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BOOK REVIEW

M. J. SHARPE, *General Theory of Markov Processes*, Academic, San Diego, 1988, 410 pages, \$49.50. ISBN: 0-12-639060-6

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Sharpe's book provides researchers with the first comprehensive presentation of the melding of Markov processes with the general theory of stochastic processes and will serve as a standard of style for the emerging generation of students of Markov process theory. The title of the book is a *double entendre*: "General Theory" signifies not only the foundations of Markov processes, but also the "general theory of stochastic processes," a subject which has assumed a central role in modern probability and whose development can be traced largely through the *Séminaire de Probabilités* series. Explaining the ingredients in the meld consumes several paragraphs and touches several major lines of thought in the last four decades of probability theory.

Markov processes. Occasional readers of the Markov process literature may be puzzled by the profusion of seemingly different Markov processes which appear. There are Feller processes, Hunt processes, standard processes, Ray processes and right processes. All of these processes are simply variations of one basic structure. It consists of a measurable space $(\Omega, \mathscr{F})$, a family of functions X_t mapping Ω into a state space E and a family $(P^x)_{x \in E}$ of probability measures on $(\Omega, \mathscr{F})$. The process X_t generates a filtration $(\mathscr{F}_t)$, namely, the appropriate completion of $\sigma\{X_s\colon s \le t\}$. Finally, there is a shift operator $\theta_t\colon \Omega \to \Omega$ which is tied to the process by the property $X_t \circ \theta_s = X_{t+s}$. These objects constitute a basic Markov process sextuplet $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ if the strong Markov property is satisfied: for every $(\mathscr{F}_t)$-stopping time T, for every positive $\mathscr{F}$-measurable random variable F and for every positive $\mathscr{F}_T$-measurable random variable G, we have

$$E^x[G \cdot F \circ \theta_T; T < \infty] = E^x\big[G \cdot E^{X(T)}[F]; T < \infty\big].$$

All of the processes mentioned above have the property that $t \to X_t$ is right continuous almost surely. Additional assumptions are peculiar to the individual processes. For example, Feller and Ray processes assume certain continuity properties about the associated semigroup P_t, resolvent U^α and state space E. Hunt processes are characterized by the quasi-left-continuity property, namely, $X_{T(n)}$ converges to X_T almost surely whenever $T(n)$ is a sequence of $(\mathscr{F}_t)$-stopping times increasing to T. With the exception of right processes, all of these variations existed in the 1950s. Right processes were introduced later by Meyer (but let us not get ahead of the story).

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In the beginning. If a book can be said to have grandparents, Sharpe's book would certainly have Doob and Hunt for its. Both the general theory of stochastic processes and the modern theory of Markov processes are rooted in their work. While the general theory really is "general" in the sense that it applies to any stochastic process, its origins and inspiration are tied closely to the development of Markov processes, and the reuniting of these two subjects is what Sharpe's book is about. However, the story begins with Markov processes in the 1940s and 1950s long before there was a general theory of stochastic processes.

Doob discovered many of the principal results about martingales in the 1940s and 1950s. Many of his articles feature the interplay between Markov processes and supermartingales, notably the role which superharmonic functions play in the study of Brownian motion. He is credited also with the first explicit case of the strong Markov property appearing in the literature. In 1957–1958, Hunt's trilogy on Markov processes appeared, and it would be difficult to overstate the impact of Hunt's work [5]. His identification of a Markov process with a potential theory is a major result which still animates large parts of the two subjects. With each transient Markov process X can be associated its potential,

$$U = \int_0^\infty P_t \, dt, \tag{1}$$

and its cone of excessive functions,

$$\mathscr{S} = \{ f\colon E \to \mathbb{R}^+ \colon E^x[f(X_t)] \text{ increases to } f(x) \text{ as } t \to 0\}.$$

For example, if X_t is Brownian motion in $\mathbb{R}^3$, then $\mathscr{S}$ is the set of positive superharmonic functions. Conversely, Hunt showed that a "sufficiently nice" kernel U satisfying the complete maximum principle may be represented in the form (1), where (P_t) is the semigroup of some Markov process. This remarkable result tied each analytic theory to a probabilistic one and, in conjunction with Doob's earlier work, set the stage for the probabilistic techniques in analysis which have proved to be so successful.

Hunt also used Choquet capacities in these articles to prove measurability of hitting times $T_A = \inf\{t > 0\colon X_t \in A\}$. This introduction of capacities to probability theory was the precursor of a major part of the general theory of stochastic processes and has become an indispensable tool.

The general theory of stochastic processes. The development of Markov processes and martingale theory led to a deeper investigation of the foundations of stochastic processes, whence came the general theory of stochastic processes. Its development was guided in large part by P.-A. Meyer, and its successful applications extend far beyond Markov process theory. An early reference to the theory is Chapters III through VI of Dellacherie [2]. An elaborated version can be found in the recent series of books *Probabilités et Potentiel* by Dellacherie and Meyer [3]. (Volume 4 of this series concerns Markov processes and is a nice complement to Sharpe's book.)

It is difficult to summarize briefly the general theory and its ramifications, but it may help to touch on certain salient points. Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $(\mathscr{F}_t)$ which is right continuous (i.e., $\mathscr{F}_t = \bigcap_{s>t}\mathscr{F}_s$) and such that $\mathscr{F}_0$ is P-complete. Let $\mathscr{M}$ be the collection of stochastic processes $Z_s(\omega)$ which are adapted to $(\mathscr{F}_s)$. We define two σ-algebras on $\mathbb{R}^+ \times \Omega$ by setting

$$\mathscr{O} = \sigma\{Z \in \mathscr{M}: s \to Z_s(\omega) \text{ is right continuous for every } \omega\},$$

$$\mathscr{P} = \sigma\{Z \in \mathscr{M}: s \to Z_s(\omega) \text{ is left continuous for every } \omega\}.$$

$\mathscr{O}$ and $\mathscr{P}$ are called the optional and predictable σ-algebras, respectively, and they form the basis for the theory. These σ-algebras are subtly different. For example, if T is an $(\mathscr{F}_t)$-stopping time, let $Z_s(\omega) = 1_{\{T(\omega)=s\}}$. Then Z_s is always $\mathscr{O}$-measurable, but is $\mathscr{P}$-measurable if and only if there is a sequence (T_n) of stopping times increasing to T with $T_n < T$ a.s. on $\{T > 0\}$. Such a time is called predictable.

There are at least two classes of results in the general theory which have had a profound effect on modern probability. The first is Meyer's section theorem, which is a direct outgrowth of Hunt's introduction of capacity theory to probability. Let Z_t be a nonnegative optional (resp., predictable) process. For every $\varepsilon > 0$, there is a stopping time T (resp., predictable time T) such that

$$Z_{T(\omega)}(\omega) > 0 \quad \text{on } \{\omega: T(\omega) < \infty\},$$

$$P\Big[\sup_s Z_s(\omega) > 0\Big] \le P[T < \infty] + \varepsilon.$$

The second collection consists of the projection theorems, developed by Meyer, Dellacherie, Doléans-Dade, and Azéma. Let Y_s be a positive measurable process. There are an optional process ${}^oY_s$ and a predictable process ${}^pY_s$ such that

$$E\big[Y_T 1_{\{T<\infty\}} | \mathscr{F}_T\big] = {}^oY_T 1_{\{T<\infty\}} \quad \text{a.s.},$$

$$E\big[Y_S 1_{\{S<\infty\}} | \mathscr{F}_{S-}\big] = {}^pY_S 1_{\{S<\infty\}} \quad \text{a.s.},$$

whenever T is a stopping time and S is a predictable time. ${}^oY$ and ${}^pY$ are called the optional and predictable projections of Y, respectively. Let A_t be a measurable increasing process (by which we mean $A_0 = 0$, $A_{t+s} \ge A_t$ and $A_{t+} = A_t$). If $E[A_\infty] < \infty$, then there is an optional increasing process A_t^o and a predictable increasing process A_t^p such that

$$E\int Z_s\, dA_s = E\int Z_s\, dA_s^o,$$

$$E\int Y_s\, dA_s = E\int Y_s\, dA_s^p,$$

whenever Z_s is a positive optional process and Y_s is a positive predictable process. A_t^o and A_t^p are called the dual optional and dual predictable projections of A_t, and these last two projections figure prominently in Markov process theory as well as in other areas of probability, such as stochastic integration and point processes. For example, one often used fact is that $A_t - A_t^p$ is a martingale whenever A_t is an adapted increasing process.

The general theory of Markov processes. Dual optional and predictable projections play a tremendous role in analyzing additive and multiplicative functionals of Markov processes. For example, if X_t is a transient diffusion on $\mathbb{R}$, and if L is the last time X_t visits a point a, then the dual predictable projection of the increasing process $A_t = 1_{\{0 < L \leq t\}}$ is the local time L_t^a at a. Technically, the general theory of stochastic processes only allows us to take the dual predictable projection relative to a fixed probability P, while here we need to take it simultaneously for each law P^x. Furthermore, we expect the result L_t^a to be an additive functional. Such a result with the added Markov structure is not a part of the general theory. In an unpublished set of notes entitled *Fonctionnelles Additives de Markov* which date from the early 1970s, Sharpe began to develop a version of the general theory within the Markov framework. The fruition of his efforts in this direction pervades and unifies the book: It is one of very few places that researchers and students can access the tools and language which have become standard in Markov process theory during the last 15 years. Having recently finished writing an article with Sharpe's book at hand, I can testify to the usefulness as a reference of his chapters concerning the unified formulation of the general theory of stochastic processes, homogeneity, random measures and additive and multiplicative functionals. The comprehensive subject and notation indices make the book especially easy to use for this purpose.

A lot of Sharpe's thought has gone into producing a consistent unification of the Markov framework itself, also. The book is concerned mainly with analyzing right processes and indicating their connection with potential theory. (The accepted name "right process" is not a very descriptive one, since it does not even advertise its Markov property. It would be more helpful to casual readers for authors to refer to a "right Markov process," but this is rarely done. There is no confusion in the original French description since one always takes "*un processus de Markov satisfaisant aux hypothèses droites*.") This class of processes was introduced 20 years ago by Meyer. It subsumes Hunt, Feller and standard processes, and seems sufficiently general to include most Markov processes with a decent potential theory. The right process formulations profferred by Meyer and later by Getoor are not invariant under some of the classical Markov process transformations. Sharpe's new version of these hypotheses is invariant under a multitude of transformations, including killing, time-change, maps of the state space, Doob's h-transforms and others, and he devotes one chapter to discussing these transformations.

There is also a chapter on Ray–Knight methods, in which Sharpe discusses the compactification technique which has become an indispensable tool in studying right processes. Finally, there is a brief chapter on stochastic calculus in the Markov context. This outline of the chapters cannot do justice to the wealth of material each contains, and each is definitely worth browsing through. For example, the chapter on additive functionals contains sections on time-changing, local times, Lévy systems and excursions. Each of these subjects is rich enough to deserve book-length treatment.

This book offers to students and researchers the state of the art in Markov processes. Sharpe has successfully meshed the general theory of stochastic processes with Markov theory and has presented the fundamentals of modern Markov processes in a unified and powerful way. Not only does it work, but it works beautifully and represents the next leap in mathematical technology in this field. There is much that Sharpe could not present in his book: forty years of vigorous work cannot be compressed into 400 pages. (Indeed, the beginner may wish to consult texts such as Blumenthal and Getoor [1], Dellacherie and Meyer [3], and Dynkin [4] before tackling the intricacies of Sharpe's text—technical power has its price! As with most first editions, there are misprints. A list of these can be obtained from the author of the book by writing to him at UCSD.) In particular, discussions of time reversal of Markov processes and Azéma's cooptional and copredictable projections were omitted. Neither was there room for a discussion of duality of Markov processes. But there is plenty in the book for the inquiring mind at an affordable price, and I recommend it heartily.

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