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A PROBABILISTIC PROOF OF S.-Y. CHENG'S LIOUVILLE THEOREM

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Let $f\colon M \to N$ be a harmonic map between complete Riemannian manifolds M and N, and suppose the Ricci curvature of M is nonnegative definite, the sectional curvature of N is nonpositive, and N is simply connected. Then if f has sublinear asymptotic growth, f must be a constant map. This result was first proved analytically by S.-Y. Cheng. This paper describes a probabilistic proof under the same hypotheses.

1. Introduction. The classical Liouville theorem states that a bounded harmonic function on the entire plane is a constant. S.-T. Yau (1975) generalized this by proving that nonnegative harmonic functions on Riemannian manifolds with nonnegative Ricci curvature are constant. S.-Y. Cheng (1980) extended it still further, by replacing the harmonic functions with maps into manifolds other than R and showing that even sublinear growth is impossible. In the case where the codomain is R^n, a harmonic map amounts to a vector of harmonic functions, but for more general codomains the theory reflects a subtle dependence on the curvature tensor of the codomain.

Along probabilistic lines, Debiard, Gaveau and Mazet (1976) proved Yau's result in the case of *bounded* functions. In this paper we will prove Cheng's generalization as it appears in the summary above. As yet there has been no probabilistic proof given of Yau's result in the case of *nonnegative* harmonic functions.

We prove the theorem by establishing two estimates which conflict in the event that a harmonic *function* is both nonconstant and sublinear. Both of these estimates appear in Debiard, Gaveau and Mazet (1976) in raw form, and we merely adapt them to our context. We describe these two estimates in Sections 2 and 3. In Section 4, we tackle the case of harmonic functions, and in Section 5 we show how to extend the argument to the case of a general harmonic map ($N \neq R$).

2. Lower bound. We first consider the case of a harmonic function f on M. Thus N is taken to be R and $\Delta f = 0$, where Δ is the Laplace–Beltrami operator of M (i.e., the positive trace of the Hessian of f). Our first estimate is a lower bound on $Ef^2(X_t)$, where E denotes expectation with respect to the Brownian motion X_t started at a point X_0. To prove this, we start with the following Bochner formula, which holds for any smooth function h on M:

$$\tfrac{1}{2}\Delta(\|dh\|^2) = \|\text{Hess}\, h\|^2 - (dh, d\,\Delta h) + \text{Ric}(dh, dh),$$

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where Ric denotes the Ricci curvature tensor of M. Analogous formulas can be derived with h replaced by a more general tensor field at the expense of making sense of Hess and replacing Ric with a more complicated curvature term [cf. Gallot, Hulin and Lafontaine (1987), page 159, or Tóth (1984), page 46].

The main probabilistic ingredient is Itô's formula. If h is a C^2 function on M, and X_t denotes the Brownian motion started at a point X_0, then

$$h(X_t) - h(X_0) - \frac{1}{2}\int_0^t \Delta h(X_s)\, ds = \text{martingale}.$$

Actually, this martingale can be described as a stochastic integral with respect to the "stochastic development" of the process X_t onto R^n. See for example Kendall (1987, 1988). Many times, the exact form of the stochastic integral is of no consequence, since upon taking expectation it reduces to zero. Because we wish to consider such martingales on noncompact manifolds, the distance of X_t from X_0 is an unbounded random variable. To ensure that the expectation of the random variables $h(X_t)$ exists, we restrict attention to those functions h which grow at most like a polynomial in the distance function. This is enough, since the transition density of the Brownian motion will decay exponentially in the distance [Li and Yau (1986)], while the volume will grow at a polynomial rate. Moreover, this will not complicate our proofs since we consider only functions of at most quadratic growth. Actually, the Li and Yau (1986) estimate is more than we need. The radial part of Brownian motion on M will be stochastically bounded by the radial part of the Euclidean Brownian motion of the same dimension. That is, $P(\text{dist}(X_t, X_0) > y) \leq P(\text{dist}(B_t, B_0) > y)$. This quantity then will decay exponentially in y, which is sufficient for our purposes.

Our first estimate is then given in Lemma 2.1.

LEMMA 2.1. *Let* $\text{Ric}^M \geq 0$ *and* $\Delta f = 0$. *Then*

$$E\big(f^2(X_t)\big) \geq f^2(X_0) + \|df(X_0)\|^2 t.$$

PROOF. Since $\Delta f = 0$, it follows that $\Delta f^2 = 2\|df\|^2$, and by Itô's formula,

$$E\big(f^2(X_t) - f^2(X_0)\big) = E\int_0^t \|df(X_s)\|^2\, ds.$$

Now we want to use Itô's formula a second time to find out what the integrand on the right-hand side is doing. Since Δf vanishes, and $\text{Ric}^M \geq 0$, the Bochner formula above yields

$$E\big(\|df(X_s)\|^2\big) = \|df(X_0)\|^2 + E\int_0^s \|\text{Hess } f(X_r)\|^2 + \text{Ric}\big(df(X_r), df(X_r)\big)\, dr,$$

and since the integrand on the right-hand side is nonnegative,

$$E\big(\|df(X_s)\|^2\big) \geq \|df(X_0)\|^2.$$

Observe now that we may interchange the expectation and integration with respect to s, and the result follows. □

To consider the general case, we need to make two alterations in the above equations. First we need to use a generalized Bochner formula, replacing the 1-form df with a TN-valued (or "twisted") 1-form so that the formula remains valid with df the differential of a map into N. Second, since f^2 no longer makes sense, we will apply Itô's formula to $\rho^2 \circ f$, where ρ is the metric distance on N from a reference point. This is done in Section 5.

3. Upper bound. If f has nonvanishing gradient at X_0, Lemma 2.1 gives us a linear lower bound on the growth of $E(f^2(X_t))$. In this section and the next we will show that if f has sublinear growth, so does $E(f^2(X_t))$. To begin this we recall that Brownian motion in R^n "moves at a rate of $t^{1/2}$," in the sense that $|B_t|^2 - nt$ is a martingale, and consequently $E|B_t|^2 = nt$. This fact can be generalized to our context by an application of Bishop's inequality.

LEMMA 3.1. *Suppose* $\mathrm{Ric}^M \geq 0$ *and let* r *denote the distance on* M *from a point* X_0. *Then,*

$$E\big(r^2(X_t)\big) \leq nt,$$

where n *is the dimension of* M.

PROOF. We apply Itô's formula to the function r^2. Since r^2 may not be C^2 on the cut locus of X_0, the formula stated above may not apply. Hence we assume at first that X_0 is a pole of M. After proving the lemma in this case, we will indicate how to generalize Itô's formula to cover the case of nonempty cut locus. By Itô's formula,

$$E\big(r^2(X_t)\big) = E\int_0^t \tfrac{1}{2}\Delta r^2(X_s)\,ds.$$

In polar coordinates at X_0,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{b'}{b}\frac{\partial}{\partial r} + \Delta^{\Theta},$$

where Δ^{Θ} denotes terms which do not contain any radial derivatives, b denotes the density of the Riemannian measure ($\sqrt{g}$) and b' denotes $\partial b/\partial r$. Observe that

$$\Delta r^2 = 2 + 2\frac{b'}{b}r.$$

Bishop's inequality allows us to compare b'/b with the corresponding function for flat space, since $\mathrm{Ric}^M \geq 0$. For flat R^n the volume form in polar coordinates is $r^{n-1}\,dr(dS^{n-1})$. Thus, $b = r^{n-1}$ times a function of variables other than r and b'/b is $(n-1)/r$. Bishop's inequality relates volumes of metric balls in M and R^n. To get this conclusion, one proves the inequality $b'/b \leq (n-1)/r$ and then integrates to obtain a comparison of volumes [Gallot, Hulin and Lafontaine (1987), page 161]. Applying this "differential" Bishop's inequality, we conclude that $\Delta r^2 \leq 2n$, so,

$$\tfrac{1}{2}\int_0^t \Delta r^2(X_s)\,ds \leq \int_0^t n\,ds = nt$$

for all paths, and the estimate follows.

Now we eliminate the requirement that X_0 be a pole. The tool we need is found in Corollary 1.2 of Kendall (1987), which states that

$$r(X_t) - \tfrac{1}{2}\int_0^t \Delta r(X_s)\,ds + L_t = \text{martingale},$$

where L_t is an increasing process which only increases when X_t crosses the cut locus of X_0, and $\Delta r = \nabla r = 0$ wherever r is not C^2. In particular, $r(X_t)$ is a semimartingale.

Since X_t is almost surely continuous and r is continuous, we may apply Itô's formula for continuous real-valued semimartingales to the C^2 function z^2 to conclude that

$$r^2(X_t) = \int_0^t 2r(X_s)\,d(r(X_s)) + \tfrac{1}{2}\int_0^t 2\,d\langle r(X_s)\rangle.$$

The first term on the right-hand side is an Itô integral with respect to $r(X_s)$, and the second term is a Lebesgue integral with respect to the quadratic variation process of $r(X_s)$, i.e., the quadratic variation of its martingale part. Therefore,

$$\langle r(X_t)\rangle = \int_0^t |\nabla r(X_s)|^2\,ds$$

[Kendall (1987)] and substituting we obtain

$$r^2(X_t) - \int_0^t r\,\Delta r\,ds - \int_0^t |\nabla r|^2\,ds + \int_0^t 2r\,dL_s = \text{martingale}.$$

Since $\Delta r^2 = 2r\,\Delta r + 2|\nabla r|^2$, taking expections gives

$$E(r^2(X_t)) = E\left(\tfrac{1}{2}\int_0^t \Delta r^2(X_s)\,ds - \int_0^t 2r(X_s)\,dL_s\right) \leq E\left(\tfrac{1}{2}\int_0^t \Delta r^2(X_s)\,ds\right) \leq nt.$$

We get the first inequality because L_s is nondecreasing. The second inequality is the content of the first part of the proof. □

4. Functions. When $\|df(X_0)\|^2 \neq 0$, Lemmas 2.1 and 3.1 give us upper and lower bounds on $Eh^2(X_t)$ for $h = f$ and $h = r$, respectively. We now show how this gives the conclusion of the theorem.

PROOF OF THEOREM (functions). Denote $B_a = \{x \in M | r(x) \leq a\}$, and define $m_f(a) = \max_{x \in B_a} |f(x)|$. Suppose f has sublinear growth, i.e.,

$$\limsup_{a \to \infty} a^{-1} m_f(a) = 0.$$

We show that this conflicts with Lemma 2.1 unless f is constant. It is easy to see that we need only prove the following lemma.

LEMMA 4.1. *If* $\limsup_{a \to \infty} a^{-1} m_f(a) = 0$, *then for all* $c > 0$, *there exists some* $C > 0$ *such that for every* $t > 0$, $Ef^2(X_t) \leq ct + C$. *That is,* $Ef^2(X_t)$ *is sublinear in* t.

PROOF. For every $\varepsilon > 0$, there exists some A such that for every $a > A$, $m_f(a) < a\varepsilon$. Moreover, for $a \leq A$, $m_{f^2} \leq C = m_{f^2}(A)$. Note that $(m_f)^2 = m_{f^2}$. Now, dividing the expectation into expectation over the two disjoint events $\{r(X_t) \leq A\}$ and $\{r(X_t) > A\}$, we find

$$Ef^2(X_t) = E\big(f^2(X_t); r(X_t) \leq A\big) + E\big(f^2(X_t); r(X_t) > A\big),$$

where $E(h; S)$ denotes $\int_S h\,dP$. Clearly, the first term is $\leq C$. In the second term, $f^2(x) \leq m_{f^2}(r(x)) = (m_f)^2(r(x)) \leq (\varepsilon r(x))^2$, hence,

$$\begin{aligned} E\big(f^2(X_t); r(X_t) > A\big) &\leq E\big(\varepsilon^2 r^2(X_t); r(X_t) > A\big) \\ &\leq (\varepsilon^2) E\big(r^2(X_t)\big) \leq (\varepsilon^2 n)t, \end{aligned}$$

so choosing $\varepsilon = \sqrt{c/n}$ completes the proof. □

5. Harmonic maps. In this section we generalize the estimates of Sections 2 and 3 to include the case where f is a harmonic map and f^2 is replaced by $\rho^2 \circ f$, where ρ^2 denotes the metric distance from $f(X_0)$ on N.

In order to apply Itô's formula to $\rho^2 \circ f$, we need to know what $\Delta(\rho^2 \circ f)$ is. In the case where $N = R^n$, $\rho^2 = \Sigma_i x_i^2$, $\rho^2 \circ f$ is $\Sigma_i f_i^2$, and $\text{Hess}_{i,j}(\rho^2 \circ f)$ is $2\Sigma_k f_k \partial_{i,j} f_k + 2\Sigma_k \partial_i f_k \partial_j f_k$. Applying $\text{tr}_{i,j}$ eliminates the first term of the Hessian since f is harmonic. The result is that

$$\Delta(\rho^2 \circ f) = 2\sum_{i,k} \left(\frac{\partial f_k}{\partial x_i}\right)^2 = 2\sum_i \left| df\left(\frac{\partial}{\partial x_i}\right)\right|^2 \equiv 2\|df\|^2,$$

which coincides with the case $N = R^1$.

For a general codomain $\text{Hess}_{i,j}(\rho^2 \circ f)$ again falls into two parts. The trace of the first part is zero because f is harmonic. The trace of the second part is the sum over i of the Hessian of ρ^2 applied as a quadratic form to the vectors $df(\partial_i)$, where ∂_i denotes an orthonormal basis of TM in some coordinates. Thus, estimating $\Delta(\rho^2 \circ f)$ amounts to estimating the quadratic form $\text{Hess}_{i,j}\,\rho^2$ on N [Cheng (1980)].

In the flat case $\text{Hess}_{i,j}\,\rho^2 = 2\delta_{i,j}$. Because $\text{Sect}^N \leq 0$, an application of the Hessian comparison theorem in Greene and Wu (1979) shows that

$\text{Hess}_{i,j}\,\rho^2 \succcurlyeq 2\delta_{i,j}$ as a quadratic form. From this we may conclude that

$$\Delta(\rho^2 \circ f) \geq 2\|df\|^2.$$

A complete proof of this appears as a special case of Lemma 4 in Jäger and Kaul (1979).

For the second application of Itô's formula in the proof of Lemma 2.1, we need an analog of the Bochner formula given above. To understand how this formula generalizes, consider how it is derived.

We wish to study $\Delta\|df\|^2$. This is just $\text{tr}_{i,j}\, D_i D_j \langle df, df \rangle$, where df is a section of the bundle $T^*M \otimes f^{-1}TN$ and $\langle \cdot, \cdot \rangle$ and D_i denote the inner product and covariant derivatives induced by the metrics on M and N. We are allowed to "distribute" the covariant derivatives over the inner product, giving $2\,\text{tr}_{i,j}[\langle D_i D_j\, df, df \rangle + \langle D_i\, df, D_j\, df \rangle]$. The second term is clearly nonnegative. Now we want to be able to pass $\text{tr}_{i,j}\, D_i D_j$ through the d operator, so that we can make use of the condition that f is harmonic. In exchanging the order of differentiations, one produces curvature terms which measure the failure of the derivatives to commute. The operator $\text{tr}_{i,j}\, D_i D_j$ is the "rough" Laplacian, which treats tensor fields as merely vectors (or matrices, etc.) of separate *scalar* fields (i.e., functions), without regard to their special *nonscalar* transformation properties. The first term then becomes $\langle d\,\Delta f, df \rangle + \langle R(df), df \rangle$ where Δ is the operator defining harmonic maps (depending on the metrics of both M and N) and $R(\cdot)$ is an endomorphism of the vector bundle $T^*M \otimes f^{-1}TN$. This endormorphism is essentially the Ricci curvature of M minus the sectional curvature of N. Note that in the simplest case—when the harmonic map is simply a function on M—the term $\text{tr}_{i,j}\langle D_i\, df, D_j\, df \rangle$ is what we called $\|\text{Hess}\, f\|^2$, and $R(\cdot)$ reduces to $\text{Ric}(df, df)$. This argument is standard in geometry, particularly in the study of harmonic maps. Cheng [(1980), page 149] derives the relevant Bochner formula and Eells and Lemaire [(1983), page 13] give a more general discussion.

The result is that the argument of Section 2 goes through with no further difficulty, and Lemma 2.1 extends to

$$E\big(\rho^2 \circ f(X_t)\big) \geq \|df(X_0)\|^2 t.$$

Lemma 3.1 requires no change, so we need only show that the proof for functions generalizes. That is, if $\limsup_{a\to\infty} a^{-1} m_\rho(a) = 0$ then $E(\rho^2 \circ f(X_t))$ is sublinear in t. This argument is exactly the same as that given in Section 4.

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