The Annals of Probability
1990, Vol. 18, No. 4, 1806–1815

ON PRODUCTS OF NONNEGATIVE MATRICES

By Harry Cohn and Olle Nerman

University of Melbourne, and Chalmers University of Technology and University of Goteborg

A representation for products of finite nonnegative matrices is given in terms of products of stochastic matrices and as a result Markov chain arguments are used to derive ratio limit properties. In particular, we obtain necessary and sufficient conditions for weak ergodicity and give a probabilistic proof of the Coale–Lopez theorem. In the general case, there are several sequences of sets of partitions of the state space corresponding to an associated nonhomogeneous Markov chain which lead to a number of ratio product limits. Asymptotic column proportionality, characteristic of weak ergodicity, may occur only inside each sequence of sets with one possible exception.

1. Introduction. Let $\{M_1, M_2, \dots\}$ be a sequence of $p \times p$ nonnegative matrices, ${}^mM^n = M_m M_{m+1} \cdots M_n$ and write ${}^mM^n(i, j)$ for the (i, j)th entry of the matrix ${}^mM^n$. Similar notation will be used with respect to a sequence of stochastic matrices $\{P_n\}$, i.e., matrices with $\sum_{j=1}^{p} P_n(i, j) = 1$ for $i = 1, \dots, p$. We shall make the blanket assumptions that $\{M_n\}$ are *allowable*, i.e., have at least one positive entry in each row and column, and that for any m there exist n (which may depend on m) such that ${}^mM^n(i, j) > 0$ for all i and j. It is easy to see that ${}^mM^{n_0}(i, j) > 0$ for all i and j entails ${}^mM^n(i, j) > 0$ for $n \geq n_0$ and all i and j, in which case we may consider the limit behavior of the ratios ${}^mM^n(i, j)/{}^mM^n(l, j)$ as $n \to \infty$ for all m, i, j and l.

We shall say that $\{M_n\}$ is *weakly ergodic* if for all m, i, j and l,

$$\lim_{n \to \infty} \frac{{}^mM^n(i, j)}{{}^mM^n(l, j)} = \gamma_m(i, l). \tag{1}$$

For stochastic $\{M_n\}$ weak ergodicity necessarily implies $\gamma_m(i, l) = 1$.

There are two questions arising in connection with ratios of the type $\{{}^mM^n(i, j)/{}^mM^n(l, j)\}$ which we address in this paper:

1. Are there some tractable minimal conditions to ensure (1)?
2. What is, in general, the asymptotic behavior of such ratios?

We shall see that these two questions are related and that the answer to the first may be provided by the second. In the weak convergence literature there are a number of sufficient conditions for (1) (see [1], [2], [6] and [11]), but the case when (1) fails does not appear to have been discussed.

Received February 1989; revised August 1989.

AMS 1980 *subject classifications*. Primary 15A48, 60F99; secondary 60J05, 60J45.

Key words and phrases. Nonnegative matrix, stochastic matrix, Markov chain, space–time chain, Harmonic function, tail σ-field, atomic set, ratio limit, weak ergodicity.

There is another notion of weak ergodicity, known in relation to Markov chains, which we shall refer to as *stochastic ergodicity*:

$$\lim_{n\to\infty} ({}^mM^n(i,j) - {}^mM^n(l,j)) = 0 \tag{2}$$

for all m, i, j and l. It is clear that for stochastic matrices $\{M_n\}$ weak ergodicity implies stochastic ergodicity but not vice versa. Indeed, (2) does not impose any restriction on the ${}^mM^n(i,j)$'s which tend to 0 as $n \to \infty$, whereas (1) requires all the ratios to tend to 1. It is well known that (2) is equivalent to the triviality of the tail σ-field of any Markov chain with transition probability matrices $\{M_n;\ n \geq m\}$ (see [3], [7] and [9]) which may be established by a number of well-known methods in the Markov chain literature. We shall show that there is a relationship between (1) and (2) which will allow us to use probabilistic arguments to derive criteria for the behavior of matrix product ratios in both the general case and in particular for weak ergodicity. To be more specific, a positive column vector h will be said to be an harmonic function for $M = (M(i,j)) \geq 0$ if $Mh = h$, i.e.,

$$\sum_{j=1}^{p} M(i,j)h(j) = h(i), \qquad i = 1,\dots,p.$$

It is easy to see that the matrix $\hat{M}$, where $\hat{M}(i,j) = M(i,j)h(j)/h(i)$, is stochastic. Similarly, $h(n,i)$ is said to be *space–time harmonic* for $\{M_n\}$ if $M_n h_{n+1} = h_n$ for $h_n = h(n,\cdot\,)$ and $n = 1,2,\dots$. In what follows we shall only be concerned with space–time harmonic functions which we shall denote by h and refer to as harmonic. A nonhomogeneous Markov chain with transition probability matrices ${}_hP_n(i,j) = h(n+1,j)M_n(i,j)/h(n,i)$ will be said to be an h-chain. The idea of turning nonnegative matrices into stochastic ones goes back to Perron. Feller [5] applied it in the setting of potential theory, but we do not know of any attempt to use space–time harmonic functions to describe the limit behavior of products of nonidentical matrices. Reducing the problem to the study of stochastic matrices will enable us to obtain our results in terms of the tail σ-field structure of a nonhomogeneous Markov chain. It will appear that if (1) fails there exist at least two sequences of sets $\{E_n^{(1)}\}$ and $\{E_n^{(2)}\}$ such that for $j_n \in E_n^{(k)}$, $\{{}^mM^n(i,j_n)/{}^mM^n(l,j_n)\}$ converges to a limit which depends on m, i, l and k. In this case we shall see that there must be an h-chain which is not stochastically ergodic. Thus stochastic ergodicity for all h-chains is a necessary and sufficient condition for weak ergodicity. Using this result, we show that a weak ergodicity criterion due to Hajnal [6] follows as a simple corollary. We conclude the paper with a probabilistic proof of the Coale–Lopez weak ergodic theorem relying only on Theorem 1 and standard criteria for stochastic ergodicity.

2. A representation for products of matrices. Next we shall identify some harmonic functions that will be of use in what follows.

THEOREM 1. *Let k be any number in $\{1,\dots,p\}$. Then*:

(i) *There is a subsequence $\{n'\}$ such that*

$$h(m,i) := \lim_{n'\to\infty} \frac{{}^mM^{n'}(i,k)}{{}^1M^{n'}(1,k)} \tag{3}$$

exists for all and i, and $0 < h(m,i) < \infty$.

(ii) *The matrices $\{{}_hP_n\}$ with ${}_hP_n(i,j) = h(n+1,j)M_n(i,j)/h(n,i)$ are stochastic.*

(iii) *For every m, n, i and j,*

$${}^mM^n(i,j) = \frac{h(m,i)}{h(n+1,j)} {}^m_hP^n(i,j). \tag{4}$$

PROOF. Fix m and let n_0 (which may depend on m) be such that ${}^mM^{n_0}(i,j) > 0$ for all i and j. Write $A_m = \min_{i,j} {}^mM^{n_0}(i,j)/\max_{i,j} {}^mM^{n_0}(i,j)$ and notice that for $n > n_0$ and any i, j and l, the matrix product definition leads to

$$0 < A_m \le {}^mM^n(i,j)/{}^mM^n(l,j) \le A_m^{-1} < \infty. \tag{5}$$

Choose by a diagonal procedure a subsequence $\{n'\}$ such that ${}^mM^{n'}(i,k)/{}^mM^{n'}(1,k)$ converges as $n' \to \infty$ for all m and i to limits (say) $\alpha(m,i)$, which by (5) are positive and finite. It is easy to see that the limits defined by (3) exist and

$$h(m,i) = \left(\sum_{j=1}^{p} {}^1M^{m-1}(1,j)\alpha(m,j)\right)^{-1},$$

which proves (i). The decomposition ${}^mM^{n'}(i,k) = \sum_{j=1}^{p} M_m(i,j)^{m+1}M^{n'}(j,k)$ yields that

$$h(m,i) = \sum_{j=1}^{p} h(m+1,j)M_m(i,j)$$

and this is equivalent to (ii). Finally, (iii) follows from (ii) and simple algebraic manipulations with products of matrices. □

REMARK. Theorem 1 identifies space–time harmonic functions h by using in fact the Martin kernel attached to the space–time chain (for an account of the Martin boundary theory see [8], Chapter 9). However, we shall avoid in this paper arguments involving the Martin boundary and rely instead on probabilistic considerations in terms of the tail σ-field structure. It will appear that both $\{h(n,i)\}$ and the limits of matrix product ratios are related to the atomic sets of the tail σ-field.

3. Tail σ-field and ratio limits.

Theorem 1 allows products of nonnegative matrices to be expressed in terms of products of stochastic matrices. For ratios of the form ${}^mM^n(i,j)/{}^mM^n(l,j)$ we get

(6) $${}^mM^n(i,j)/{}^mM^n(l,j) = h(m,i)/h(m,l){}_h^mP^n(i,j)/{}_h^mP^n(l,j).$$

Thus the asymptotic behavior of such ratios as $n \to \infty$ may be expressed by means of the associated stochastic matrices. Notice that (4) and (6) hold for any space–time harmonic function h, not only the ones described by Theorem 1. These properties may allow us to use rather powerful probabilistic methods, provided we can extract sufficient information about $\{{}_hP_n\}$ from $\{M_n\}$. The main probabilistic tool used to describe the asymptotics of such ratios in general is the h-chain.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and Λ a set in $\mathscr{F}$. Let $\mathscr{G}$ be a sub σ-field of $\mathscr{F}$. We shall say that Λ is an *atomic* set of $\mathscr{G}$ if $P(\Lambda) > 0$ and Λ does not contain any subset Λ' in $\mathscr{G}$ such that $0 < P(\Lambda') < P(\Lambda)$. If the only atomic set of $\mathscr{G}$ has probability 1 we say that $\mathscr{G}$ is *trivial*. If $\mathscr{G}$ contains a finite number of atomic sets, i.e., equivalence classes of atomic sets which are P-a.s. equal, and no other sets of positive probability we say that $\mathscr{G}$ is *finite*. Write $S = \{1,\dots,p\}$, $\Omega = S \times S \times \cdots$, $X_n(\omega) = \omega_n$ for $\omega = (\omega_1,\dots,\omega_n,\dots)$ and $\mathscr{F}_n$ for the σ-field generated by $\{X_k;\ k \ge n\}$. A positive probability vector $\pi^{(m)} = (\pi_1^{(m)},\dots,\pi_p^{(m)})$ and a sequence of $p \times p$ stochastic matrices $\{P_n;\ n \ge m\}$ uniquely determine a probability measure $P^{(m)}$ on $\mathscr{F}_m$ such that $\{X_n;\ n \ge m\}$ is a nonhomogeneous Markov chain on $(\Omega, \mathscr{F}_m, P^{(m)})$ with $P^{(m)}(X_m = i) = \pi_i^{(m)}$ and $P^{(m)}(X_{n+1} = j|X_n = i) = P_n(i,j)$ for $i, j \in S$ and $n \ge m$. Write now $\{A_n \text{ i.o.}\}$ for $\bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$ and $\{A_n \text{ ult.}\}$ for $\bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m$. Further $A = B$ a.s. will stand for $P(A \vartriangle B) = 0$, where $\vartriangle$ is the symbol of the symmetric difference of two sets. We shall write $\lim_{n\to\infty} A_n = A$ a.s. if $\{A_n \text{ i.o.}\} = \{A_n \text{ ult.}\} = A$ a.s. Consider now the tail σ-field of $\{X_n;\ n \ge m\}$ defined as $\mathscr{T} = \bigcap_{n=m}^\infty \mathscr{F}_n$. It is known (see [3] or [4]) that $\mathscr{T}$ is finite with respect to $P^{(m)}$ and the number t of atomic sets $T_1,\dots,T_t$ does not exceed p. In what follows we shall refer to the atomic sets of $\mathscr{T}$ as atomic. Under the assumption that the transition probabilities $\{{}^mP^n\}$ become positive for n large enough, the atomic sets of $\mathscr{T}$ are independent of m. To ease the notation, we shall also write P for $P^{(m)}$. It follows from the martingale convergence theorem that $\lim_{n\to\infty} P(X_m = i|X_n) = P(T_k|X_m = i)P(X_m = i)/P(T_k)$ for almost all $\omega \in T_k$. Since $P(X_m = i|X_{n+1} = j) = P(X_m = i)/P(X_{n+1} = j){}^mP^n(i,j)$ for $\omega \in \{X_{n+1} = j\}$, we are led to the following result (for details see [3] or [4]).

PROPOSITION. *Let $\{X_n;\ n \ge m\}$ be a Markov chain. Then there exist sequences of eventually disjoint subsets of S $\{E_n^{(k)};\ n = 1,2\dots\}$ for $k = 1,\dots,t$ such that for any i, l, k and m:*

(7) (i) $$\lim_{n\to\infty} \{X_{n+1} \in E_n^{(k)}\} = T_k \quad a.s.$$

(8)

(ii) $$\lim_{n\to\infty} \frac{{}^mP^n(i,j_n)}{{}^mP^n(l,j_n)} = \frac{P(T_k|X_m = i)}{P(T_k|X_m = l)} \quad \text{for } j_n \in E_n^{(k)}.$$

(iii) *If $E_n := S \setminus \bigcup_{k=1}^t E_n^{(k)}$ are not empty, then $P(\{X_{n+1} \in E_n\ i.o.\}) = 0$.*

We shall say that an harmonic function h is *extremal* if for every harmonic function h' such that $0 \le h' \le h$ there is a real number c such that $h' = ch$. To avoid confusion, we shall say that h' is M-harmonic (or ${}_hP$-harmonic) if it is harmonic for $\{M_n\}$ (or $\{{}_hP_n\}$). The probability measure of the Markov chain corresponding to $\{{}_hP_n\}$ constructed as above will be referred to as the ${}_hP$-measure.

LEMMA. *Suppose that h is an M-harmonic function. Then the tail σ-field $\mathcal{T}$ is ${}_hP$-trivial if and only if h is extremal.*

PROOF. Suppose first that $\mathcal{T}$ is not trivial under ${}_hP$ and that there are t atomic sets $T_1, \dots, T_t$ with $t \ge 2$ and ${}_hP(T_1) + \cdots + {}_hP(T_t) = 1$. It is easy to check that $g_1, \dots, g_t$ are ${}_hP$-harmonic, where $g_1(n, i) = {}_hP(T_1 | X_n = i), \dots, h_t(n, i) = {}_hP(T_t | X_n = i)$. From the definition of ${}_hP$ we get that $h_1 = hg_1, \dots, g_t = hg_t$ are M-harmonic. But $\sum_{i=1}^t h_i = h \sum_{i=1}^t g_i = h$. Thus $h_1, \dots, h_t$ are not proportional (meaning having nonconstant ratios) if $g_1, \dots, g_t$ are not proportional. However, it is easily seen from (7) and (8) that $g_k(n, j_n)$ tends to 1 as $n \to \infty$, for some $\{j_n\}$ with $j_n \in E_n^{(k)}$ and $\liminf_{n \to \infty} {}_hP(X_n = j_n) > 0$. Since for the same sequence $\{j_n\}$, $g_{k'}(n, j_n)$ tends to 0 as $n \to \infty$ for $k' \ne k$, we conclude that $g_1, \dots, g_t$ are not proportional which makes $h_1, \dots, h_t$ nonproportional as well and implies that h is not extremal. Therefore a nontrivial tail σ-field leads to a nonextremal h. Suppose now that h is not extremal. Then there are two nonproportional harmonic functions h_1 and h_2 such that $h = h_1 + h_2$. Suppose that for some r, $\mathcal{T}$ is ${}_{h_r}P$-nontrivial. Then $\mathcal{T}$ should be nontrivial with respect to ${}_hP$ as well. To prove this, notice first that by (4)

$$(9) \qquad {}_h^mP^n(i,j) \ge \frac{h_r(n+1,j)}{h(m,i)} {}^mM^n(i,j) \ge \frac{h_r(m,i)}{h(m,i)} {}_{h_r}^mP^n(i,j).$$

If for some r and $\{j_n\}$ $\liminf_{n \to \infty} {}_{h_r}P(X_n = j_n) > 0$, then (9) and (4) with h_r replacing h imply that $\liminf_{n \to \infty} {}_hP(X_n = j_n) > 0$ as well. If T_k is an atomic set for ${}_{h_r}P$, then such a sequence $\{j_n\}$ with $j_n \in E_n^{(k)}$ must exist. It follows that ${}_hP(X_n = j_n \text{ i.o.}) > 0$ which necessarily implies ${}_hP(T_k) > 0$. Notice that in view of (6) and the proposition, an atomic set with respect to ${}_{h'}P$ for an harmonic function h' is either atomic or has probability 0 with respect to ${}_{h''}P$, where h'' is another harmonic function. Thus an atomic set T_k with respect to either ${}_{h_1}P$ or ${}_{h_2}P$ will also be atomic with respect to ${}_hP$, and $\mathcal{T}$ will be ${}_hP$-nontrivial. Assume therefore that $\mathcal{T}$ is ${}_{h_r}P$-trivial for $r = 1, 2$ and write $\{E_n^{(1)}\}$ and $\{\tilde{E}_n^{(1)}\}$ for the sequences of sets described by (7) and (8) for $r = 1$ and $r = 2$, respectively. According to (6) these sequences are either eventually disjoint or identical. In the first case, the above reasoning would lead to a nontrivial $\mathcal{T}$ with respect to ${}_hP$ as each of these sequences of sets corresponds to a set in $\mathcal{T}$ of positive ${}_hP$-probability. In the second case, the limit in (8) is 1 for both ${}_{h_1}P$ and ${}_{h_2}P$ for the same sequence $\{j_n\}$, and using (6) again we get that h_1 and h_2 must be proportional, a contradiction which completes the proof. □

REMARK. The equivalence of extremality of an harmonic function h to tail σ-field triviality of the h-chain is well known and is standard procedure for identifying extremal harmonic functions in the Martin boundary theory. The probabilistic proof included above is based on the proposition and pertains to the structure of finite nonhomogeneous Markov chains.

We are now in a position to give a description of the limit behavior of matrix product ratios in the general case.

THEOREM 2. *Suppose that there are t extremal M-harmonic functions $h_1, \dots, h_t$. Let $h = h_1 + \cdots + h_t$. Then $1 \le t \le p$ and:*

(i) *The tail σ-field $\mathscr{T}$ of the ${}_hP$-chain has t atomic sets $T_1, \dots, T_t$ such that*

$$h_k(n, i) = {}_hP(T_k | X_n = i) h(n, i) \tag{10}$$

and to each h_k, $k = 1, \dots, t$, there corresponds a nonempty sequence of eventually disjoint sets $\{E_n^{(k)}\}$ such that $\lim_{n \to \infty} {}_hP(\bigcup_{k=1}^t E_n^{(k)}) = 1$, and

$$\lim_{n \to \infty} \frac{{}^mM^n(i, j_n)}{{}^mM^n(l, j_n)} = \frac{h_k(m, i)}{h_k(m, l)} \quad \textit{for } j_n \in E_n^{(k)} \tag{11}$$

for any m, i and l.

(ii) *If $E_n := S \setminus \bigcup_{k=1}^t E_n^{(k)}$ are not empty, then there are some nonnegative functions $c_k(n, i)$ with $k = 1, \dots, t$ such that $\sum_{k=1}^t c_k(n, j) = 1$ and for any m, i, l and j,*

$$\frac{{}^mM^n(i, j)}{{}^mM^n(l, j)} \sim \frac{\sum_{k=1}^t c_k(n, j) h_k(m, i)}{\sum_{k=1}^t c_k(n, j) h_k(m, l)}, \tag{12}$$

where $\sim$ means that the ratio of the left and right side tends to 1 as $n \to \infty$.

PROOF. If h_k is M-extremal, by the lemma we must have a trivial $\mathscr{T}$ with respect to ${}_{h_k}P$. Write $\{E_n^{(k)}\}$ for the sequence of sets corresponding to ${}_{h_k}P$ by the proposition. It is easy to see that (6) and (8) yield (11). By a reasoning already used in the proof of the lemma, for $h = h_1 + \cdots + h_t$ $\mathscr{T}$ has exactly t ${}_hP$-atomic sets, $T_1, \dots, T_t$, corresponding to the sequences $\{E_n^{(k)}\}$ for $k = 1, \dots, t$. By (6) and (8) we get

$$\lim_{n \to \infty} \frac{{}^mM^n(i, j_n)}{{}^mM^n(l, j_n)} = \frac{{}_hP(T_k | X_m = i) h(m, i)}{{}_hP(T_k | X_m = l) h(m, l)} \quad \text{for } j_n \in E_n^{(k)}. \tag{13}$$

Comparing now (11) with (13) yields (10) and completes the proof of (i).

To prove (ii), assume that $h_1(1,1) = \cdots = h_t(1,1) = 1$. Since $h_1, \dots, h_t$ are extremal and therefore linearly independent we can find and fix a set of t pairs of indexes $\mathbf{F}$ such that $(1,1) \in \mathbf{F}$ and the vectors $(h_1(s, l), \dots, h_t(s, l))$, with $(s, l) \in \mathbf{F}$ are linearly independent. Now define for n large enough

$c_1(n,j),\dots,c_t(n,j)$ to be the unique solution of the linear system

$$\begin{aligned}(14)\qquad &{}^sM^n(l,j)/{}^1M^n(1,j)\\ &\quad = c_1(n,j)h_1(s,l) + \cdots + c_t(n,j)h_t(s,l), \qquad (s,l) \in \mathbf{F}.\end{aligned}$$

For the values of n which are too small for (14) to make sense, define $c_k(n,j)$ arbitrarily. We proceed by first proving (12) with these $\{c_k\}$ and then argue that we can modify them to become both nonnegative and add to 1. To prove (12), it will suffice to show that for any j_n replacing j the ratio of the left and the right side tends to 1. We shall show that from any subsequence $\{j_{n'}\}$ one can extract a further subsequence, say $\{j_{n''}\}$, with such a property. Choose a subsequence $\{n''\}$ such that $\{c_k(n'', j_{n''})\}$ converges to a limit c_k as n'' tends to ∞, for $k = 1,\dots,t$. Since $\sup_n {}^sM^n(l,j)/{}^1M^n(i,j) < \infty$, $c_1,\dots,c_t$ must be finite. We use next a variant of Theorem 1 to obtain an harmonic function by extracting, if necessary, another subsequence of $\{n''\}$, which we shall also denote by $\{n''\}$, such that for all m and i,

$$\frac{{}^mM^{n''}(i, j_{n''})}{{}^1M^{n''}(1, j_{n''})} \to \tilde{h}(m,i) \quad \text{as } n'' \to \infty.$$

It follows that

$$\frac{{}^mM^{n''}(i, j_{n''})}{{}^mM^{n''}(l, j_{n''})} \to \frac{\tilde{h}(m,i)}{\tilde{h}(m,l)} \quad \text{as } n'' \to \infty.$$

It is easy to see from (14) that $\tilde{h}$ satisfies the system of equations

$$(15)\qquad \tilde{h}(s,l) = c_1h_1(s,l) + \cdots + c_th_t(s,j), \qquad (s,l) \in \mathbf{F}.$$

As (15) uniquely determines $\tilde{h}$ in the class of harmonic functions h such that $h = c_1h_1 + \cdots + c_th_t$, and since any nonnegative harmonic function is of this form the stated ratio property follows. Consider the same subsequence $\{n''\}$. Since $(1,1) \in \mathbf{F}$ (15) implies $c_1 + \cdots + c_t = 1$. But any nonnegative harmonic function belongs to the convex hull of the extremal harmonic functions which makes $c_i \ge 0$ or $i = 1,\dots,t$. It follows that (12) is still satisfied with $c_k(n,j)$ redefined as $\max(c_k(n,j),0)/\sum_{k=1}^t \max(c_k(n,j),0)$, which finishes the proof of (ii). □

We next give an example of a sequence of stochastic matrices for which nontrivial $\mathscr{T}$ and the sequence $\{E_n\}$ are both existent. Consider

$$P_n = \begin{pmatrix} 1 - 1/n^2 & 0 & 1/n^2 \\ 0 & 1 - \gamma_n/n^2 & \gamma_n/n^2 \\ 1/2 & 1/2 & 0 \end{pmatrix},$$

where $1/2 \le \gamma_n \le 1$. It is clear that there are two atomic sets, one corresponding to $E_n^{(1)} = \{1\}$ and the other corresponding to $E_n^{(2)} = \{2\}$, and a sequence

$\{E_n\}$ with $E_n = \{3\}$. Simple calculations show that

$$\frac{{}^mP^n(1,3)}{{}^mP^n(2,3)} = \frac{{}^mP^{n-1}(1,1)\frac{1}{n^2} + {}^mP^{n-1}(1,2)\frac{\gamma_n}{n^2}}{{}^mP^{n-1}(2,1)\frac{1}{n^2} + {}^mP^{n-1}(1,1)\frac{\gamma_n}{n^2}} \sim \frac{\frac{1}{n^2}}{\frac{\gamma_n}{n^2}} = \frac{1}{\gamma_n},$$

if both m and n are large. Thus choosing a large m the above ratios will get as many limit points as the sequence $\{\gamma_n\}$. In view of the proposition, we deduce that $\{3\}$ cannot correspond to an atomic set whatever the choice of the harmonic function h.

4. Weak ergodicity. We recall that $\{M_n\}$ is said to be weakly ergodic if for all m, i, j and l,

$$\lim_{n\to\infty} \frac{{}^mM^n(i,j)}{{}^mM^n(l,j)} = \gamma_m(i,l),$$

and stochastically ergodic if

$$\lim_{n\to\infty} \left({}^mM^n(i,j) - {}^mM^n(l,j)\right) = 0$$

for all m, i, j and l. A number of papers and monographs have dealt with weak ergodicity (see [1], [2], [6] and [11]). Clearly, weak ergodicity is a case of Theorem 2 when there exists only one sequence of sets $\{E_n^{(k)}\}$, say $\{E_n^{(1)}\}$, and $\{E_n\}$ is absent. In terms of the associated Markov chains described by the proposition it amounts to a case of trivial $\mathscr{T}$ for all h and absent $\{E_n\}$. It will appear that for the ${}_hP$-chain described by Theorem 2 weak ergodicity is equivalent to stochastic ergodicity. This will turn out to follow from some necessary and sufficient conditions for weak ergodicity which are the object of the next Theorem 3.

THEOREM 3. *The following conditions are equivalent*:

(i) *The sequence $\{M_n\}$ is weakly ergodic.*
(ii) *The ${}_hP$-chain has a trivial tail σ-field for any M-harmonic function h.*
(iii) *The ${}_hP$-chain is stochastically ergodic for any M-harmonic function h.*

PROOF. By (6) and the proposition weak ergodicity makes $\mathscr{T}\,{}_hP$ trivial for any harmonic function h. We show now that if the ${}_hP$-chain described by Theorem 2 has a trivial tail σ-field, then weak ergodicity holds. Indeed, if $\mathscr{T}$ is trivial there must be only one sequence $\{E_n^{(k)}\}$ and possibly a sequence $\{E_n\}$. However, according to Theorem 2(ii) the sequence $\{E_n\}$ could not exist if there are less than two atomic sets, and this proves weak ergodicity. A more direct proof, based on Theorem 2(i), is obtained by noticing that if an exceptional sequence $\{E_n\}$ were present, then choosing $j \in E_n$ and proceeding as in Theorem 1 would lead to an M-harmonic function with ratios different from the ones attached to $\{j_n\}$ in $\{E_n^{(1)}\}$ described by (10). But this may happen only if there are at least two nonproportional harmonic functions which by Theorem 2(i) would be incompatible with the triviality of $\mathscr{T}$ for all h. The proof of the equivalence of (ii) and (iii) is standard and will be omitted (see, e.g., [10]).

Let T be a nonnegative matrix with entries $t(i,j)$. Define

$$\phi(T) = \min_{i,j,k,l} \frac{t(i,k)t(j,l)}{t(j,k)t(i,l)},$$

$$\alpha(T) = 1 - \min_{i,j} \sum_{l=1}^{p} \min((t(i,l), t(j,l)).$$

The functions ϕ and α are called ergodic coefficients. The coefficient ϕ has been used in connection with weak ergodicity and is related to Birkhoff's contraction coefficient, whereas variants of α have proved useful in deriving stochastic ergodicity (see [6] and [11]).

Hajnal [6] has proven that $\sum_{n=1}^{\infty}\sqrt{\phi(M_n)} = \infty$ implies weak ergodicity. We show now that this result follows from Theorem 3. Indeed, notice that for any M-harmonic function h,

$$\frac{{}_hP_n(i,k)\,{}_hP_n(j,l)}{{}_hP_n(i,l)\,{}_hP_n(j,k)} = \frac{M_n(i,k)M_n(j,l)}{M_n(i,l)M_n(j,k)}.$$

Thus if we write P_n for ${}_hP_n$ we only need to show that

$$\sum_{n=1}^{\infty} \sqrt{\min_{i,j,k,l} \frac{P_n(i,k)P_n(j,l)}{P_n(i,l)P_n(j,k)}} = \infty$$

entails stochastic ergodicity since (iii) implies (i) in Theorem 3. Fix a matrix P_n and let l_i and k_j be chosen such that $P_n(i,l_i) \ge 1/p$ and $P_n(j,k_j) \ge 1/p$. Then

$$\begin{aligned}
&\min_{i,j,k,l} \frac{P_n(i,k)P_n(j,l)}{P_n(i,l)P_n(j,k)} \\
&\quad \le p^2 \min_{i,j} \big(P_n(i,k_j), P_n(j,l_i)\big) \\
&\quad \le p^4 \min_{i,j} \big(\min\big(P_n(i,k_j), P_n(j,k_j)\big)\min\big(P_n(j,l_i), P_n(i,l_i)\big)\big) \\
&\quad \le p^4 \left(\min_{i,j} \left(\sum_{k=1}^{p} \min(P_n(i,k), P_n(j,k)) \right) \right)^2.
\end{aligned}$$

Thus we deduce that $\sum_{n=1}^{\infty}(1 - \alpha(P_n)) = \infty$, which is known to imply stochastic ergodicity (see, e.g., [11]) for $\{P_n\}$ and weak ergodicity for $\{M_n\}$. □

Finally, we shall try a Markov chain approach to a well-known result—the Coale–Lopez theorem.

THEOREM 4. *Suppose that* $\{M_n\}$ *are nonnegative allowable matrices and*:

(i) $\min^+_{i,j} M_n(i,j)/\max_{i,j} M_n(i,j) \geq \gamma$ *for some* $\gamma > 0$ *and all* n, *where* $\min^+$ *refers to the positive elements.*

(ii) *There exists* r_0 *such that* ${}^nM^{n+r_0} > 0$ *for all* n.

Then $\{M_n\}$ *is weakly ergodic at a geometric rate.*

PROOF. It will suffice to consider the stochastic matrices $\{{}_hP_n\}$ for an arbitrary h defined in the manner of Theorem 1. We shall write P_n for ${}_hP_n$. The allowability of M_n in conjunction with (i) and (ii) yields

$$P_n(i,j) = \lim_{n' \to \infty} \left(\frac{{}^nM^{n'}(i,1)}{M_n(i,j)^{n+1}M^{n'}(j,1)} \right)^{-1}$$

$$= \lim_{n' \to \infty} \left(\sum_{l=1}^{p} \frac{M_n(i,l)^{n+1}M^{n'}(l,1)}{M_n(i,j)^{n+1}M^{n'}(j,1)} \right)^{-1} \geq \delta,$$

where $\delta = (1/p)\gamma^{r_0+1}$. Simply algebraic manipulations lead to ${}^n\dot{P}^{n+r_0}(i,j) \geq \delta^{r_0}$ and ${}^nP^m(i,j) \geq \delta^{r_0}$ for $m \geq n + r_0$. Thus $\{{}^nP^m(i,j)\}$ stay away from 0 as $m \to \infty$, a case when stochastic ergodicity is equivalent to weak ergodicity. It is now easy to see that we are in a situation where we can use one of the ergodic coefficients for Markov chains (e.g., α) and simple manipulations as in [11] to conclude both weak ergodicity and geometric rate of convergence. □

REFERENCES

[1] COHEN, J. E. (1979). Contractive inhomogeneous products of non-negative matrices. *Math. Proc. Cambridge Philos. Soc.* **86** 351–364.

[2] COHEN, J. E. (1979). Ergodic theorems in demography. *Bull. Amer. Math. Soc.* **1** 275–295.

[3] COHN, H. (1976). Finite non-homogeneous Markov chains: Asymptotic behaviour. *Adv. in Appl. Probab.* **8** 502–516.

[4] COHN, H. (1989). Products of stochastic matrices. *Internat. J. Math.* **12** 209–233.

[5] FELLER, W. (1956). Boundaries induced by non-negative matrices. *Trans. Amer. Math. Soc.* **83** 19–54.

[6] HAJNAL, J. (1976). On products of non-negative matrices. *Math. Proc. Cambridge Philos. Soc.* **79** 521–530.

[7] IOSIFESCU, M. (1972). On two recent papers on ergodicity in non-homogeneous Markov chains. *Ann. Math. Statist.* **43** 1732–1736.

[8] KEMENY, J. G., SNELL, J. L. and KNAPP, A. W. (1976). *Denumerable Markov Chains.* Springer, New York.

[9] KINGMAN, J. F. C. (1975). Geometrical aspects of the theory of non-homogeneous Markov chains. *Math. Proc. Cambridge Philos. Soc.* **77** 171–183.

[10] OREY, S. (1971). *Limit Theorems for Markov Chains Transition Probabilities.* Van Nostrand, London.

[11] SENETA, E. (1981). *Non-Negative Matrices and Markov Chains.* Springer, New York.

DEPARTMENT OF STATISTICS
UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA 3052
AUSTRALIA

DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY
AND UNIVERSITY OF GOTEBORG
S-412 96 GOTEBORG
SWEDEN