in U-probability. Since $\hat{H}_n \to \hat{H}$ uniformly on compact subsets of $M(X) \times \mathbf{R}$ (Appendix A), choosing $\tau = \tau_C$ in (2.18) and noting that $\hat{L} - pu + \hat{H} \geq 0$ (Appendix A), letting $n \uparrow \infty$, $C \uparrow \infty$, in that order, the result follows. □

COROLLARY 2.9. *Let $m \in M(X)$ be arbitrary and let Q be such that the derivatives $\hat{B}v(\mu_m(t))$, $t \geq 0$, exist a.s.-$(dt \times Q)$. Then* (2.23) *holds.*

PROOF. For $\delta > 0$, let $Q_{\delta,\omega'}$ be the r.c.p.d. satisfying

$$E^{Q_{\delta,\cdot}}(\Phi) = E^{Q}(\Phi(x(\cdot+\delta), y(\cdot+\delta) - y(\delta), u(\cdot+\delta))|\mathscr{Y}_{\delta}').$$

For $\omega' = (y, u)$, set $S_{\delta}\omega'(t) = (y(t+\delta) - y(\delta), u(t+\delta))$. Since Q starts at m, it follows that $Q_{\delta,\omega'}$ starts at $\mu_m(\delta, \omega')$, a.s.-Q. It also follows from the definition (Section 1) that $Q_{\delta,\omega'}$ is a generalized control for Q-a.a. ω'. Thus

$$v(Q) = E^{Q}\left(\int_0^{\delta} e^{-t}L(x(t), u(t))\,dt + e^{-\delta}v(Q_{\delta,\cdot})\right). \tag{2.24}$$

Since μ_m is the unique solution of (2.6) the semigroup property holds,

$$\mu_m(t+\delta, \omega') = \mu_{\mu_m(\delta,\omega')}(t, S_{\delta}\omega'), \tag{2.25}$$

for Q-a.a. ω'. This plus the fact that $\hat{B}v(\mu_m(t))$ exists for $t \geq \delta$ implies the existence of $\hat{B}v(\mu_{\mu_m(\delta,\omega')}(t))$, a.s.-$Q_{\delta,\omega'}$ for $t \geq 0$, for Q-a.a. ω'. Now the nondegeneracy of A implies that of B_t^*; this implies that $\mu_m(\delta, \omega')$ is smooth for Q-a.a. ω', since $\pi^{u,y}$ (Section 2) satisfies a nondegenerate parabolic integrodifferential equation with C^{∞} coefficients. Thus Proposition 2.8 applies with $Q_{\delta,\omega'}$ replacing Q; this yields

$$\begin{aligned} v(Q_{\delta,\omega'}) = v(\mu_m(\delta,\omega')) + E^{Q_{\delta,\omega'}}\bigg(\int_0^{\infty} e^{-t}\big(\hat{L}(\mu_z(t), u(t)) \\ -u(t)\hat{B}v(\mu_z(t)) + \hat{H}\big(\mu_z(t), \hat{B}v(\mu_z(t))\big)\big)\,dt\bigg), \end{aligned} \tag{2.26}$$

where $z = \mu_m(\delta, \omega')$,

for Q-a.a. ω'. Now combining (2.24) and (2.25) together with E^Q of (2.26) and letting $\delta \downarrow 0$, (2.23) follows. □

By the remarks at the beginning of this section, this concludes the proof of Theorem B. □

3. Bellman equation. Here we establish that the value function v is a viscosity solution of the Bellman equation

$$-\hat{A}v + \hat{H}(\mu, \hat{B}v) + v = 0 \tag{3.1}$$

on $M(X)$, relative to the test function space $\hat{\mathscr{D}}$. We begin by recalling the definition [Crandall and Lions (1983)].

Let $v \in C(M(X))$. We say v is a *viscosity subsolution* of (3.1) if for all $\Phi \in \hat{\mathscr{D}}$, $m \in M(X)$, the equality $0 = v(m) - \Phi(m) = \max_{\mu \in M(X)}(v(\mu) - \Phi(\mu))$ implies

$$-\hat{A}\Phi(m) + \hat{H}(m, \hat{B}\Phi(m)) + \Phi(m) \leq 0. \tag{3.2}$$

We say v is a *viscosity supersolution* of (3.1) if for all $\Phi \in \hat{\mathscr{D}}$, $m \in M(X)$, the equality $0 = v(m) - \Phi(m) = \min_{\mu \in M(X)}(v(\mu) - \Phi(\mu))$ implies

$$-\hat{A}\Phi(m) + \hat{H}(m, \hat{B}\Phi(m)) + \Phi(m) \geq 0. \tag{3.3}$$

A *viscosity solution* of (3.1) is a function that is both a subsolution and a supersolution of (3.1). In Part II it is shown that $v \in C(M(X))$.

THEOREM C. *The value function $v \in C(M(X))$ is a viscosity solution of* (3.1).

Before we prove this result, we make some comments on the choice of test function space $\hat{\mathscr{D}}$. Certainly we expect Theorem C to hold for a larger class of functions, as long as it is reasonable, i.e., $\hat{A}$, etc., are naturally defined on the class. We choose $\hat{\mathscr{D}}$ because it is the smallest possible such class and arises naturally in this context. It is an open problem whether a uniqueness result for viscosity solutions holds relative to the above $\hat{\mathscr{D}}$. Relative to a larger class, existence and uniqueness were established in Lions (1988).

The proof of Theorem C is standard and is based on the following *dynamic programming principle*.

PROPOSITION. *For $T > 0$, $m \in M(X)$,*

$$v(m) = \inf_Q E^Q\left(\int_0^T e^{-t} L(x(t), u(t))\,dt + e^{-T} v(\mu_m(T))\right),$$

where the infimum is over all generalized controls starting at m.

PROOF. By (2.24),

$$\begin{aligned} v(Q) &= E^Q\left(\int_0^T e^{-t} L(x(t), u(t))\,dt + e^{-T} v(Q_{T,\cdot})\right) \\ &\geq E^Q\left(\int_0^T e^{-t} L(x(t), u(t))\,dt + e^{-T} v(\mu_m(T))\right). \end{aligned}$$

Taking the infimum over Q and recalling that $v = v'$ (Section 1) yields

$$v(m) \geq E\left(\int_0^T e^{-t} L(x(t), u(t))\,dt + e^{-T} v(\mu_m(T))\right).$$

Recall that Fleming and Pardoux (1982) established the existence of optimal generalized controls. In this setting, this is established as a corollary of results in Part II. Now for the reverse inequality, let $\mu \to Q_\mu$ be a Borel map $M(X) \to M(\Omega')$ such that Q_μ is an optimal generalized control starting from μ [Stroock and Varadhan (1979), Section 12.1]. Then $Q_{\mu_m(T)}$ is $\mathscr{Y}'_T$ measurable. Given a generalized control P starting at m, set $Q = P \otimes_{\mathscr{Y}'_T} Q_{\mu_m(T)}$ [Stroock and Varadhan, Section 6.1] and let $Q_{T,\cdot}$ be as in Corollary 2.9. Then $Q_{T,\cdot} = Q_{\mu_m(T)}$ a.s.-Q and so

$$\begin{aligned} v(m) \leq v(Q) &= E^Q\left(\int_0^T e^{-t} L\big(x(t), u(t)\,dt + e^{-T} v\big(Q_{\mu_m(T)}\big)\big)\right) \\ &= E^Q\left(\int_0^T e^{-t} L\big(x(t), u(t)\,dt + e^{-T} v\big(\mu_m(T)\big)\big)\right). \end{aligned}$$

This completes the proof. $\square$

We now show that v is a subsolution of (3.1). Suppose $m \in M(X)$, $\Phi \in \hat{\mathscr{D}}$, are such that $0 = v(m) - \Phi(m) = \max_\mu(v(\mu) - \Phi(\mu))$. Then for all $u \in U$ constant,

$$\Phi(m) \leq E^u\left(\int_0^T e^{-t} L(x(t), u)\, dt + e^{-T}\Phi(\mu_m^u(T))\right).$$

Using (2.7) and (2.21) we obtain

$$0 \leq E^u\left(\int_0^T e^{-t}(\hat{L} - \Phi + \hat{A}\Phi - u\hat{B}\Phi)\, dt\right).$$

Dividing by T and letting $T \downarrow 0$ yields

$$0 \leq \hat{L}(m, u) - \Phi(m) + \hat{A}\Phi(m) - u\hat{B}\Phi(m).$$

Taking the infimum over $u \in U$ yields (3.2). To show v is a supersolution, suppose $m \in M(X)$, $\Phi \in \hat{\mathscr{D}}$, are such that $0 = v(m) - \Phi(m) = \min_\mu(v(\mu) - \Phi(\mu))$. Then there exists Q such that

$$\Phi(m) \geq E^Q\left(\int_0^T e^{-t} L(x(t), u(t))\, dt + e^{-T}\Phi(\mu_m(T))\right).$$

This implies

$$0 \geq E^Q\left(\int_0^T e^{-t}(\hat{L} - \Phi + \hat{A}\Phi - u(t)\hat{B}\Phi)\, dt\right)$$

$$\geq E^Q\left(\int_0^T e^{-t}(-\hat{H} - \Phi + \hat{A}\Phi)\, dt\right).$$

Dividing by T and letting $T \downarrow 0$ yields (3.3).

APPENDIX A

Let X be a Polish space, $U \subset \mathbf{R}^d$ closed and convex, and let $L: X \times U \to \mathbf{R}$ satisfy:

1. $L \in C_b(X \times \{u \mid |u| \leq r\})$ for all r.
2. $L(x, \cdot\,)$ is strictly convex on U.
3. $\inf_x L(x, u)/|u| \to +\infty$ as $|u| \to \infty$.

Set $\hat{L}(\mu, u) = \int_X L(x, u)\, d\mu(x)$,

$$\hat{H}(\mu, p) = \sup_{u \in U}\left(p \cdot u - \hat{L}(\mu, u)\right). \tag{A.1}$$

Then $\hat{L} \in C_b(M(X) \times \{u \mid |u| \leq r\})$, for all r.

THEOREM. *$\hat{H} \in C_b^{0,1}(M(X) \times \{p \mid |p| \leq r\})$ for all r, and*

$$\hat{L}(\mu, u) - p \cdot u + \hat{H}(\mu, p) \geq 0$$

with equality iff $u = \hat{H}_p(\mu, p)$. *Moreover suppose L_n are uniformly bounded* *on* $X \times \{u \mid |u| \le r\}$ *for all* r. *Then* $L_n \to L$ *uniformly on compact subsets of* $X \times \mathbf{R}^d$ *implies* $\hat{H}_n \to \hat{H}$ *uniformly on compact subsets of* $M(X) \times \mathbf{R}^d$.

PROOF. We prove only the first part and omit the proof of the convergence statement. Clearly assumptions (2) and (3) on L imply the corresponding conditions on $\hat{L}$. Given μ, p let u_n be a sequence in U such that $p \cdot u_n - \hat{L}(\mu, u_n)$ approaches the supremum in (A.1). Then because $\hat{L}$ grows faster than linearly at infinity, u_n must lie in a bounded subset of $\mathbf{R}^d$. Hence the supremum is attained at some point in U, call it $u^*(\mu, p)$. Now strict convexity of $\hat{L}$ implies that the supremum is attained at only u^*. Since $\hat{H}$ is a supremum of continuous functions, it follows that $\hat{H}$ is lower semicontinuous. To establish upper semicontinuity, let $\mu_n \to \mu$, $p_n \to p$. Then by the growth of $\hat{L}$ at infinity, $u^*(\mu_n, p_n)$ lies in a bounded subset of $\mathbf{R}^d$. Hence by passing to a subsequence $\hat{H}(\mu_n, p_n) = p_n \cdot u_n^* - \hat{L}(\mu_n, u_n^*) \to p \cdot u' - \hat{L}(\mu, u') \le \hat{H}(\mu, p)$. This implies upper semicontinuity, and also implies the continuity of u^*. The boundedness of $\hat{H}, u^*$ is straightforward; we conclude that $\hat{H} \in C_b(M(X) \times \{p \mid |p| \le r\})$ *for all* r.

Now let $v \in \mathbf{R}^d$ and set $p_t = p + tv$, $u_t^* = u^*(\mu, p_t)$. Then $\hat{H}(\mu, p_t) \ge p_t \cdot u_0^* - \hat{L}(\mu, u_0^*)$, $\hat{H}(\mu, p_0) = p_0 \cdot u_0^* - \hat{L}(\mu, u_0^*)$. Subtracting these two expressions, dividing by t and letting $t \downarrow 0$ yields

$$\liminf_{t \downarrow 0} \frac{\hat{H}(\mu, p + tv) - \hat{H}(\mu, p)}{t} \ge u^*(\mu, p) \cdot v.$$

Reversing the roles of p_t, p_0 yields

$$\limsup_{t \downarrow 0} \frac{\hat{H}(\mu, p + tv) - \hat{H}(\mu, p)}{t} \le u^*(\mu, p) \cdot v,$$

where we have used the fact that $u_t^* \to u_0^*$ as $t \downarrow 0$. This shows $\hat{H}_p(\mu, p) = u^*(\mu, p)$. The result follows. □

APPENDIX B

Here we sketch the heuristic derivation leading to the Bellman equation (0.5). Let μ denote the conditional distribution process. Then (2.7) for $\Phi \in \hat{\mathscr{D}}$,

$$\text{(B.1)} \quad e^{-t}\Phi(\mu(t)) - \int_0^t e^{-s}\big(-\Phi(\mu(s)) + \hat{A}^{u(s)}\Phi(\mu(s))\big)\,ds, \qquad t \ge 0,$$

is a martingale. Suppose F: $M(X) \to \mathbf{R}$ is a bounded solution of

$$\text{(B.2)} \qquad -\hat{A}F + \hat{H}(\mu, \hat{B}F) + F = 0,$$

such that (B.1) applies to F. Then

$$\text{(B.3)} \quad \begin{aligned} &E(e^{-T}F(\mu(T))) \\ &\quad = F(m) + E\bigg(\int_0^T e^{-t}\big(-u(t)\hat{B}F(\mu(t)) + \hat{H}\big(\mu(t), \hat{B}F(\mu(t))\big)\big)\,ds\bigg). \end{aligned}$$

Since

$$v^u(m) = E\left(\int_0^\infty e^{-t}\hat{L}(\mu(t), u(t))\,dt\right), \tag{B.4}$$

letting $T \uparrow \infty$ in (B.3) and combining (B.3) and (B.4) yields

$$\begin{aligned} v^u(m) &= F(m) \\ &\quad + E\left(\int_0^\infty e^{-t}\big(\hat{L}(\mu, u) - u\hat{B}F(\mu) + \hat{H}(\mu, \hat{B}F(\mu))\big)\,dt\right). \end{aligned} \tag{B.5}$$

Suppose now there existed a control u satisfying the feedback $u = \hat{H}_p(\mu, \hat{B}F(\mu))$. Since (Appendix A)

$$\hat{L}(\mu, u) - pu + \hat{H}(\mu, p) \geq 0,$$

with equality iff $u = \hat{H}_p(\mu, p)$, (B.5) implies:

1. $F = v$,
2. v satisfies (0.5).
3. Optimal controls are characterized by $u = \hat{H}_p(\mu, \hat{B}v)$.

This concludes the nonrigorous derivation of the results described in Section 0.

APPENDIX C

The purpose here is to establish (2.13). To this end we need to recall some notions from pseudo-differential operator (ψDO) theory. Suppose that $\dim(X) = d$. We begin in $\mathbf{R}^d$.

A *symbol* of order m is a C^∞ map p: $\mathbf{R}^d \times \mathbf{R}^d \to \mathbf{C}$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m - |\beta|}, \tag{C.1}$$

for all multi-indices α, β. Let $S^m = S^m(\mathbf{R}^d)$ denote the set of all symbols of order m on $\mathbf{R}^d$. The best constants $C_{\alpha\beta}$ that fit in (C.1) are seminorms on $S^m(\mathbf{R}^d)$; with these seminorms, $S^m(\mathbf{R}^d)$ is a Frechet space. A linear operator A: $C_0^\infty(\mathbf{R}^d) \to C^\infty(\mathbf{R}^d)$ is a *pseudo-differential operator* (ΦDO) of order m if

$$A\phi(x) = \int_{\mathbf{R}^d} p(x, \xi)\hat{\phi}(\xi)e^{i\xi\cdot x}\,d\xi, \qquad i = \sqrt{-1}. \tag{C.2}$$

Here $\hat{\phi}$ is the Fourier transform of ϕ. In this case we write $A = p(x, D)$. Let $H^s = H^s(\mathbf{R}^d)$ denote the *Sobolev space* of order s. Then $A \in S^m$ implies A: $H^s \to H^{s-m}$ continuously for all s, and the map $p \to p(x, D)$, $S^m \to \mathscr{L}(H^s, H^{s-m})$ is continuous for all s. Multiplication $(a, b) \to ab$ is continuous $S^m \times S^n \to S^{m+n}$ and there is a continuous bilinear map $(a, b) \to c$, $S^m \times S^n \to S^{m+n-1}$, such that

$$c(x, D) = a(x, D)b(x, D) - ab(x, D).$$

In particular there are continuous maps $(a, b) \to c, (a, b) \to d$, $S^m \times S^n \to S^{m+n}, S^m \times S^n \to S^{m+n-1}$, satisfying $c(x, D) = a(x, D)b(x, D), d(x, D) = [a(x, D), b(x, D)]$ [Hörmander (1984)].

Let X be a compact C^∞ manifold. It can be shown that under change of coordinates the class S^m remains invariant and so it is meaningful to say a linear operator A: $C^\infty(X) \to C^\infty(X)$ is in $\mathrm{OPS}^m(X)$ if in each chart A can be represented as in (C.2). Then the topology on S^m induces, via a fixed choice of finite atlas on X, a Frechet topology on $\mathrm{OPS}^m(X)$. It follows that composition and bracketing are continuous bilinear maps $\mathrm{OPS}^m(X) \times \mathrm{OPS}^n(X) \to \mathrm{OPS}^{m+n}(X), \mathrm{OPS}^{m+n-1}(X)$, respectively. It also follows that $A \in \mathrm{OPS}^0(X)$ implies A is bounded on $L^2(X)$ [Hörmander (1984)].

The Laplacian Δ is a negative-definite self-adjoint (densely defined) operator on $L^2(X)$. By the functional calculus $\Lambda = \sqrt{I - \Delta}$ is positive definite and self-adjoint. It is well known that $\Lambda \in \mathrm{OPS}^1(X)$ and, in fact, $H^s(X)$ is the domain of the unbounded operator Λ^s in $L^2(X)$, $0 \le s < \infty$. In particular $A \in \mathrm{OPS}^m$ iff $A\Lambda^{-m} \in \mathrm{OPS}^0$. Now suppose $p \in S^0(\mathbf{R})$. Then $p(\Lambda) \in \mathrm{OPS}^0(X)$. More generally, $p_\varepsilon(\xi) = p(\varepsilon\xi)$, $0 < \varepsilon \le 1$, is bounded in $S^0(\mathbf{R})$ and hence $p_\varepsilon(\Lambda)$, $0 < \epsilon \le 1$, is bounded in $\mathrm{OPS}^0(X)$ [Taylor (1981)]. Choosing $p(\xi) = e^{1-\xi^2}$ and applying the above facts yields

LEMMA C.1. *The family* $\{e^{\varepsilon\Delta}, 0 \le \varepsilon \le 1\}$ *is bounded in* $\mathrm{OPS}^0(X)$. *Moreover setting* $\Delta_0 = \Delta$ *and*

$$\Delta_\varepsilon = \frac{e^{\varepsilon\Delta} - I}{\varepsilon} = \Delta \int_0^1 e^{\varepsilon t \Delta}\, dt, \qquad 0 < \varepsilon \le 1,$$

$\{\Delta_\varepsilon, 0 \le \varepsilon \le 1\}$ *is bounded in* $\mathrm{OPS}^2(X)$ *and*

$$[[\Delta_\varepsilon, A], B], \qquad 0 \le \varepsilon \le 1, \tag{C.3}$$

is bounded in $\mathrm{OPS}^0(X)$ *for* $A \in \mathrm{OPS}^m(X)$, $B \in \mathrm{OPS}^{-m}(X)$. *In particular* (C.3) *is a uniformly bounded family of operators on* $L^2(X)$.

Let H be a Hilbert space and suppose A: $\mathscr{D} \subset H \to H$. Let A^* denote the adjoint (in H!) of A and define $\mathrm{Re}\, A$ by setting $2\,\mathrm{Re}\, A = A + A^*$. Then $\mathrm{Re}\, A$ is a symmetric operator on H. We say $\mathrm{Re}\, A \le C$ if $\langle \mathrm{Re}\, A\phi, \phi\rangle_H = \mathrm{Re}\langle A\phi, \phi\rangle_H \le C|\phi|_H^2$ for all $\phi \in \mathscr{D}$.

Let Δ_ε be as in Lemma C.1. Let $\psi \in C^\infty(X)$ be such that $\psi^{-1} \in C^\infty(X)$. Set $\Delta_\varepsilon^\psi = \psi \Delta_\varepsilon \psi^{-1}$. Recall that the inner product on $H^s(X)$ is given by $\langle \phi, \psi\rangle_{H^s} = \langle \Lambda^s\phi, \Lambda^s\psi\rangle_{L^2}$.

LEMMA C.2. *For all* s, $\mathrm{Re}\, \Delta_\varepsilon^\psi \le C_s$, $0 \le \varepsilon \le 1$, *on* $H^s(X)$.

PROOF. Note that

$$\langle \Delta_\varepsilon^\psi \phi, \phi\rangle_{H^s} = \langle \Lambda^s \Delta_\varepsilon^\psi \Lambda^{-s}\Lambda^s\phi, \Lambda^s\phi\rangle_{L^2} = \langle B_s \Delta_\varepsilon B_s^{-1}\Lambda^s\phi, \Lambda^s\phi\rangle_{L^2},$$

where $B_s = \Lambda^s\psi\Lambda^s \in \mathrm{OPS}^{2s}(X)$. Thus it is enough to establish $\mathrm{Re}(B_s \Delta_\varepsilon B_s^{-1}) \le C_s$ on $L^2(X)$.

Now

$$\mathrm{Re}\big(B_s \Delta_\varepsilon B_s^{-1}\big) = B_s \Delta_\varepsilon B_s^{-1} + B_s^{-1} \Delta_\varepsilon B_s = \Delta_\varepsilon - \tfrac{1}{2}\big[[\Delta_\varepsilon, B_s], B_s^{-1}\big].$$

But $\Delta_\varepsilon \leq 0$ on $L^2(X)$ by the spectral theorem. Hence by Lemma C.1 the result follows. □

We also need to establish the analog of Lemma C.2 for a first-order differential operator. Let B be a first-order differential operator that is skew-symmetric on $L^2(X)$, $B^* = -B$. Let e^{tB} denote the corresponding group (see Section 2). Then e^{tB} is unitary on $L^2(X)$. Set $B_\varepsilon = (e^{\varepsilon B} - I)/\varepsilon$, $\varepsilon > 0$, $B_0 = B$, $B_\varepsilon^\psi = \psi B_\varepsilon \psi^{-1}$.

LEMMA C.3. Re $B_\varepsilon^\psi \leq C_s$, $0 \leq \varepsilon \leq 1$, *on* $H^s(X)$, *for any even nonnegative integer* s.

PROOF. Note $e^{\varepsilon B} - e^{-\varepsilon B}$ is skew-adjoint on $L^2(X)$ so

$$\begin{aligned} 2\varepsilon \operatorname{Re} B_\varepsilon &= e^{\varepsilon B} - 2I + e^{-\varepsilon B} \\ &= \left(e^{\varepsilon B/2} - e^{-\varepsilon B/2}\right)^2 \leq 0 \end{aligned}$$

on $L^2(X)$. Set $D_s = \Lambda^s \psi \Lambda^s$. Then D_s is a differential operator of order $2s$ and $e^{\varepsilon B}$ is not a ψDO. However, appealing to the explicit formula for $e^{\varepsilon B}$ in Section 2, using the chain rule and noting that D_s is a differential operator, we see that $[D_s, e^{\varepsilon B}] = \varepsilon e^{\varepsilon B} P_{s,\varepsilon}$ with $P_{s,\varepsilon}$, $-1 \leq \varepsilon \leq 1$, bounded in $\mathrm{OPS}^{2s}(X)$. We conclude that $T_{s,\varepsilon} = [D_s, B_\varepsilon] D_s^{-1}$, $-1 \leq \varepsilon \leq 1$, is uniformly bounded on $L^2(X)$. Since $\Lambda^s B_\varepsilon^\psi \Lambda^{-s} = D^s B_\varepsilon D_s^{-1} = B_\varepsilon + T_{s,\varepsilon}$, it follows that $\operatorname{Re}(\Lambda^s B_\varepsilon^\psi \Lambda^{-s}) \leq C_s$ on $L^2(X)$. From this it follows that Re $B_\varepsilon^\psi \leq C_s$ on $H^s(X)$, completing the proof. □

We return to our setting. As a warm-up to (2.13) we first establish (2.8). Let B_t be as in Section 2. Then (remember $u \leq 0$) since B_t is a second-order differential operator with C^∞ coefficients, it follows that Re $B_t^* =$ Re $B_t \leq C_s$ on $H^s(X)$. Differentiating $|p(t)|_{H^s}^2$ (the standard trick) yields

$$\frac{d}{dt}|p(t)|_{H^s}^2 = 2\langle \operatorname{Re} B_t p(t), p(t)\rangle_{H^s} \leq C_s(t, y_1)|p(t)|_{H^s}^2$$

and hence $\sup_{0 \leq t \leq T}|p(t)|_{H^s} \leq k_s(T, y_1)$. (2.5) follows then by the Sobolev embedding theorem. Since by Feynman–Kac

$$p(t) \geq \inf_x p_0(x) \exp\left(-\int_0^t |B_s^* 1|\, ds\right),$$

$p(t)$ is bounded away from zero and (2.8) follows readily.

We proceed with a similar argument to establish (2.13). We begin with an intermediate case. Set (see the definition of A_n)

$$\begin{aligned} \tilde{A}_n &= n(\varepsilon^{\Delta/n} - I) + n(e^{B_0'/n} - e^{B_0'/n}1) + A_1, \\ \tilde{B}_n &= n(e^{B'/n} - e^{B'/n}1), \\ \tilde{A}_n^u &= \tilde{A}_n - \tilde{B}_n u, \\ \tilde{B}_n^{u,y} &= e^{cy}\tilde{A}_n^u e^{-cy} - \tfrac{1}{2}c^2, \\ B_n^{u,y} &= e^{c_n y} A_n^u e^{-c_n y} - \tfrac{1}{2}c_n^2. \end{aligned}$$

Let $\tilde{B}_{nt}, B_{nt}$ denote the time-dependent operators obtained by plugging $u \in L^2$, $y \in C([0,\infty); \mathbf{R}^\infty)$ into $\tilde{B}_n^{u,y}$, $B_n^{u,y}$. Let $\tilde{p}_n(t)$, $p_n(t)$ solve the initial value problems

$$\frac{\partial \tilde{p}_n}{\partial t} = \tilde{B}_{nt}^* \tilde{p}_n, \qquad t > 0,\ \tilde{p}_n(0) = p_0,$$

$$\frac{\partial p_n}{\partial t} = B_{nt}^* p_n, \qquad t > 0,\ p_n(0) = p_0,$$

as linear ordinary differential equations valued in $L^2(X)$.

Then $\tilde{B}_{nt}$ is not a differential operator, but it is close to one. In fact it is a ΨDO in S^2 and the whole point of bringing in ΨDO theory is that the above lemmas imply the following crucial fact.

Lemma C.4. *Re $\tilde{B}_{nt}$ = Re $\tilde{B}_{nt}^* \le k_s(y_1(t))$ on $H^s(X)$. Hence by the standard trick (see above), it follows that*

$$\sup_{0 \le t \le T} |\tilde{p}_n(t)|_{H^s} \le k_s(T, y_1)$$

for all even nonnegative s.

Comparing the definitions of $\tilde{B}_n^{u,y}$ and $B_n^{u,y}$ above we see we must estimate the effect of the error between c and c_n and the projection operator P_n.

Lemma C.5. *The following estimates hold (recall $\varepsilon_n N_n^3 = \delta_n \le 1$):*

$$|e^{\pm c_n y} - e^{\pm c y_1}| \le \left(\varepsilon_n N_n^3 Y + |y_1||P_n c - c|\right)\exp\left(3|c||y_1| + 3\varepsilon_n N_n^3 Y\right)$$
$$\le (Y + |y_1||c|_{C^1})\, \delta_n \exp(3|c||y_1| + 3Y),$$

where

$$Y = Y(y) = \sup_{n \ge 1} \frac{1}{n^3}\Big(\max_{1 \le k \le n} |y_k|\Big),$$

$$|\tilde{A}_n^{u*}\phi - A_n^{u*}\phi| = O(n\,\delta_n)|\phi|_{C^4},$$

$$|\tilde{B}_n^{u,y*}\phi - B_n^{u,y*}\phi| = O(n\,\delta_n)|\phi|_{C^4} k(Y).$$

These estimates are straightforward and omitted.

Lemma C.6. *Let W denote Wiener measure on $C([0,\infty); \mathbf{R}^\infty)$. Then*

$$W\Big(\sup_{0 \le t \le T} Y(y(t)) < \infty\Big) = 1.$$

Again this lemma is an easy exercise since $y_1, y_2, \dots$ are i.i.d. We omit the proof.

Let $T^n_{t,s}$ denote the semigroup generated by B^*_{nt} on $C(X)$. Estimating as in Lemma C.5 yields $|B^*_{nt}1| \le k(t,Y)$. This implies:

1. The norm of $T^n_{t,s}$ on $C(X)$, $0 \le s \le t \le T$, is bounded by some $k(T,Y)$.
2. $p_n(t)$ is uniformly bounded away from zero,

$$\log \inf_{0 \le t \le T} p_n(t) \ge -k(T,Y).$$

Now let $q_n = p_n - \tilde{p}_n$. Then q_n satisfies

$$\frac{\partial q_n}{\partial t} = B^*_{nt} q_n + \left(B^*_{nt} - \tilde{B}^*_{nt}\right)\tilde{p}_n, q_n(0) = 0.$$

Integrating this last equation yields

$$\begin{aligned} q_n(t) &= \int_0^t T^n_{t,s}\left(B^*_{ns} - \tilde{B}^*_{ns}\right)\tilde{p}_n \, ds \\ &= O(n\,\delta_n) \sup_{0 \le t \le T} |\tilde{p}_n(t)|_{C^4} k(T,Y) = O(n\,\delta_n) k(T, y_1, y_2, \dots). \end{aligned}$$

Recalling that $n^3\,\delta_n \le 1$ (Section 2), $|B^*_n\phi| = O(n)|\phi|$ with B_n as in Section 2, we conclude

$$\sup_u \sup_{0 \le t \le T} |B^*_n p_n(t)| \le k(T, y_1, y_2, \dots),$$

$$\sup_u \sup_{0 \le t \le T} |B^{2*}_n p_n(t)| \le k(T, y_1, y_2, \dots).$$

Now let $\pi_n(t)$ be the probability measure whose density is $p_n(t)$ as in Section 2. Then the above estimate on $p_n(t)$ yields

$$\sup_u \sup_{0 \le t \le T} |\pi_n(t) B_n|_{\text{var}} \le k(T, y_1, y_2, \dots),$$

$$\sup_u \sup_{0 \le t \le T} |\pi_n(t) B^2_n|_{\text{var}} \le k(T, y_1, y_2, \dots),$$

almost surely W. Now note that B_n satisfies the Leibnitz rule,

$$B_n(\phi\psi) = (B_n\phi)\psi + e^{B/n}\phi B_n\psi + O(n\,\delta_n)|\phi\psi|.$$

Moreover, as in Section 2, $\mu_{m,n}$ and π_n are related via

$$\mu_{m,n}(t)(\phi) = \pi_n(t)\left(e^{c_n y(t)}\phi\right)\mu_{m,n}(t)\left(e^{-c_n y(t)}\right).$$

Combining the last three equations yields (2.13).

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