The Annals of Probability
1990, Vol. 18, No. 4, 1765–1783

A STABILITY RESULT FOR THE PERIODOGRAM[1]

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Let $\{X_t\}_{t=1}^{\infty}$ be a stationary Gaussian time series with zero mean, unit variance, absolutely summable autocorrelation function and at least once differentiable spectral density function which is strictly positive in $[0, \pi]$. In this paper it is shown that, if M_n denotes the maximum of the normalized periodogram of $\{X_1, \dots, X_n\}$ over the interval $[0, \pi]$, then, almost surely,

$$\liminf_{n\to\infty}[M_n - 2\log n + \log\log n] \geq 0 \tag{1}$$

and

$$\limsup_{n\to\infty}\left[M_n - 2\log n - 2(\log n)^{\delta}\right] = -\infty \tag{2}$$

for any $\delta > 0$.

1. Introduction. Let $\{X_n\}_{t=1}^{\infty}$ be a stationary time series with autocorrelation function $r(u)$ and the spectral density function $h(\omega)$. The periodogram of $\{X_1, \dots, X_n\}$ is defined by

$$I_n(\omega) = \frac{2}{n}\left|\sum_{t=1}^{n} X_t e^{i\omega t}\right|^2, \qquad \omega \in [0, \pi].$$

Then $I_n(\omega)/4\pi$ appears to be the natural estimator of $h(\omega)$. Yet it is inconsistent and its erratic behaviour is well known. In fact under quite general conditions, An, Chen and Hannan (1983) showed that, almost surely,

$$\lim_{n\to\infty}\max_{\omega\in[0,\pi]}\frac{I_n(\omega)}{4\pi h(\omega)\log n} = 1. \tag{1.1}$$

In this paper, a stronger result than (1.1) is obtained at the cost of imposing stronger conditions on $\{X_t\}_{t=1}^{\infty}$. Namely, let $\{X_t\}_{t=1}^{\infty}$ be a stationary Gaussian time series with $E[X_t] = 0$, $E[X_t^2] = 1$ and the autocorrelation function $r(u)$ such that $\sum_0^{\infty}|r(u)| < \infty$ [and hence $h(\omega)$ is continuous]. Further assume that $h(\omega)$ is strictly positive and has bounded first derivative in $[0, \pi]$.

Let

$$M_n = \max_{\omega\in[0,\pi]}\frac{I_n(\omega)}{2\pi h(\omega)}.$$

Received November 1987; revised September 1989.
[1]This work was supported in part by Junta Nacional de Investigacao Cientifica e Tecnologica of Portugal.
AMS 1980 *subject classifications.* 60F15, 62F15.
Key words and phrases. Periodogram, trigonometric polynomials, spectral density function.

Then, almost surely,

$$\liminf_{n\to\infty}[M_n - 2\log n + \log\log n] \geq 0, \tag{1.2}$$

and for any $\delta > 0$,

$$\limsup_{n\to\infty}\left[M_n - 2\log n - 2(\log n)^{\delta}\right] = -\infty. \tag{1.3}$$

The outline of the paper is as follows. In Section 2 some preliminary results on the covariance structure of the periodogram ordinates will be given. The proofs of (1.3) and (1.2) will be given, respectively, in Sections 3 and 4.

2. Some preliminary results. Let

$$X_n(\omega) = \sqrt{\frac{2}{n}}\sum_{t=1}^{n} X_t\cos\omega t, \qquad Y_n(\omega) = \sqrt{\frac{2}{n}}\sum_{t=1}^{n} X_t\sin\omega t \tag{2.1}$$

[so that $I_n(\omega) = X_n^2(\omega) + Y_n^2(\omega)$], and denote the variances of $X_n(\omega)$ and $Y_n(\omega)$ by $\sigma_{X_n}^2(\omega)$ and $\sigma_{Y_n}^2(\omega)$, respectively.

LEMMA 2.1. *Let $\varepsilon > 0$ be arbitrarily small and fixed. Then as $n \to \infty$, uniformly in $\omega \in [\varepsilon, \pi - \varepsilon]$,*

$$\sigma_{X_n}^2(\omega) = \sum_{|u|\leq n-1}\left(1 - \frac{|u|}{n}\right)r(u)\cos\omega u + O(n^{-1}), \tag{2.2}$$

$$\sigma_{Y_n}^2(\omega) = \sum_{|u|\leq n-1}\left(1 - \frac{|u|}{n}\right)r(u)\cos\omega u + O(n^{-1}), \tag{2.3}$$

$$\operatorname{Cov}(X_n(\omega), Y_n(\omega)) = O(n^{-1}). \tag{2.4}$$

PROOF.

$$\sigma_{X_n}^2(\omega) = \frac{2}{n}\sum_{s=1}^{n}\sum_{t=1}^{n} E[X_tX_s]\cos\omega t\cos\omega s.$$

If $u = s - t$ then $1 \leq t \leq n - u$ when $u \geq 0$ and $1 - u \leq t \leq n$ when $u < 0$. Thus

$$\sigma_{X_n}^2(\omega) = \frac{1}{n}\sum_{|u|\leq n-1}(n - |u|)r(u)\cos\omega u + T_n, \tag{2.5}$$

where

$$nT_n = \sum_{u=0}^{n-1}\left[r(u)\sum_{t=1}^{n-u}\cos\omega(u + 2t)\right] + \sum_{u=-(n-1)}^{-1}\left[r(u)\sum_{t=1-u}^{n}\cos\omega(u + 2t)\right].$$

Now

$$\left|\sum_{t=1}^{n-u} \cos \omega(u+2t)\right| \le \left|\sum_{t=1}^{n-u} e^{2i\omega t}\right| \le \frac{1}{|\sin \omega|} \le c(\varepsilon)$$

and

$$\left|\sum_{t=1-u}^{n} \cos \omega(u+2t)\right| \le c(\varepsilon)$$

for some constant $c(\varepsilon)$, uniformly for $\omega \in [\varepsilon, \pi-\varepsilon]$, $u \in (-\infty, \infty)$. (2.2) now follows from (2.5). (2.3) can be established similarly.

$$\operatorname{Cov}(X_n(\omega), Y_n(\omega)) = \frac{1}{n} \sum_{|u| \le n-1} (n-|u|) r(u) \sin \omega u + T_n^*,$$

where

$$nT_n^* = \sum_{u=0}^{n-1}\left[r(u) \sum_{t=1}^{n-u} \sin \omega(u+2t)\right] + \sum_{u=-(n-1)}^{-1}\left[r(u) \sum_{t=1-u}^{n} \sin \omega(u+2t)\right].$$

Now

$$\sum_{|u| \le n-1} (n-|u|) r(u) \sin \omega u = 0,$$

and the sums

$$\left|\sum_{t=1}^{n-u} \sin \omega(u+2t)\right| \quad \text{and} \quad \left|\sum_{t=1-u}^{n} \sin \omega(u+2t)\right|$$

are uniformly bounded for $\omega \in [\varepsilon, \pi-\varepsilon]$ and $u \in (-\infty, \infty)$. Hence (2.4) follows from the absolute summability of $r(u)$. □

LEMMA 2.2. *For any $\eta > 0$, let $m(n)$ be the integer part of $n/(\log n)^\eta$ and let $\{\omega_1, \dots, \omega_{m(n)}\}$ be an equally spaced partition of $[\varepsilon, \pi-\varepsilon]$, where $\varepsilon > 0$ is arbitrarily small. Then when $i \ne j$,*

$$[1]\ \left|\operatorname{Cov}(X_n(\omega_i), X_n(\omega_j))\right| \le \frac{C_1}{(\log n)^\eta |i-j|},$$

$$(2.6)\qquad [2]\ \left|\operatorname{Cov}(X_n(\omega_i), Y_n(\omega_j))\right| \le \frac{C_2}{(\log n)^\eta |i-j|},$$

$$[3]\ \left|\operatorname{Cov}(Y_n(\omega_i), Y_n(\omega_j))\right| \le \frac{C_3}{(\log n)^\eta |i-j|}$$

for some constants C_1, C_2, C_3.

PROOF.

$$
\begin{aligned}
\operatorname{Cov}(X_n(\omega_i), X_n(\omega_j)) &= \frac{1}{n}\sum_{u=0}^{n-1} r(u) \sum_{t=1}^{n-u} \cos[(\omega_i+\omega_j)t+\omega_i u] \\
&\quad + \frac{1}{n}\sum_{u=-(n-1)}^{-1} r(u) \sum_{t=1-u}^{n} \cos[(\omega_i+\omega_j)t+\omega_i u] \\
&\quad + \frac{1}{n}\sum_{u=0}^{n-1} r(u) \sum_{t=1}^{n-u} \cos[(\omega_i-\omega_j)t+\omega_i u] \\
&\quad + \frac{1}{n}\sum_{u=-(n-1)}^{-1} r(u) \sum_{t=1-u}^{n} \cos[(\omega_i-\omega_j)t+\omega_i u].
\end{aligned} \tag{2.7}
$$

Now

$$
\begin{aligned}
\left|\sum_{t=1}^{n-u} \cos[(\omega_i+\omega_j)t+\omega_i u]\right| &\le \left|\sum_{t=1}^{n-u} e^{i\omega_i u + i(\omega_i+\omega_j)t}\right| \\
&\le \frac{1}{\left|\sin\frac{1}{2}(\omega_i+\omega_j)\right|} \\
&\le c
\end{aligned}
$$

for some constant c. Similarly,

$$
\left|\sum_{t=1-u}^{n} \cos[(\omega_i+\omega_j)t+\omega_i u]\right| \le \frac{1}{\left|\sin\frac{1}{2}(\omega_i+\omega_j)\right|} \le c.
$$

Thus the first two sums in (2.7) are of order $O(n^{-1})$ uniformly in u, i, j. However,

$$
\left|\sum_{t=1}^{n-u} \cos[(\omega_i-\omega_j)t+\omega_i u]\right| \le \left|\sum_{t=1}^{n-u} e^{i(\omega_i-\omega_j)t}\right| \le \frac{1}{\left|\sin\frac{1}{2}(\omega_i-\omega_j)\right|} \le \frac{c_1 m(n)}{|i-j|}
$$

for some constant c_1.

$$
\left|\sum_{t=1-u}^{n} \cos[(\omega_i-\omega_j)t+\omega_j u]\right| \le \frac{c_1 m(n)}{|i-j|}.
$$

Thus the last two sums in (2.7) are bounded by

$$
\frac{C_1}{(\log n)^{\eta}|i-j|}.
$$

Now [1] of (2.6) follows. [2] and [3] of (2.6) can be established similarly. □

3. Upper bound.

DEFINITION. A finite trigonometric sum

$$T_n(x) = \sum_{t=1}^{n} (a_t \cos tx + b_t \sin tx) = \sum_{k=-n}^{n} c_k e^{ikx},$$

where x is real and a_t, b_t, c_k are independent of x, is called a trigonometric polynomial of order n.

Note that the periodogram of $\{X_1, \dots, X_n\}$ is a trigonometric polynomial of order n, since

$$I_n(\omega) = 2 \sum_{k=-n}^{n} c_k e^{ik\omega},$$

where

$$c_k = \frac{1}{n} \sum_{t=1}^{n-|k|} X_t X_{t+|k|}.$$

THEOREM 3.1. *Let $T_n(x)$ be a trigonometric polynomial of order n and let*

$$M = \max_{x \in [a,b]} |T_n(x)|.$$

Then

$$\max_{x \in [a,b]} |T_n'(x)| \le nM,$$

where $T_n'(x)$ is the derivative of $T_n(x)$ with respect to x.

[See, for example, Zygmund (1959), Volume 2, page 11.]

In the subsequent lemmas we make use of the ideas suggested by Salem and Zygmund (1954).

LEMMA 3.1. *For any $\theta \in (0,1)$, there exist a constant c and an interval $A_n = [a_n, b_n] \subseteq [0, \pi]$ of length at least $c(1-\theta)/n$ such that for every $\omega \in A_n$, almost surely,*

$$\theta M_n \le \frac{I_n(\omega)}{2\pi h(\omega)}.$$

PROOF. Let $\omega_0 \in [0, \pi]$ be such that

$$M_n = \frac{I_n(\omega_0)}{2\pi h(\omega_0)}.$$

If $I_n(\omega) \ge \theta M_n$ for all $\omega \in [\omega_0, \pi]$, we may clearly take $A_n = [\omega_0, \pi]$. Otherwise we may take $A_n = [\omega_0, \omega_1]$, where $\omega_1 > \omega_0$ is the first point to the right of ω_0 such that

$$\theta M_n = \frac{I_n(\omega_1)}{2\pi h(\omega_1)}.$$

To show that then $|\omega_1 - \omega_0| \geq c(1-\theta)/n$, for some constant c, we proceed as follows.

By the mean value theorem

$$\begin{aligned} M_n(1-\theta) &= \frac{I_n(\omega_0)}{2\pi h(\omega_0)} - \frac{I_n(\omega_1)}{2\pi h(\omega_1)} \\ &\leq |\omega_1 - \omega_0| \max_{\omega\in[0,\pi]} \left| \frac{d}{d\omega} \frac{I_n(\omega)}{2\pi h(\omega)} \right| \\ &\leq |\omega_1 - \omega_0| \left[\max_{\omega\in[0,\pi]} \left| \frac{I_n'(\omega)}{2\pi h(\omega)} \right| + \max_{\omega\in[0,\pi]} \left| \frac{I_n(\omega) h'(\omega)}{2\pi h^2(\omega)} \right| \right] \\ &\leq |\omega_1 - \omega_0| \left[c_1 \max_{\omega\in[0,\pi]} |I_n'(\omega)| + c_2 \max_{\omega\in[0,\pi]} I_n(\omega) \right], \end{aligned}$$

where

$$c_1 = \max_{\omega\in[0,\pi]} \left| \frac{1}{2\pi h(\omega)} \right|,$$

$$c_2 = \max_{\omega\in[0,\pi]} \left| \frac{h'(\omega)}{2\pi h^2(\omega)} \right|.$$

Hence using Theorem 3.1, we have

$$M_n(1-\theta) \leq (\omega_1 - \omega_0)(c_1 n + c_2) M_n$$

so

$$(\omega_1 - \omega_0) \geq \frac{(1-\theta)}{n} (c_1 n + c_2)^{-1} \geq \frac{1-\theta}{n(c_1 + c_2)}. \qquad \square$$

COROLLARY. *Let $M_n^{(1)}, M_n^{(2)}, M_n^{(3)}$ be, respectively, the maxima of $I_n(\omega)/2\pi h(\omega)$ over the intervals*

$$\left[\frac{\log n}{n}, \pi - \frac{\log n}{n}\right], \left[0, \frac{\log n}{n}\right] \quad \textit{and} \quad \left[\pi - \frac{\log n}{n}, \pi\right]. \tag{3.1}$$

Given any $\theta \in (0,1)$, there exist intervals $A_n^{(1)}, A_n^{(2)}, A_n^{(3)}$, respectively, contained in the intervals defined in (3.1), each of length at least $c(1-\theta)/n$ and such that for every $\omega \in A_n^{(i)}$,

$$\theta M_n^{(i)} \leq \frac{I_n(\omega)}{2\pi h(\omega)}, \qquad i = 1, 2, 3. \tag{3.2}$$

The proof of (3.2) is exactly the same as that of Lemma 3.1 except that $[0, \pi]$ is replaced by the intervals defined in (3.1) in each case.

LEMMA 3.2. *Let k be a fixed positive integer and let n_1 be the integer part of $e^{(\log n)^{k+1}}$. Then, almost surely,*

$$\limsup_{n\to\infty}\left[M_{n_1} - 2\log n_1 - 2(\log n_1)^{1/k}\right] = -\infty. \tag{3.3}$$

PROOF. From the corollary to Lemma 3.1, for any $\theta \in (0,1)$ there exists $A_{n_1}^{(1)} \subseteq [\log n_1/n_1, \pi - \log n_1/n_1]$ of length at least $c(1-\theta)/n_1$, for some constant positive c, such that

$$\theta M_{n_1}^{(1)} \le \frac{I_{n_1}(\omega)}{2\pi h(\omega)}$$

for every $\omega \in A_{n_1}^{(1)}$. Hence, for every $\gamma > 0$,

$$\begin{aligned} c\frac{(1-\theta)}{n_1}\exp\left[\theta\gamma M_{n_1}^{(1)}\right] &\le \int_{A_{n_1}^{(1)}}\exp\left[\theta\gamma M_{n_1}^{(1)}\right]d\omega \\ &\le \int_{A_{n_1}^{(1)}}\exp\left[\gamma\frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right]d\omega \\ &\le \int_{\log n_1/n_1}^{\pi-\log n_1/n_1}\exp\left[\gamma\frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right]d\omega. \end{aligned}$$

Thus, denoting by $P(\cdot)$ the probability measure over the σ-field generated by $\{X_t\}$, we have

$$\int\exp\left[\theta\gamma M_{n_1}^{(1)}\right]dP \le c^{-1}\frac{n_1}{(1-\theta)}\int_{\log n_1/n_1}^{\pi-\log n_1/n_1}\int\exp\left[\gamma\frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right]dP\,d\omega. \tag{3.4}$$

Now for every $\omega \in [\log n/n, \pi - \log n/n]$,

$$\left\{\frac{X_n(\omega)}{\sqrt{2\pi h(\omega)}}, \frac{Y_n(\omega)}{\sqrt{2\pi h(\omega)}}\right\} \sim N(0, \Sigma_n(\omega)),$$

where

$$\Sigma_n(\omega) = I_2 + O\big((\log n)^{-1}\big),$$

uniformly in ω, I_2 being the 2-dimensional identity matrix. This can be shown by proceeding as in the proof of Lemma 2.1 with ε being replaced by $\log n/n$ and using the fact that as $n \to \infty$,

$$\sum_{|u|\le n-1}\left(1 - \frac{|u|}{n}\right)r(u)\cos\omega u = 2\pi h(\omega) + O\left(\frac{\log n}{n}\right),$$

uniformly in ω. [See, for example, Priestley (1981), pages 417 and 418.]

It follows by taking a polar transformation that the density function of

$$\frac{I_{n_1}(\omega)}{2\pi h(\omega)}$$

is given by

$$p_{n_1}(z) = \tfrac{1}{2}\big(1 + O((\log n_1)^{-1})\big)\exp\big[-\tfrac{1}{2}z\big(1 + O((\log n_1)^{-1})\big)\big]$$

as $n \to \infty$, uniformly in ω. Hence from (3.4), for any $\gamma_n \in (0, \frac{1}{2})$,

$$\int \exp\big[\theta\gamma_n M_{n_1}^{(1)}\big]\,dP \le \frac{n_1}{2c(1-\theta)}\int_{\log n_1/n_1}^{\pi - \log n_1/n_1} d\omega \int_0^\infty \left(1 + \frac{c^*}{\log n_1}\right) \times \exp\left[\gamma_n z - \tfrac{1}{2}z\left(1 - \frac{c^*}{\log n_1}\right)\right] dz$$

for some constant c^*. Thus for all n such that $c^* < \frac{1}{2}\log n_1$ ($n \ge n_0$, say), we have, on taking $\gamma_n = \frac{1}{2} - c^*/\log n_1$,

$$\int \exp\big[\theta\gamma_n M_{n_1}^{(1)}\big]\,dP \le n_1\pi\left(1 + \frac{c^*}{\log n_1}\right)\left(2c(1-\theta)\left(\frac{1}{2} - \frac{c^*}{2\log n_1} - \gamma_n\right)\right)^{-1}$$
$$= O\left(\frac{n_1 \log n_1}{1-\theta}\right).$$

Therefore, with $\theta = \theta_n = 1 - n^{-m}$, $m > 0$,

$$\sum_{n=n_0}^{\infty} \int \exp\big[\theta_n\gamma_n M_{n_1}^{(1)} - \log n_1 - (\log n_1)^{1/k}\big]\,dP$$
$$= O\left(\sum_{n=n_0}^{\infty} n^m \log n_1 \exp\big[-(\log n_1)^{1/k}\big]\right) < \infty.$$

From the known result that

$$\sum_{n=1}^{\infty} \int f_n\,dP < \infty \Rightarrow \sum_{n=1}^{\infty} f_n < \infty, \quad \text{a.s.},$$

where $\{f_n\}$ is a sequence of positive random variables, it follows that, almost surely,

$$\sum_{n=n_0}^{\infty} \exp\big[\theta_n\gamma_n M_{n_1}^{(1)} - \log n_1 - (\log n_1)^{1/k}\big] < \infty,$$

which implies that

$$\limsup_{n\to\infty}\big[\theta_n\gamma_n M_{n_1}^{(1)} - \log n_1 - (\log n_1)^{1/k}\big] = -\infty,$$

and thus

$$\limsup_{n\to\infty}\big[M_{n_1}^{(1)} - 2\log n_1 - 2(\log n_1)^{1/k}\big] = -\infty. \tag{3.5}$$

Now let $M_{n_1}^{(2)}$ be as in (3.1). Then from the corollary to Lemma 3.1, by using arguments similar to those leading to (3.4), we have

$$(3.6)\quad \int \exp\left[\theta\gamma M_{n_1}^{(2)}\right] dP < \frac{n_1}{c(1-\theta)} \int_0^{\log n_1/n_1} d\omega \int \exp\left[\gamma \frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right] dP.$$

For $\gamma < \frac{1}{8}$, the right-hand side of (3.6) will be finite for sufficiently large n and an upper bound for it can be obtained as follows.

Let

$$U_n(\omega) = \frac{X_n^2(\omega)}{2\pi h(\omega)}, \qquad V_n(\omega) = \frac{Y_n^2(\omega)}{2\pi h(\omega)}.$$

Then

$$\begin{aligned}\exp\left[\gamma \frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right] &= \exp\left[\gamma\left(U_{n_1}(\omega) + V_{n_1}(\omega)\right)\right] \\ &\le \tfrac{1}{2}\left[\exp\left[2\gamma U_{n_1}(\omega)\right] + \exp\left[2\gamma V_{n_1}(\omega)\right]\right].\end{aligned}$$

Now writing

$$\sigma_{u,n}^2 = \operatorname{Var}\left(\frac{X_n(\omega)}{2\pi h(\omega)}\right),$$

we have

$$\begin{aligned}\int \exp\left[2\gamma U_{n_1}(\omega)\right] dP &= \frac{1}{\sqrt{2\pi\sigma_{u,n_1}^2}} \int_{-\infty}^{\infty} \exp\left[2\gamma x^2 - \frac{x^2}{2\sigma_{u,n_1}^2}\right] dx \\ &= \left(1 - 4\gamma\sigma_{u,n_1}^2\right)^{-1/2},\end{aligned}$$

provided that $4\gamma\sigma_{u,n_1}^2 < 1$. Also

$$\begin{aligned}\sigma_{u,n_1}^2 \le E\left[\frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right] &= \frac{2}{2\pi h(\omega)} \sum_{|u|\le n_1-1} \left(1 - \frac{|u|}{n_1}\right) r(u)\cos\omega u \\ &= 2 + O\left(\frac{\log n_1}{n_1}\right),\end{aligned}$$

uniformly for $0 \le \omega \le \pi$, so that

$$\liminf_{n_1\to\infty}\left(1 - 4\gamma\sigma_{u,n_1}^2\right) \ge 1 - 8\gamma.$$

Thus for sufficiently large n_1,

$$\int \exp\left[2\gamma U_{n_1}(\omega)\right] dP = O(1),$$

and similarly

$$\int \exp\left[2\gamma V_{n_1}(\omega)\right] dP = O(1),$$

uniformly in ω. Hence

$$\int \exp\left[\theta\gamma M_{n_1}^{(2)}\right] dP \le \frac{n_1}{c(1-\theta)} \int_0^{\log n_1/n_1} d\omega \int \exp\left[\frac{I_{n_1}(\omega)}{2\pi h(\omega)}\right] dP$$
$$= O\left(\frac{\log n_1}{1-\theta}\right).$$

So taking $\theta = \theta_n = 1 - n^{-m}$, $m > 0$, we have

$$\int \exp\left[\theta_n \gamma M_{n_1}^{(2)} - 2\gamma \log n_1 - 2\gamma(\log n_1)^{1/k}\right] dP$$
$$= O\left[(\log n_1)^{k+1} n^{m-2\gamma(\log n)^k}\right].$$

Since

$$\sum_{n=1}^{\infty} (\log n)^{k+1} n^{m-2\gamma(\log n)^k} < \infty,$$

it follows, by the same argument used to obtain (3.5), that

$$\limsup_{n\to\infty}\left[M_{n_1}^{(2)} - 2\log n_1 - 2(\log n_1)^{1/k}\right] = -\infty. \tag{3.7}$$

Let $M_{n_1}^{(3)}$ be as in (3.1). Then using exactly the same arguments as those leading to (3.7), it can be shown that, almost surely,

$$\limsup_{n\to\infty}\left[M_{n_1}^{(3)} - 2\log n_1 - 2(\log n_1)^{1/k}\right] = -\infty.$$

(3.3) now follows on observing that

$$M_{n_1} = \max\left[M_{n_1}^{(1)}, M_{n_1}^{(2)}, M_{n_1}^{(3)}\right]. \qquad \square$$

THEOREM 3.2. *For any $\delta > 0$, almost surely,*

$$\limsup_{n\to\infty}\left[M_n - 2\log n - 2(\log n)^{\delta}\right] = -\infty. \tag{3.8}$$

PROOF. Let k be a fixed integer and let $d(n)$ be the integer part of $\exp[(\log h(n))^{k+1}]$, where $h(n)$ is the integer part of $\exp[(\log n)^{1/(k+1)}]$.

Then

$$M_n = M_{d(n)} + M_n - M_{d(n)}$$
$$\le M_{d(n)} + |M_n - M_{d(n)}|$$
$$\le M_{d(n)} + M_n^*,$$

where

$$M_n^* = \max_{0\le\omega\le\pi}\left|\frac{I_n(\omega)}{2\pi h(\omega)} - \frac{I_{d(n)}(\omega)}{2\pi h(\omega)}\right|.$$

Let $u_n = 2\log n + 2(\log n)^{1/k}$. Then

$$M_n - u_n \le M_{d(n)} - u_{d(n)} + u_{d(n)} - u_n + M_n^*$$

and

$$\limsup_{n\to\infty}(M_n - u_n) \le \limsup_{n\to\infty}(M_{d(n)} - u_{d(n)}) + \limsup_{n\to\infty}(u_{d(n)} - u_n) + \limsup_{n\to\infty} M_n^*.$$

Note that $d(n)$ is related to $h(n)$ in the same way as n_1 is related to n. An argument analogous to that used to prove Lemma 3.2 can thus be applied to show that

$$\limsup_{n\to\infty}(M_{d(n)} - u_{d(n)}) = -\infty. \tag{3.9}$$

Also,

$$\limsup_{n\to\infty}(u_{d(n)} - u_n) = 0$$

and hence (3.8) follows upon showing that

$$\limsup_{n\to\infty} M_n^* = 0. \tag{3.10}$$

First note that $d(n) \le n$ and clearly we can suppose that $d(n) < n$.

Now

$$\left|\sum_{t=1}^{n} X_t e^{i\omega t}\right|^2 \le \left|\sum_{t=1}^{d(n)} X_t e^{i\omega t}\right|^2 + \left|\sum_{t=d(n)+1}^{n} X_t e^{i\omega t}\right|^2 + 2\left|\sum_{t=1}^{d(n)} X_t e^{i\omega t}\right|\left|\sum_{t=d(n)+1}^{n} X_t e^{i\omega t}\right|.$$

Thus

$$\begin{aligned}|I_n(\omega) - I_{d(n)}(\omega)| &\le \left|\frac{2}{n} - \frac{2}{d(n)}\right|\left|\sum_{t=1}^{d(n)} X_t e^{i\omega t}\right|^2 + \frac{2}{n}\left|\sum_{t=d(n)+1}^{n} X_t e^{i\omega t}\right|^2 \\ &\quad + \frac{4}{n}\left|\sum_{t=1}^{d(n)} X_t e^{i\omega t}\right|\left|\sum_{t=d(n)+1}^{n} X_t e^{i\omega t}\right|.\end{aligned} \tag{3.11}$$

Hence, on writing

$$M_{d(n),n} = \max_{0\le\omega\le\pi} \frac{2}{n - d(n)} \frac{\left|\sum_{t=d(n)+1}^{n} X_t e^{i\omega t}\right|^2}{2\pi h(\omega)}$$

and dividing both sides of (3.11) by $2\pi h(\omega)$,

$$M_n^* \le \left(1 - \frac{d(n)}{n}\right) M_{d(n)} + \left(1 - \frac{d(n)}{n}\right) M_{d(n),n} + 2\frac{\sqrt{d(n)}\sqrt{n-d(n)}}{n}\sqrt{M_{d(n)}}\sqrt{M_{d(n),n}}. \tag{3.12}$$

From (3.9) $M_{d(n)} = O(\log n)$ and also, $M_{d(n),n}$ has the same distribution as

$$\max_{0\le\omega\le\pi} \frac{2}{n-d(n)}\left|\sum_{t=1}^{n-d(n)} X_t e^{i\omega t}\right|^2$$

and we can show that almost surely as $n \to \infty$, $M_{d(n),n} = O(\log n)$ by applying arguments of the same type as those of Lemma 3.2. Then from (3.12),

$$M_n^* \le O\left(\left(1 - \frac{d(n)}{n}\right)\log n\right) + O\left(\left(1 - \frac{d(n)}{n}\right)^{1/2} \log n\right). \tag{3.13}$$

Now

$$\frac{d(n)}{n} = \exp\left[(\log h(n))^{k+1} - \log n\right],$$

and it is easily seen that

$$(\log h(n))^{k+1} - \log n = -\delta_n + O(\delta_n),$$

where

$$\delta_n = (k+1)(\log n)^{k/(k+1)} \exp\left[-(\log n)^{1/(k+1)}\right].$$

Hence

$$1 - \frac{d(n)}{n} = \delta_n + O(\delta_n),$$

and as $n \to \infty$,

$$\left(1 - \frac{d(n)}{n}\right)(\log n)^2 \to 0.$$

Hence (3.10) follows from (3.13). □

4. The lower bound. The proof of the lower bound (1.2) is slightly more complicated, so in order to give a greater degree of clarity, we first summarize the steps involved in the proof.

Let $(\omega_1, \omega_2, \dots, \omega_{m(n)})$ be the equally spaced partition of $[\varepsilon, \pi - \varepsilon]$ as defined in Lemma 2.2.

Let

$$M_{n,m(n)} = \max_{1 \le i \le m(n)} \left[\frac{X_n^2(\omega_i)}{\sigma_n^2(\omega_i)} + \frac{Y_n^2(\omega_i)}{\sigma_n^2(\omega_i)} \right],$$

$$M_{n,m(n)}^{(1)} = \max_{1 \le i \le m(n)} \frac{I_n(\omega_i)}{2\pi h(\omega_i)} \tag{4.1}$$

and

$$u_n = 2\log n - \log\log n.$$

1. Show that for sufficiently large n,

$$P(M_{n,m(n)} \le u_n) \le \prod_{i=1}^{m(n)} P\left(\left[\frac{X_n^2(\omega_i)}{\sigma_{X_n}^2(\omega_i)} + \frac{Y_n^2(\omega_i)}{\sigma_{Y_n}^2(\omega_i)} \right] \le u_n \right) + d_n$$

for some d_n such that $\Sigma_{n=1}^{\infty} d_n < \infty$ (Lemma 4.1).

2. Show that

$$\sum_{n=1}^{\infty} \prod_{i=1}^{m(n)} P\left(\left[\frac{X_n^2(\omega_i)}{\sigma_{X_n}^2(\omega_i)} + \frac{Y_n^2(\omega_i)}{\sigma_{Y_n}^2(\omega_i)} \right] \le u_n \right) < \infty$$

(Lemma 4.2).

3. Hence by the Borel–Cantelli lemma,

$$\liminf_{n \to \infty} \left[M_{n,m(n)} - u_n \right] \ge 0.$$

4. Show that $M_{n,m(n)}$ and $M_{n,m(n)}^{(1)}$ are asymptotically of the same order. (1.2) now follows on observing that, almost surely, $M_n \ge M_{n,m(n)}^{(1)}$ (Theorem 4.1).

As in Section 3, the above steps will first be proved for the particular subsequence n_1 and then it will be shown that the result in fact holds for any n. We now have a more detailed study of the steps involved.

LEMMA 4.1. *For sufficiently large* n,

$$\left| P(M_{n_1,m(n_1)} \le u_{n_1}) - \prod_{i=1}^{m(n_1)} P\left(\frac{X_{n_1}^2(\omega_i)}{\sigma_{X_{n_1}}^2(\omega_i)} + \frac{Y_{n_1}^2(\omega_i)}{\sigma_{Y_{n_1}}^2(\omega_i)} \le u_{n_1} \right) \right| \tag{4.2}$$

$$\le c e^{-(1-\delta)(\log n)^{k+1}}$$

for some constants $c > 0$ *and* $\delta \in (0,1)$.

PROOF. Let

$$Z_{n_1}(\omega_i) = \left\{ \frac{X_{n_1}(\omega_i)}{\sigma_{X_{n_1}}(\omega_i)}, \frac{Y_{n_1}(\omega_i)}{\sigma_{Y_{n_1}}(\omega_i)} \right\}$$

and

$$F_{n_1}(1) = P\left(\prod_{i=1}^{m(n_1)} \left(Z_{n_1}(\omega_i) \in A \right) \right),$$

where

$$A = \left\{ (x, y) : x^2 + y^2 \le u_{n_1} \right\}.$$

Then

$$F_{n_1}(1) = \int_A \cdots \int_A f_1(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y}, \tag{4.3}$$

where f_1 is $N(0, \Sigma_1)$, Σ_1 being the covariance matrix of the random variables

$$\{Z_{n_1}(\omega_i),\, i = 1, \dots, m(n_1)\} \tag{4.4}$$

and

$$\mathbf{x} = (x_1, \dots, x_{m(n_1)}), \qquad \mathbf{y} = (y_1, \dots, y_{m(n_1)}).$$

Also let

$$F_{n_1}(0) = \prod_{i=1}^{m(n_1)} P\left(Z_{n_1}(\omega_i) \in A \right) = \int_A \cdots \int_A f_0(\mathbf{x}, \mathbf{y})\, d\mathbf{x}, d\mathbf{y}, \tag{4.5}$$

where f_0 is $N(0, \Sigma_0)$, Σ_0 being of the form $\operatorname{diag}(V_1, \dots, V_{m(n_1)})$, where V_i is the matrix

$$\begin{bmatrix} 1 & r_{x_i y_i} \\ r_{x_i y_i} & 1 \end{bmatrix}.$$

Then, by using arguments similar to those given by Leadbetter, Lindgren and Rootzén (1983), pages 81–83, we have

$$\begin{aligned} |F_{n_1}(1) - F_{n_1}(0)| &\le \int_0^1 \left[\sum_{1 \le i < j \le m(n_1)} |r_{x_i, x_j}| \left| \int_A \cdots \int_A \frac{\partial f_h(\mathbf{x}, \mathbf{y})}{\partial x_i\, \partial x_j}\, d\mathbf{x}\, d\mathbf{y} \right| \right] dh \\ &\quad + \int_0^1 \left[\sum_{1 \le i < j \le m(n_1)} |r_{x_i, y_j}| \left| \int_A \cdots \int_A \frac{\partial f_h(\mathbf{x}, \mathbf{y})}{\partial x_i\, \partial y_j}\, d\mathbf{x}\, d\mathbf{y} \right| \right] dh \\ &\quad + \int_0^1 \left[\sum_{1 \le i < j \le m(n_1)} |r_{y_i, y_j}| \left| \int_A \cdots \int_A \frac{\partial f_h(\mathbf{x}, \mathbf{y})}{\partial y_i\, \partial y_j}\, d\mathbf{x}\, d\mathbf{y} \right| \right] dh \\ &= S_{xx}^{(n_1)} + S_{xy}^{(n_1)} + S_{yy}^{(n_1)}, \quad \text{say,} \end{aligned} \tag{4.6}$$

where f_h is $N(0, \Sigma_h)$, Σ_h being a covariance matrix of the form $\Sigma_h = h\Sigma_1 + (1 - h)\Sigma_0$, where $0 < h < 1$, and $r_{x_i, x_j}, r_{x_i, y_j}, r_{y_i, y_j}$ are, respectively, the correlation coefficients of the pairs $\{X_{n_1}(\omega_i), X_{n_1}(\omega_j)\}$, $\{X_{n_1}(\omega_i), Y_{n_1}(\omega_j)\}$, $\{Y_{n_1}(\omega_i), Y_{n_1}(\omega_j)\}$.

We now obtain an upper bound for $S_{xx}^{(n_1)}$. First we note that we can write

$$\left|\int_A \cdots \int_A \frac{\partial f_h(\mathbf{x},\mathbf{y})}{\partial x_i\, \partial x_j}\, d\mathbf{x}\, d\mathbf{y}\right| \tag{4.7}$$
$$= \left|\int_A \cdots \int_A \left[\int_A \int_A \frac{\partial f_h(\mathbf{x},\mathbf{y})}{\partial x_i\, \partial x_j}\, dx_i\, dy_i\, dx_j\, dy_j\right] d\mathbf{x}^*\, d\mathbf{y}^*\right|,$$

where $\mathbf{x}^*, \mathbf{y}^*$ are $(m(n_1) - 2)$-dimensional vectors.

On integrating with respect to x_i and x_j in the inner integral, we see that the right-hand side of (4.7) is equal to

$$\left|\int_A \cdots \int_A \left[J_{n_1,1} - J_{n_1,2} - J_{n_1,3} + J_{n_1,4}\right] d\mathbf{x}^*\, d\mathbf{y}^*\right|, \tag{4.8}$$

where

$$J_{n_1,1} = \int_{|y_i| \le \sqrt{u_{n_1}}} \int_{|y_j| \le \sqrt{u_{n_1}}} f_h\left(\sqrt{u_{n_1} - y_i^2}, \sqrt{u_{n_1} - y_j^2}, \mathbf{x}^*, y_i, y_j, \mathbf{y}^*\right) dy_i\, dy_j$$

and $J_{n_1,2}, J_{n_1,3}, J_{n_1,4}$ are similar integrals except that $(\sqrt{u_{n_1} - y_i^2}, \sqrt{u_{n_1} - y_j^2})$ are, respectively, replaced by

$$\left(-\sqrt{u_{n_1} - y_i^2}, \sqrt{u_{n_1} - y_j^2}\right), \left(\sqrt{u_{n_1} - y_i^2}, -\sqrt{u_{n_1} - y_j^2}\right)$$

and

$$\left(-\sqrt{u_{n_1} - y_i^2}, -\sqrt{u_{n_1} - y_j^2}\right).$$

Now clearly

$$\int_A \cdots \int_A J_{n_1,i}\, d\mathbf{x}^*\, d\mathbf{y}^* \le \int_{R_2} \cdots \int_{R_2} J_{n_1,i}\, d\mathbf{x}^*\, d\mathbf{y}^*, \qquad i = 1, \dots, 4,$$

where

$$R_2 = \{(x,y): -\infty < x < \infty,\ -\infty < y < \infty\}.$$

Thus (4.8) is not greater than

$$\sum_{1 \le j \le 4} J_{n_1,j}^*,$$

where

$$J_{n_1,1}^* = \int_{|y_i| \le \sqrt{u_{n_1}}} \int_{|y_j| \le \sqrt{u_{n_1}}} \tilde{f}_h\left(\sqrt{u_{n_1} - y_i^2}, \sqrt{u_{n_1} - y_j^2}, y_i, y_j\right) dy_i\, dy_j,$$

$\tilde{f}_h$ being the marginal density of

$$\left\{\frac{X_{n_1}(\omega_i)}{\sigma_{X_{n1}}(\omega_i)}, \frac{X_n(\omega_j)}{\sigma_{X_n}(\omega_j)}, \frac{Y_{n_1}(\omega_i)}{\sigma_{Y_{n1}}(\omega_i)}, \frac{Y_n(\omega_j)}{\sigma_{Y_n}(\omega_j)}\right\}$$

under $N(0, \Sigma_h)$ and $J^*_{n_1,2}, J^*_{n_1,3}, J^*_{n_1,4}$ are similarly defined. The covariance matrix associated with $\tilde{f}_h$ is of the form $I_4 + O((\log n_1)^{-\eta})$, I_4 being the four-dimensional identity matrix. Hence

$$\tilde{f}_h\left(\sqrt{u_{n_1} - y_i^2}, \sqrt{u_{n_1} - y_j^2}, y_i, y_j\right) = \frac{1}{(2\pi)^2}\left[1 + O\left((\log n_1)^{-\eta}\right)\right] e^{-u_{n_1} + O(u_{n_1}(\log n_1)^{-\eta})},$$

uniformly for $|y_i| \le \sqrt{u_{n_1}}$, $|y_j| \le \sqrt{u_{n_1}}$. Thus

$$J^*_{n_1,1} \le c_1 u_{n_1} e^{-u_{n_1} + k_1 u_{n_1}(\log n_1)^{-\eta}},$$

k_1 and c_1 being positive constants. With similar arguments, bounds of the same form for $J^*_{n_1,i}$, $i = 2, 3, 4$, can be obtained.

Also, since $u_{n_1} = 2\log n_1 - \log\log n_1 < 2\log n_1$,

$$J^*_{n_1,i} \le 2c_1 n_1^{-2} (\log n_1)^2 e^{u_{n_1}/k_2},$$

where k_2 may be arbitrarily large. Hence

$$J^*_{n_1,i} \le 2c_1 n_1^{-2+\delta}, \tag{4.9}$$

where $\delta > 0$ may be arbitrarily small.

From (4.9), result [1] of (2.6) in Lemma 2.2 and (4.6), we see that for sufficiently large n_1,

$$\begin{aligned} S_{xx}^{(n_1)} \le C_1 \sum_{1 \le i < j \le m(n_1)} \frac{1}{|i-j|} (\log n_1)^{-\eta} n_1^{-2+\delta} &\le C_1 n_1^{-1+\delta} \sum_{1 \le u \le m(n_1)} \frac{1}{u} \\ &\le C_1 n_1^{-1+\delta}(1 + \log n_1) \\ &\le C_1 n_1^{-1+\delta^*}, \end{aligned}$$

where $\delta < \delta^* < 1$, since we may assume that $1 + \log n_1 \le n_1^{\delta^* - \delta}$.

Hence

$$\begin{aligned} S_{xx}^{(n_1)} &\le C_1 e^{-(1-\delta^*)(\log n)^{k+1}}[1 + O(1)] \\ &\le c e^{-(1-\delta^*)(\log n)^{k+1}}. \end{aligned} \tag{4.10}$$

With similar arguments, it can be shown that $S_{xy}^{(n_1)}$, $S_{yy}^{(n_1)}$ have bounds of the same form as (4.10). (4.2) follows on observing that its left-hand side is equal to $|F_{n_1}(1) - F_{n_1}(0)|$. □

LEMMA 4.2. *Let $k \ge 2$ and $\eta < \frac{1}{2} - 1/(k+1)$, so that $(k+1)(\frac{1}{2} - \eta) > 1$. Then, almost surely,*

$$\liminf_{n\to\infty}\left[M_{n_1, m(n_1)} - u_{n_1}\right] \ge 0. \tag{4.11}$$

PROOF. From (4.2) we have, for sufficiently large n,

$$P\left(M_{n_1, m(n_1)} \le u_{n_1}\right) \le \prod_{i=1}^{m(n_1)} P\left(\frac{X_{n_1}^2(\omega_i)}{\sigma_{X_{n_1}}^2(\omega_i)} + \frac{Y_{n_1}^2(\omega_i)}{\sigma_{Y_{n_1}}^2(\omega_i)} \le u_{n_1}\right) + c e^{-(1-\delta)(\log n)^{k+1}},$$

where c, δ are constants such that $c > 0$ and $0 < \delta < 1$. Using the fact that $\{X_n(\omega), Y_n(\omega)\}$ has a normal distribution and the fact that $\operatorname{Cov}(X_n(\omega), Y_n(\omega)) = O(n^{-1})$ uniformly in $\omega \in [\varepsilon, \pi - \varepsilon]$, we have

$$\begin{aligned}
&\prod_{i=1}^{m(n_1)} P\left(\frac{X_{n_1}^2(\omega_i)}{\sigma_{X_{n_1}}^2(\omega_i)} + \frac{Y_{n_1}^2(\omega_i)}{\sigma_{Y_{n_1}}^2(\omega_i)} \le u_{n_1}\right)\\
&\quad= \prod_{i=1}^{m(n_1)} \left[1 - e^{-(1/2)u_{n_1}}(1 + o(1))\right]\\
&\quad\le \exp\left[-m(n_1)e^{-(1/2)u_{n_1}}(1 + o(1))\right]\\
&\quad= \exp\left[-\frac{n_1}{(\log n_1)^{\eta}} e^{-\log n + (1/2)\log\log n_1}(1 + o(1))\right]\\
&\quad\le \exp\left[-(\log n_1)^{1/2-\eta}(1 + o(1))\right].
\end{aligned} \tag{4.12}$$

Since

$$\sum_{n=1}^{\infty} e^{-(1-\delta)(\log n)^{k+1}} < \infty$$

and

$$\sum_{n=1}^{\infty} e^{-(\log n)^{(k+1)(1/2-\eta)}}(1 + o(1)) < \infty,$$

when $(k + 1)(\frac{1}{2} - \eta) > 1$, it follows from (4.2) and (4.12) that

$$\sum_{n=1}^{\infty} P(M_{n_1, m(n_1)} \le u_{n_1}) < \infty,$$

which implies (4.11) by the Borel–Cantelli lemma. □

THEOREM 4.1. *Let*

$$M_n = \max_{\omega \in [0, \pi]} \frac{I_n(\omega)}{2\pi h(\omega)}.$$

Then

$$\liminf_{n \to \infty} [M_n - 2\log n + \log\log n] \ge 0. \tag{4.13}$$

PROOF. Let

$$M_{n_1, m(n_1)}^{(1)} = \max_{1 \le i \le m(n_1)} \frac{I_{n_1}(\omega_i)}{2\pi h(\omega_i)}.$$

Since

$$\sigma_{X_{n_1}}^2(\omega_i) = 2\pi h(\omega_i) + O\left(\frac{\log n_1}{n_1}\right),$$

$$\sigma_{Y_{n_1}}^2(\omega_i) = 2\pi h(\omega_i) + O\left(\frac{\log n_1}{n_1}\right),$$

we have

$$\frac{X_{n_1}^2(\omega_i)}{\sigma_{X_{n_1}}^2(\omega_i)} + \frac{Y_{n_1}^2(\omega_i)}{\sigma_{Y_{n_1}}^2(\omega_i)} = \frac{I_{n_1}(\omega_i)}{2\pi h(\omega_i)}\left(1 + O\left(\frac{\log n_1}{n_1}\right)\right),$$

giving

$$M_{n_1, m(n_1)}^{(1)}\left(1 + O\left(\frac{\log n_1}{n_1}\right)\right) = M_{n_1, m(n_1)}. \tag{4.14}$$

Now let $(k+1)(\frac{1}{2} - \eta) > 1$, so that (4.11) holds. Then from (4.14),

$$\lim_{n\to\infty}\left[M_{n_1, m(n_1)}^{(1)} - u_{n_1}\right] \geq 0.$$

Clearly

$$M_{n_1} \geq M_{n_1, m(n_1)}^{(1)}.$$

Hence

$$\liminf_{n\to\infty}\left[M_{n_1} - u_{n_1}\right] \geq 0. \tag{4.15}$$

Let $d(n)$ and M_n^* be defined as in Lemma 3.3. Then, as $|M_n - M_{d(n)}| \geq M_n^*$,

$$M_n = M_{d(n)} + M_n - M_{d(n)} \geq M_{d(n)} - M_n^*.$$

Hence, if $u_n = 2\log n - \log\log n$, then

$$M_n - u_n \geq M_{d(n)} - u_{d(n)} + u_{d(n)} - u_n - M_n^*$$

and

$$\liminf_{n\to\infty}(M_n - u_n) \geq \liminf_{n\to\infty}(M_{d(n)} - u_{d(n)}) + \liminf_{n\to\infty}(u_{d(n)} - u_n) + \liminf_{n\to\infty}(-M_n^*).$$

Since $d(n)$ is related to $h(n)$ in the same way as n_1 is related to n, an argument analogous to that used to obtain (4.15) can be applied to show that, almost surely,

$$\liminf_{n\to\infty}(M_{d(n)} - u_{d(n)}) \geq 0.$$

Also from (3.10) we have

$$\liminf_{n\to\infty}(-M_n^*) = 0,$$

and it is easily seen that

$$\lim_{n\to\infty}(u_{d(n)} - u_n) = 0.$$

Hence

$$\liminf_{n\to\infty}(M_n - u_n) \geq 0. \qquad \square$$

Some further comments. Let $\{X_t\}_{t=0}^{\infty}$ be a stationary Gaussian time series with $E(X_t) = 0$, $E(X_t^2) = 1$ and the autocorrelation function $r(u)$ not necessarily absolutely summable. Then we have the following conjecture: If

$$M_n = \max_{\omega \in [0,\pi]} \frac{I_n(\omega)}{\sum_{|u| \le n-1}(1 - |u|/n) r(u) \cos \omega u},$$

then, almost surely,

$$\lim_{n \to \infty} \frac{M_n}{2 \log n} = 1.$$

The proof of the above conjecture can probably be given in a similar way. However, there would be quite a few technical complications.

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