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LOWER BOUNDS ON THE CONNECTIVITY FUNCTION IN ALL DIRECTIONS FOR BERNOULLI PERCOLATION IN TWO AND THREE DIMENSIONS[1]

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The probability $P[0 \leftrightarrow x]$ of connection of 0 to x by a path of occupied bonds for Bernoulli percolation at density p below the critical point is known to decay exponentially for each direction $x \in \mathbb{Z}^d$, in that $P[0 \leftrightarrow nx] \approx e^{-n\sigma g(x)}$ as $n \to \infty$ for some $\sigma > 0$ and $g(x)$ of order $\|x\|$. This approximation is also an upper bound: $P[0 \leftrightarrow x] \le e^{-\sigma g(x)}$ for all x. Here a complementary power-law lower bound is established for $d = 2$ and 3: $P[0 \leftrightarrow x] \ge c\|x\|^{-r}e^{-\sigma g(x)}$ for some $r = r(d)$ and $c = c(p, d)$.

1. Preliminaries. Let us consider Bernoulli bond percolation on the d-dimensional integer lattice, with $d = 2$ or 3. Elements of $\mathbb{Z}^d$ are called *sites*; sites x and y are *adjacent* if $\|x - y\|_1 = 1$. The corresponding *bonds* (i.e., pairs of adjacent sites) are independently occupied with probability p and vacant with probability $1 - p$. The *cluster* $C(x)$ of a site x consists of those sites y such that x is connected to y by a path of occupied bonds, an event denoted by $x \leftrightarrow y$. Broadbent and Hammersley (1957) showed that there is a critical probability $0 < p_c(d) < 1$ such that $P_\infty(p) := P_p[|C(0)| = \infty] = 0$ if $p < p_c(d)$ and is positive if $p > p_c(d)$; here $|A|$ denotes the number of sites in a subset A of $\mathbb{R}^d$. The function

$$\tau_{xy} = \tau_{xy}(p) := P_p[x \leftrightarrow y] \tag{1.1}$$

is called the *connectivity function*. Our interest here is in lower bounds for this function, when $p < p_c(d)$.

Throughout this paper we will be working with a fixed but arbitrary p, so we will frequently suppress the p in our notation, as in (1.1). Also, our results are valid for more general lattices, but we will restrict ourselves to the integer lattice to keep the exposition simple.

In the nonpercolating phase $p < p_c(d)$, it is known that $P_p[0 \leftrightarrow ne_1]$ decays exponentially in n:

$$P_p[0 \leftrightarrow ne_1] \approx e^{-\sigma n} \quad \text{for some } 0 < \sigma(p) < \infty, \tag{1.2}$$

where $a_n \approx b_n$ means the ratio of the logarithms converges to 1 and e_i denotes the ith coordinate vector in $\mathbb{R}^d$. $\sigma(p)$ is of course the inverse of the correlation length. (1.2) is due to Hammersley (1957) when $E_p|C(0)| < \infty$; that this is

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equivalent to $p < p_c(d)$ is due to Aizenman and Barsky (1987) and independently to Men'shikov, Molchanov and Sidorenko (1986). There is also exponential decay in off-axis directions: Given $x \in \mathbb{Q}^d$, we may let $n \to \infty$ through those values for which $nx \in \mathbb{Z}^d$ and obtain

$$(1.3) \qquad P_p[0 \leftrightarrow nx] \approx e^{-\sigma g(x)n} \quad \text{for some } g(x) = g^{(p)}(x).$$

The special case

$$(1.4) \qquad P_p[0 \leftrightarrow n(x+y)] \geq P_p[0 \leftrightarrow nx]P_p[0 \leftrightarrow ny]$$

of the Harris–FKG inequality [Harris (1960)], along with (1.2) and the normalization by σ, ensure that

$$(1.5) \qquad \|x\|_\infty \leq g(x) \leq \|x\|_1 \quad \text{and} \quad g(e_1) = 1.$$

In Alexander, Chayes and Chayes (1989), it is shown that g is convex, continuous and homogeneous and extends by continuity to a norm on $\mathbb{R}^d$ equivalent to the Euclidean norm. The convexity is a consequence of (1.4) and homogeneity. The subadditivity of $-\log P_p[0 \leftrightarrow nx]$ in n, also a consequence of the Harris–FKG inequality, leads to the a priori upper bound

$$(1.6) \qquad P_p[0 \leftrightarrow x] \leq e^{-\sigma g(x)} \quad \text{for all sides } x.$$

For x on an axis, say $x = ne_1$, a standard short argument (see Lemma 2.1) making strong use of symmetry about the axis shows that for all dimensions d,

$$(1.7) \qquad P_p[0 \leftrightarrow x] \geq cn^{-2(d-1)}e^{-\sigma n} = c\|x\|^{-2(d-1)}e^{-\sigma g(x)}.$$

(Here and throughout this paper $c, c_1, c_2, \dots$ stand for unspecified constants which may depend on p and d. $\|\cdot\|$ denotes the Euclidean norm.) In Campanino, Chayes and Chayes (1988) a highly complex argument shows that for all x "near the axis" in the sense that $x = (n, a)$ for some $a \in \mathbb{Z}^{d-1}$ with $\|a\| \leq n^{3/4-\varepsilon}$, one has

$$(1.8) \qquad \begin{aligned} &P_p[0 \leftrightarrow (n,a)] \\ &\quad = c_1 n^{-(d-1)/2} e^{-\|a\|^2/c_2 n} e^{-\sigma n}\big(1 + O(\max(n^{-1}, n^{-4\varepsilon}))\big), \end{aligned}$$

with the O uniform in a. Again, symmetry is a crucial element of the proof.

Now (1.8) shows that the proper form for a lower bound on $P_p[0 \leftrightarrow x]$ is the upper bound (1.6) multiplied by a negative power of $\|x\|$, i.e., a power law lower bound. Ornstein and Zernike (1914) predicted for certain models that the analog of τ_{0x} should behave like $e^{-m\|x\|}/\|x\|^{(d-1)/2}$ for some constant m. For self-avoiding random walk, for x near an axis, such behavior was established by Chayes and Chayes (1986b). For general models at high temperatures, see Bricmont and Fröhlich (1985a, b).

For general off-axis x, there is no symmetry but the existence of good lower bounds remains a natural question. In dimension $d = 2$, such bounds could be used to obtain information about the shape and the probability of large finite clusters in the percolating phase, roughly as follows, as is discussed in

Alexander, Chayes and Chayes (1989). Let $\mathscr{W} = \mathscr{W}(p)$ denote the minimal $g^{(1-p)}$-length of any loop in the plane enclosing unit area and let $W = W(p)$ denote the region (unique up to translation) whose boundary achieves this minimum. Then for each $p > \frac{1}{2} = p_c(2)$,

$$P_p[N \le |C(0)| < \infty] \\ = \exp\left(-\sigma(1-p)\mathscr{W}(p)P_\infty(p)^{-1/2}N^{1/2}(1+\varepsilon(N))\right) \tag{1.9}$$

for some $\varepsilon(N) \to 0$ as $N \to \infty$. Furthermore, conditionally on $[N \le |C(0)| < \infty]$, with probability approaching 1 as $N \to \infty$, the shape of $C(0)$ approximates W to within a factor of $1 + \eta(N)$ for some $\eta(N) \to 0$. Good lower bounds on the connectivity function facilitate estimates of the errors $\eta(N)$ and $\varepsilon(N)$. The relevant connections are made by dual bonds, which are in the nonpercolating phase when $p > \frac{1}{2}$. An analogous problem for the Ising magnet at very low temperatures, in which a droplet of one phase is immersed in another phase and takes on a near-deterministic shape and the corresponding error $\eta(N)$ is estimated, has been investigated by Dobrushin, Kotecky and Shlosman (1989) and Shlosman (1989).

With this motivation in mind, let us state our main result.

THEOREM 1.1. *For Bernoulli bond percolation on the integer lattice in $d = 2$ and 3 dimensions, there exist positive finite constants $c = c(p, d)$ and $r = r(d)$ such that for all $x \in \mathbb{Z}^d$ and $p < p_c(d)$,*

$$P_p[0 \leftrightarrow x] \ge c\|x\|^{-r}e^{-\sigma g(x)}. \tag{1.10}$$

Further, $r(2) \le 420$ *and* $r(3) \le 2328$.

Theorem 1.1 will be proved in Section 2. Of course one could as well replace $\|x\|^{-r}$ with $g(x)^{-r}$ in (1.10), by (1.5).

Our bounds for $r(2)$ and $r(3)$ are obviously crude, even more so that (1.7); (1.8) suggests the right value of $r(d)$ may be $(d-1)/2$. Thus far, this crudeness seems to be the price we pay for leaving the symmetry-induced comfort of the region near the axes. Further, we have no bounds in dimension 4 and higher, but this seems potentially more repairable: The only use of the assumption $d \le 3$ is in the purely geometric Proposition 2.7, which we suspect is true for all d but have only been able to prove for low dimensions.

Let U_g denote the unit ball of the norm g in $\mathbb{R}^d$. Let H_x denote a hyperplane tangent to $g(x)U_g$ at x. Combining Theorem 1.1 with the Campanino, Chayes and Chayes (1988) result (1.8), we will prove the following.

COROLLARY 1.2. *Let $d = 2$ or 3 and let ζ denote an arbitrary point of $\mathbb{R}^{d-1}$, so $(1, \zeta) \in H_{e_1}$. For each $0 < p < p_c(d)$, for $c_2(p, d)$ as in* (1.8),

$$g(1,\zeta) = 1 + (\sigma c_2)^{-1}\|\zeta\|^2 + o\left(\|\zeta\|^2\right) \quad \text{as } \zeta \to 0.$$

Thus ∂U_g cannot have zero or infinite curvature at e_1.

2. Proof of the theorem. Throughout this section p is fixed but arbitrary with $p < p_c(d)$.

Let us begin with a result along the coordinate axes. Though it is much weaker that (1.8), we include it here because it clearly illustrates the role of symmetry in results along the axes. A slightly weaker result appears in Grimmett (1989), and very analogous proofs have appeared for other systems [see Chayes and Chayes (1986a)]. The proof here was provided by J. T. Chayes and L. Chayes.

LEMMA 2.1. *For all* $0 < p < p_c(d)$, *for some* $c_3 = c_3(p, d)$ *and all* $n \geq 1$,

$$P_p[0 \leftrightarrow ne_1] \geq c_3 n^{-2(d-1)} e^{-\sigma n}.$$

PROOF. Let $H(n) := H_{ne_1}$ and $G(n) := \sum_{y \in H(n)} P_p[0 \leftrightarrow y]$. It follows from the Hammersley–Simon inequality [Hammersley (1957); Simon (1980)] that $\log G(n)$ is subadditive and from (1.6) and (1.3) that $G(n) \approx e^{-\sigma n}$. Hence

$$G(n) \geq e^{-\sigma n}.$$

Now

$$\sum_{y \in H(n) \cap (2nU_g)^c} P_p[0 \leftrightarrow y] = o(e^{-\sigma n}) \quad \text{as } n \to \infty$$

and $|H(n) \cap (2nU_g)| \leq c_4 n^{(d-1)}$. Therefore, for large n, there exists $y_n \in H(n) \cap (2nU_g)$ with $P_p[0 \leftrightarrow y_n] \geq c_5 n^{-(d-1)} e^{-\sigma n}$. But from symmetry, since every lattice path $0 \leftrightarrow y_n$ is a reflection through $H(n)$ of a lattice path $y_n \leftrightarrow 2ne_1$, we have $P_p[y_n \leftrightarrow 2ne_1] = P_p[0 \leftrightarrow y_n]$. Hence from the Harris–FKG inequality [Harris (1960)],

$$P_p[0 \leftrightarrow 2ne_1] \geq P_p[0 \leftrightarrow y_n] P_p[y_n \leftrightarrow 2ne_1] \geq c_6 n^{-2(d-1)} e^{-2\sigma n}.$$

The lemma now follows easily. □

This proof does not work for general off-axis directions because the reflection of a lattice path through a general plane does not result in a lattice path.

Here is a rough outline of the main ideas in the proof of Theorem 1.1. It is a useful standard heuristic to think of $P_p[0 \leftrightarrow y]$, or sometimes of $|\log P_p[0 \leftrightarrow y]|$, as the cost of a path from 0 to y. Given a self-avoiding path from 0 to nx which contributes to $P_p[0 \leftrightarrow nx]$, we may divide it at some vertices v_i into n or more segments of length of order $\|x\|$. These vertices form a *skeleton* of the path; conversely given a skeleton a path to nx is formed if each consecutive pair of vertices is connected by occupied bonds. Most of the probability $P_p[0 \leftrightarrow nx]$ is shown to come from skeletons of $3n + 1$ or fewer vertices, with corresponding segments each of reasonable cost. For fixed x, there are only finitely many possible values of the increments $v_{i+1} - v_i$ and x is in a multiple (at most 3) of the convex hull of these values. Therefore some $d + 1$ of these values, say $y_1, \dots, y_{d+1}$, satisfy $\sum_{i=1}^{d+1} \alpha_i y_i = x$ for some $\alpha_i \geq 0$ with $\sum_{i=1}^{d+1} \alpha_i \leq 3$. The method of selecting a skeleton ensures each cost $P_p[0 \leftrightarrow y_i]$

is reasonable; using a purely geometric fact about curves in $\mathbb{R}^d$, this is shown to imply that each cost $P_p[0 \leftrightarrow \alpha_i y_i]$ is also reasonable. This enables us to construct a path $0 \leftrightarrow \alpha_1 y_1 \leftrightarrow \alpha_1 y_1 + \alpha_2 y_2 \leftrightarrow \cdots \leftrightarrow \sum_{i=1}^{d+1} \alpha_i y_i = x$, still at a reasonable cost, which proves the theorem.

We need to quantify the notion of cost that we will use. In view of (1.6), one might think of $e^{-\sigma g(y)}$ as the base cost of a path from 0 to y and the ratio $P_p[0 \leftrightarrow y]/e^{-\sigma g(y)}$ (or its logarithm) as a surcharge for inefficiency in reaching y. For our purposes it is best to modify this somewhat to take account of the fact that we are interested in the efficiency of a path to any y in terms of the progress it makes toward a specific site x. Therefore, we replace g above with a linear functional g_x on $\mathbb{R}^d$ which measures such progress, defined as follows. Recall that H_x denotes a hyperplane tangent to $g(x)U_g$ at x and let H_x^0 be the hyperplane through 0 parallel to H_x. Define g_x on $\mathbb{R}^d$ by

$$g_x(x) = g(x), \qquad g_x = 0 \quad \text{on } H_x^0 \quad \text{and} \quad g_x \text{ is linear}.$$

Note that by convexity and homogeneity of g,

$$|g_x(y)| \le g(y) \quad \text{for all } y \in \mathbb{R}^d. \tag{2.1}$$

Then define the *x-surcharge function* s_x on $\mathbb{Z}^d$ by the expression

$$P_p[0 \leftrightarrow y] = e^{-s_x(y)} e^{-\sigma g_x(y)}. \tag{2.2}$$

By (1.6) and (2.1), s_x is nonnegative. From the Harris–FKG inequality [cf. (1.4) with $n = 1$] we have

$$s_x(y + z) \le s_x(y) + s_x(z) \quad \text{for all } y, z \in \mathbb{Z}^d, \tag{2.3}$$

while from (1.6),

$$s_x(y) + \sigma g_x(y) \ge \sigma g(y) \quad \text{for all } y \in \mathbb{Z}^d. \tag{2.4}$$

The property (2.3) is the concrete reason for linearizing g before defining the surcharge; it ensures that when paths from 0 to y and 0 to z are strung together to make a path from 0 to $y + z$, the "reasonableness" (i.e., low surcharge) of the costs $P_p[0 \leftrightarrow y]$ and $P_p[0 \leftrightarrow z]$ implies the reasonableness of $P_p[0 \leftrightarrow y + z]$.

It is clear that

$$|2g(x)U_g| \le c_7 \|x\|^d \quad \text{for all } x, \text{ for some } c_7(d). \tag{2.5}$$

Let us define η_x by

$$e^{-\eta_x} = \left(c_7 \|x\|^d\right)^{-1}. \tag{2.6}$$

Let

$$Q_x := \{y \in \mathbb{Z}^d : s_x(y) \le 4\eta_x \text{ and } g_x(y) \le g(x)\}$$

be the set of sites of reasonable cost which are not beyond x in the g_x direction and let

$$B_x := \{y \in \mathbb{Z}^d : y \notin Q_x, \ y \text{ adjacent to } Q_x\}$$

be its boundary. Given a self-avoiding lattice path γ from 0 to any site z, we can now define its *x-cost skeleton*, a finite sequence (v_i) of sites in γ, iteratively as follows: Let $v_0 := 0$ and let v_{i+1} be the first site in γ after v_i which is not in the translate $v_i + Q_x$. If there is no such site after v_i, then let $v_{i+1} = z$ and end the construction. Clearly each increment $v_{i+1} - v_i$, except possibly the last, is in B_x. (The last increment is in Q_x and may even be 0.) Let

$$\delta = \delta(p) := \log 1/p.$$

LEMMA 2.2. (i) *If* $\|x\| \geq c_8$, *then* $B_x \subset 2g(x)U_g$.
(ii) *If* $y \in B_x$, *then* $-2\sigma^{-1}\eta_x - 1 \leq g_x(y) \leq g(x) + 1$ *and* $s_x(y) \leq 4\eta_x + \delta$.

PROOF. (i) Suppose $y \notin (3g(x)/2)U_g$ and $g_x(y) \leq g(x)$. Then

$$e^{-s_x(y)}e^{-\sigma g(x)} \leq e^{-s_x(y)}e^{-\sigma g_x(y)} = P[0 \leftrightarrow y] \leq e^{-\sigma g(y)} \leq e^{-3\sigma g(x)/2},$$

so $s_x(y) \geq \sigma g(x)/2 > 4\eta_x$, provided $\|x\| \geq c_8$. Thus $Q_x \subset (3g(x)/2)U_g$ from which (i) follows easily.

(ii) If $y \in B_x$, then $y = z + e_i$ for some $z \in Q_x$ and $i \leq d$ so

$$g_x(y) = g_x(z) + g_x(e_i) \leq g(x) + g(e_i) = g(x) + 1.$$

By (2.1) and (2.4),

$$2\sigma g_x(z) \geq \sigma g_x(z) - \sigma g(z) \geq -s_x(z) \geq -4\eta_x, \tag{2.7}$$

so that

$$g_x(y) \geq g_x(z) - g(e_i) \geq -2\sigma^{-1}\eta_x - 1.$$

The bound on s_x follows from (2.3) and

$$p \leq P[0 \leftrightarrow e_i] \leq e^{-s_x(e_i)},$$

which shows $s_x(e_i) \leq \delta$. □

Given a cost skeleton $(v_0, \dots, v_m)$, abbreviated (v_i), we may divide the corresponding increments into two classes, according to which of the conditions defining Q_x is violated: the short increments

$$S((v_i)) := \{i: i < m - 1, g_x(v_{i+1} - v_i) \leq g(x)\}$$

corresponding to segments which ended because the surcharge s_x exceeded $4\eta_x$ and the full-length increments

$$L((v_i)) := \{i: i < m - 1, g_x(v_{i+1} - v_i) > g(x)\}$$

corresponding to segments which ended because they were long enough in the g_x direction. Note that the final increment is in neither class. Let Γ^x be the set of all x-cost skeletons and

$$\Gamma_{jk}^x(z) := \{(v_0,\dots,v_m) : (v_0,\dots,v_m) \in \Gamma^x,\ v_m = z,\ |S((v_i))| = j,\ |L((v_i))| = k\}$$

for $j, k \geq 0$.

We want to show that most of the probability of a path from 0 to nx comes from x-cost skeletons of $3n+1$ or fewer vertices for n large. For this we need the following special case of the van den Berg–Kesten (1985) inequality:

$$\begin{aligned} &P_p[0 \leftrightarrow z \text{ via a self-avoiding lattice path} \\ &\qquad \text{with } x\text{-cost skeleton } (v_0,\dots,v_m)] \\ &\qquad \leq \prod_{i=0}^{m-1} P_p[0 \leftrightarrow v_{i+1} - v_i]. \end{aligned} \tag{2.8}$$

LEMMA 2.3. *If* $\|x\| \geq c_9$, *then*

$$\begin{aligned} &P_p[0 \leftrightarrow nx \text{ via a self-avoiding lattice path with } x\text{-cost skeleton} \\ &\qquad \text{of more than } 3n+1 \text{ vertices}] = o(P_p[0 \leftrightarrow nx]) \quad \text{as } n \to \infty. \end{aligned} \tag{2.9}$$

PROOF. By (2.8), the probability on the left side of (2.9) is bounded above by

$$\begin{aligned} &\sum_{j+k \geq 3n} P_p\big[0 \leftrightarrow nx \text{ via a self-avoiding lattice path} \\ &\qquad \text{with } x\text{-cost skeleton } (v_i) \in \Gamma_{jk}^x(nx)\big] \\ &\leq \sum_{j+k\geq 3n} \sum_{(v_i)\in\Gamma_{jk}^x(nx)} \prod_{i=0}^{j+k} P_p[0 \leftrightarrow v_{i+1} - v_i] \\ &\leq \sum_{j+k\geq 3n} \sum_{(v_i)\in\Gamma_{jk}^x(nx)} \left(\prod_{i \notin L((v_i))} e^{-s_x(v_{i+1}-v_i)} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\ &\qquad \times \left(\prod_{i\in L((v_i))} e^{-\sigma g_x(v_{i+1}-v_i)} \right), \end{aligned} \tag{2.10}$$

while by (1.3) the probability on the right side of (2.9) can be bounded below:

$$P_p[0 \leftrightarrow nx] \geq 2^{-n} e^{-\sigma n g(x)} \quad \text{for } n \text{ large}. \tag{2.11}$$

We will bound (2.10) in three parts, according to the number of full-length increments: $k \geq 3n$, $n \leq k < 3n$ and $k < n$. Note that by (2.5), (2.6) and Lemma 2.2(i),

$$|\Gamma_{jk}^x(nx)| \leq |B_x|^{j+k} \leq e^{(j+k)\eta_x} \quad \text{for all } \|x\| \geq c_8. \tag{2.12}$$

Next, for $\|x\| \geq$ some c_{12}, similar calculations give

$$
\begin{aligned}
&\sum_{n \le k < 3n} \sum_{j \ge 3n-k} \sum_{(v_i) \in \Gamma_{jk}^x(nx)} \left(\prod_{i \notin L((v_i))} e^{-s_x(v_{i+1}-v_i)} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\
&\qquad \times \left(\prod_{i \in L((v_i))} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\
&\le \sum_{k \ge 3n} \sum_{j \ge 0} e^{(j+k)\eta_x} e^{-4j\eta_x} e^{(2\eta_x+\sigma)j} e^{-\sigma k g(x)} \qquad (2.13)\\
&\le c_{11} e^{3n\eta_x} e^{-3\sigma n g(x)} \\
&= o\left(2^{-n} e^{-\sigma n g(x)}\right) \quad \text{as } n \to \infty.
\end{aligned}
$$

Next, for $\|x\| \geq$ some c_{12}, similar calculations give

$$
\begin{aligned}
&\sum_{n \le k < 3n} \sum_{j \ge 3n-k} \sum_{(v_i) \in \Gamma_{jk}^x(nx)} \left(\prod_{i \notin L((v_i))} e^{-s_x(v_{i+1}-v_i)} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\
&\qquad \times \left(\prod_{i \in L((v_i))} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\
&\le \sum_{k \ge n} \sum_{j \ge 3n-k} e^{(j+k)\eta_x} e^{-4j\eta_x} e^{(2\eta_x+\sigma)j} e^{-\sigma k g(x)} \qquad (2.14)\\
&\le \sum_{k \ge n} c_{13} e^{k\eta_x} e^{-\sigma k g(x)} e^{-(\eta_x-\sigma)(3n-k)} \\
&\le c_{14} e^{-(\eta_x-3\sigma)n} e^{-\sigma n g(x)} \\
&= o\left(2^{-n} e^{-\sigma n g(x)}\right) \text{ as } n \to \infty.
\end{aligned}
$$

Finally, again for large $\|x\|$,

$$
\begin{aligned}
&\sum_{0 \le k < n} \sum_{j \ge 3n-k} \sum_{(v_i) \in \Gamma_{jk}^x(nx)} \left(\prod_{i \notin L((v_i))} e^{-s_x(v_{i+1}-v_i)} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\
&\qquad \times \left(\prod_{i \in L((v_i))} e^{-\sigma g_x(v_{i+1}-v_i)} \right) \\
&\le \sum_{0 \le k < n} \sum_{j \ge 3n-k} e^{(j+k)\eta_x} e^{-4j\eta_x} e^{-\sigma g_x(nx)} \qquad (2.15)\\
&\le c_{15} n e^{-5n\eta_x} e^{-\sigma n g(x)} \\
&= o\left(2^{-n} e^{-\sigma n g(x)}\right) \quad \text{as } n \to \infty.
\end{aligned}
$$

Now (2.10), (2.11) and (2.13)–(2.15) establish the lemma. □

We can now show that our target x is in a multiple of the convex hull of some reasonable-cost sites.

LEMMA 2.4. *Given* $x \in \mathbb{Z}^d$, *there exist* $y_i \in B_x$ *and* α_i, $i = 1, \dots, d+1$, *satisfying*

$$\alpha_i \geq 0, \qquad \sum_{i=1}^{d+1} \alpha_i y_i = x, \qquad \sum_{i=1}^{d+1} \alpha_i \leq 3. \tag{2.16}$$

PROOF. By Lemma 2.3, for large n there exists an x-cost skeleton of $3n+1$ or fewer vertices for some path from 0 to nx. For $y \in B_x$, let $m_y(n)$ be the number of increments in this skeleton which are equal to y and let $z_n \in Q_x$ be the last increment of the skeleton. Then

$$\sum_{y \in B_x} m_y(n) y + z_n = nx \quad \text{and} \quad \sum_{y \in B_x} m_y(n) < 3n.$$

Taking a subsequence along which $n^{-1} m_y(n)$ converges to some a_y for all y, we see that the a_y satisfy

$$\sum_{y \in B_x} a_y y = x, \qquad s := \sum_{y \in B_x} a_y \leq 3, \qquad a_y \geq 0. \tag{2.17}$$

Thus x/s is in the convex hull of B_x in $\mathbb{R}^d$. But this implies x/s is in the convex hull of some $d+1$ points in B_x, which proves the lemma. □

We say a self-avoiding lattice path from some y to z is *x-clean* if for every pair of sites u, v in the path with u preceding v, we have $s_x(v-u) \leq 7\eta_x$. Thus an x-clean path has no expensive segments, so its segments can be used to help build a path to x. Note that being x-clean is a deterministic property of a path and does not involve the configuration of vacant and occupied bonds.

LEMMA 2.5. *Uniformly in* $y \in B_x$,

$$P_p[0 \leftrightarrow y \text{ via a self-avoiding lattice path which is not } x\text{-clean}]$$
$$= o\big(P_p[0 \leftrightarrow y]\big) \text{ as } \|x\| \to \infty.$$

PROOF. Let $R_x := \{z \in 2g(x)U_g\colon z \text{ adjacent to } (2g(x)U_g)^c\}$; note that $g(z) \geq 2g(x) - 1$ for $z \in R_x$. Then by (2.5), (2.6), the van den Berg-Kesten (1985) inequality, linearity of g_x and Lemma 2.2(ii),

$$\begin{aligned}
&P_p[0 \leftrightarrow y \text{ via a self-avoiding lattice path which is not } x\text{-clean}] \\
&\leq P_p[0 \leftrightarrow R_x] + \sum_{u, v \in 2g(x)U_g,\, s_x(v-u) > 7\eta_x} P_p[0 \leftrightarrow u] P_p[u \leftrightarrow v] P_p[v \leftrightarrow y] \\
&\leq |R_x| e^{-(2g(x)-1)\sigma} + |2g(x)U_g|^2 e^{-7\eta_x} e^{-\sigma g_x(y)} \\
&= o\big(e^{-(4\eta_x + \delta)} e^{-\sigma g_x(y)}\big) \\
&= o\big(P_p[0 \leftrightarrow y]\big).
\end{aligned}$$

□

We say that a self-avoiding lattice path from some y to z *x-backtracks* by t ($t \geq 0$) if there exist sites u, v in the path with u preceding v but $g_x(v - u) \leq -t$.

LEMMA 2.6. *An x-clean self-avoiding lattice path does not x-backtrack by* $4\sigma^{-1}\eta_x$.

PROOF. If u precedes v in an x-clean path, then as in (2.7), $2\sigma g_x(v - u) \geq -s_x(v - u) \geq -7\eta_x$. $\square$

As previously mentioned, we need to show that if the cost $P_p[0 \leftrightarrow y]$ is reasonable, then so is $P_p[0 \leftrightarrow \alpha y]$ for $0 \leq \alpha \leq 3$. (If $\alpha y \notin \mathbb{Z}^d$, replace it here with any nearby site.) This will be done by assembling a path from 0 to αy from segments of an x-clean path from 0 to y. To accomplish this assembly we need a purely geometric fact about curves in $\mathbb{R}^d$. Given a curve $\gamma: [0, T] \to \mathbb{R}^d$ and $s \in [0, T]$, define the *cyclic s-permutation* γ_s of γ to be the curve formed by interchanging the segment of γ from $\gamma(0)$ to $\gamma(s)$ with the segment from $\gamma(s)$ to $\gamma(T)$; more formally,

$$\gamma_s(t) := \begin{cases} \gamma(0) + \gamma(s+t) - \gamma(s) & \text{if } 0 \leq t \leq T - s, \\ \gamma(T) - \gamma(s) + \gamma(s + t - T) & \text{if } T - s < t \leq T. \end{cases}$$

Note that γ_s is continuous and has the same endpoints as γ.

We now show that a path from 0 to αy can be assembled from at most six segments of any nonbacktracking path from 0 to y.

PROPOSITION 2.7. *Let $d = 2$ or 3 and let $\gamma: [0,1] \to \mathbb{R}^d$, with $\gamma(0) = 0$, $\gamma(1) = y$, be a curve such that $f(\gamma(t))$ is nondecreasing for some linear functional f. Let $I := \{\gamma(t) - \gamma(s): 0 \leq s \leq t \leq 1\}$ be the set of vector increments of segments of γ. There exist constants k_d (not depending on γ) such that the Minkowski sum $I + \cdots + I$ of k_d copies of I contains $\{\alpha y: 0 \leq \alpha \leq 1\}$. Further, $k_2 = 2$ and $k_3 \leq 6$.*

PROOF. Since the set of all curves satisfying the conclusion of the proposition is uniformly closed we may assume that $f(\gamma(t))$ is strictly increasing. Since this set of curves is also invariant under invertible linear transformations, we may then assume that $f(z) = z_1$ (the first coordinate) for all z, that $y = e_1$ and that the curve is parametrized by the first coordinate, i.e., $f(\gamma(t)) = t$.

Fix $\alpha \in [0,1]$ and let H be the hyperplane $\{z: z_1 = \alpha\}$. Then

$$\beta(t) := \gamma_t(\alpha), \qquad 0 \leq t \leq 1,$$

defines a continuous curve in H; the values of β are the cyclic increments of γ over intervals of length α. Let $\overline{\beta}$ denote the image of β; we claim that αe_1 is in the convex hull of $\bar{\beta}$. In fact, αe_1 is the average of β:

$$\begin{aligned}\int_0^1 \beta(t)\,dt &= \int_0^{1-\alpha}(\gamma(t+\alpha)-\gamma(t))\,dt \\ &\quad + \int_{1-\alpha}^1 (\gamma(1)-\gamma(t)+\gamma(t+\alpha-1))\,dt \\ &= \int_\alpha^1 \gamma(t)\,dt - \int_0^{1-\alpha}\gamma(t)\,dt + \alpha e_1 - \int_{1-\alpha}^1 \gamma(t)\,dt + \int_0^\alpha \gamma(t)\,dt \\ &= \alpha e_1.\end{aligned}$$

In dimension $d = 2$, $\bar{\beta}$ is convex since H is one-dimensional, so we have $\alpha e_1 \in \bar{\beta}$. Every point of $\bar{\beta}$ is a cyclic increment of γ and therefore is in $I + I$. It is easily verified that $k_2 > 1$ and the proposition follows for $d = 2$.

Thus let us work in dimension $d = 3$. There then exists a line l in H through αe_1 and two points $\beta(u)$ and $\beta(v)$ in l on opposite sides of αe_1. By rotating, we may assume $l = H \cap \{z\colon z_3 = 0\}$ and $\beta(u) = (\alpha, a, 0)$, $\beta(v) = (\alpha, -b, 0)$ for some $a, b \geq 0$. Now

$$\eta(t) := \gamma_u(t), \qquad 0 \leq t \leq \alpha,$$

$$\xi(t) := \gamma_v(t) + \alpha e_1 - \beta(v), \qquad 0 \leq t \leq \alpha,$$

are curves from 0 to $(\alpha, a, 0) = \beta(u)$ and from $(0, b, 0)$ to αe_1, respectively, and the four endpoints are all in the plane $\{z\colon z_3 = 0\}$. Note that both η and ξ are translates of segments of cyclic permutations of γ. Their orthogonal projections into the plane $\{z\colon z_3 = 0\}$ necessarily intersect; we would like to force the unprojected curves to intersect, creating a path from 0 to αe_1. To do this we will cyclically permute both η and ξ. Let $t_m(\eta)$ denote a value of $t \in [0, \alpha]$ for which the third coordinate of $\eta(t)$ is minimized and $t_M(\eta)$ a value for which it is maximized. Then $\eta_{t_m(\eta)}$ lies entirely in the halfspace $\{z\colon z_3 \geq 0\}$, and $\eta_{t_M(\eta)}$ entirely in $\{z\colon z_3 \leq 0\}$. All curves η_s lie in the slab $\{z\colon 0 \leq z_1 \leq \alpha\}$, have increasing first coordinate and have the same endpoints. Similar definitions and statements apply to ξ in place of η.

Let q vary between $t_m(\eta)$ and $t_M(\eta)$ and r between $t_m(\xi)$ and $t_M(\xi)$. When $(q, r) = (t_m(\eta), t_M(\xi))$, η_q is entirely above (or intersects) ξ_r, as they lie in opposite half spaces; when $(q, r) = (t_M(\eta), t_m(\xi))$, η_q is entirely below (or intersects) ξ_r. Letting (q, r) follow a straight line from $(t_m(\eta), t_M(\xi))$ to $(t_M(\eta), t_m(\xi))$, we see that there must exist q and r such that η_q intersects ξ_r, i.e.,

$$\eta_q(t_1) = \xi_r(t_2) \quad \text{for some } t_1, t_2 \in [0, \alpha].$$

(Note we are making strong use here of $d = 3$ and monotonicity of f.) Then

$$(2.18) \qquad \big(\eta_q(t_1) - \eta_q(0)\big) + \big(\xi_r(\alpha) - \xi_r(t_2)\big) = \xi(\alpha) - \eta(0) = \alpha e_1.$$

Now each increment of η_q is a sum of at most two disjoint increments of η and hence (since η is a segment of a cyclic permutation of γ) is a sum of at most three increments of γ. Thus

$$\eta_q(t_1) - \eta_q(0) \in I + I + I$$

and similarly,

$$\xi_r(\alpha) - \xi_r(t_2) \in I + I + I,$$

which with (2.18) proves the proposition □

The next step is to prove a lower bound for the connectivity function which is much cruder than the one in Theorem 1.1.

LEMMA 2.8. *For each $\varepsilon > 0$, there exists a $c_\varepsilon = c_\varepsilon(p, d)$ such that*

$$P_p[0 \leftrightarrow y] \geq c_\varepsilon e^{-(1+\varepsilon)\sigma g(y)} \quad \text{for all } y \in \mathbb{Z}^d.$$

PROOF. Let M be large enough so

$$(1 + \varepsilon/2)\big(1 + 2\varepsilon/M + 2\varepsilon(\log p^{-1})/M\sigma\big) \leq 1 + \varepsilon,$$

let $k \geq dM/\epsilon$ be a positive integer and let $S := (k^{-1}\mathbb{Z}^d \setminus \{0\}) \cap 2U_g$. Let $n_0 \geq 2dM/\epsilon$ be such that

$$P_p[0 \leftrightarrow nku] \geq e^{-(1+\varepsilon/2)\sigma g(nku)} \quad \text{for all } u \in S \text{ and } n \geq n_0.$$

Such an n_0 exists because S is finite.

Suppose $y \in \mathbb{Z}^d$, with $g(y) \geq n_0 k$ and let $n := [k^{-1}g(y)] \geq n_0$. Let $u \in S$ with $\|y/g(y) - u\|_1 \leq dk^{-1}$. Then since $\|u\|_1 \leq dg(u) \leq 2d$,

$$\begin{aligned}\|y - nku\|_1 &\leq \|y - g(y)u\|_1 + \|g(y)u - nku\|_1 \leq dk^{-1}g(y) + 2dk \\ &\leq \big(dk^{-1} + 2dn_0^{-1}\big)g(y) \leq 2\varepsilon g(y)/M,\end{aligned}$$

so that

$$g(nku) \leq g(y)(1 + 2\varepsilon/M).$$

Therefore by the Harris–FKG inequality [Harris (1960)],

$$\begin{aligned}P_p[0 \leftrightarrow y] &\geq P_p[0 \leftrightarrow nku]P_p[nku \leftrightarrow y] \\ &\geq \exp(-(1 + \varepsilon/2)\sigma g(nku))p^{2\varepsilon g(y)/M} \\ &\geq \exp\big(-(1 + \varepsilon/2)\sigma g(y)\big(1 + 2\varepsilon/M + 2\varepsilon(\log p^{-1})/M\sigma\big)\big) \\ &\geq \exp(-(1 + \varepsilon)\sigma g(y)),\end{aligned}$$

and the lemma follows. □

LEMMA 2.9. *In dimension $d = 2$ or 3, let $\|x\| \geq c_{16}$, $y \in B_x$, $\alpha \in [0, 3]$ and $z \in \mathbb{Z}^d$ with $\|z - \alpha y\|_1 \leq d$. Then*

$$P_p[0 \leftrightarrow z] \geq c_{17}e^{(31k_d+8)\eta_x}e^{-\sigma g_x(\alpha y)}.$$

PROOF. By Lemma 2.5, there exists a self-avoiding lattice path γ from 0 to y which is x-clean. We may assume γ is parametrized by $[0, 1]$. We need to approximate γ by a curve which does not x-backtrack. Let $v_1 := x/g(x)$ and let $\{v_2, \dots, v_d\}$ be a basis for H_x^0. Let $\gamma_i(t)$ denote the ith coordinate of γ in the basis $\{v_1, \dots, v_d\}$; note $\gamma_1(t) = g_x(\gamma(t))$. Define a new curve $\tilde{\gamma}$ from 0 to y (not in general a lattice path) by its coordinates

$$\tilde{\gamma}_1(t) := \sup_{s \le t} (\gamma_1(s) \vee 0) \wedge g_x(y),$$
$$\tilde{\gamma}_i(t) := \gamma_i(t), \qquad i = 2, \dots, d,$$

in the basis $\{v_1, \dots, v_d\}$. Then $g_x(\tilde{\gamma}(t))$ is nondecreasing, and for all t, $\gamma(t) - \tilde{\gamma}(t)$ is a scalar multiple of x. From this and Lemma 2.5 it follows that

$$g(\gamma(t) - \tilde{\gamma}(t)) = |g_x(\gamma(t) - \tilde{\gamma}(t))| \le 4\sigma^{-1}\eta_x. \tag{2.19}$$

Suppose first that $\alpha \le 1$. By Proposition 2.7, there exist $0 \le s_i \le t_i \le 1$ for $i = 1, \dots, k_d$ such that

$$\begin{aligned} \alpha y &= \sum_{i=1}^{k_d} (\tilde{\gamma}(t_i) - \tilde{\gamma}(s_i)) \\ &= \sum_{i=1}^{k_d} [(\tilde{\gamma}(t_i) - \gamma(t_i)) + (\gamma(t_i) - \gamma(s_i)) + (\gamma(s_i) - \tilde{\gamma}(s_i))]. \end{aligned} \tag{2.20}$$

Let r_i, u_i, v_i and w_i be points of $\mathbb{Z}^d$ such that u_i and v_i are sites in γ with $\gamma^{-1}(v_i) \ge \gamma^{-1}(u_i)$ and

$$\begin{aligned} \|(\tilde{\gamma}(t_i) - \gamma(t_i)) - r_i\|_1 &\le d/2, \\ \|(\gamma(s_i) - \tilde{\gamma}(s_i)) - w_i\|_1 &\le d/2, \\ \|\gamma(t_i) - v_i\|_1 &\le \tfrac{1}{2}, \\ \|\gamma(s_i) - u_i\|_1 &\le \tfrac{1}{2}. \end{aligned} \tag{2.21}$$

Then by (1.5), (2.19) and (2.21),

$$g(r_i) \le 4\sigma^{-1}\eta_x + d/2 \quad \text{and} \quad g(w_i) \le 4\sigma^{-1}\eta_x + d/2, \tag{2.22}$$

and since γ is x-clean,

$$s_x(v_i - u_i) \le 7\eta_x. \tag{2.23}$$

From the Harris–FKG inequality [Harris (1960)], Lemma 2.8, (2.22), (2.23) and (2.1), we obtain

$$\begin{aligned} P_p[0 \leftrightarrow r_i + (v_i - u_i) + w_i] &\ge P_p[0 \leftrightarrow r_i] P_p[0 \leftrightarrow (v_i - u_i)] P_p[0 \leftrightarrow w_i] \\ &\ge c_{18} e^{-2\sigma g(r_i)} e^{-2\sigma g(w_i)} e^{-s_x(v_i - u_i)} e^{-\sigma g_x(v_i - u_i)} \\ &\ge c_{19} e^{-23\eta_x} e^{-\sigma g_x(v_i - u_i)} \\ &\ge c_{20} e^{-31\eta_x} e^{-\sigma g_x(r_i + (v_i - u_i) + w_i)}. \end{aligned} \tag{2.24}$$

Let $q := \sum_{i=1}^{k_d}(r_i + (v_i - u_i) + w_i)$; then by (2.20) and (2.21),

$$\|z - q\|_1 \le (d+1)k_d + d.$$

This, the Harris–FKG inequality again and (2.24) show that

$$\begin{aligned} P_p[0 \leftrightarrow z] &\ge P_p[z \leftrightarrow q]\prod_{i=1}^{k_d} P_p[0 \leftrightarrow r_i + (v_i - u_i) + w_i] \\ &\ge p^{(d+1)k_d+d} c_{21} e^{-31k_d\eta_x} e^{-\sigma g_x(q)} \\ &\ge c_{22} e^{-31k_d\eta_x} e^{-\sigma g_x(\alpha y)}. \end{aligned} \tag{2.25}$$

This completes the proof for $\alpha \le 1$.

If $1 < \alpha \le 3$, let $[\alpha]$ and β be the integer and fractional parts of α, respectively. Then by the Harris–FKG inequality once more, Lemma 2.2(ii) and (2.25),

$$\begin{aligned} P_p[0 \leftrightarrow z] &\ge P_p[0 \leftrightarrow y]^{[\alpha]} P_p[0 \leftrightarrow z - [\alpha]y] \\ &\ge c_{23} e^{-8\eta_x} e^{-\sigma g_x([\alpha]y)} e^{-31k_d\eta_x} e^{-\sigma g_x(\beta y)} \\ &= c_{23} e^{-(31k_d+8)\eta_x} e^{-\sigma g_x(\alpha y)}. \end{aligned}$$

□

PROOF OF THEOREM 1.1. Fix $x \in \mathbb{Z}^d$ and let α_i and y_i, $i = 1, \dots, d+1$, be as in Lemma 2.4. Let $z_i \in \mathbb{Z}^d$ with $\|z_i - \alpha_i y_i\|_1 \le d$ and $w := \sum_{i=1}^{d+1} z_i$. Then $\|w - x\|_1 \le d(d+1)$ so by the Harris–FKG inequality [Harris (1960)] and Lemma 2.9,

$$\begin{aligned} P_p[0 \leftrightarrow x] &\ge P_p[x \leftrightarrow w]\prod_{i=1}^{d+1} P_p[0 \leftrightarrow z_i] \\ &\ge c_{24} p^{d(d+1)} e^{-(31k_d+8)(d+1)\eta_x} e^{-\sigma g_x(x)} \end{aligned}$$

and the theorem follows; the bound on $r(d)$ is $(31k_d + 8)(d+1)$. □

PROOF OF COROLLARY 1.2. Let $\zeta_i \to 0$ in $\mathbb{R}^{d-1} \setminus \{0\}$ and let $n_k \to \infty$ with $\|\zeta_k\| \sim n_k^{-3/8}$. Let $a_k \in \mathbb{Z}^{d-1}$ with $\|a_k - n_k\zeta_k\|_1 \le d$. Then

$$|g(1, n_k^{-1}a_k) - g(1, \zeta_k)| \le d/n_k,$$

so for large k, by Theorem 1.1 and (1.8),

$$\begin{aligned} c_{25} n_k^{-r} e^{-\sigma n_k g(1,\zeta_k)} &\le c_{26}\|(n_k, a_k)\|^{-r} e^{-\sigma n_k g(1, n_k^{-1}a_k)} \\ &\le P_p[0 \leftrightarrow (n_k, a_k)] \\ &\le c_{27} e^{-\|a_k\|^2/c_2 n_k} e^{-\sigma n_k}. \end{aligned}$$

Therefore for large k,

$$\begin{aligned} g(1,\zeta_k) &\ge 1 - r\sigma^{-1}n_k^{-1}(c_{28} + \log n_k) + \|a_k\|^2/c_2\sigma n_k^2 \\ &\ge 1 - r\sigma^{-1}n_k^{-1}(c_{28} + \log n_k) + \|\zeta_k\|^2/c_2\sigma - 2d\|\zeta_k\|/c_2\sigma n_k \\ &= 1 + \|\zeta_k\|^2/c_2\sigma + o(\|\zeta_k\|^2). \end{aligned}$$

In the other direction, by (1.6) and (1.8),

$$\begin{aligned} e^{-\sigma n_k g(1,\zeta_k)} &\geq c_{29} e^{-\sigma n_k g(1, n_k^{-1} a_k)} \\ &\geq c_{29} P_p[0 \leftrightarrow (n_k, a_k)] \\ &\geq c_{30} n_k^{-(d-1)/2} e^{-\|a_k\|^2/c_2 n_k} e^{-\sigma n_k}, \end{aligned}$$

so that for large k,

$$\begin{aligned} g(1, \zeta_k) &\leq 1 + \|a_k\|^2/c_2\sigma n_k^2 + c_{31} n_k^{-1} \log n_k \\ &\leq 1 + \|\zeta_k\|^2/c_2\sigma + c_{32}\|\zeta_k\|/n_k + c_{31} n_k^{-1} \log n_k \\ &= 1 + \|\zeta_k\|^2/c_2\sigma + o\big(\|\zeta_k\|^2\big). \end{aligned}$$

Since the sequence $\zeta_k \to 0$ is arbitrary, the corollary follows. □

Note that the proof of Corollary 1.2 does not use the full strength of Theorem 1.1. In fact, for the upper bound on g, Theorem 1.1 is not used at all. For the lower bound, it would be enough to know that $P_p[0 \leftrightarrow x] \geq ce^{-h(\|x\|)} e^{-\sigma g(x)}$ for some $h(t) = O(t^{1/2-\varepsilon})$, where $\varepsilon > 0$. The exponent $\frac{3}{8}$ appearing in the definition of n_k can be replaced by $\frac{1}{4} + \tau$ for arbitrarily small τ.

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