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A BOREL MEASURABLE VERSION OF KÖNIG'S LEMMA FOR RANDOM PATHS

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Starting at x in a Polish space X, a player selects the distribution σ_0 of the next state x_1 from the collection $\Gamma(x)$ of those distributions available and then selects the distribution $\sigma_1(x_1)$ for x_2 from $\Gamma(x_1)$ and so on. Suppose the player wins if every x_i in the stochastic process $x_1, x_2, \dots$ lies in a given Borel subset A of X, that is, if the process stays in A forever. If $\{(x, \gamma): \gamma \in \Gamma(x)\}$ is a Borel subset of $X \times \mathbb{P}(X)$, where $\mathbb{P}(X)$ is the natural Polish space of probability measures on X, and if $0 \le p \le 1$, then a player can stay in A forever with probability at least p if and only if the player can stay in A up to time t with probability at least p for every Borel stop rule t. A similar result holds when the object of the game is to visit A infinitely often.

1. Introduction. A random version of König's lemma was proved by Purves and Sudderth [17] in the general finitely additive setting of Dubins and Savage [5]. In the gambling theory of Dubins and Savage, the state space X is an arbitrary nonempty set and probability measures are finitely additive and defined on all subsets. The theory permits the treatment of interesting problems while avoiding many issues relating to measurability. Our main result (Theorem 1.2) is a Borel-measurable (hence, more constructive) version of the generalized König's lemma of [17]. Quite unsurprisingly, the major new difficulties in the proof revolve around problems of measurability, which are surmounted through the methods of effective descriptive set theory.

Let X be a nonempty Borel subset of a Polish space and let $\mathbb{P}(X)$ be the collection of countably additive probability measures defined on the Borel subsets of X. Equip $\mathbb{P}(X)$ with its usual weak topology so that it, too, has the structure of a Borel subset of a Polish space [see, for example, Chapter II of Parthasarathy [15] for information about the weak topology on $\mathbb{P}(X)$]. An *analytic* (*Borel*) *gambling house* Γ on X is a mapping which assigns to each $x \in X$ a nonempty set $\Gamma(x) \subseteq \mathbb{P}(X)$ and such that the set

$$\Gamma = \{(x, \gamma) \in X \times \mathbb{P}(X): \gamma \in \Gamma(x)\}$$

is an analytic (Borel) subset of the product space $X \times \mathbb{P}(X)$. A player in the house Γ starts at some initial state x and chooses a *measurable strategy σ at x*, which means a sequence $\sigma = (\sigma_0, \sigma_1, \dots)$, where $\sigma_0 \in \Gamma(x)$ and, for $n = 1, 2, \dots, \sigma_n$ is a universally measurable mapping from X^n to $\mathbb{P}(X)$ such that

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$\sigma_n(x_1, x_2, \dots, x_n) \in \Gamma(x_n)$ for every $(x_1, x_2, \dots, x_n) \in X^n$. In case each σ_n is analytically measurable from X^n to $\mathbb{P}(X)$ (i.e., σ_n is measurable when X^n is endowed with the σ-field generated by analytic subsets of X^n and $\mathbb{P}(X)$ is given its usual Borel σ-field) and $\sigma_n(x_1, x_2, \dots, x_n) \in \Gamma(x_n)$ for every $(x_1, x_2, \dots, x_n) \in X^n$, we say that σ is an *analytically measurable* strategy at x. Every measurable strategy σ determines a probability measure, also denoted by σ, on the Borel subsets of

$$H = X \times X \times \cdots .$$

This probability measure can be regarded as the distribution of the coordinate process $h_1, h_2, \dots,$ where h_1 has distribution σ_0 and h_{n+1} has conditional distribution $\sigma_n(x_1, x_2, \dots, x_n)$ given $h_1 = x_1,\ h_2 = x_2, \dots, h_n = x_n$. It is not hard to verify that if σ is a probability measure on the Borel subsets of H induced by a measurable strategy at x, then σ is already induced by an analytically measurable strategy at x. For $x \in X$, let $\Sigma(x)$ be the collection of all measurable strategies σ at x. In the sequel, we shall frequently regard $\Sigma(x)$ as a set of probability measures on H, viz., the probability measures induced by measurable strategies at x.

Let g: $H \to (-\infty, \infty)$ be a bounded, upper analytic function, that is, a function such that for every real number c, the set $\{g > c\}$ is analytic. The properties of such functions that will be used in this paper may be found in [1], [7] and [12]. Now regard $g(h)$ as the payoff for a player who experiences the history $h = (h_1, h_2, \dots)$. The *optimal reward operator* M assigns to each such g the function Mg on X defined by

$$Mg(x) = \sup\{\sigma g \colon \sigma \in \Sigma(x)\}.$$

We will follow de Finetti's convention of identifying a set B with its indicator function 1_B. We write $M(B)$ for $M1_B$ when B is an analytic subset of H.

Let now Γ be an analytic gambling house on X and A a Borel subset of X. Define

$$A^\infty = \{h \in H \colon h_i \in A,\ i = 1, 2, \dots\},$$

$$A^n = \{h \in H \colon h_i \in A,\ i = 1, 2, \dots, n\},$$

for $n = 1, 2, \dots$.

THEOREM 1.1. *If $\Gamma(x)$ is finite for every $x \in X$, then*

$$M(A^\infty) = \inf\{M(A^n) \colon n \geq 1\}.$$

This result can be viewed as a random form of König's lemma. Indeed, it reduces to König's lemma if every probability measure available in Γ is a point mass.

To state our main result, we first introduce some notation and terminology. For $h, h' \in H$ and for a natural number n, we write $h \equiv_n h'$ if $h_i = h'_i$, $i = 1, 2, \dots, n$. A mapping τ from H to $N \cup \{\infty\}$, where $N = \{1, 2, \dots\}$, is called a *stopping time* if

$$\tau(h) = n \in N \quad \text{and} \quad h' \equiv_n h \quad \text{imply } \tau(h') = n.$$

A stopping time τ is *Borel* if for each $n \in N$, the set $\{\tau \leq n\}$ is a Borel subset of H. A *stop rule* is an everywhere finite stopping time. For a stop rule t, let

$$A^t = \{h \in H: h \in A^{t(h)}\}.$$

THEOREM 1.2. *For every analytic house Γ and Borel set $A \subseteq X$,*

$$M(A^\infty) = \inf\{M(A^t): t \text{ a Borel stop rule}\}.$$

This result is the Borel analogue of Theorem 2 in [17]. To state our final result, let

$$\{A \text{ i.o.}\} = \{h: h_i \in A \text{ for infinitely many } i\}.$$

THEOREM 1.3. *For every analytic house Γ and Borel set $A \subseteq X$,*

$$M(\{A \text{ i.o.}\}) = \inf\{M(\{\tau < \infty\}): \tau \text{ a Borel stopping time and } \{A \text{ i.o.}\} \subseteq \{\tau < \infty\}\}.$$

This is, of course, the Borel analogue of Theorem 3 of [17].

We now fix some notation in respect of strategies and stopping times. Suppose σ is a measurable strategy at x and $p \in X^m$. We define a measurable strategy $\sigma[p]$ at $(p)_m$, the mth coordinate of p, as follows:

$$(\sigma[p])_0 = \sigma_m(p)$$

and, for $n \geq 1$,

$$(\sigma[p])_n(x_1, x_2, \dots, x_n) = \sigma_{m+n}(px_1x_2 \cdots x_n),$$

where $px_1x_2 \cdots x_n$ is the element of X^{m+n} obtained by catenating p and $(x_1, x_2, \dots, x_n)$. It is easy to verify that the measures (induced by) $\sigma[p]$, $p \in X^m$ are a version of the conditional σ-distribution of $(h_{m+1}, h_{m+2}, \dots)$ given $(h_1, h_2, \dots, h_m) = p$. If τ is a Borel stopping time and $\tau(h) < \infty$, we write $h_\tau = h_\tau(h)$ for $h_{\tau(h)}$; set $p_\tau = p_\tau(h) = (h_1, h_2, \dots, h_\tau)$ and $\sigma[p_\tau] = \sigma[p_\tau](h) = \sigma[p_\tau(h)]$.

If $B \subseteq H$ and $p \in X^m$, then Bp will denote the set of $h \in H$ such that $ph \in B$, where ph is the element of H obtained by catenating p and h. Similarly, if τ is a Borel stopping time and $\tau(h) < \infty$, we write Bp_τ for the function $(Bp_\tau)(h) = Bp_\tau(h)$.

If τ is a stopping time and $x \in X$, define $\tau[x]$ on H by

$$\tau[x](h) = \tau(xh) - 1.$$

Then either $\tau[x]$ is again a stopping time or is identically equal to zero.

There is a natural way to associate with every stop rule an ordinal number $j(t)$, called the *index* of t, by setting $j(0) = 0$ and requiring that

$$j(t) = \sup\{j(t[x]) + 1: x \in X\},$$

for every stop rule t. This terminology is due to Dellacherie and Meyer [4], but

$j(t)$ is familiar to students of Dubins and Savage as being the structure of the finitary function h_t (see [5], pages 14–15 and [11], Proposition 4.1) except in the case when X is a singleton.

This is how the rest of the paper is organized. In Section 2 we establish an integration formula for the optimal reward operator M. Section 3 deals with the optimality equation which arises out of the problem of staying in a set forever and contains the proof of Theorem 1.1. Sections 4 and 5 contain the proofs of Theorem 1.2 and 1.3, respectively. In Section 6 we take up the regularity properties of the optimal reward operator M and extend Theorems 1.2 and 1.3 to the case where A is an analytic subset of X. Section 7 reconciles the countably additive setting of the present paper with the finitely additive framework of [17]. In particular, we prove that even if nonmeasurable strategies are allowed in the context of Theorems 1.2 and 1.3, the optimal reward operator assigns the same value to all the sets under consideration. Section 8 contains further remarks about the optimality equation introduced in Section 3.

2. Basic integration formula. We begin with a result on measurable selections which will be used several times in this article.

LEMMA 2.1. *Let Y and Z be Borel subsets of Polish spaces and let $E \subseteq Y \times Z$ be analytic with nonempty vertical sections. Suppose u: $E \to [0, 1]$ is an upper analytic function and set $v(y) = \sup\{u(y, z)\colon z \in E_y\}$, $y \in Y$, where E_y is the vertical section of E at y. Then v is upper analytic and, for each $\varepsilon > 0$, there is an analytically measurable selector f: $Y \to Z$ of E such that*

$$u(y, f(y)) > v(y) - \varepsilon, \qquad y \in Y.$$

PROOF. The first assertion is clear. For the second, choose $n \geq 1$ such that $1/n < \varepsilon$. Let

$$A_k = \{y \in Y\colon v(y) > (k-1)/n\}, \qquad k = 0, 1, \dots, n.$$

Since v is upper analytic, the sets A_k are analytic. Define

$$B_k = \{(y, z) \in Y \times Z\colon (y, z) \in E \quad \text{and} \quad u(y, z) > (k-1)/n\}, \qquad k = 0, 1, \dots, n.$$

Again the sets B_k are analytic. By the von Neumann selection theorem ([14], 4E.9, page 240), for each k, there is an analytically measurable function f_k: $A_k \to Z$ such that $(y, f_k(y)) \in B_k$ for all $y \in A_k$. Define

$$f(y) = f_k(y) \quad \text{if } y \in A_k - A_{k+1},\ k = 0, 1, \dots, n,$$

where $A_{n+1} = \varnothing$. Since $Y = \bigcup_{k=0}^{n}(A_k - A_{k+1})$, the function f is defined everywhere on Y. Plainly, f is analytically measurable. Finally, if $y \in A_k - A_{k+1}$, then

$$u(y, f(y)) = u(y, f_k(y)) > (k-1)/n \geq v(y) - 1/n > v(y) - \varepsilon. \qquad \square$$

LEMMA 2.2. *For each analytic set $B \subseteq H$, $M(B)$ is an upper analytic function.*

PROOF. Let $u(x, \sigma) = \sigma(B)$ for $\sigma \in \Sigma(x)$. Then, for $x \in X$,

$$M(B)(x) = \sup\{u(x, \sigma): \sigma \in \Sigma(x)\}.$$

Since u is upper analytic ([1]) and the set $\Sigma = \{(x, \sigma): \sigma \in \Sigma(x)\}$ is analytic ([3] or [19]), the first assertion of Lemma 2.1 implies that $M(B)$ is upper analytic. □

For each Borel stopping time τ and every h for which $\tau(h) < \infty$, define

$$M(B, \tau) = M(B, \tau)(h) = M(Bp_\tau(h))(h_\tau(h)),$$

which can be interpreted as the optimal return given the past up to time τ.

LEMMA 2.3. *If τ is a Borel stopping time and B a Borel subset of H, then the function $M(B, \tau)$ is upper analytic on the set $\{\tau < \infty\}$.*

PROOF. Let $u(h, \sigma) = \sigma(Bp_\tau(h))$ if $\tau(h) < \infty$ and $\sigma \in \Sigma(h_\tau)$. It is easy to verify that u is Borel measurable on the analytic set $E = \{(h, \sigma) \in H \times \mathbb{P}(H): \tau(h) < \infty \text{ and } \sigma \in \Sigma(h_\tau)\}$. Plainly,

$$M(B, \tau)(h) = \sup\{u(h, \sigma): \sigma \in E_h\} \quad \text{if } \tau(h) < \infty.$$

The conclusion now follows from the first assertion of Lemma 2.1 with $Y = \{\tau < \infty\}$. □

LEMMA 2.4. *There is a Borel measurable mapping ν: $\mathbb{P}(H) \times X \to \mathbb{P}(H)$ such that for each $\sigma \in \mathbb{P}(H)$, $\nu(\sigma, x)$ is a regular conditional σ-distribution of $(h_2, h_3, \dots)$ given $h_1 = x$.*

PROOF. See Lemma 2.2 of [12]. □

For $\sigma \in \mathbb{P}(H)$ and $x \in X$, we write $\sigma[x]$ for $\nu(\sigma, x)$; more generally, if $x_1, x_2, \dots, x_n \in X$, we shall write $\sigma[x_1, x_2, \dots, x_n]$ for $\sigma[x_1][x_2] \cdots [x_n]$. If $\sigma \in \mathbb{P}(H)$, σ_0 will denote the marginal distribution of σ on the first coordinate of H. This notation is identical with what we introduced in Section 1 for strategies, but it should be clear from the context which entity we have in mind.

We are now ready to establish the basic integration formula for the optimal reward operator M.

THEOREM 2.5. *Let B be a Borel subset of H and τ a Borel measurable stopping time such that $B \subseteq \{\tau < \infty\}$. Then*

$$M(B)(x) = \sup_{\sigma \in \Sigma(x)} \int_{\{\tau < \infty\}} M(B, \tau)\, d\sigma,$$

for all $x \in X$.

PROOF. (i) Lemma 2.3 ensures that the integral above on the right side is well defined.

(ii) Let $\sigma \in \Sigma(x)$. Then

$$\sigma(B) = \sigma(B \cap \{\tau < \infty\}) = \int_{\{\tau < \infty\}} \sigma[p_\tau](Bp_\tau)\, d\sigma \le \int_{\{\tau < \infty\}} M(B, \tau)\, d\sigma.$$

Take sup over $\sigma \in \Sigma(x)$ to get

$$M(B)(x) \le \sup_{\sigma \in \Sigma(x)} \int_{\{\tau < \infty\}} M(B, \tau)\, d\sigma.$$

(iii) Let $\varepsilon > 0$ and fix $x \in X$. In order to prove the opposite inequality, it suffices to find $\sigma^* \in \Sigma(x)$ such that

$$\sigma^*(B) \ge \sup_{\sigma \in \Sigma(x)} \int_{\{\tau < \infty\}} M(B, \tau)\, d\sigma - \varepsilon.$$

Let u and E be as in the proof of Lemma 2.3. An application of Lemma 2.1 yields an analytically measurable selector $\bar{\sigma}\colon \{\tau < \infty\} \to \mathbb{P}(H)$ of E such that

$$\bar{\sigma}(h)(Bp_\tau(h)) > M(B, \tau)(h) - \varepsilon/2,$$

for every $h \in \{\tau < \infty\}$.

Next choose $\hat{\sigma} \in \Sigma(x)$ such that

$$\int_{\{\tau < \infty\}} M(B, \tau)\, d\hat{\sigma} > \sup_{\sigma \in \Sigma(x)} \int_{\{\tau < \infty\}} M(B, \tau)\, d\sigma - \varepsilon/2.$$

Set $h^* = (x^*, x^*, \dots)$, where x^* is a fixed element of X and let

$$\tilde{\sigma}(p_\tau(h)) = \bar{\sigma}(p_\tau(h)h^*).$$

Then define σ^* to be the composition of $(\hat{\sigma}, \tau)$ with $\tilde{\sigma}$ as in [5]; that is,

$$\begin{aligned} \sigma_0^* &= \hat{\sigma}_0, \\ \sigma_n^*(h_1, h_2, \dots, h_n) &= \hat{\sigma}_n(h_1, h_2, \dots, h_n) \quad \text{if } 1 \le n < \tau(h_1, h_2, \dots, h_n, \dots), \\ &= (\tilde{\sigma}(p_\tau(h))[h_{\tau+1}, h_{\tau+2}, \dots, h_n])_0 \\ &\qquad \text{if } \tau(h_1, h_2, \dots, h_n, \dots) \le n \text{ and } (\tilde{\sigma}(p_\tau(h))[h_{\tau+1}, h_{\tau+2}, \dots h_n])_0 \in \Gamma(h_n), \\ &= g(h_n) \\ &\qquad \text{if } \tau(h_1, h_2, \dots, h_n, \dots) \le n \text{ and } (\tilde{\sigma}(p_\tau(h))[h_{\tau+1}, h_{\tau+2}, \dots, h_n])_0 \notin \Gamma(h_n), \end{aligned}$$

where $g\colon X \to \mathbb{P}(X)$ is an analytically measurable selector of Γ. Then σ^* is measurable, so $\sigma^* \in \Sigma(x)$. Finally,

$$\begin{aligned}\sigma^*(B) &= \int_{\{\tau<\infty\}} \sigma^*[p_\tau](Bp_\tau)\, d\sigma^* \\ &> \int_{\{\tau<\infty\}} M(B,\tau)\, d\sigma^* - \varepsilon/2 \\ &= \int_{\{\tau<\infty\}} M(B,\tau)\, d\hat{\sigma} - \varepsilon/2 \\ &> \sup_{\sigma\in\Sigma(x)} \int_{\{\tau<\infty\}} M(B,\tau)\, d\sigma - \varepsilon.\end{aligned}$$

This completes the proof. □

3. The optimality equation. If we set $B = A^\infty$ and $\tau \equiv 1$ in Theorem 2.5 and use the change of variable theorem, we get Lemma 3.1.

LEMMA 3.1.

$$M(A^\infty)(x) = \sup_{\gamma\in\Gamma(x)} \int_A M(A^\infty)(x')\, d\gamma(x'),$$

for each $x \in X$. *Moreover*, $0 \le M(A^\infty) \le 1$.

The optimal reward function for A^∞, therefore, satisfies the equation

$$u(x) = \sup_{\gamma\in\Gamma(x)} \int_A u(x')\, d\gamma(x'), \qquad x \in X. \tag{3.1}$$

We call (3.1) the optimality equation for the "staying in A forever" problem. We shall only concern ourselves here with solutions u of (3.1) such that $0 \le u \le 1$; so, in the sequel, a solution u of (3.1) will be assumed to satisfy $0 \le u \le 1$. We now show that $M(A^\infty)$ is the largest upper analytic solution of (3.1). Indeed, Lemma 3.2 establishes a slightly stronger result.

LEMMA 3.2. *Suppose* w: $X \to [0,1]$ *is an upper analytic function such that*

$$w(x) \le \sup_{\gamma\in\Gamma(x)} \int_A w(x')\, d\gamma(x'), \qquad x \in X. \tag{3.2}$$

Then, for any $x_0 \in X$ *and* $\varepsilon > 0$, *there is an analytically measurable* (*Markov*) *strategy* σ *at* x_0 *such that*

$$\sigma(A^\infty) \ge w(x_0) - \varepsilon.$$

Consequently, $M(A^\infty) \ge w$.

PROOF. Let $u(x,\gamma) = \int_A w(x')\, d\gamma(x')$, for $\gamma \in \Gamma(x)$, and let $E = \Gamma$. Then u is an upper analytic function (see, for instance, [1]) on the analytic set E. By Lemma 2.1, for each $m \ge 0$, there is an analytically measurable selector g_m:

$X \to \mathbb{P}(X)$ of Γ such that

$$u(x, g_m(x)) > \sup_{\gamma \in \Gamma(x)} \int_A w(x')\, d\gamma(x') - \varepsilon/2^{m+1},$$

for all $x \in X$, so that, by virtue of (3.2),

$$u(x, g_m(x)) > w(x) - \varepsilon/2^{m+1},$$

for all $x \in X$. Define a strategy σ at x_0 as follows:

$$\sigma_0 = g_0(x_0),$$
$$\sigma_m(x_1, x_2, \dots, x_m) = g_m(x_m), \qquad m \ge 1.$$

Then σ is an analytically measurable Markov strategy at x_0. Next define a sequence of random variables S_n on H as

$$S_0 = w(x_0),$$
$$S_n(h) = A^n(h) w(h_n) + \varepsilon(1 - 1/2^n), \qquad n \ge 1.$$

It is straightforward to verify that $\sigma S_n \ge S_0$ for all $n \ge 1$ (indeed, under σ, the sequence $\{S_n\}$ is a submartingale). Thus

$$\sigma(A^n(h) w(h_n)) + \varepsilon(1 - 1/2^n) \ge w(x_0), \qquad n \ge 1,$$

so

$$\sigma(A^n) + \varepsilon(1 - 1/2^n) \ge w(x_0), \qquad n \ge 1,$$

since $0 \le w \le 1$. Now let $n \to \infty$ and note $A^n \downarrow A^\infty$, so

$$\sigma(A^\infty) + \varepsilon \ge w(x_0). \qquad \square$$

COROLLARY 3.3. *$M(A^\infty)$ is the largest upper analytic solution of the optimality equation* (3.1).

We are now in a position to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Since $A^n \supseteq A^{n+1}$ and the optimal reward operator M is monotone, it follows that $M(A^n) \ge M(A^{n+1})$. Let $Q = \lim_n M(A^n)$. By Lemma 2.2, $M(A^n)$ is upper analytic. Consequently, since Q is the limit of a nonincreasing sequence of upper analytic functions, Q is also upper analytic ([7]). Furthermore, $0 \le Q \le 1$.

Again, as M is monotone, $Q \ge M(A^\infty)$. To prove the reverse inequality, it suffices by virtue of Lemma 3.2 to prove that Q satisfies (2). To see this, fix $x \in X$. Since $\Gamma(x)$ is finite, there is $\gamma^* \in \Gamma(x)$ and an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\sup_{\gamma \in \Gamma(x)} \int_A M(A^{n_i})(x')\, d\gamma(x') = \int_A M(A^{n_i})(x')\, d\gamma^*(x'), \qquad i \ge 1.$$

Now putting $B = A^{n_i+1}$ and $\tau \equiv 1$ in Theorem 2.5, we get

$$M(A^{n_i+1})(x) = \sup_{\gamma \in \Gamma(x)} \int_A M(A^{n_i})(x')\, d\gamma(x'), \qquad i \geq 1,$$

so that

$$M(A^{n_i+1})(x) = \int_A M(A^{n_i})(x')\, d\gamma^*(x'), \qquad i \geq 1.$$

Now let $i \to \infty$ and use dominated convergence to obtain

$$\begin{aligned} Q(x) &= \int_A Q(x')\, d\gamma^*(x') \\ &\leq \sup_{\gamma \in \Gamma(x)} \int_A Q(x')\, d\gamma(x'). \end{aligned}$$

Thus, Q satisfies (3.2) and the proof is complete. □

4. The proof of Theorem 1.2. In this section, we set

$$Q(x) = \inf\{M(A^t)(x) : t \text{ a Borel stop rule}\}, \qquad x \in X.$$

Since M is monotone and $A^\infty \subset A^t$ for each Borel stop rule t, it follows that $M(A^\infty) \leq Q$. So it remains only to prove that $M(A^\infty) \geq Q$.

We first consider the case when the fortune space X is countable. To prove $M(A^\infty) \geq Q$, it suffices to show, by virtue of Lemma 3.2, that Q is upper analytic and satisfies (3.2) of the aforementioned lemma. Since X is countable, Q is, in fact, Borel measurable. Now let $\varepsilon > 0$. For each $x \in X$, choose a Borel stop rule t_x such that

$$M(A^{t_x})(x) \leq Q(x) + \varepsilon.$$

Define a stop rule t^* by setting

$$t^*[x] = t_x.$$

Since X is countable, t^* is Borel measurable on H. Moreover, for any $x \in X$,

$$\begin{aligned} Q(x) \leq M(A^{t^*})(x) &= \sup_{\gamma \in \Gamma(x)} \int_A M(A^{t^*[x']})(x')\, d\gamma(x') \\ &= \sup_{\gamma \in \Gamma(x)} \int_A M(A^{t_{x'}})(x')\, d\gamma(x') \\ &\leq \sup_{\gamma \in \Gamma(x)} \int_A Q(x')\, d\gamma(x') + \varepsilon, \end{aligned}$$

where the first equality is by virtue of Theorem 2.5. Letting $\varepsilon \to 0$, we get (3.2) of Lemma 3.2, and the proof is complete.

When the fortune space X is uncountable, the proof given above runs into difficulties. First, it is far from clear that Q is (universally) measurable, far less that it is upper analytic. Second, the rough-and-ready method employed above to construct the stop rule t^* will not yield a Borel stop rule, unless the stop rules t_x are chosen with a good deal more circumspection and care. To get around the first difficulty, we shall define an upper analytic function on X, which satisfies (3.2) of Lemma 3.2 and majorizes Q. As for the second, fortunately, the methods of effective descriptive set theory are tailor-made for making careful choices. The rest of this section is, therefore, an excursion into descriptive set theory.

But first we recast the function Q into a more tractable form. So far in the paper we have worked with the product topology on $H = X^N$, when X is assigned the topology under which it is a Borel subset of a Polish space. We now introduce a second topology on H, viz., the product topology when X is assigned the discrete topology. In the sequel, the words "clopen," "open," "closed" and G_δ, when used to qualify subsets of H, will refer to the second topology; while the words Borel, analytic and coanalytic will refer to the first topology.

LEMMA 4.1. *Let E be an analytic subset of X and O a coanalytic, open subset of H such that $E^\infty \subseteq O$. Then there is a Borel stop rule t such that $E^t \subseteq O$.*

PROOF. Set $O_i = (E^i)^c \cup O$, $i \geq 1$. Then the sets O_i are coanalytic and open. Furthermore, $O_i \uparrow H$. Hence, by Proposition 2.6 in [11], there is a Borel stop rule t such that $O_t = H$. It follows that $E^t \subset O$. □

LEMMA 4.2. *For each $x \in X$,*

$$Q(x) = \inf\{M(K)(x) \colon K \text{ Borel, clopen and } K \supseteq A^\infty\}.$$

PROOF. The inequality $\geq$ from left to right follows from the fact that A^s is Borel and clopen for each Borel stop rule s. In the opposite direction, the inequality follows easily from Lemma 4.1. □

The characterization of Q above eliminates stop rules. The reason for doing this is that Borel, clopen subsets of H admit nice parametrizations, at least for suitable X.

Effective descriptive set theory takes place in Polish spaces which admit a smooth recursion theory. This is made precise in the next definition.

We say that a topological space Z is *Δ_1^1-recursively presented* if Z admits a complete metric d and a dense sequence $(r_n)_{n \in \omega}$ such that the relations

$$d(r_n, r_m) \leq p/q + 1 \quad \text{and} \quad d(r_n, r_m) < p/q + 1$$

are Δ_1^1 in ω^4. Examples of such spaces are 2, 2^ω, ω^ω, $[0,1]$ $[0,1]^\omega$, etc. (see Moschovakis [14], pages 128–135).

Suppose now that Z_1 and Z_2 are Δ_1^1-recursively presented compact metric spaces. Then $Z_1 \times Z_2$ and $\mathbb{P}(Z_1)$, the set of probability measures on Z_1, are again Δ_1^1-recursively presented compact metric spaces (Louveau [8], [10]).

In what follows, our terminology and notation, pertaining to concepts in effective descriptive set theory, are taken from Moschovakis [14].

LEMMA 4.3. *Let Z be a Δ_1^1-recursively presented compact metric space and let $\mathbf{N}$ be a nonempty Σ_1^1 set of probability measures on Z. For each subset E of Z, let*

$$\Phi(E) = \sup\{\mu^*(E): \mu \in \mathbf{N}\},$$

where μ^* is the outer measure induced by μ. If P is a Σ_1^1 subset of Z, then

$$\Phi(P) = \inf\{\Phi(B): B \text{ is } \Delta_1^1 \text{ and } B \supseteq P\}.$$

PROOF. The inequality from left to right being trivial, it remains to prove the inequality in the opposite direction. Assume that P is not Δ_1^1 and $\Phi(P) < a$, where a is a rational in $(0, 1)$. Now there is a function ϕ on $Z - P$ into the ordinals such that:

(a) The range of ϕ is an ordinal.

(b) For each $\xi < \omega_1^{\mathrm{CK}}$ (the Church–Kleene ω_1), the set $\{z \in Z - P: \phi(z) \leq \xi\}$ is Δ_1^1.

(c) The relation

$$z <_\phi^* z' \leftrightarrow z \in Z - P \quad \text{and} \quad [z' \in P \text{ or } \phi(z) < \phi(z')] \text{ is } \Pi_1^1.$$

(d) The relation

$$z \leq_\phi^* z' \Leftrightarrow z \in Z - P \quad \text{and} \quad [z' \in P \text{ or } \phi(z) \leq \phi(z')] \text{ is } \Pi_1^1.$$

The existence of ϕ can be proved by using the basic representation theorem for Π_1^1 sets ([14], 4A.3) and transferring the canonical norm on the set of ordinal codes ([14], 4A.2) to the set $Z - P$. Now fix such a ϕ.

Now consider the set

$$C = \left\{z \in Z: (\exists\, \mu \in \mathbf{N})\left(\mu\left(\{z': z' <_\phi^* z\}\right) \leq 1 - a\right)\right\}.$$

To see that C is Σ_1^1, imitate the computation in the proof of Theorem 2.2.3(a) in [6]. Furthermore, $C \subseteq Z - P$. For, if $z \in P$, then $\{z' \in Z: z' <_\phi^* z\} = Z - P$, so that $\mu(\{z' \in Z: z' <_\phi^* z\}) = \mu(Z - P) > 1 - a$ for every $\mu \in \mathbf{N}$, hence $z \notin C$. Consequently, by the first principle of separation for Σ_1^1 sets ([14], page 204), there is a Δ_1^1 set D such that $C \subseteq D \subseteq Z - P$. Now use the proof of the boundedness theorem ([14], 4A.4) to get $\xi < \omega_1^{\mathrm{CK}}$ such that $D \subseteq \{z \in Z - P: \phi(z) \leq \xi\}$. Since $Z - P$ is not Δ_1^1, it follows from (b) that the length of ϕ is at least ω_1^{CK}. Consequently, since ω_1^{CK} is a limit ordinal and the range of ϕ is an ordinal, $\xi + 1$ is in the range of ϕ. So there is $z_0 \in Z - P$ such that $\phi(z_0) = \xi + 1$, hence $z_0 \notin C$. It follows that

$$\mu(\{z \in Z - P: \phi(z) \leq \xi\}) = \mu(\{z \in Z: z <_\phi^* z_0\}) > 1 - a,$$

for every $\mu \in \mathbf{N}$.

Set $B^* = Z - \{z \in Z - P: \phi(z) \leq \xi\}$. So, by (b) above, B^* is Δ_1^1. Furthermore, $P \subseteq B^*$, and finally, $\mu(B^*) < a$ for every $\mu \in \mathbf{N}$. Hence, $\Phi(B^*) \leq a$. Since a is an arbitrary rational in (0, 1), this establishes the inequality. □

The analogous result for capacities was established by Louveau ([10], Corollary 2.5). It is not hard to see that Φ is, in general, not a capacity; it lacks, in general, the going down property of capacities on decreasing sequences of compact sets. Lemma 4.3 can also be deduced directly from a result of Barua on definable hereditary families of sets in his 1986 Ph.D. thesis.

We record for future use the boldface version of Lemma 4.3. The transition to the boldface result is routine and we omit the proof.

COROLLARY 4.4. *Suppose Z is a Borel subset of a Polish space and let $\mathbf{N}$ be a nonempty, analytic set of probability measures on Z. Let Φ be defined as in Lemma 4.3. If P is an analytic subset of Z, then*

$$\Phi(P) = \inf\{\Phi(B): B \textit{ is Borel and } B \supseteq P\}.$$

We now work in 2^ω, the space of sequences of 0's and 1's. Then, since 2^ω is a Δ_1^1-recursively presented compact metric space, so are $H = 2^\omega \times 2^\omega \times 2^\omega \times \cdots, 2^\omega \times [0,1]$, $\mathbb{P}(H)$ and $\mathbb{P}(2^\omega \times [0,1])$ (see [8], [10]).

Following Louveau ([9], page 13), we say that the pair (W, C) is a *coding of Borel subsets* of a Δ_1^1-recursively presented compact metric space Y if (a) W is a Π_1^1 subset of $\omega^\omega \times \omega$; (b) C is a Π_1^1 subset of $\omega^\omega \times \omega \times Y$ whose projection on $\omega^\omega \times \omega$ is W; (c) the set $\{(\alpha, n, y) \in \omega^\omega \times \omega \times Y: (\alpha, n) \in W \text{ and } (\alpha, n, y) \notin C\}$ is Π_1^1; (d) for fixed $(\alpha, n) \in \omega^\omega \times \omega$, the section $C_{\alpha,n} = \{y \in Y: (\alpha, n, y) \in C\}$ is $\Delta_1^1(\alpha)$; (e) if $\varnothing \neq P \subseteq Y$ is $\Delta_1^1(\alpha)$, then there is an n such that $(\alpha, n) \in W$ and $P = C_{\alpha,n}$.

Fix a coding (W, C) of Borel subsets of the Δ_1^1-recursively presented compact metric space H. We will be interested in $(\alpha, n) \in W$ such that $C_{\alpha,n}$ is Borel and clopen. Define $W^* \subseteq \omega^\omega \times \omega$ as follows:

$$(\alpha, n) \in W^* \leftrightarrow (\alpha, n) \in W,$$

$$\text{and } (\forall h)[(\alpha, n, h) \notin C \text{ or } (\exists m)(\forall h')(h' \equiv_m h \to (\alpha, n, h') \in C)],$$

$$\text{and } (\forall h)[(\alpha, n, h) \in C \text{ or } (\exists m)(\forall h')(h' \equiv_m h \to (\alpha, n, h') \notin C)].$$

Plainly, W^* is Π_1^1 and if $(\alpha, n) \in W$, then $(\alpha, n) \in W^*$ if and only if $C_{\alpha,n}$ is clopen. Set

$$C^* = C \cap (W^* \times H).$$

Then (i) C^* is Π_1^1 and the projection of C^* to $\omega^\omega \times \omega$ is W^*, (ii) the set $\{(\alpha, n, y) \in \omega^\omega \times \omega \times H: (\alpha, n) \in W^* \text{ and } (\alpha, n, h) \notin C^*\}$ is Π_1^1; (iii) $\varnothing \neq P \subseteq H$ is $\Delta_1^1(\alpha)$ and clopen if and only if $P = C_{\alpha,n}^*$ for some $(\alpha, n) \in W^*$.

Suppose now $\mathbf{N}$ is a nonempty, Σ_1^1 subset of $2^\omega \times \mathbb{P}(2^\omega)$ and $\mathbf{M}$ a nonempty, Σ_1^1 subset of $2^\omega \times \mathbb{P}(H)$ such that whenever $\mu \in \mathbf{M}_{x_0}$, $x_0 \in 2^\omega$, we have

$$\mu_0(\{x \in 2^\omega : \mu[x] \in \mathbf{M}_x\}) = 1 \quad \text{and} \quad \mu_0 \in \mathbf{N}_{x_0},$$

where $\mu[x]$, $x \in X$, is a version of the regular conditional distribution of $(h_2, h_3, \dots)$ given $h_1 = x$ (by courtesy of Lemma 2.4) and μ_0 is the distribution of h_1 under μ. For each Borel subset B of H, let

$$\Psi(B)(x) = \sup\{\mu(B) : \mu \in \mathbf{M}_x\}, \qquad x \in 2^\omega.$$

Finally, let A be a Δ_1^1 subset of 2^ω and define, for $x \in 2^\omega$,

$$R(x) = \inf\{\Psi(K)(x) : K\ \Delta_1^1(x), \text{clopen and } K \supseteq A^\infty\}.$$

LEMMA 4.5. *The set* $\{(x, a) \in 2^\omega \times [0,1] : R(x) \geq a\}$ *is* Σ_1^1 *in* $2^\omega \times [0,1]$.

PROOF. Observe that

$$R(x) \geq a \leftrightarrow (\forall\, r)(\forall\, n)\big[(r < a \text{ and } (x,n) \in W^* \text{ and } A^\infty \subset C^*_{x,n}) \\ \to (\exists\, \mu \in \mathbf{M}_x)(\mu(C^*_{x,n}) > r)\big],$$

where r runs through the rationals in $[0,1]$, and we think of 2^ω as being imbedded as a Π_1^0 subset of ω^ω. Using again the proof of Theorem 2.2.3(a) in [6], the facts recorded above about C^* and W^* and the hypothesis that $\mathbf{M}$ is Σ_1^1, it is easy to verify that the condition on the right of $\leftrightarrow$ is Σ_1^1, which completes the proof. □

We are now ready to establish the key result which will help us prove Theorem 1.2 when the fortune space X is uncountable.

THEOREM 4.6. *For each* $x \in 2^\omega$,

$$R(x) \leq \sup_{\gamma \in \mathbf{N}_x} \int_A R(x')\, d\gamma(x').$$

PROOF. Let $E = \{(x, a) \in 2^\omega \times [0,1] : R(x) \geq a\}$. By Lemma 4.5, E is Σ_1^1, hence so is $E \cap (A \times [0,1])$.

Fix $x_0 \in 2^\omega$ and $\varepsilon > 0$ rational. Now the set $\{\gamma \times \lambda : \gamma \in \mathbf{N}_{x_0}\}$ is a $\Sigma_1^1(x_0)$ set in $\mathbb{P}(2^\omega \times [0,1])$, where λ is Lebesgue measure on $[0,1]$. It now follows from the relativization of Lemma 4.3 to x_0 that there is a $\Delta_1^1(x_0)$ set $B \subseteq 2^\omega \times [0,1]$ such that $E \cap (A \times [0,1]) \subseteq B$ and

$$\begin{aligned} \sup_{\gamma \in \mathbf{N}_{x_0}} (\gamma \times \lambda)(B) &\leq \sup_{\gamma \in \mathbf{N}_{x_0}} (\gamma \times \lambda)(E \cap (A \times [0,1])) + \varepsilon/3 \\ &= \sup_{\gamma \in \mathbf{N}_{x_0}} \int_A R(x)\, d\gamma(x) + \varepsilon/3. \end{aligned} \tag{4.1}$$

Define $g\colon 2^\omega \to [0,1]$ by

$$g(x) = \lambda(B_x).$$

Note that g is $\Delta_1^1(x_0)$-recursive, $g \geq R$ on A and, from (4.1),

$$\sup_{\gamma \in \mathbf{N}_{x_0}} \int g(x)\, d\gamma(x) \leq \sup_{\gamma \in \mathbf{N}_{x_0}} \int_A R(x)\, d\gamma(x) + \varepsilon/3. \tag{4.2}$$

We now define a set $P \subset 2^\omega \times \omega$ as

$$(x,n) \in P \leftrightarrow (x,n) \in W^*, \quad \text{and} \quad A^\infty \subseteq C^*_{x,n},$$
$$\text{and} \quad (\forall\, \mu \in \mathbf{M}_x)(\mu(C^*_{x,n}) \leq g(x) + \varepsilon/3).$$

Using results from Kechris [6], it is straightforward to verify that P is a $\Pi_1^1(x_0)$ set. It now follows from the definition of R and the fact that $g \geq R$ on A that

$$(\forall\, x \in A)(\exists\, n)((x,n) \in P).$$

Hence, by the Kreisel selection theorem ([14], page 203), there is a $\Delta_1^1(x_0)$-recursive function $f\colon 2^\omega \to \omega$ such that

$$(\forall\, x \in A)((x, f(x)) \in P).$$

Define $K^* \subseteq H$ by

$$h \in K^* \leftrightarrow h_1 \in A \quad \text{and} \quad (h_1, f(h_1), h^*) \in C^*,$$

where $h^* = (h_2, h_3, \dots)$. Plainly, K^* is a $\Delta_1^1(x_0)$, clopen subset of H containing A^∞. Consequently, $R(x_0) \leq \Psi(K^*)(x_0)$. Choose $\mu \in \mathbf{M}_{x_0}$ such that

$$\mu(K^*) \geq R(x_0) - \varepsilon/3. \tag{4.3}$$

Now

$$\mu(K^*) = \int \mu[x](K^*x)\, d\mu_0(x). \tag{4.4}$$

If $x \notin A$, then $K^*x = \varnothing$, so $\mu[x](K^*x) = 0 \leq g(x)$. If $x \in A$ and if $\mu[x] \in \mathbf{M}_x$, then it follows from the definition of K^* that $\mu[x](K^*x) \leq g(x) + \varepsilon/3$. Consequently, by virtue of our hypothesis on $\mathbf{M}$, we have

$$\mu_0(\{x \in 2^\omega\colon \mu[x](K^*x) \leq g(x) + \varepsilon/3\}) = 1. \tag{4.5}$$

Using (4.2), (4.3), (4.4) and (4.5), we have

$$\begin{aligned} R(x_0) - \varepsilon/3 &\leq \int g(x)\, d\mu_0(x) + \varepsilon/3 \\ &\leq \sup_{\gamma \in \mathbf{N}_{x_0}} \int g(x)\, d\gamma(x) + \varepsilon/3 \\ &\leq \sup_{\gamma \in \mathbf{N}_{x_0}} \int_A R(x)\, d\gamma(x) + 2\varepsilon/3, \end{aligned}$$

where the second inequality is by virtue of the fact that $\mu_0 \in \mathbf{N}_{x_0}$. Thus,

$$R(x_0) \leq \sup_{\gamma \in \mathbf{N}_{x_0}} \int_A R(x)\, d\gamma(x) + \varepsilon.$$

The proof is completed by letting $\varepsilon \to 0$ and observing that x_0 is an arbitrary element of 2^ω. □

We now return to our gambling problem where we assume that our fortune space X is 2^ω. Recall that Γ is an analytic gambling house on X and that A is a Borel subset of X.

COROLLARY 4.7. *If $X = 2^\omega$, then there is an upper analytic function $\tilde{R}$: $X \to [0, 1]$ such that $\tilde{R} \geq Q$ and*

$$\tilde{R}(x) \leq \sup_{\gamma \in \Gamma(x)} \int_A \tilde{R}(x')\, d\gamma(x'),$$

for every $x \in X$.

PROOF. In Theorem 4.6, we take $\mathbf{N} = \Gamma$, $\mathbf{M} = \Sigma$ and $A = A$. Since Γ and Σ are analytic and A is Borel, there is an $\alpha \in \omega^\omega$ (indeed, α can be chosen to be in 2^ω) such that Γ and Σ are both $\Sigma_1^1(\alpha)$ and A is $\Delta_1^1(\alpha)$ (see [14], page 160). It is easy to verify that $\mathbf{N}$, $\mathbf{M}$ and A satisfy the hypotheses of Theorem 4.6 except that the definability conditions are satisfied with a parameter α. So, by relativizing Theorem 4.6 to the parameter α, we get the second inequality of the present corollary, if we define $\tilde{R}$ as

$$\tilde{R}(x) = \inf\{M(K)(x) \colon K\Delta_1^1(\alpha, x), \text{clopen and } K \supseteq A^\infty\},$$

for $x \in X$. Recall that, with our present choice of $\mathbf{N}$ and $\mathbf{M}$, Ψ is just M. That $\tilde{R}$ is upper analytic follows from a relativization of Lemma 4.5. Finally, since each $\Delta_1^1(\alpha, x)$ subset of H is a Borel set in H, the inequality $Q \leq \tilde{R}$ is clear and the proof is complete. □

It is now easy to complete the proof of Theorem 1.2. It only remains to establish the theorem when the fortune space X is uncountable. By the Borel isomorphism theorem ([15]), we may assume without loss of generality that $X = 2^\omega$. By Corollary 4.7 and Lemma 3.2, we then have

$$M(A^\infty) \geq \tilde{R} \geq Q.$$

This completes the proof of Theorem 1.2.

In the finitely additive setting of [17], a stronger statement is true (see Theorem 2 in [17]). The analogue of this statement in the present setting would assert that for every $\varepsilon > 0$, there is a Borel stop rule s such that

$$M(A^s)(x) \leq M(A^\infty)(x) + \varepsilon,$$

for every $x \in X$.

We conclude this section with an example which shows that the above statement does not hold in general. Our example is related to the basic example of [12].

Let $N = \{1, 2, \dots\}$ and let T be the collection of all stopping times on N^N. Give $N \cup \{\infty\}$ the usual compact topology and give T its topology of pointwise convergence. Then T is a compact metric space ([4]).

In our example, the fortune space X is $T \cup \{0\}$. The gambling house Γ is defined as

$$\begin{aligned} \Gamma(t) &= \{\delta(t[n]) : n \in N\} \quad &&\text{if } t \in T, \\ &= \{\delta(0)\} &&\text{if } t = 0. \end{aligned}$$

It is not difficult to verify that (X, Γ) is a Borel gambling problem. Take the set A to be T, so A is a Borel subset of X.

Let $\varepsilon = \frac{1}{2}$. Toward a contradiction, assume that τ is a Borel stop rule on $H = X^N$ such that

$$M(A^\tau)(x) \le M(A^\infty)(x) + \tfrac{1}{2},$$

for every $x \in X$.

Since $M(A^\infty)(t) = 0$ for every stop rule $t \in T$, the above inequality implies that $M(A^\tau)(t) \le \frac{1}{2}$ for every stop rule $t \in T$. We will arrive at a contradiction as soon as we show that there is a stop rule $t^* \in T$ such that $M(A^\tau)(t^*) = 1$.

To see this, note that, by a result of Dellacherie [2], the index $j(\tau) < \omega_1$. Choose a stop rule $t^* \in T$ such that $j(\tau) < j(t^*)$. Using the definition of index, one can define inductively positive integers $n_1, n_2, \dots, n_k$ such that

$$\begin{aligned} &j(\tau[t^*[n_1]][t^*[n_1][n_2]] \cdots [t^*[n_1][n_2] \cdots [n_i]]) \\ &\qquad < j(t^*[n_1][n_2] \cdots [n_i]), \qquad 1 \le i \le k, \end{aligned}$$

where k is the least positive integer i for which the left side is zero. It follows that

$$\tau(t^*[n_1], t^*[n_1][n_2], \dots, t^*[n_1][n_2] \cdots [n_k], \dots) = k.$$

Since $j(t^*[n_1][n_2] \cdots [n_k]) > 0$, each $t^*[n_1][n_2] \cdots [n_i]$ is in T. So $M(A^\tau)(t^*) = 1$.

5. The proof of Theorem 1.3. We assume that $X - A$ is nonempty. We define a new gambling house in which the problem of staying in A forever is equivalent to the problem of visiting A infinitely often in the original gambling house.

Fix an element $x^* \in X - A$ and denote by τ^* the time of first entrance into A. Define a mapping ψ: $H \to X$ by

$$\begin{aligned} \psi(h) &= h_{\tau^*} \quad &&\text{if } \tau^*(h) < \infty, \\ &= x^* &&\text{otherwise}. \end{aligned}$$

Let

$$\tilde{\Gamma}(x) = \{\sigma\psi^{-1}: \sigma \in \Sigma(x)\}, \qquad x \in X.$$

Since Σ is analytic ([3]) and the mapping $\sigma \to \sigma\psi^{-1}$ is Borel, it follows that $\tilde{\Gamma}$ is an analytic gambling house. We will denote by $\tilde{M}$ the optimal reward operator for $\tilde{\Gamma}$.

Let τ_n be the time of nth entrance into A, $n \geq 1$, so that $\tau_1 = \tau^*$. Define ϕ: $H \to H$ by

$$\begin{aligned} \phi(h) &= (x^*, x^*, \dots) && \text{if } \tau_1(h) = \infty, \\ &= (h_{\tau_1}, h_{\tau_2}, \dots, h_{\tau_n}, x^*, x^*, \dots) && \text{if } \tau_n(h) < \infty \text{ and } \tau_{n+1}(h) = \infty, \\ &= (h_{\tau_1}, h_{\tau_2}, \dots) && \text{if } \tau_i(h) < \infty, \text{ all } i \geq 1. \end{aligned}$$

Suppose now t is a Borel stop rule and let

$$O(t) = \phi^{-1}(A^t).$$

We record a few simple properties of ϕ and $O(t)$.

LEMMA 5.1. (a) *If* $\tau^*(h) < \infty$, *then* $\phi(p_{\tau^*}(h)h') = h_{\tau^*}\phi(h')$ *and* $O(t)p_{\tau^*}(h) = O(t[h_{\tau^*}])$.
(b) $O(t)$ *is Borel and open.*
(c) $\{A \text{ i.o.}\} \subseteq O(t) \subseteq \{\tau^* < \infty\}$.

The proof is straightforward and is omitted.

LEMMA 5.2. $M(O(t)) = \tilde{M}(A^t)$.

PROOF. We prove this by induction on $j(t)$. For $t \equiv 1$, $O(t) = \phi^{-1}(A \times X \times X \times \cdots) = \psi^{-1}(A)$, so the assertion of the lemma is true by virtue of the definition of $\tilde{\Gamma}$. Suppose now that the assertion is true for all Borel stop rules s such that $j(s) < j(t)$. Then, for any $x \in X$,

$$\begin{aligned} M(O(t))(x) &= \sup_{\sigma \in \Sigma(x)} \int_{\{\tau^* < \infty\}} M(O(t)p_{\tau^*})(h_{\tau^*})\, d\sigma \\ &= \sup_{\sigma \in \Sigma(x)} \int_{\{\tau^* < \infty\}} M(O(t[h_{\tau^*}]))(h_{\tau^*})\, d\sigma \\ &= \sup_{\sigma \in \Sigma(x)} \int_{\{\tau^* < \infty\}} \tilde{M}(A^{t[h_{\tau^*}]})(h_{\tau^*})\, d\sigma \\ &= \sup_{\gamma \in \tilde{\Gamma}(x)} \int_A \tilde{M}(A^{t[x']})\, d\gamma \\ &= \tilde{M}(A^t)(x). \end{aligned}$$

The first equality is by virtue of Theorem 2.5, the second is by Lemma 5.1, the third is courtesy of the induction hypothesis, the fourth is by the change of variable theorem and the definition of $\tilde{\Gamma}$ and the last equality is by virtue of one more application of Theorem 2.5. □

Lemma 5.3. *$M(\{A\ i.o.\}) \le \tilde{M}(A^\infty)$.*

Proof. By Theorem 2.5, for any $x \in X$,

$$M(\{A \text{ i.o.}\})(x) = \sup_{\sigma \in \Sigma(x)} \int_{\{\tau^* < \infty\}} M(\{A \text{ i.o}\})(h_{\tau^*})\, d\sigma$$

$$= \sup_{\gamma \in \tilde{\Gamma}(x)} \int_A M(\{A \text{ i.o.}\})\, d\gamma,$$

the second equality being a consequence of the change of variable theorem and the definition of $\tilde{\Gamma}$.

Thus, $M(\{A \text{ i.o.}\})$ is a solution of the optimality equation (3.1) of Section 3. Moreover, $0 \le M(\{A \text{ i.o}\}) \le 1$. Hence, by Corollary 3.3 applied to the gambling house $\tilde{\Gamma}$,

$$M(\{A \text{ i.o.}\}) \le \tilde{M}(A^\infty). \qquad \square$$

Lemma 5.4. *For every $\varepsilon > 0$ and $x_0 \in X$, there is a $\sigma^* \in \Sigma(x_0)$ such that*

$$\sigma^*(\{A\ i.o\}) \ge \tilde{M}(A^\infty)(x_0) - \varepsilon.$$

Consequently, $M(\{A\ i.o.\}) \ge \tilde{M}(A^\infty)$.

Proof. By Lemma 3.2 applied to the gambling house $\tilde{\Gamma}$, there is an analytically measurable (Markov) strategy σ at x_0 such that

$$\sigma(A^\infty) \ge \tilde{M}(A^\infty)(x_0) - \varepsilon.$$

Suppose $\sigma_m(x_1, x_2, \dots, x_m) = g_m(x_m)$, $m \ge 1$, where $g_m\colon X \to \mathbb{P}(X)$ is an analytically measurable selector of $\tilde{\Gamma}$. To complete the proof, we need to unravel σ into a strategy for the original gambling house Γ.

Let $f\colon \Sigma \to \tilde{\Gamma}$ be defined by

$$f(x, \sigma) = (x, \sigma\psi^{-1}).$$

Then f is Borel measurable and, since Σ is analytic, it follows from the von Neumann selection theorem that there is an analytically measurable function $\tilde{f}\colon \tilde{\Gamma} \to \Sigma$ such that $f(\tilde{f}(x, \gamma)) = (x, \gamma)$, for all $\gamma \in \tilde{\Gamma}(x)$, $x \in X$. Let $\tilde{f}_2(x, \gamma)$ be the second coordinate of $\tilde{f}(x, \gamma)$ for $\gamma \in \tilde{\Gamma}(x)$, $x \in X$. Choose a measurable strategy $\hat{\sigma}$ at x_0 such that $\hat{\sigma}\psi^{-1} = \sigma_0$.

Now define a strategy σ^* at x as follows:

$$\begin{aligned}
\sigma_0^* &= \hat{\sigma}_0,\\
\sigma_n^*(h_1, h_2, \dots, h_n) &= \hat{\sigma}_n(h_1, h_2, \dots, h_n) \quad \text{if } 1 \le n < \tau_1(h_1, h_2, \dots, h_n, \dots),\\
&= \left(\tilde{f}_2\left(h_{\tau_k}, g_k(h_{\tau_k})\right)\left[h_{\tau_k+1}, h_{\tau_k+2}, \dots, h_n\right]\right)_0\\
&\quad \text{if } \tau_k(h_1, h_2, \dots, h_n, \dots) \le n < \tau_{k+1}(h_1, h_2, \dots, h_n, \dots)\\
&\quad \text{and } \left(\tilde{f}_2\left(h_{\tau_k}, g_k(h_{\tau_k})\right)\left[h_{\tau_k+1}, h_{\tau_k+2}, \dots, h_n\right]\right)_0 \in \Gamma(h_n),\\
&= g(h_n)\\
&\quad \text{if } \tau_k(h_1, h_2, \dots, h_n, \dots) \le n < \tau_{k+1}(h_1, h_2, \dots, h_n, \dots)\\
&\quad \text{and } \left(\tilde{f}_2\left(h_{\tau_k}, g_k(h_{\tau_k})\right)\left[h_{\tau_k+1}, h_{\tau_k+2}, \dots, h_n\right]\right)_0 \notin \Gamma(h_n),
\end{aligned}$$

where g: $X \to \mathbb{P}(X)$ is an analytically measurable selector of Γ. It is easy to check that $\sigma^* \in \Sigma(x_0)$.

In order to complete the proof, we will verify that

$$\sigma^*(\{A \text{ i.o.}\}) = \sigma(A^\infty), \tag{5.1}$$

from which it will follow that $\sigma^*(\{A \text{ i.o}\}) \ge \tilde{M}(A^\infty)(x_0) - \varepsilon$. The second assertion of the lemma is an immediate consequence of this.

Now, for each $n \ge 1$, define ψ_n: $H \to X^n$ by

$$\begin{aligned}
\psi_n(h) &= \left(h_{\tau_1}, h_{\tau_2}, \dots, h_{\tau_n}\right) \quad \text{if } \tau_n(h) < \infty,\\
&= \left(h_{\tau_1}, h_{\tau_2}, \dots, h_{\tau_{i-1}}, x^*, \dots, x^*\right)\\
&\qquad \text{if } (\exists\, i)(1 \le i \le n, \tau_{i-1}(h) < \infty \text{ and } \tau_i(h) = \infty),
\end{aligned}$$

where $\tau_0 \equiv 0$. We claim that for each $n \ge 1$ and Borel set

$$E \subseteq \underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = A_n,$$

$$\sigma^*\left(\psi_n^{-1}(E)\right) = \sigma(E \times X \times \cdots). \tag{5.2}$$

Let us first deduce (5.1) from (5.2). Putting $E = A_n$ in (5.2), we get

$$\begin{aligned}
&\sigma^*(\{h \in H \colon h_i \in A \text{ for at least } n\ i\text{'s}\})\\
&\quad = \sigma^*\left(\psi_n^{-1}(A_n)\right)\\
&\quad = \sigma(A^n), \qquad n \ge 1.
\end{aligned}$$

Now let $n \to \infty$ and (5.1) falls out.

We prove (5.2) by induction on n. For $n = 1$,

$$\sigma^*\left(\psi_1^{-1}(E)\right) = \hat{\sigma}\left(\psi^{-1}(E)\right) = \sigma_0(E) = \sigma(E \times X \times X \times \cdots).$$

Suppose now (5.2) is true for $n = m$. It suffices to prove (5.2) with $n = m + 1$ for sets E of the form $E_1 \times E_2 \times \cdots \times E_m \times E_{m+1}$, where E_i is a Borel subset of A. Now calculate as follows:

$$\begin{aligned}\sigma^*(\psi_{m+1}^{-1}(E)) &= \int_{\{\tau_m<\infty\}} \sigma^*[p_{\tau_m}](\psi_{m+1}^{-1}(E)p_{\tau_m})\,d\sigma^* \\ &= \int_{\{\tau_m<\infty\}} E_1(h_{\tau_1})E_2(h_{\tau_2})\cdots E_m(h_{\tau_m}) \\ &\qquad \times \tilde{f}_2(h_{\tau_m}, g_m(h_{\tau_m}))(\psi^{-1}(E_{m+1}))\,d\sigma^* \\ &= \int_{\{\tau_m<\infty\}} E_1(h_{\tau_1})E_2(h_{\tau_2})\cdots E_m(h_{\tau_m})g_m(h_{\tau_m})(E_{m+1})\,d\sigma^* \\ &= \int_{A^m} E_1(h_1)E_2(h_2)\cdots E_m(h_m)g_m(h_m)(E_{m+1})\,d\sigma^*\psi_m^{-1} \\ &= \int_{A^m} E_1(h_1)E_2(h_2)\cdots E_m(h_m)g_m(h_m)(E_{m+1})\,d\sigma \\ &= \sigma(E_1\times E_2\times\cdots\times E_m\times E_{m+1}\times X\times X\times\cdots) \\ &= \sigma(E\times X\times X\times\cdots).\end{aligned}$$

The second equality is justified by using the definition of σ^*, the third by virtue of the fact that $f(h_{\tau_m}, \tilde{f}_2(h_{\tau_m}, g_m(h_{\tau_m}))) = (h_{\tau_m}, g_m(h_{\tau_m}))$, the fourth involves an application of the change of variable theorem and the fifth is by virtue of the induction hypothesis. □

We can now complete the proof of Theorem 1.3. Using Lemma 5.3, Lemma 5.4, Theorem 1.2 and Lemma 5.2, we have

$$\begin{aligned}M(\{A \text{ i.o.}\}) &= \tilde{M}(A^\infty) \\ &= \inf\{\tilde{M}(A^t): t \text{ a Borel stop rule}\} \\ &= \inf\{M(O(t)): t \text{ a Borel stop rule}\}.\end{aligned}$$

But by Lemma 5.1(b), each $O(t)$ above is Borel and open, so

$$M(\{A \text{ i.o.}\}) \geq \inf\{M(O): O \text{ Borel, open and } O \supseteq \{A \text{ i.o.}\}\}.$$

Clearly, if τ is a Borel stopping time, then $\{\tau < \infty\}$ is Borel and open; conversely, if O is Borel and open in H, then there exists a Borel stopping time τ such that $O = \{\tau < \infty\}$ (see [11]). It follows that

$$\begin{aligned}&M(\{A \text{ i.o.}\}) \\ &\qquad \geq \inf\{M(\{\tau<\infty\}): \tau \text{ a Borel stopping time and } \{A \text{ i.o.}\} \subset \{\tau<\infty\}\}.\end{aligned}$$

Since the inequality in the opposite direction is trivial, the proof of Theorem 1.3 is complete.

In the case of Theorem 1.3 as well, the following stronger statement, the analogue of which is true in the finitely additive setting (see Theorem 3 of [17]) fails to hold, in general, in the present setting.

For every $\varepsilon > 0$, there is a Borel, open set O containing $\{A \text{ i.o.}\}$ such that

$$M(O)(x) \le M(\{A \text{ i.o.}\})(x) + \varepsilon, \tag{5.3}$$

for every $x \in X$.

One sees that this may fail by considering the example given at the end of Section 4. Toward a contradiction, assume that (5.3) holds for some Borel, open set O containing $\{A \text{ i.o.}\}$ and $\varepsilon = 1/2$. Now $A^\infty \subseteq O$ and since $\delta(0)$ is the only gamble available at 0, $M(\{A \text{ i.o.}\}) = M(A^\infty)$. Hence, by Lemma 4.1, there is a Borel, stop rule τ on $H = X^N$ such that $A^\tau \subset O$. It now follows from (5.3) that

$$M(A^\tau) \le M(A^\infty) + \varepsilon,$$

the impossibility of which was established in Section 4.

6. Regularity. Theorems 1.2 and 1.3 can be viewed as regularity results for the optimal reward operator M. For instance, Theorem 1.2 states that, for each $x \in X$, the value at x of M on the Borel, closed set A^∞ is the infimum of the values at x of M on Borel, clopen sets containing A^∞. Similarly, Theorem 1.3 states that the value at x of M on the Borel, G_δ set $\{A \text{ i.o.}\}$ is the infimum of the values at x of M on Borel, open sets containing $\{A \text{ i.o.}\}$. A natural question then, is whether, given an arbitrary Borel subset B of H, the value at x of M on B is the infimum of its values at x on Borel, open sets containing B. We have partial answers to this question, which we now present in this section.

We begin by defining a new gambling problem. Let $\bar{X}$ be a disjoint copy of X. We will denote by $\bar{x} \in \bar{X}$ the copy of $x \in X$. Set

$$X^* = \bar{X} \cup X \cup X^2 \cup \cdots,$$

where the union above is a disjoint union and X^* is endowed with the union topology. Under this topology, X^* is a Borel subset of a Polish space. In the new gambling problem, X^* will serve as the space of fortunes. Next, we define a gambling house Γ^* on X^* as follows, where certain measures have been identified in an obvious way:

$$\Gamma^*(\bar{x}) = \Gamma(x) \quad \text{if } \bar{x} \in \bar{X};$$

$$\Gamma^*(p) = \left\{\gamma\phi_p^{-1} \colon \gamma \in \Gamma((p)_n)\right\} \quad \text{if } p \in X^n,\ n \ge 1,$$

where $(p)_i$ denotes the ith coordinate of p and $\phi_p \colon X \to X^{n+1}$ is defined by $\phi_p(x) = px$. It is easy to verify that Γ^* is an analytic gambling house on X^*.

Let M^* denote the optimal reward operator for Γ^* and for $p \in X^*$, let $\Sigma^*(p)$ denote the set of measurable strategies available at p in the new gambling problem. Set $H^* = X^* \times X^* \times \cdots$. Finally, define $\psi \colon H \to H^*$ by

$$\psi(h) = ((h_1), (h_1, h_2), (h_1, h_2, h_3), \ldots).$$

LEMMA 6.1. *Let B be a Borel subset of H^*. Then*

$$M^*(B)(\bar{x}) = M(\psi^{-1}(B))(x),$$

for each $\bar{x} \in \bar{X}$.

PROOF. Let $\sigma \in \Sigma(x)$. Then σ induces a strategy $\sigma^* \in \Sigma^*(\bar{x})$ as follows:

$$\sigma_0^* = \sigma_0,$$

and for $n \geq 1$,

$$\begin{aligned}\sigma_n^*(p_1, p_2, \dots, p_n) &= \sigma_{1(p_n)}(p_n)\phi_{p_n}^{-1} \quad \text{if } p_n \in \bigcup_{i=1}^{\infty} X^i,\\ &= \sigma_1(x) \qquad\qquad\quad \text{if } p_n = \bar{x} \in \bar{X},\end{aligned}$$

where $1(p_n)$ denotes the length of the finite sequence p_n. Plainly $\sigma^* = \sigma\psi^{-1}$ so that

$$\sigma(\psi^{-1}(B)) = \sigma^*(B) \leq M^*(B)(\bar{x}).$$

Take the sup over all $\sigma \in \Sigma(x)$ to obtain

$$M(\psi^{-1}(B))(x) \leq M^*(B)(\bar{x}).$$

Conversely, let $\sigma^* \in \Sigma^*(\bar{x})$. Thus σ^* induces a strategy $\sigma \in \Sigma(x)$ as follows:

$$\sigma_0 = \sigma_0^*,$$

and for $n \geq 1$,

$$\sigma_n(x_1, x_2, \dots, x_n) = \sigma_n^*((x_1), (x_1, x_2), \dots, (x_1, x_2, \dots, x_n))\pi_{n+1}^{-1},$$

where π_{n+1} is the projection function on X^{n+1} to the last coordinate. It is easy to check that $\sigma\psi^{-1} = \sigma^*$; consequently,

$$\sigma^*(B) = \sigma(\psi^{-1}(B)) \leq M(\psi^{-1}(B))(x).$$

Taking the sup over all $\sigma^* \in \Sigma^*(\bar{x})$, we get

$$M^*(B)(\bar{x}) \leq M(\psi^{-1}(B))(x). \qquad \square$$

LEMMA 6.2. *Let C be a Borel, closed subset of H. Then there is a Borel subset P of $\bigcup_{i=1}^{\infty} X^i$ such that*

$$C = \psi^{-1}(P^\infty).$$

PROOF. If $C = \varnothing$ or $C = H$, take $P = \varnothing$ or $\bigcup_{i=1}^{\infty} X^i$. Suppose then that $\varnothing \neq C \neq H$. Let $O = H - C$. By a result in [11], there is a Borel stopping time τ such that $O = \{\tau < \infty\}$. Let $S = \{p_\tau(h)\colon h \in O\}$. Then S is a Borel subset of $\bigcup_{i=1}^{\infty} X^i$ and $O = \psi^{-1}(\{h^* \in H^*\colon h_n^* \in S \text{ for some } n \geq 1\})$. Consequently, if we set $P = \bigcup_{i=1}^{\infty} X^i - S$, we get $C = \psi^{-1}(P^\infty)$. $\square$

THEOREM 6.3. *Let C be a Borel, closed subset of H. Then*

$$M(C)(x) = \inf\{M(K)(x): K \text{ Borel, clopen and } K \supseteq C\},$$

for each $x \in X$.

PROOF. By Lemma 6.2, there is a Borel set $P \subseteq \bigcup_{i=1}^{\infty} X^i$ such that $C = \psi^{-1}(P^\infty)$. Let $x \in X$. Then

$$\begin{aligned} M(C)(x) &= M(\psi^{-1}(P^\infty))(x) \\ &= M^*(P^\infty)(\bar{x}) \\ &= \inf\{M^*(K^*)(\bar{x}): K^* \text{ Borel, clopen and } K^* \supseteq P^\infty\} \\ &= \inf\{M(\psi^{-1}(K^*))(x): K^* \text{ Borel, clopen and } K^* \supseteq P^\infty\} \\ &\geq \inf\{M(K)(x): K \text{ Borel, clopen and } K \supseteq C\}. \end{aligned}$$

The second equality is by virtue of Lemma 6.1, the third is an application of Theorem 1.2, the fourth is yet another application of Lemma 6.1 and the last inequality is by virtue of the fact that ψ is continuous when both H and H^* are endowed with the product of discrete topologies on X and X^*, respectively. Since the inequality in the opposite direction is trivial, we are done. □

COROLLARY 6.4. *Let B be an analytic, closed subset of H. Then*

$$M(B)(x) = \inf\{M(K)(x): K \text{ Borel, clopen and } K \supseteq B\},$$

for each $x \in X$.

PROOF. Plainly, the left side is less than or equal to the right side. For the reverse inequality, fix $\varepsilon > 0$ and $x \in X$. By Corollary 4.4, find a Borel set D such that $B \subseteq D$ and

$$M(D)(x) < M(B)(x) + \varepsilon/2.$$

Now the analytic set B can be separated from the analytic set D^c by a closed set, viz., B itself. So, by Theorem 3.1 in [11], there is a Borel, closed set C such that $B \subseteq C \subseteq D$. Apply Theorem 6.3 to the Borel, closed set C to get a Borel, clopen set K_0 containing C such that

$$M(K_0)(x) < M(C)(x) + \varepsilon/2.$$

Consequently, $B \subseteq K_0$ and, since $M(C) \leq M(D)$,

$$M(K_0)(x) < M(B)(x) + \varepsilon.$$

So

$$\inf\{M(K)(x): K \text{ Borel, clopen and } K \supseteq B\} < M(B)(x) + \varepsilon.$$

As ε is arbitrary, the proof is complete. □

We now specialize Corollary 6.4 to obtain an extension of Theorem 1.2 to the case where A is an analytic subset of X.

COROLLARY 6.5. *Let A be an analytic subset of X. Then*

$$M(A^\infty)(x) = \inf\{M(A^t)(x): t \text{ a Borel stop rule}\},$$

for each $x \in X$.

PROOF. It suffices to prove that $M(A^\infty)$ is at least as large as the right side. Fix $\varepsilon > 0$ and $x \in X$. By Corollary 6.4, there is a Borel, clopen set K containing A^∞ such that

$$M(K)(x) < M(A^\infty)(x) + \varepsilon.$$

So, by Lemma 4.1, there is a Borel stop rule s such that $A^s \subset K$. Hence

$$M(A^s)(x) < M(A^\infty)(x) + \varepsilon.$$

The conclusion of the corollary now follows easily. □

We turn our attention now to special Borel, G_δ sets.

LEMMA 6.6. *Let $\{G_n\}$ be a nonincreasing sequence of Borel, open subsets of H. Set $G = \bigcap_{n=1}^\infty G_n$. Then there is a Borel subset P of $\bigcup_{i=1}^\infty X^i$ such that*

$$G = \psi^{-1}(\{P\ i.o.\}).$$

PROOF. For each n, let t_n be a Borel stopping time such that $G_n = \{t_n < \infty\}$. The existence of such a stopping time is proved in [11]. Set $\tau_n = \max\{t_1, t_2, \dots, t_n\} + n$. Then τ_n is a Borel stopping time, $G_n = \{\tau_n < \infty\}$ and $\tau_n < \tau_{n+1}$ on G_n.

Let $P_n = \{p_{\tau_n}(h)\colon h \in G_n\}$, $n \ge 1$. Then each P_n is a Borel subset of $\bigcup_{i=1}^\infty X^i$. We claim that the P_n's are disjoint. Toward a contradiction, assume that $P_k \cap P_m \neq \varnothing$ for some k, m with $k < m$. It follows that $p_{\tau_k}(h) = p_{\tau_m}(h')$ for some $h, h' \in H$, hence that $\tau_k(h) = \tau_m(h') = q$(say). So $h_i = h_i'$, $1 \le i \le q$, and therefore, $\tau_k(h) = \tau_m(h') = \tau_m(h)$, contradicting $\tau_k < \tau_m$. Set $P = \bigcup_{n=1}^\infty P_n$, so P is a Borel subset of $\bigcup_{i=1}^\infty X^i$.

To complete the proof, we have to show that $G = \psi^{-1}(\{P \text{ i.o.}\})$. Let, then, $h \in G$. Observe that, since $\tau_1 < \tau_2 < \cdots$ on G, $p_{\tau_i}(h)$ are all distinct and belong to P, so $(h_1, h_2, \dots, h_m) \in P$ for infinitely many m. Conversely, let $(h_1, h_2, \dots, h_{n_i}) \in P$ for all $i \ge 1$, where $1 \le n_1 < n_2 < \cdots$. Since the P_n's are disjoint, for each $i \ge 1$, there is a unique m_i such that $(h_1, h_2, \dots, h_{n_i}) \in P_{m_i}$, that is, $h \in G_{m_i}$. Furthermore, since $\tau_{m_i}(h) = n_i$, the m_i's are all distinct. Consequently, $h \in G_n$ for infinitely many n, so $h \in G$. This completes the proof. □

THEOREM 6.7. *Suppose that G is a countable intersection of Borel, open subsets of H. Then*

$$M(G)(x) = \inf\{M(O)(x): O \text{ Borel, open and } O \supseteq G\},$$

for every $x \in X$.

PROOF. As usual, it suffices to prove that the left side is at least as large as the right side. By Lemma 6.6, obtain a Borel subset P of $\bigcup_{i=1}^{\infty} X^i$ such that $G = \psi^{-1}(\{P \text{ i.o}\})$. Then, for any $x \in X$,

$$\begin{aligned} M(G)(x) &= M(\psi^{-1}(\{P \text{ i.o.}\}))(x) \\ &= M^*(\{P \text{ i.o.}\})(\bar{x}) \\ &= \inf\{M^*(O)(\bar{x}) \colon O \text{ Borel, open and } O \supseteq \{P \text{ i.o.}\}\} \\ &= \inf\{M(\psi^{-1}(O))(\bar{x}) \colon O \text{ Borel, open and } O \supseteq \{P \text{ i.o.}\}\} \\ &\geq \inf\{M(O)(x) \colon O \text{ Borel, open and } O \supseteq G\}. \end{aligned}$$

The second and fourth equalities are by virtue of Lemma 6.1 and the third involves an application of Theorem 1.3. □

We do not know if the result above is true for an arbitrary Borel, G_δ set. It would be true and would follow from Theorem 6.7 if a Borel, G_δ set could be written as a countable intersection of Borel, open sets. We do not know if this latter statement is true. However, we have the following extension of Theorem 6.7.

COROLLARY 6.8. *Let B be a countable intersection of analytic, open subsets of H. Then*

$$M(B)(x) = \inf\{M(O)(x) \colon O \textit{ Borel, open and } O \supseteq B\},$$

for each $x \in X$.

PROOF. Let $\varepsilon > 0$, $x \in X$. By Corollary 4.4, find a Borel set D such that $B \subseteq D$ and

$$M(D)(x) < M(B)(x) + \varepsilon/2.$$

Write $B = \bigcap_{n=1}^{\infty} B_n$, where each B_n is an analytic, open subset of H. It follows that $\bigcup_{n=1}^{\infty} B_n^c \cap D^c = D^c$. The sets $B_n^c \cap D^c$ are coanalytic. So, by the reduction principle [7], there exist (disjoint) Borel sets C_n such that $C_n \subseteq B_n^c \cap D^c$ and $\bigcup_{n=1}^{\infty} C_n = D^c$. Hence, for each n, B_n is an analytic set which can be separated from the analytic set C_n by an open set, viz., B_n itself. Consequently, by Theorem 3.1 in [11], there is a Borel, open set G_n such that $B_n \subseteq G_n$ and $G_n \cap C_n = \varnothing$. It follows that $B = \bigcap_{n=1}^{\infty} B_n \subseteq \bigcap_{n=1}^{\infty} G_n \subset D$.

Set $G = \bigcap_{n=1}^{\infty} G_n$. By Theorem 6.7, there is a Borel, open set O_0 containing G such that

$$M(O_0)(x) < M(G)(x) + \varepsilon/2.$$

Consequently, $B \subset O_0$ and

$$M(O_0)(x) < M(D)(x) + \varepsilon/2 < M(B)(x) + \varepsilon. \qquad \square$$

An immediate consequence is an extension of Theorem 1.3.

COROLLARY 6.9. *Let A be an analytic subset of X. Then*

$$M(\{A\, i.o\})(x) = \inf\{M(O)(x) : O \text{ Borel, open and } O \supseteq \{A\, i.o\}\},$$

for every $x \in X$.

7. Nonmeasurable strategies. As was explained in the introduction, this paper is a sequel to an earlier paper [17]. In fact, the two papers establish analogous results but in somewhat different mathematical settings. The earlier work took place in the general finitely additive framework of Dubins and Savage [5] and imposed no requirements of measurability on either sets or strategies. In this section, we study the relationship between the finitely additive results of [17] and the countably additive theory of this paper.

Assume that every gamble γ available in Γ is defined on all subsets of X as a finitely additive probability measure. We continue to assume that γ restricted to the Borel subsets of X is countably additive. Thus we can evaluate such quantities as $\sigma(A^\infty)$ and $\sigma(\{A \text{ i.o.}\})$ for strategies σ which are not necessarily measurable by using the Dubins and Savage definition of the measure σ as extended by Purves and Sudderth [16]. If the strategy σ is measurable and A is analytic, then these quantities have the same value when calculated in either setting (Theorem 6.1 of [16]).

An interesting question, which was posed by Dubins and Savage [5], is whether a gambler can do better if allowed to use nonmeasurable strategies or whether, to the contrary, measurable strategies are adequate. The next two theorems establish (for Borel problems in which the goal is to stay forever in an analytic set or visit an analytic set infinitely open) that a gambler can without loss restrict attention to measurable strategies.

Denote by Γ^∞ the optimal reward operator when both measurable and nonmeasurable strategies are allowed.

THEOREM 7.1. *If X is a Borel subset of a Polish space, Γ an analytic gambling house on X and A an analytic subset of X, then*

$$\Gamma^\infty(A^\infty) = M(A^\infty).$$

PROOF. Plainly, $\Gamma^\infty(A^\infty) \geq M(A^\infty)$. For the opposite inequality, we claim that for any Borel, clopen set K,

$$\Gamma^\infty(K) = M(K).$$

The claim is easily established by induction on the structure of K (see [5] for the definition of structure) and the formulas

$$\Gamma^\infty(K)(x) = \sup_{\gamma \in \Gamma(x)} \int \Gamma^\infty(Kx')(x')\, d\gamma(x'),$$

$$M(K)(x) = \sup_{\gamma \in \Gamma(x)} \int M(Kx')(x')\, d\gamma(x'),$$

where the first is proved in [17] and the second is a special case of Theorem 2.5

of this paper (see, also, Theorem 6.4 in [20]). Consequently, we have

$$\begin{aligned}\Gamma^\infty(A^\infty) &\le \inf\{\Gamma^\infty(K): K \text{ clopen and } K \supseteq A^\infty\}\\ &\le \inf\{\Gamma^\infty(K): K \text{ Borel, clopen and } K \supseteq A^\infty\}\\ &= \inf\{M(K): K \text{ Borel, clopen and } K \supseteq A^\infty\}\\ &= M(A^\infty),\end{aligned}$$

where the last equality is by virtue of Corollary 6.4. □

THEOREM 7.2. *Under the same hypotheses as in Theorem* 7.1,

$$\Gamma^\infty(\{A\ i.o.\}) = M(\{A\ i.o.\}).$$

PROOF. Again it is obvious that $\Gamma^\infty(\{A \text{ i.o.}\}) \ge M(\{A \text{ i.o.}\})$. The proof of the inequality in the opposite direction is similar to the proof in Theorem 7.1, except that we use Corollary 6.9 instead of Corollary 6.4 and verify the equality

$$\Gamma^\infty(O) = M(O),$$

for any Borel set O.

To prove this, consider the gambling problem (X^*, Γ^*) of Section 6. Use the proof of Lemma 6.2 to find a Borel subset S of $\bigcup_{i=1}^\infty X^i$ such that

$$O = \psi^{-1}(\{h^* \in H^*: h_n^* \in S \text{ for some } n \ge 1\}).$$

The above equality then becomes

$$\begin{aligned}&\Gamma^{*\infty}(\{h^* \in H^*: h_n^* \in S \text{ for some } n \ge 1\})(\bar{x})\\ &\qquad = M^*(\{h^* \in H^*: h_n^* \in S \text{ for some } n \ge 1\})(\bar{x}),\end{aligned}$$

for every $\bar{x} \in \overline{X}$.

But this last equality is true, because it concerns a leavable, analytic gambling problem of the type considered by Strauch [18] (see also Sudderth [19]) with utility function u equal to the indicator of S, where it is known that measurable strategies are adequate. □

For an interesting consequence of Theorem 7.2, the reader is referred to Monticino ([13], Theorem 4.2).

8. The optimality equation (continued). As in Section 7, we will assume in this section that we have an analytic gambling house Γ on a Borel subset X of a Polish space, where each gamble γ is defined as a finitely additive probability measure on all subsets of X. Consequently, we are able to admit nonmeasurable solutions of the optimality equation of Section 3, since the integral $\int u\, d\gamma$ is well defined for any gamble γ and any function u: $X \to [0, 1]$. Using the results of Section 7, we now sharpen Corollary 3.3. As in Section 3, we consider only solutions u of the optimality equation which satisfy $0 \le u \le 1$.

THEOREM 8.1. *Let A be an analytic subset of X. Then $M(A^\infty)$ is the largest solution of the equation*

$$(8.1) \qquad u(x) = \sup_{\gamma \in \Gamma(x)} \int_A u(x')\, d\gamma(x'), \qquad x \in X.$$

PROOF. It is proved in [17] that $\Gamma^\infty(A^\infty)$ is the largest solution of (8.1). Now use Theorem 7.1 to complete the proof. □

There is also an optimality equation for the visiting A infinitely often problem. Consider the equation

$$(8.2) \qquad u(x) = \sup_{\sigma \in \overline{\Sigma}(x)} \int_{\{\tau^* < \infty\}} u(h_{\tau^*})\, d\sigma, \qquad x \in X,$$

where $u\colon X \to [0, 1]$, τ^* is the time of first entrance into A and $\overline{\Sigma}(x)$ is the set of all strategies (including nonmeasurable ones) available at x. We then have a result analogous to Theorem 8.1.

THEOREM 8.2. *Let A be an analytic subset of X. Then $M(\{A\ i.o\})$ is the largest solution of* (2).

PROOF. From [17], we know that $\Gamma^\infty(\{A \text{ i.o.}\})$ is the largest solution of (2). But, by Theorem 7.2, $M(\{A \text{ i.o}\}) = \Gamma^\infty(\{A \text{ i.o}\})$, so we are done. □

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