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A COMPOUND POISSON CONVERGENCE THEOREM

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In 1971, Simons and Johnson showed that the classical theorem of binomial to Poisson convergence is actually stronger than in the usual sense. Their result was proved valid also for the distributions of sums of independent, but not necessarily identically distributed, Bernoulli random variables by Chen in 1974. Here we show that their result is indeed true for a much larger class of random variables. The limiting distribution is generalized to a compound Poisson distribution.

1. Introduction. Let $X_{n1}, \dots, X_{nn}$ be $n \geq 1$ independent random variables (r.v.'s) taking values in the set of nonnegative integers and define $S_n = \sum_{i=1}^n X_{ni}$. If for each n, X_{ni} are identically distributed Bernoulli r.v.'s with $p_n = P(X_{ni} = 1)$, it is well known that

$$\lim_{n \to \infty} P(S_n = k) = e^{-\lambda}\lambda^k/k!, \qquad k = 0, 1, \dots, \tag{1}$$

if $np_n \to \lambda$ as $n \to \infty$.

Simons and Johnson (1971) showed that if $np_n = \lambda$, then the pointwise convergence (1) can be strengthened to

$$\lim_{n \to \infty} \sum_{n=0}^{\infty} h(k)|P(S_n = k) - e^{-\lambda}\lambda^k/k!| = 0, \tag{2}$$

for all nonnegative functions h with $\sum_{n=0}^{\infty} h(k)e^{-\lambda}\lambda^k/k! < \infty$.

The convergence (1) for nonidentically distributed Bernoulli r.v.'s was established by von Mises (1921). [See Wang (1988) for a simple combinatorial proof.] Chen (1974) pointed out that (2) also holds for the nonidentically distributed case provided that $\sum_{i=1}^n p_{ni} = \lambda$ and $\max_{1 \leq i \leq n} p_{ni} \to 0$ as $n \to \infty$.

In this paper, we show that (2) is also true for a much larger class of r.v.'s and the limiting distribution is generalized to a compound Poisson distribution.

2. The theorem. Denote $p_{ni}(k) = P(X_{ni} = k)$, $p_{ni} = 1 - p_{ni}(0)$, $\lambda_n = \sum_{i=1}^n p_{ni}$, for $i = 1, \dots, n$, $n = 1, 2, \dots$, $k = 0, 1, 2, \dots$.

THEOREM. *If $\lambda_n = \lambda + O(1/n)$ for some real $\lambda > 0$, $p_{ni}(k)/p_{ni} = \alpha(k)$ are independent of i and n for all $k \geq 1$ and $\max_{1 \leq i \leq n} p_{ni} \to 0$ as $n \to \infty$, then*

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} h(k)|P(S_n = k) - P(Y = k)| = 0, \tag{3}$$

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for all nonnegative functions h with $\sum_{k=0}^{\infty} h(k)P(Y=k) < \infty$, *where Y has a compound Poisson distribution with parameter* λ *and compounding distribution* $\{\alpha(k);\ k = 1, 2, \dots\}$.

PROOF. It follows as a special case of Theorem 7 in Wang (1989) that

$$\lim_{n\to\infty} P(S_n = k) = P(Y = k) \quad \text{for all } k = 0, 1, \dots . \tag{4}$$

Thus, by virtue of the Lebesgue dominated convergence theorem, to establish (3) it is sufficient to show that there exists a constant $S > 0$ such that

$$P(S_n = k) \le SP(Y = k) \quad \text{for all } k \text{ and } n. \tag{5}$$

Denote $\lambda_{ni} = (p_{ni}/\lambda_n)\lambda$. Since a compound Poisson distribution is additive, for each integer n there exist independent Y_{ni} (compound Poisson), N_{ni} (Poisson) with parameters λ_{ni} and identically distributed r.v.'s $Z_{i1}, Z_{i2}, \dots$ with $P(Z_{ij} = k) = \alpha(k)$, such that

$$Y_{ni} = \sum_{j=1}^{N_{ni}} Z_{ij}, \qquad Y = \sum_{i=1}^{n} Y_{ni}$$

and $\{N_{ni}\}$ and $\{Z_{ij}\}$ are independent.

Now for $i = 1, 2, \dots, n$,

$$P(Y_{ni} = 0) = P(N_{ni} = 0) = e^{-\lambda_{ni}} \ge e^{-\lambda_{ni}} P(X_{ni} = 0). \tag{6}$$

For $k \ge 1$ and $i = 1, 2, \dots, n$,

$$P(Y_{ni} = k) \ge \alpha(k)P(N_{ni} = 1) = e^{-\lambda_{ni}}(\lambda/\lambda_n)P(X_{ni} = k). \tag{7}$$

Letting $C_n = \max(\ln(\lambda_n/\lambda), 0)$, we can combine (6) and (7) as

$$P(X_{ni} = k) \le e^{C_n + \lambda_{ni}} P(Y_{ni} = k).$$

For $i \ne j$ and all $k = 0, 1, \dots,$

$$\begin{aligned} P(X_{ni} + X_{nj} = k) &= \sum_{m=0}^{k} P(X_{ni} = m)P(X_{nj} = k - m) \\ &\le e^{2C_n + \lambda_{ni} + \lambda_{nj}} P\big(Y_{ni} + Y_{nj} = k\big). \end{aligned}$$

It then follows by induction that

$$P(S_n = k) \le e^{nC_n + \lambda} P(Y = k) \quad \text{for all } k \text{ and } n.$$

By assumption $\lambda_n = \lambda + O(1/n)$, there exists $C > 0$ such that $n \ge n_0$ implies $|n \ln(\lambda_n/\lambda)| < C$. (5) can thus be obtained by letting $S = Ke^{C+\lambda}$, for some appropriate constant K. □

An interesting example is as follows: Let

$$P(X_{ni} = 0) = 1 - r_{ni}, \qquad 0 \le r_{ni} < 1,$$

$$P(X_{ni} = k) = (1 - q_n)q_n^{k-1} r_{ni}, \qquad 0 \le q_n < 1,\ k = 1, 2, \dots .$$

If $q_n = q$ for all n and $\lambda_n = \lambda + O(1/n)$, where $\lambda_n = \sum_{i=1}^n r_{ni}$, then our theorem says that the distribution of S_n converges in the sense (3) to a compound Poisson distribution with parameter λ and geometric compounding distribution $\{(1-q)q^{k-1};\ k = 1, 2, \dots\}$.

If $r_{ni} = q_n$ for all i, then it is well known that

$$P(S_n = k) = \binom{n+k-1}{k}(1-q_n)^n q_n^k, \qquad k = 0, 1, \dots,$$

and the pointwise convergence (1) holds, if $nq_n \to \lambda$ as $n \to \infty$. Now let $0 < \lambda < 1$, $q_n = \lambda/n$ and $h(k) = k!$. Then $\sum_{k=0}^{\infty} h(k)e^{-\lambda}\lambda^k/k! = e^{-\lambda}/(1-\lambda) < \infty$. Rewriting,

$$\begin{aligned}&\sum_{k=0}^{\infty} h(k)\left|\binom{n+k-1}{k}(1-\lambda/n)^n(\lambda/n)^k - e^{-\lambda}\lambda^k/k!\right| \\ &\qquad = e^{-\lambda}\sum_{k=0}^{\infty} a_n(k)\lambda^k,\end{aligned} \tag{8}$$

where $a_n(0) = 1 - e^{\lambda}(1-\lambda/n)^n$ and

$$a_n(k) = \left|e^{\lambda}(1-\lambda/n)^n \prod_{i=0}^{k-1}(1+i/n) - 1\right|.$$

Since $\lim_{k\to\infty} a_n(k+1)/a_n(k) = \lim_{k\to\infty}(1+k/n) = \infty$, the series (8) diverges to infinity for all $n \geq 1$. In this case, the ratios $p_{ni}(k)/p_{ni} = (1-\lambda/n)(\lambda/n)^{k-1}$ are not independent of n, hence our theorem is not applicable.

3. Discussion. Let $\{Y_n\colon n = 1, 2, \dots\}$ be a sequence of discrete r.v.'s taking values in $\mathbf{D} \subseteq \{0, \pm 1, \pm 2, \dots\}$. Denote $\{f_n\colon n = 1, 2, \dots\}$ and $\{F_n\colon n = 1, 2, \dots\}$ its corresponding probability density function (pdf) and cumulative distribution function (cdf), respectively. Let Y be a discrete r.v. also taking values in $\mathbf{D}$, with pdf f and cdf F. The statement that the distribution of Y_n converges to that of Y can be defined in one of the following modes:

(a) $$\lim_{n\to\infty} f_n(k) = f(k) \quad \text{for each } k \in \mathbf{D}.$$

(b) $$\lim_{n\to\infty} f_n(k) = f(k) \quad \text{uniformly in } \mathbf{D}.$$

(c) $$\lim_{n\to\infty} \sup_{k\in\mathbf{D}} |F_n(k) - F(k)| = 0.$$

(d) $$\lim_{n\to\infty} \sup_{\mathbf{A}\subseteq\mathbf{D}} |P(Y_n \in \mathbf{A}) - P(Y \in \mathbf{A})| = 0.$$

(e) $$\lim_{n\to\infty} \sum_{k\in\mathbf{D}} |f_n(k) - f(k)| = 0.$$

(f) $$\lim_{n\to\infty} \sup_{|g|\leq 1} |E(g(Y_n)) - E(g(Y))| = 0.$$

(g) $$\lim_{n\to\infty} \sum_{k\in\mathbf{D}} h(k)|f_n(k) - f(k)| = 0,$$

for all nonnegative functions h with $\sum_{k\in\mathbf{D}} h(k)f(k) < \infty$.

In the problem of binomial to Poisson convergence, all the modes have been used at one time or another with $Y_n = S_n$. There seems to have been a misconception among researchers in the field that one mode might be stronger than another. We hope the following proposition will clarify relations among them.

PROPOSITION. *Among all the modes of convergence,* (g) *is the strongest while all the others are equivalent.*

PROOF. It is evident that (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a). It follows from Remark 1 in Wang [(1989), page 41] that (d), (e) and (f) are equivalent. Hence to show (a)–(f) are all equivalent, it is sufficient to show (a) $\Rightarrow$ (e). To this end, define $\mathbf{B}_n = \{k\colon f_n(k) < f(k)\}$ and $d_n(k) = 2(f(k) - f_n(k))$ if $k \in \mathbf{B}_n$ and equal to 0 otherwise. Then

$$\sum_{k \in \mathbf{D}} |f_n(k) - f(k)| = 2 \sum_{k \in \mathbf{B}_n} (f(k) - f_n(k)) = \sum_{k \in \mathbf{D}} d_n(k).$$

Since $d_n(k) \le 2f(k)$ for all k and n, then, applying the Lebesgue dominated convergence theorem to the series $\sum_{k \in \mathbf{D}} d_n(k)$, we conclude that (a) implies (e).

Hence all the modes from (a) to (f) are equivalent.

If we let $h(k) = 1$ for all k, then (g) is reduced to (e). Therefore (g) implies all the others. The example in Section 2 with $r_{ni} = q_n = \lambda/n$, $0 < \lambda < 1$, shows that (g) is stronger than (a) and hence is the strongest. In that example, $E(h(S_n)) = \infty$.

Here is another simpler and more general example. Let f be a pdf defined on $\{0, 1, 2, \dots\}$, with $f(0) > 0$ and finite first moment μ. Define $h(k) = k$ and $f_n(0) = f(0)(1 - 1/n)$, $f_n(n) = f(n) + f(0)/n$ and $f_n(k) = f(k)$ for all other k. Then f_n is a pdf for all n with finite first moment $\mu + f(0)$ and $\sum_{k=0}^{\infty} |f_n(k) - f(k)| = 2f(0)/n$. But $\sum_{k=0}^{\infty} h(k)|f_n(k) - f(k)| = f(0) > 0$ for all n. □

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