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ON THE NONLINEAR RENEWAL THEOREM[1]

BY MICHAEL WOODROOFE

The University of Michigan

Let $Z_1, Z_2, \ldots$ be jointly distributed random variables for which $\sup_k Z_k = \infty$ w.p.1 and let $t = t_a = \inf(n \geq 1\colon Z_n > a)$ and $R_a = Z_t - a$ for $a \geq 0$. Conditions under which R_a has a limiting distribution as $a \to \infty$ are developed. These require that the finite dimensional, conditional distributions of the increments $Z_{t+k} - Z_t$, $k \geq 1$, converge to the finite dimensional distributions of a process for which the result is known, thus weakening the slow change condition in earlier work. The main result is applied to some sequences for which the limiting distributions are those of the partial sums of an exchangeable process. These include the Euclidean norms of a driftless random walk in several dimensions and sequences for which the conditional distribution of $Z_{n+1} - Z_n$ given the past has a limit w.p.1 as $n \to \infty$.

1. Introduction. Let $(\Omega, \mathscr{A}, P)$ denote a probability space, let $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \ldots$ denote subsigma-algebras of $\mathscr{A}$ and let $Z_1, Z_2, \ldots$ be random variables, defined on $(\Omega, \mathscr{A}, P)$, for which Z_k is $\mathscr{A}_k$ measurable for all $k = 1, 2, \ldots$ and

$$\sup_{k \geq 1} Z_k = \infty \quad \text{w.p.1.} \tag{1}$$

Such a sequence may be called an *infinite supremum process*. For any such process, the first passage times and excesses

$$t_a = \inf\{k \geq 1\colon Z_k > a\} \tag{2}$$

and

$$R_a = Z_{t_a} - a, \tag{3}$$

may be defined for all $a \geq 0$ w.p.1. Let H_a denote the distribution function of R_a; that is,

$$H_a(r) = P\{t_a < \infty, R_a \leq r\}, \qquad \forall\, a, r \geq 0. \tag{4}$$

The problem considered is to find conditions under which R_a has a limiting distribution H as $a \to \infty$; that is, $H_a \Rightarrow H$ as $a \to \infty$, where $\Rightarrow$ denotes weak

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convergence. The process $Z_1, Z_2, \dots$ may be called a *renewal process* in this case.

The best known examples of renewal processes are random walks (the partial sums of i.i.d. random variables). If $Z_1, Z_2, \dots$ is a random walk with a nonarithmetic step distribution and either a finite positive drift or zero drift and finite second moments, then R_a has limiting distribution H, where

$$(5) \qquad H(dr) = \frac{1}{E(Z_{t_0})} P\{Z_{t_0} > r\}\, dr, \qquad r > 0.$$

See, for example, Feller (1971), Section 11.3. (Here and below the same symbol is used to denote a distribution function and the associated distribution.)

There has been recent interest in processes called perturbed random walks. These are processes of the form

$$(6) \qquad Z_n = S_n + \xi_n, \qquad n \geq 1,$$

where $S_1, S_2, \dots$ is a nonarithmetic random walk with a positive drift μ, ξ_n is independent of the sequence $S_{n+k} - S_n$, $k \geq 1$, for all n and $\xi_1, \xi_2, \dots$ are slowly changing in the sense that there exists a ρ for which $\frac{1}{2} < \rho \leq 1$,

$$(7) \qquad a^{-\rho}\left\{t_a - \frac{a}{\mu}\right\} \to 0 \quad \text{in probability as } a \to \infty$$

and

$$(8) \qquad \lim_{\delta \to 0} \sup_{n \geq 1} P\left\{\max_{k \leq \delta n^{\rho}} |\xi_{n+k} - \xi_n| \geq \varepsilon\right\} = 0$$

for all $\varepsilon > 0$. If $Z_1, Z_2, \dots$ is a perturbed random walk, then the nonlinear renewal theorem of Lai and Siegmund (1977, 1979) asserts that R_a has the same limiting distribution as if $\xi_n = 0$ for all $n = 1, 2, \dots$ [obtainable from (5) and Z replaced by S]. Woodroofe (1982) and Siegmund (1985) describe applications of this result to sequential analysis. For recent extensions, see Zhang (1988, 1989).

The goal of this paper is to present an alternative formulation of the nonlinear renewal theorem in which the conditions (6), (7) and (8) are relaxed.

A particular example to which existing nonlinear renewal theorems are not applicable is that in which $Z_n = \|S_n\|$, $n \geq 1$, where $S_n = X_1 + \cdots + X_n$, $n \geq 1$, $X_1, X_2, \dots$ are i.i.d. random vectors with mean vector 0 and a nonsingular convariance matrix, and $\|\cdot\|$ denotes the Euclidean norm. As an application of the main theorem, it is shown that R_a has the same limiting distribution as R_a^*, where $Z_n^* = \langle U, S_n \rangle$, $n = 1, 2, \dots$, $\langle \cdot , \cdot \rangle$ denotes inner product, and U denotes a random unit vector which is independent of $X_1, X_2, \dots$. As a second application, the theorem is applied to a class of processes for which the conditional distribution of $Z_{n+1} - Z_n$ given $\mathscr{A}_n$ has a (possibly random) limit w.p.1 as $n \to \infty$. In both applications, the limiting process is formed from the partial sums of an exchangeable process, so that the random walk theory may be applied conditionally.

2. The main theorem. To state the conditions, it is convenient to introduce the prior sigma algebras and delayed processes,

$$\mathscr{F}_a = \mathscr{A}_{t_a} \quad \text{and} \quad Z_{a,k} = Z_{t_a+k} - Z_{t_a},$$

where $k = 1, 2, \dots$ and $a \geq 0$ (and $\mathscr{A}_t$ denotes the collection of $A \in \mathscr{A}$ for which $A \cap \{t \leq n\} \in \mathscr{A}_n$ for all $n = 1, 2, \dots$ for any stopping time t). Let $Q_{a,m}$ denote (versions of) the conditional distributions of $(Z_{a,1}, \dots, Z_{a,m})$, given $\mathscr{F}_a$; that is,

$$Q_{a,m}(\omega; B) = P\{(Z_{a,1}, \dots, Z_{a,m}) \in B | \mathscr{F}_a\}(\omega)$$

for Borel sets $B \subseteq R^m$, $\omega \in \Omega$, $m = 1, 2, \dots$ and $a \geq 0$. The conditions require that these (finite dimensional) distributions converge to those of an appropriate renewal process $Z_1^*, Z_2^*, \dots$.

Here $Z_1^*, Z_2^*, \dots$ is required to be an infinite supremum process, defined on a probability space $(\Omega^*, \mathscr{A}^*, P^*)$ along with a random element W with values in a complete separable metric space $\mathscr{W}$. It is assumed that there is a consistent set of weakly continuous versions of the conditional probabilities

$$Q_m^*(w; B) = P\{(Z_1^*, \dots, Z_m^*) \in B | W = w\},$$

defined for Borel sets $B \subseteq R^m$, $w \in \mathscr{W}$ and $m = 1, 2, \dots$. Let Γ denote the distribution of W and let

$$\begin{aligned} H_a^*(w; r) &= P^*\{t_a^* < \infty, R_a^* \leq r | W = w\} \\ &:= \sum_{k=1}^{\infty} Q_k^*\Big(w; \big\{z \in R^k \colon z_j \leq a, \forall\, j < k, a < z_k \leq a + r\big\}\Big) \end{aligned} \tag{9}$$

for $a, r \geq 0$ and $w \in \mathscr{W}$, where t_a^* and R_a^* are defined by (2) and (3) with $Z_1, Z_2, \dots$ replaced by $Z_1^*, Z_2^*, \dots$. It is required that there exist distributions $H(w; \cdot)$, $w \in \mathscr{W}$, for which $H_a^*(w; \cdot) \Rightarrow H(w; \cdot)$ for a.e. w (Γ) as $a \to \infty$. Then (W, R_a^*) has limiting distribution K as $a \to \infty$, where

$$K(Fx[0, r]) = \int_F H(w; r) \Gamma(dw) \tag{10}$$

for Borel sets $F \subseteq \mathscr{W}$ and $r \geq 0$. R_a^* has limiting distribution $H(r) = K(\mathscr{W} \times [0, r])$, $r \geq 0$. A sequence $W, Z_1^*, Z_2^*, \dots$ which satisfies the conditions of this paragraph is called a *conditional renewal process*.

EXAMPLE 1. If $Z_k^* - Z_{k-1}^*$, $k \geq 1$, are conditionally i.i.d. given W and if the conditional distributions satisfy the requirements for (5) w.p.1, then $W, Z_1^*, Z_2^*, \dots$ is a conditional renewal process.

Let Δ_m denote the Prokhorov metric for probability measures on the Borel sets of R^m. Thus, if μ and ν are two such measures, then $\Delta_m(\mu, \nu)$ is the infimum of $\varepsilon > 0$ for which $\mu(B) < \nu(B_\varepsilon) + \varepsilon$ and $\nu(B) < \mu(B_\varepsilon) + \varepsilon$ for all Borel sets $B \subseteq R^m$, where $B_\varepsilon = \{x \in R^m \colon \operatorname{dist}(x, B) \leq \varepsilon\}$ for $B \subseteq R^m$ and $\varepsilon > 0$. Weak convergence of probability measures in R^m is equivalent to convergence in the Prokhorov metric.

THE CONVERGENCE CONDITION. Suppose that there are $\mathscr{F}_a$-measurable random elements W_a and a conditional renewal process $W, Z_1^*, Z_2^*, \dots$ for which

$$W_a \Rightarrow W \tag{11}$$

and

$$\lim_{a \to \infty} \int \Delta_m [Q_{a,m}, Q_m^*(W_a; \cdot)]\, dP = 0, \tag{12}$$

where (now) $\Rightarrow$ denotes convergence in distribution.

THEOREM 1. *If the convergence condition is satisfied and if R_a is stochastically bounded as $a \to \infty$, then R_a has the same limiting distribution H as R_a^*. If also $d(W_a, W_{a+b}) \to 0$ in probability as $a \to \infty$ for every $b > 0$, then (W_a, R_a) has the same limiting distribution K as (W, R_a^*).*

The theorem is proved in the next section. The remainder of this section is devoted to discussion of the conditions.

Since Δ_m is a bounded metric, (12) is equivalent to the convergence of $\Delta_m[Q_{a,m}, Q_m^*(W_a; \cdot)]$ to zero in probability as $a \to \infty$ for all $m \ge 1$. This condition may be replaced by a slightly stronger one which involves the first passage times less directly. If there are random elements V_n, $n \ge 1$, for which the Prokhorov distance between the conditional distribution of $Z_{n+k} - Z_n$, $k = 1, \dots, m$, given A_n, and $Q_m^*(V_n; \cdot)$ approaches zero w.p.1 as $n \to \infty$ for all $m = 1, 2, \dots$, then (12) holds, with $W_a = V_{t_a}$, since $Q_{a,m}$ is obtained by substituting t_a for n.

PROPOSITION 1. *Let $Z_1, Z_2, \dots$ be an infinite supremum process and $W, Z_1^*, Z_2^*, \dots$ be a conditional renewal process. Suppose that there are $\mathscr{F}_a$-measurable random elements W_a and random variables $Z_{a,1}^*, Z_{a,2}^*, \dots$ and $r_{a,1}, r_{a,2}, \dots$ for which $Z_{a,k} = Z_{a,k}^* + r_{a,k}$ for $a \ge 0$ and $k \ge 1$. Suppose further that the Prokhorov distance between the conditional distribution of $(Z_{a,1}^*, \dots, Z_{a,m}^*)$ given $\mathscr{F}_a$ and $Q_m^*(W_a; \cdot)$ approaches zero in probability as $a \to \infty$ for all $m = 1, 2, \dots$ and that $r_{a,k} \to 0$ in probability as $a \to \infty$ for all $k = 1, 2, \dots$. Then (12) holds.*

PROOF. If $m \ge 1$ and $B \subseteq R^m$ is a Borel set, then

$$P\{(Z_{a,1}, \dots, Z_{a,m}) \in B | \mathscr{F}_a\} \le P\{(Z_{a,1}^*, \dots, Z_{a,m}^*) \in B_\delta | \mathscr{F}_a\} + P\Big\{\max_{k \le m} |r_{a,k}| \ge \delta | \mathscr{F}_a\Big\},$$

and the roles of $(Z_{a,1}, \dots, Z_{a,m})$ and $(Z_{a,1}^*, \dots, Z_{a,m}^*)$ may be reversed. Letting $Q_{a,m}^*$ denote the conditional distribution of $(Z_{a,1}^*, \dots, Z_{a,m}^*)$, it follows that

$$\Delta_m[Q_{a,m}, Q_{a,m}^*] \le \delta + P\Big\{\max_{k \le m} |r_{a,k}| \ge \delta | \mathscr{F}_a\Big\},$$

which approaches zero in the first mean as first $a \to \infty$ and then $\delta \to 0$. The proposition follows easily. □

COROLLARY 1. *If* (6), (7) *and* (8) *hold, then the convergence condition is satisfied with* $W_a = W = 0$ *for* $a \ge 0$ *and* $Z_k^* = S_k$, *for* $k = 1, 2, \dots$.

PROOF. In this case, $Z_{a,k} = S_{a,k} + \xi_{a,k}$ for all $a \ge 0$ and $k \ge 1$, where $S_{a,k} = S_{t_a+k} - S_{t_a}$, $k \ge 1$, are independent of $\mathscr{F}_a$ and have the same distribution as S_k, $k \ge 1$, for all $a \ge 0$, by the strong Markov property, and $\xi_{a,k} = \xi_{t_a+k} - \xi_{t_a} \to 0$ in probability as $a \to \infty$ for all $k \ge 1$, as in Lai and Siegmund (1977). □

3. The proof.

The broad brush strokes of the proof follow those of Lai and Siegmund (1977): For large b and much larger a, the conditional distribution of R_{a+b} given $\mathscr{F}_a$ may be approximated by $H^*_{b-R_a}(W_a; \cdot)$, the expectation of which may be approximated by H. The dependence on W_a and the use of finite-dimensional distributions complicates the argument, however.

The convergence condition is assumed throughout this section.

LEMMA 1. *Let* $W, Z_1^*, Z_2^*, \dots$ *be a conditional renewal process and let*

$$D_a = \left\{w \in \mathscr{W}: \sup_{k \ge 1} Q_k^*\left(w; \{z \in R^k: z_k = a\}\right) > 0\right\}$$
$$\cup \left\{w \in \mathscr{W}: \sup_{b \ge 0} \inf_{k \ge 1} Q_k^*\left(w; (-\infty, b]^k\right) > 0\right\}$$

for $a \ge 0$. *Then* $H_a^*(w; \cdot)$ *is weakly continuous in* w *at every* $w_0 \in D_a'$, *the complement of* D_a, *for every* $a \ge 0$, *and* $\Gamma(D_a) = 0$ *for a.e.* $a \ge 0$ (*Lebesgue*).

PROOF. The first assertion of the lemma follows easily from (9) and the Portmanteau theorem [Billingsley (1968), pages 11–14]. For the second, let λ denote Lebesgue measure and let $D = \{(a, w): w \in D_a\}$. Then D is easily seen to be a Borel set $R \times \mathscr{W}$ and D_a is the a section of D for each $a \ge 1$. Now, a.e. w-section $D^w = \{a: (a, w) \in D\}$ is countable, so that $\lambda \times \Gamma(D) = 0$, where $\lambda \times \Gamma$ denotes the product measure. So, a.e. a section has Γ measure zero, by Fubini's theorem. □

Let $\mathscr{C}$ denote the set of a for which $\Gamma(D_a) = 0$. Then $\lambda(\mathscr{C}') = 0$.

For any $0 < \varepsilon < \frac{1}{2}$ and $c > 1$, there are partitions $0 = b_0 < \cdots < b_h = c$ of $[0, c]$ and values $\varepsilon_1, \dots, \varepsilon_h \in [\varepsilon, 2\varepsilon]$ for which $h \le 2c/\varepsilon$, $b_i - b_{i-1} < \varepsilon$ and $2c - b_i \pm \varepsilon_i \in \mathscr{C}$ for all $i = 1, \dots, h$. The lists $b_1, \dots, b_h$ and $\varepsilon_1, \dots, \varepsilon_h$ will be called partitions.

Let Γ_a denote the distribution of W_a for $a \ge 0$.

LEMMA 2. *For* $0 < \varepsilon < \frac{1}{2}$, $c \in \mathscr{C} \cap (1, \infty)$, $m \ge 1$ *and fixed* (*but arbitrary*) *versions of the partitions, let*

$$G_1 = G_1(\varepsilon, c) = \left\{w \in \mathscr{W}: \Delta_1\left[H_c^*(w; \cdot), H^*_{2c-b_i \pm \varepsilon_i}(w; \cdot)\right] < \varepsilon, \forall\, i \le h\right\},$$
$$G_2 = G_2(\varepsilon, c, m) = \left\{w \in \mathscr{W}: Q_m^*\left(w; (-\infty, 3c]^m\right) < \varepsilon\right\}$$

and

$$G = G(\varepsilon, c, m) = G_1 \cap G_2.$$

Then for any $\varepsilon > 0$ *and* $c_0 > 1$, *there are* $c \in \mathscr{C} \cap (c_0, \infty)$, $m \geq 1$ *and* a_0 *for which* $\Gamma(G) \geq 1 - \varepsilon \leq \Gamma_a(G)$ *for all* $a \geq a_0$.

PROOF. For any $0 < \varepsilon < \frac{1}{2}$ and $c_0 > 1$, $\Gamma(G_1) \to 1$ as $c \to \infty$, since $H_a^*(w; \cdot) \Rightarrow H(w; \cdot)$ as $a \to \infty$ for a.e. w (Γ). So, there is a $c > c_0$ for which $\Gamma(G_1) \geq 1 - \varepsilon/4$. With this choice of c, $\Lambda(G_2) \to 1$ as $m \to \infty$, since $Z_1^*, Z_2^*, \dots$ is an infinite supremum process. So, there is an m for which $\Gamma(G_2) \geq 1 - \varepsilon/4$ and, therefore, $\Gamma(G) \geq 1 - \varepsilon/2$. With the given choices of c and m, $\liminf_{a \to \infty} \Gamma_a(G) \geq \Gamma(G)$, since G differs from its interior by a set of Γ-measure zero. See Billingsley [(1968), pages 11–14]. So, there is an a_0 for which $\Gamma_a(G) \geq 1 - \varepsilon$ for all $a \geq a_0$. □

PROPOSITION 2. *Given* $0 < \varepsilon < \frac{1}{2}$ *and* $c_0 > 1$, *let* c, m, a_0 *and* G *be as in Lemma* 2. *Then there is an* $a_1 > a_0$ *for which*

$$\begin{aligned}
&\int_{\{W_a \in F\}} [1 - H_c^*(W_a; r + 5\varepsilon)]\, dP - 5\varepsilon \\
&\quad \leq P\{W_a \in F, R_a \leq c, R_{a+2c} > r\} \\
&\quad \leq \int_{\{W_a \in F\}} [1 - H_c^*(W_a; r - 5\varepsilon)]\, dP + 5\varepsilon
\end{aligned}$$

for all $a \geq a_1$, *all* $r \geq 5\varepsilon$ *and all Borel sets* $F \subseteq G$.

PROOF. With this choice of c and m, there is an $a_1 > a_0$ for which

$$P\{\Delta_m[Q_{a,m}, Q_m^*(W_a; \cdot)] \geq \varepsilon\} \leq \frac{\varepsilon^2}{2c}, \qquad \forall\, a \geq a_1. \tag{13}$$

Now, if $a \geq 1$ and $R_a \leq c$, then $t_{a+2c} > t_a$, so that $R_{a+2c} > r$ iff $Z_j \leq a + 2c$ for all $j \in (t_a, k)$ and $Z_k > a + 2c + r$ for some $k > t_a$; that is,

$$\begin{aligned}
&\{R_a \leq c, R_{a+2c} > r\} \\
&\quad = \bigcup_{k=1}^{\infty} \{R_a \leq c, Z_{a,j} \leq 2c - R_a, \forall\, j < k, \text{and } Z_{a,k} > 2c - R_a + r\}.
\end{aligned}$$

Now let $F \subseteq G$ be a Borel set. Let $0 = b_0 < b_1 < \cdots < b_h = c$ and $\varepsilon_1, \dots, \varepsilon_h$ be the partition described in Lemma 2 and let

$$C_i = C_i(a) = \{W_a \in F\} \cap \{b_{i-1} < R_a \leq b_i\},$$

$$A_i = \{z \in R^m : z_j \leq 2c - b_i - \varepsilon_i, \forall\, j < k \text{ and } z_k > 2c - b_{i-1} + r + 2\varepsilon, \exists\, k \leq m\},$$

$$\begin{aligned}
B_i = &\{z \in R^m : z_j \leq 2c - b_{i-1}, \forall\, j < k, \text{and } z_k > 2c - b_i + r, \exists\, k \leq m\} \\
&\cup \{z \in R^m : z_j \leq 2c - b_{i-1}, \forall\, j \leq m\}
\end{aligned}$$

for all $i = 1, \dots, h$ and all $a \geq a_1$. Then for all such i and a, $C_i \in \mathscr{F}_a$,

$$\{W_a \in F, R_a \leq c, R_{a+2c} > r\} = \bigcup_{i=1}^{h} C_i \cap \{R_{a+2c} > r\} \tag{14}$$

and

$$\begin{aligned} C_i \cap \{(Z_{a,1}, \dots, Z_{a,m}) \in A_{i,\varepsilon}\} &\subseteq C_i \cap \{R_{a+2c} > r\} \\ &\subseteq C_i \cap \{(Z_{a,1}, \dots, Z_{a,m}) \in B_i\}, \end{aligned} \tag{15}$$

where $A_{i,\varepsilon}$ denotes an ε neighborhood of A_i. By relation (13),

$$\begin{aligned} P(C_i \cap \{(Z_{a,1}, \dots, Z_{a,m}) \in B_i\}) &= \int_{C_i} Q_{a,m}(\omega; B_i) P(d\omega) \\ &\leq \int_{C_i} [Q_m^*(W_a; B_{i,\varepsilon}) + \varepsilon] \, dP + \frac{\varepsilon^2}{2c} \end{aligned} \tag{16}$$

for $i = 1, \dots, h$ and $a \geq a_1$. Moreover, for all $i = 1, \dots, h$, all $a \geq a_1$ and all $w \in G$,

$$\begin{aligned} Q_m^*(w; B_{i,\varepsilon}) &\leq Q_m^*\big(w; \{z \in R^m : z_j \leq 2c - b_{i-1} + \varepsilon_{i-1}, \\ &\qquad \forall\, j < k \text{ and } z_k > 2c - b_i - \varepsilon + r, \exists\, k \leq m\}\big) \\ &\qquad + Q_m^*\big(w; (-\infty, 3c]^m\big) \\ &\leq (1 - H^*_{2c - b_{i-1} + \varepsilon_{i-1}})(w; r - 4\varepsilon) + \varepsilon \\ &\leq (1 - H_c^*)(w; r - 5\varepsilon) + 2\varepsilon, \end{aligned} \tag{17}$$

by definition of G. In particular, (17) holds when w is replaced by W_a when $W_a \in G$. So, combining (14)–(17),

$$\begin{aligned} P\{W_a \in F, R_a \leq c, R_{a+2c} > r\} &\leq \sum_{i=1}^{h} \int_{C_i} [(1 - H_c^*)(W_a; r - 5\varepsilon) + 3\varepsilon] \, dP + \varepsilon \\ &\leq \int_{\{W_a \in F\}} (1 - H_c^*)(W_a; r - 5\varepsilon) \, dP + 4\varepsilon. \end{aligned}$$

This establishes the second inequality asserted in the proposition. The first may be established similarly. □

Proof of Theorem 1. It suffices to show that for every continuity set F of Γ and every $\delta > 0$, there is a $c = c(F, \delta) > 0$ for which

$$\begin{aligned} K[Fx(r + \delta, \infty)] - \delta &\leq \liminf_{a \to \infty} P\{W_a \in F, R_{a+2c} > r\} \\ &\leq \limsup_{a \to \infty} P\{W_a \in F, R_{a+2c} > r\} \\ &\leq K[Fx(r - \delta, \infty)] + \delta \end{aligned} \tag{18}$$

for all $r \geq \delta$, where K is as in (10). The first assertion of Theorem 1 then follows directly by setting $F = \mathscr{W}$ and letting $\delta \to 0$. For the second, W_a may be replaced by W_{a+2c} for fixed $\delta > 0$ and then δ may approach zero. The proofs of the two extreme inequalities in (18) are similar, so only the last one needs to be given in detail.

Given F and $\delta > 0$, let $\varepsilon = \delta/9$. Then there is a $c_0 \geq 1$ for which

$$P\{R_a \leq c\} \geq 1 - \varepsilon,$$

$$K[Fx(s+\varepsilon,\infty)] - \varepsilon \leq \int_F [1 - H_c^*(w;s)]\Gamma(dw) \leq K[Fx(s-\varepsilon,\infty)] + \varepsilon \tag{19}$$

for all $a \geq 1$, $s \geq \varepsilon$ and $c \geq c_0$. Next, let c, m and $G = G(\varepsilon, c, m)$ be as in Lemma 2, with the given $\varepsilon > 0$ and c_0 and let $r \geq \delta$. Then there is an $a_1 > a_0$ for which

$$P\{W_a \in F \cap G, R_a \leq c, R_{a+2c} > r\} \leq \int_F [1 - H_c^*(W_a; r - 5\varepsilon)]\, dP + 5\varepsilon$$

for all $a \geq a_1$, by Proposition 2. Moreover, it follows easily from the weak continuity of $H_c(w; \cdot)$ and (19) that

$$\limsup_{a\to\infty} \int_F [1 - H_c^*(W_a; r-5\varepsilon)]\, dP \leq \int_F [1 - H_c^*(w; r - 6\varepsilon)]\Gamma(dw) + \varepsilon \leq K[Fx(r - 7\varepsilon, \infty)] + 2\varepsilon.$$

So,

$$P\{W_a \in F, R_{a+2c} > r\} \leq P\{W_a \in F \cap G, R_a \leq c \text{ and } R_{a+2c} > r\} + 2\varepsilon \leq K[Fx(r-7\varepsilon,\infty)] + 9\varepsilon \leq K[Fx(r-\delta,\infty)] + \delta$$

for all sufficiently large a. The final inequality in (18) follows and the first may be established by a similar argument. □

4. Boundedness conditions. In addition to the convergence conditions, Theorem 1 requires that R_a be stochastically bounded as $a \to \infty$. This is trivially the case if $Z_n - Z_{n-1}$, $n \geq 2$, and uniformly bounded. Some less restrictive conditions are developed in this section.

Observe that for each $a \geq 0$, $Z_{a,k}$, $k = 1, 2, \ldots$, is another infinite supremum process. So, the delayed first passage times and excesses,

$$t_{a,b} = \inf\{k \geq 1: Z_{a,k} > b\} \quad \text{and} \quad R_{a,b} = Z_{a,t_{a,b}} - b,$$

are well defined for all $a, b \geq 0$. It is convenient to write $t_{-1,b} = t_b$ and $R_{-1,b} = R_b$ for $b \geq 0$.

Proposition 3. *For all $a > 0$ and $r > 2$,*

$$P\{R_a > r\} \leq \sum_{j \leq a} P\{R_{j,1} > a + r - j - 2\}.$$

PROOF. Let

$$M_n = \max\{0, Z_1, \ldots, Z_n\}, \qquad n = 0, 1, 2, \ldots .$$

Then

$$\begin{aligned} P\{R_a > r\} &= P\{M_n \le a, Z_{n+1} > a + r, \exists\, n \ge 0\} \\ &\le \sum_{j \le a} P\{j < M_n \le j + 1, Z_{n+1} > a + r, \exists\, n \ge 0\}. \end{aligned}$$

If $r > 2$, $0 \le j \le a$, $j < M_n \le j + 1$ and $Z_{n+1} > a + r$ for some integer $n = 1, 2, \ldots,$ then, for the same n, $t_j \le n$, $j < Z_{t_j} \le j + 1$, $Z_{j,k} = Z_{t_j + k} - Z_{t_j} \le j + 1 - j = 1$, $\forall\, k \le n - t_j$ and $Z_{j, n - t_j + 1} = Z_{n+1} - Z_{t_j} > a + r - j - 1 > 1$, so that $t_{j,1} = n - t_j + 1$ and $R_{j,1} > (Z_{n+1} - Z_{t_j}) - 1 > a + r - j - 2$. So,

$$P\{j < M_n \le j + 1, Z_{n+1} > a + r, \exists\, n \ge 0\} \le P\{R_{j,1} > a + r - j - 2\}$$

for $0 \le j \le a$. The same inequality may be obtained when $j = -1$. The proposition then follows easily. □

To exploit Proposition 3, some uniform integrability of the increments $Z_n - Z_{n-1}$, $n \ge 2$, is needed, along with some uniformity in the basic condition (1).

THE UNIFORM INTEGRABILITY CONDITION. Let $Z_0 = 0$ and $Y_n = Z_n - Z_{n-1}$ for $n = 1, 2, \ldots,$ and for $p \ge 2$, let $M_{n,p} = E[(Y_n^+)^p | \mathscr{A}_n]$, $n \ge 1$. Suppose that $\|\sup_n M_{n,p}\|_\alpha < \infty$ for some $1 < \alpha \le \infty$ and some $p \ge 2$, where $\|\cdot\|_\alpha$ denotes the norm in $L^\alpha(\Omega, \mathscr{A}, P)$.

THE UNIFORM INFINITE SUPREMUM CONDITION. Suppose that there are constants $0 < C < \infty$ and $q > 0$ for which

$$\sup_{j \ge -1} P\{t_{j,1} > n\} \le C n^{-q}, \qquad \forall\, n \ge 1.$$

In the next proposition, $\beta = \alpha/(\alpha - 1)$ denotes the conjugate value to α (and $\beta = 1$ if $\alpha = \infty$).

PROPOSITION 4. *Suppose that the uniform integrability and uniform infinite supremum conditions are satisfied for some* $1 < \alpha \le \infty$, $p \ge 2$ *and* $0 < q < \beta$ *and let* $p' = \alpha p/(\alpha + q)$. *Then there is a constant C for which*

$$P\{R_{j,1} > r\} \le C r^{-qp'}, \qquad \forall\, r \ge 1, j \ge -1.$$

PROOF. In the proof, C denotes a constant which is independent of j and r, not necessarily the same from one usage to the next. For all $r \ge 1$ and

$j \geq -1$, $P\{t_{j,1} > r^{p'}\} \leq Cr^{-qp'}$, by assumption and

$$
\begin{aligned}
P\{R_{j,1} > r, t_{j,1} \leq r^{p'}\} &\leq \sum_{n \leq r^{p'}} P\{t_{j,1} \geq n, Y_{t_j+n} > r\} \\
&\leq \sum_{n \leq r^{P'}} r^{-p} \int_{\{t_{j,1} \geq n\}} M_{t_j+n,p}\, dP \\
&\leq r^{-p} \sum_{n \leq r^{P'}} P\{t_{j,1} \geq n\}^{1/\beta} \|M_{t_j+n,p}\|_{\alpha} \\
&\leq Cr^{-p} \sum_{n \leq r^{P'}} n^{-q/\beta} \leq Cr^{-qp'},
\end{aligned}
$$

so

$$
P\{R_{j,1} > r\} \leq P\{t_{j,1} > r^{p'}\} + P\{R_{j,1} > r, t_{j,1} \leq r^{p'}\} \leq Cr^{-qp'}
$$

for all $r > 1$ and $j \geq -1$. □

Corollary 2 is a direct consequence of Theorem 1 and Proposition 4.

COROLLARY 2. *Suppose that the convergence, uniform integrability and uniform infinite supremum conditions are all satisfied for some* $1 < \alpha \leq \infty$, $p \geq 2$ *and* $0 < q < \beta$ *and let* $p' = \alpha p/(\alpha + p)$. *If* $qp' > 1$, *then* R_a *has a limiting distribution as* $a \to \infty$ *and* R_a^{γ}, $a \geq 1$, *are uniformly integrable for* $0 > \gamma < qp' - 1$.

5. The norm of a driftless random walk. In this section $d \geq 2$ is an integer and $X_1, X_2, \ldots$ denote i.i.d., d-dimensional random vectors with mean vector 0, a nonsingular covariance matrix Σ and finite third moments. $\mathscr{A}_n = \sigma\{X_1, \ldots, X_n\}$ for $n = 1, 2, \ldots$ and

$$
Z_n = \|S_n\|, \qquad n \geq 1, \tag{20}
$$

where $\|\cdot\|$ denotes the Euclidean norm and $S_n = X_1 + \cdots + X_n$ for all $n = 1, 2, \ldots$. With t_a as in (2), let

$$
S_{a,k} = S_{t_a+k} - S_{t_a}, \qquad k \geq 1, a \geq 0.
$$

Then $S_{a,k}$, $k \geq 1$, is independent of $\mathscr{F}_a$ and $S_{a,k}$, $k \geq 1$, has the same joint distribution as S_k, $k \geq 1$, for all $a \geq 0$, by the strong Markov property for random walks.

LEMMA 3. *There is a constant* C *for which*

$$
G(u, b; n) := P\{\langle u, S_k \rangle \leq b, \forall\, k \leq n\} \leq \frac{Cb}{\sqrt{n}}
$$

for all $n = 1, 2, \ldots$, *all* $b \geq 1$ *and* u *for which* $\|u\| = 1$.

PROOF. This follows easily from Berry–Esseen bounds for the distributions of maxima of partial sums. See Arak (1974). □

LEMMA 4. *Let*

$$U_a = S_{t_a}/\|S_{t_a}\|, \qquad a \geq 1.$$

Then

$$W_a := \left(a^{-2}t_a, U_a\right) \Rightarrow \left[\tau, B(\tau)\right], \quad as \quad a \to \infty,$$

where $B(s)$, $0 \leq s < \infty$, *denotes a d-dimensional Brownian motion with mean* 0 *and covariance* Σ *per unit time, and* $\tau = \inf\{s \geq 0\colon \|B(s)\| \geq 1\}$. *Moreover,* $\|W_{a+b} - W_a\| \to 0$ *in probability as* $a \to \infty$ *for all* $b > 0$.

PROOF. It follows easily from the invariance principle that $(t_a/a^2, S_{t_a}) \Rightarrow (\tau, B(\tau))$, as $a \to \infty$. So, it suffices to show that $R_a/a \to 0$, $(t_{a+b} - t_a)/a^2 \to 0$ and $\|S_{t_{a+b}} - S_{t_a}\|/a \to 0$ in probability for all $b > 0$.

Since $\|X_1\|$ has a finite second moment, $\|X_n\|/\sqrt{n} \to 0$ w.p.1 as $n \to \infty$, so that $R_a/a \leq (\sqrt{t}_a/a)(\|X_{t_a}\|/\sqrt{t}_a) \to 0$ in probability. Moreover, for all $a, b > 0$ and all $n = 1, 2, \ldots,$

$$\begin{aligned} P\{t_{a+b} - t_a > n\} &\leq P\{t_{a,b} > n\} \leq P\{\langle U_a, S_{a,k}\rangle \leq b, \forall\, k \leq n\} \\ &= E\{G(U_a, b; n)\} \leq \frac{Cb}{\sqrt{n}}, \end{aligned} \tag{21}$$

by Lemma 3 and the independence of $\mathscr{F}_a$ and $S_{a,k}$, $k \geq 1$. So $t_{a+b} - t_a$ is stochastically bounded in $a \geq 1$ for all $0 < b < \infty$. The remainder of the lemma follows easily. □

In the sequel, Γ denotes the joint distribution of $[\tau, B(\tau)]$, $W = (T, U)$ denotes a random vector which has distribution Γ and is independent of $X_1, X_2, \ldots$ and $Z_k^* = \langle U, S_k\rangle$ for $k = 1, 2, \ldots$.

LEMMA 5. $W, Z_1^*, Z_2^*, \ldots$ *is a conditional process with*

$$H(u; dr) = \frac{1}{E\left[\langle u, S_{\tau_u}\rangle\right]} P\{\langle u, S_{\tau_u}\rangle > r\}\, dr \tag{22}$$

and

$$\tau_u = \inf\{k \geq 1\colon \langle u, S_k\rangle > 0\} \quad a.e.\ u.$$

PROOF. Since $\langle u, S_k\rangle$, $k = 1, 2, \ldots,$ is a one-dimensional random walk with mean 0 and finite, positive variance for $\|u\| = 1$, it suffices to show that the distribution of $\langle u, X_1\rangle$ is nonarithmetic for a.e. u (Γ). To see this, let $\varphi(\theta) = E\{\exp(i\langle\theta, X_1\rangle)\}$, $\theta \in R^d$, denote the characteristic function of X_1. Then the set of $\theta \in R^d$ for which $\varphi(\theta) = 1$ is a subgroup of R^d and since X_1 has a nonsingular convariance matrix, this subgroup must be discrete. It follows that $\varphi(tu) = 1$ for some $t \neq 0$ for at most countably many unit vectors u; that is, $\langle u, X_1 \rangle$ has a nonarithmetic distribution for at most countably many u. □

THEOREM 2. *Let $X_1, X_2, \dots$ be i.i.d. with mean vector 0, a nonsingular covariance matrix Σ and a finite pth moment for some $p \geq 3$ and define Z_n, $n \geq 1$, by (20). Then $(a^{-2}t_a, U_a, R_a) \Rightarrow (T, U, R)$, where (T, U) have distribution Γ and the conditional distribution of R given $(T, U) = (t, u)$ is $H(u; \cdot)$ a.e. (t, u), where H is as in (22). Moreover, R_a^γ, $a \geq 1$, are uniformly integrable for all $0 < \gamma < \frac{1}{2}p - 1$.*

PROOF. Since $Z_n - Z_{n-1} \leq \|X_n\|$ and X_n is independent of $Z_1, \dots, Z_{n-1}$ for all $n \geq 1$, it is clear that the uniform integrability condition is satisfied with $\alpha = \infty$ for some $p \geq 3$ and it follows directly from (21) that the uniform infinite supremum condition is satisfied with $q = \frac{1}{2}$. That the convergence condition is satisfied follows from the independence of W_a and $S_{a,k}$, $k \geq 1$, for all $a \geq 0$, Proposition 1 and the simple relation

$$\|x + y\| - \|x\| = \big(2\langle x, y \rangle + \|y\|^2\big)/\big(\|x\| + \|x + y\|\big) \quad \text{for } x \neq 0,$$

which implies

$$(Z_{a,1}, \dots, Z_{a,m}) - [\langle U_a, S_{a,1} \rangle, \dots, \langle U_a, S_{a,m} \rangle] \to 0$$

in probability as $a \to \infty$ for every $m \geq 1$. □

COROLLARY 3. *If $\|X_1\|$ has a finite pth moment for some $p > 4$, then*

$$E(t_a) = \frac{a^2 + 2\rho a}{\operatorname{tr}(\Sigma)} + o(a), \quad \text{as } a \to \infty,$$

where ρ denotes the mean of the limiting distribution of R_a.

PROOF. Since $\|S_n\| \geq |\langle u, S_n \rangle|$ for all $\|u\| = 1$ and all $n = 1, 2, \dots,$ it follows easily from Stein's lemma that $E(t_a) < \infty$ for all a. [See Woodroofe (1982), pages 29–30, for Stein's lemma.] So, by Wald's lemma,

$$\operatorname{tr}(\Sigma) E(t_a) = E\big\{\|S_{t_a}\|^2\big\} = E\big\{(a + R_a)^2\big\} = a^2 + 2aE(R_a) + E\big(R_a^2\big). \tag{23}$$

Here $E(R_a) \to \rho$ as $a \to \infty$, since R_a, $a \geq 1$, are uniformly integrable. So, it suffices to show that $E(R_a^2) = o(a)$. This may be seen as follows. Let $Y_a = \sup_{n \geq t_a} \|X_n\|^2/\sqrt{n}$. Then $E(R_a^2) \leq \sqrt{E}(t_a)\sqrt{E}(Y_a^2)$ for all a, by Schwarz' inequality and $E(Y_a^2) \to 0$ as $a \to \infty$, since $\|X_1\|$ is assumed to have a pth moment for some $p > 4$. It then follows easily from (23) that $E(t_a) = O(a^2)$. So $E(R_a^2) = o(a)$, as required. □

REMARK 1. Theorem 2 is valid when $d = 1$, if the distribution of X_1 is nonarithmetic. An analogous result holds in the arithmetic case. Simpler proofs exist for $d = 1$, however.

REMARK 2. Theorem 2 is valid under the condition that $\|X_1\|$ has a finite pth moment for some $p > 2$. The (fairly lengthy) proof uses Spitzer's identity and Stein's lemma in place of the Berry–Esseen bounds in Lemma 3. It seems reasonable to hope that Theorem 2 is valid assuming only second moments and a nonsingular convariance matrix.

REMARK 3. If the distribution of X_1 is spherically symmetric in Theorem 2, then the distributions of $Z_1^*, Z_2^*, \dots$ are those of a random walk. Otherwise, they are not, so that the theorem of Lai and Siegmund cannot be applicable, even if ladder heights are considered. See Corollary 1.

6. Asymptotically exchangeable increments. Let $\mathscr{W}$ denote a complete separable metric space and let $\{F(w; \cdot): w \in \mathscr{W}\}$ be a collection of distribution functions for which $F(w; x)$ is jointly measurable in $(w, x) \in \mathscr{W} \times R$ and $F(w; \cdot)$ is weakly continuous in $w \in \mathscr{W}$. Next, let $w_0, w_1, w_2, \dots$ denote jointly distributed random elements with values in $\mathscr{W}$; let $Z_1, Z_2, \dots$ denote random variables for which the conditional distribution of $Z_k - Z_{k-1}$ given $w_0, \dots, w_{k-1}, Z_1, \dots, Z_{k-1}$ is $F(w_{k-1}; \cdot)$ w.p.1 for all $k = 1, 2, \dots$ and let $\mathscr{A}_n = \sigma\{w_0, \dots, w_n, Z_1, \dots, Z_n\}$ for $n = 1, 2, \dots$.

Such models arise in the theory of sequentially designed experiments when subjects are allocated to treatments according to an adaptive coin design, as in Wei (1978). See Heckman (1985) and Eisele (1989) for specific examples.

In addition to the assumptions of the first paragraph it is assumed throughout this section that

$$w_\infty = \lim_{n \to \infty} w_n \qquad \exists \text{ w.p.1.} \tag{24}$$

The existence of $w_0, w_1, w_2, \dots$ for which (24) holds is equivalent to requiring that the conditional distribution of $Z_{n+1} - Z_n$ given $Z_1, \dots, Z_n$ have a weak limit w.p.1 as $n \to \infty$, since one may let w_n be that conditional distribution. Let Γ denote the distribution of w_∞. It is also assumed that for a.e. w (Γ), $F(w; \cdot)$ is nonarithmetic and

$$0 < \mu(w) := \int x F(w; dx) < \infty.$$

Let W denote a random element with distribution Γ. Let $X_1^*, X_2^*, \dots$ denote random variables which are conditionally independent and identically distributed with common distribution $F(w; \cdot)$ given $W = w$ for every $w \in \mathscr{W}$ and let $Z_n^* = X_1^* + \cdots + X_n^*$ for $n = 1, 2, \dots$.

LEMMA 6. *$W, Z_1^*, Z_2^*, \dots$ is a conditional renewal process with*

$$H(w; dr) = \frac{1}{E_w^*(Z_\tau^*)} P_w^*\{Z_\tau^* > r\}\, dr, \qquad r \geq 0,$$

where

$$\tau = \inf\{n \geq 1: Z_n^* > 0\}$$

for a.e. $w \in \mathscr{W}$ and all $r \geq 0$. P_w^* and E_w^* denote conditional probability and expectation given $W = w$. Moreover, the convergence condition is satisfied with $W_a = w_{t_a}$ for $a \geq 0$.

PROOF. That $W, Z_1^*, Z_2^*, \dots$ is a conditional renewal process follows easily from Example 1.

It is clear that $W_a \Rightarrow W$ as $a \to \infty$. For the rest of the convergence condition, write $Fg(w) = \int_R g(x)F(w; dx)$ for $w \in \mathscr{W}$ and bounded continuous functions g on R. For fixed $m \geq 1$, let $g_1, \dots, g_m$ denote bounded continuous functions on R, say $|g_k| \leq 1$ for $k = 1, \dots, m$. Then

$$\left| E\left\{ \prod_{k=1}^{m} g_k(X_{n+k}) | \mathscr{A}_n \right\} - \prod_{k=1}^{m} Fg_k(w_n) \right|$$

$$= \sum_{k=1}^{m} \left| E\left\{ \prod_{j=1}^{k-1} g_j(X_{n+j}) [Fg_k(w_{n+k-1})] - Fg_k(w_n)] \prod_{j=k+1}^{m} Fg_j(w_n) | \mathscr{A}_n \right\} \right|$$

$$\leq \sum_{k=1}^{m} \left| E\{ Fg_k(w_{n+k-1}) - Fg_k(w_n) | \mathscr{A}_n \} \right|,$$

which approaches zero w.p.1 as $n \to \infty$. The remainder of the convergence condition follows easily. □

To complete the verification of the conditions of Theorem 1, it is necessary to show that R_a is (well defined and) stochastically bounded. If the conditional means are (uniformly) bounded away from 0 and ∞ and the conditional variances are bounded above, then stochastic boundedness of R_a, $a \geq 1$, may be deduced from Proposition 2.2 of Lalley and Lorden (1986). Alternative conditions are the existence of $p \geq 2$, $\beta > 0$, $\gamma > \frac{1}{2}$ and $c > 0$ for which

$$P\left\{ \inf_{m \geq 0} \sum_{k=m+1}^{m+n} \mu(w_k) \leq 2cn^{\gamma} \right\} = O(n^{-\beta}) \quad \text{as } n \to \infty \tag{25}$$

and

$$\left\| \sup_{n \geq 0} \nu_p(w_n) \right\|_1 < \infty, \tag{26}$$

where

$$\nu_p(w) = \int |x - \mu(w)|^p F(w; dx), \qquad w \in \mathscr{W}.$$

If (26) holds, then the uniform integrability condition is (clearly) satisfied with $\alpha = 1$ and p of (26).

LEMMA 7. *If (25) and (26) hold, then the uniform infinite supremum condition is satisfied with $q = \min\{\beta, p(\gamma - \frac{1}{2})\}$.*

PROOF. For a fixed (but arbitrary) $j \geq -1$, $Z_{j,n}$ may be written in the form $Z_{j,n} = M_n + \zeta_n$, $n \geq 1$, where

$$M_n = \sum_{k=1}^{n} \left[Z_{t_j+k} - Z_{t_j+k-1} - E\left(Z_{t_j+k} - Z_{t_j+k-1} | \mathscr{A}_{t_j+k-1} \right) \right]$$

and

$$\zeta_n = \sum_{k=1}^{n} \mu(w_{t_j+k-1})$$

for all $n = 1, 2, \dots$. Let γ be as in (25) and let n be so large that $cn^{\gamma} > 1$. Then $\gamma > \frac{1}{2}$.

$$P\{t_{j,1} > n\} \leq P\{Z_{j,n} \leq 1\} \leq P\{M_n \leq -cn^{\gamma}\} + P\{\zeta_n \leq 2cn^{\gamma}\}$$

and

$$P\{\zeta_n \leq 2cn^{\gamma}\} \leq P\left\{ \inf_{m \geq 0} \sum_{k=m+1}^{m+n} \mu(w_k) \leq 2cn^{\gamma} \right\} = O(n^{-\beta}),$$

as $n \to \infty$. Next, observe that $M_1, M_2, \dots$ is a martingale and let $s_1^2, s_2^2, \dots$ denote the square functions for $M_1, M_2, \dots$. Then there is an absolute constant B for which

$$\begin{aligned} P\{|M_n| > cn^{\gamma}\} &\leq c^{-p} n^{-p\gamma} E\{|M_n|^p\} \\ &\leq Bc^{-p} n^{-p\gamma} E\{|s_n|^p\} \\ &\leq Bc^{-p} n^{-p(\gamma - 1/2)} E\left\{ \sup_{k \geq 1} \nu_p(w_{k-1}) \right\}. \end{aligned}$$

The lemma follows immediately. □

THEOREM 3. *If (24)–(26) hold with $pq/(1+q) > 1$, where q is as in Lemma 7, then (W_a, R_a) has the same limiting distribution as (W, R_a^*) and $R_a^{\gamma'}$, $a \geq 1$, are uniformly integrable for all $0 < \gamma' < pq/(1+q) - 1$.*

PROOF. This follows easily from Theorem 1, Corollary 2 and Lemmas 7 and 8. □

EXAMPLE 2. Let $u_0, u_1, u_2, \dots$ denote random variables for which $u_0 = \frac{1}{2}$ and

$$P\{u_{n+1} = 1 | u_0, \dots, u_n\} = q(w_n) = 1 - P\{u_{n+1} = 0 | u_0, \dots, u_n\}$$

with $w_n = (u_0 + \cdots + u_n)/(n+1)$ for all $n = 1, 2, \dots$, where q is a continuous function from $[0, 1]$ into $[0, 1]$ for which (24) holds. If F_0 and F_1 are two nonarithmetic distribution functions with positive means and finite pth moments, and if

$$F(w; \cdot) = [1 - q(w)] F_0 + q(w) F_1 \quad \text{for } w \in \mathscr{W} = [0, 1],$$

then the conditions of Theorem 3 are satisfied with p, $\gamma = 1$, and any $\beta > 0$ for sufficiently small $c > 0$.

(a) If $q(w) = 1 - w$ for $0 \le w \le 1$, this is an example of Wei's (1978) adaptive biased coin design and $w_\infty = \frac{1}{2}$.

(b) If $q(w) = w$, then $u_0, u_1, u_2, \dots$ is a Polya urn process and w_∞ has the arcsine distribution [that is, $P\{w_\infty \le w\} = (2/\pi)$ arcsine $(\sqrt{w})$ for all $0 \le w \le 1$].

Conditions on q under which (24) holds may be found in Hill, Lane and Sudderth (1980).

REMARK 4. It is hoped that Theorem 3 may find applications to statistical problems with adaptive biased coin designs. However, it is not applicable to the original biased coin design of Efron (1971) and Theorem 1 seems awkward for such problems, since the condition (11) is difficult to check.

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DEPARTMENT OF STATISTICS
1444 MASON HALL
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109–1027