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HYPERCONTRACTION METHODS IN MOMENT INEQUALITIES FOR SERIES OF INDEPENDENT RANDOM VARIABLES IN NORMED SPACES[1]

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We prove that if (θ_k) is a sequence of i.i.d. real random variables then, for $1 < q < p$, the linear combinations of (θ_k) have comparable pth and qth moments if and only if the joint distribution of (θ_k) is (p, q)-hypercontractive. We elaborate hypercontraction methods in a new proof of the inequality

$$\left(E\left\|\sum_i X_i\right\|^p\right)^{1/p} \le C_p\left(E\left\|\sum_i X_i\right\| + \left(E\sup_i\|X_i\|^p\right)^{1/p}\right),$$

where (X_i) is a sequence of independent zero-mean random variables with values in a normed space, and $C_p \approx p/\ln p$.

1. Introduction. A method of comparing moments of sums of random variables that we are dealing with stems from the theory of hypercontractive semigroups.

We shall use a convexity-type inequality

$$(1.1)\qquad \left(\int_S\left(\int_T|f(s,t)|^q\mu(dt)\right)^{p/q}\nu(ds)\right)^{1/p} \le \left(\int_T\left(\int_S|f(s,t)|^p\nu(ds)\right)^{q/p}\mu(dt)\right)^{1/q},$$

which holds for every positive measure space $(S, \mathscr{S}, \mu), (T, \mathscr{T}, \nu)$, and any measurable function $f(s,t)$ whenever $0 < q < p < \infty$. (1.1) can be easily seen since, substituting $g(s,t) = |f(s,t)|^q$ and $r = p/q > 1$, we have

$$\begin{aligned}\left(\int_S\left(\int_T g(s,t)\mu(dt)\right)^r\nu(ds)\right)^{1/r} &= \sup_{\int_S|h(s)|^{r'}\nu(ds)\le 1}\int_S\int_T g(s,t)\mu(dt)\cdot h(s)\nu(ds)\\ &\le \int_T\left\{\sup_h\int_S g(s,t)\cdot h(s)\nu(ds)\right\}\mu(dt)\\ &= \int_T\left(\int_S g(s,t)^r\nu(ds)\right)^{1/r}\mu(dt),\end{aligned}$$

where $1/r + 1/r' = 1$.

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For a couple of random variables X and Y with values in a Banach space F let us write $X \prec_{p,q} Y$ if, for every $x \in F$,

$$(E\|x + X\|^p)^{1/p} \le (E\|x + Y\|^q)^{1/q}.$$

If $q < p$ and (X_1, Y_1) and (X_2, Y_2) are independent random vectors with values in F^2, and if $X_i \prec_{p,q} Y_i$, $i = 1, 2$, then

$$X_1 + X_2 \prec_{p,q} Y_1 + Y_2.$$

To see this, we put $E_i = E[\cdot|X_i, Y_i]$, and using the inequality (1.1) combined with Fubini's theorem, we infer that

$$\begin{aligned}(E\|x + X_1 + X_2\|^p)^{1/p} &= (E_1E_2\|x + X_1 + X_2\|^p)^{1/p}\\ &\le \left(E_1(E_2\|x + Y_1 + X_2\|^q)^{p/q}\right)^{1/p}\\ &\le \left(E_2(E_1\|x + Y_1 + X_2\|^p)^{q/p}\right)^{1/q}\\ &\le (E_2E_1\|x + Y_1 + X_2\|^q)^{1/q}\\ &= (E\|x + Y_1 + Y_2\|^q)^{1/q}.\end{aligned}$$

Thus, for a sequence of independent couples of random variables such that $X_i \prec_{p,q} Y_i$, $i = 1, \dots, n$, we have

$$\sum_{i=1}^n X_i \prec_{p,q} \sum_{i=1}^n Y_i.$$

In particular, if there is a constant σ such that $\sigma X_i \prec_{p,q} X_i$, $i = 1, \dots, n$, we obtain a comparison inequality for moments

$$\left(E\left\|\sum_{i=1}^n X_i\right\|^p\right)^{1/p} \le \sigma^{-1}\left(E\left\|\sum_{i=1}^n X_i\right\|^q\right)^{1/q}.$$

This justifies the following definitions [introduced by Krakowiak and Szulga (1988)]. A real random variable $\theta \in \mathrm{HC}(p, q, F, \sigma)$ if $\sigma y\theta \prec_{p,q} y\theta$ for all $y \in F$, or equivalently,

$$(E\|x + \sigma y\theta\|^p)^{1/p} \le (E\|x + y\theta\|^q)^{1/q} \quad \text{for all } x, y \in F. \tag{1.2}$$

We shall write $\theta \in \mathrm{HC}(p, q, F)$ if $\theta \in \mathrm{HC}(p, q, F, \sigma)$ for some $\sigma > 0$.

Therefore, the preceding argument shows that whenever (1.2) holds, and if (θ_i) is a sequence of independent copies of θ, then

$$\left(E\left\|\sum_i y_i\theta_i\right\|^p\right)^{1/p} \le \sigma^{-1}\left(E\left\|\sum_i y_i\theta_i\right\|^q\right)^{1/q} \quad \text{for all } y_i \in F. \tag{1.3}$$

Moreover, the same argument yields

$$(1.4)\qquad \begin{aligned} &\left(E\left\| y_0 + \sum_{k=1}^{\infty} \sum_{1 \le i_1 < \cdots < i_k} y_{i_1,\dots,i_k}\theta_{i_1} \cdots \theta_{i_k}\right\|^p\right)^{1/p} \\ &\quad \le \left(E\left\| y_0 + \sum_{k=1}^{\infty} \sigma^{-k} \sum_{1 \le i_1 < \cdots < i_k} y_{i_1,\dots,i_k}\theta_{i_1} \cdots \theta_{i_k}\right\|^q\right)^{1/q}, \end{aligned}$$

for all $\{y_{i_1,\dots,i_k}\} \subset F$. This method can be traced to Nelson (1966, 1973) who proved that (1.4) holds if θ is a centered Gaussian random variable, $F = \mathbb{R}$ and the hypercontractivity constant $\sigma = \sqrt{(q-1)/(p-1)}$. For a two-point symmetric random variable, (1.2) and (1.4) were proved by Bonami (1970) and Gross (1973) with the same constant σ, from which Nelson's result follows by CLT. Borell (1979) observed that these results can be carried over to an arbitrary normed space F. Later on, Borell (1984), and then Krakowiak and Szulga (1988), extended the discussed inequalities to a larger class of random variables.

In Section 2 we shall prove, in particular, that (1.2) holds in every normed space F (respectively, for $F = \mathbb{R}$) if and only if (1.3) is fulfilled in all normed spaces F (respectively, for $F = \mathbb{R}$) and we shall characterize the underlying distribution of θ.

In Section 3 we shall apply hypercontraction methods in a simple proof of the inequality

$$(1.5)\qquad \left(E\left\|\sum_i X_i\right\|^p\right)^{1/p} \le C_p\left(\left(E\left\|\sum_i X_i\right\|^2\right)^{1/2} + \left(E\sup_i \|X_i\|^p\right)^{1/p}\right),$$

where (X_i) is a sequence of independent random variables with values in a Banach space F with $EX_i = 0$, $i = 1, 2, \dots$, and $C_p \approx p/\ln p$ when $p \to \infty$. Inequality (1.5) is due to Johnson, Schechtman and Zinn (1985) when $F = \mathbb{R}$, and to Talagrand (1989) when F is a Banach space.

As an immediate consequence we obtain the condition

$$(1.6)\qquad E\exp\{\alpha W \ln_+ W\} < \infty \quad \text{or} \quad E\Gamma(1 + \alpha W) < \infty,$$

where $W = \sup_n \|S_n\|$, X_i's are uniformly bounded and $\alpha \cdot \sup_i \|X_i\|_\infty < 1$. Talagrand (1989) proved (1.6) by employing isoperimetric-type inequalities.

2. Linear combinations of independent random variables. In the sequel, (θ_k) is a sequence of independent copies of a random variable θ.

THEOREM 2.1. *Let $p > 1$ and θ be a real symmetric random variable. Then the following conditions are equivalent*:

(i) *$\theta \in HC(p, q, \mathbb{R})$ for some (equivalently, for all) $q \in (1, p)$.*

(ii) *For each (equivalently, for some) $q \in (0, p)$ there is a constant $C > 0$ such that for every sequence (a_k) of real numbers*

$$\left(E\left|\sum a_k\theta_k\right|^p\right)^{1/p} \le C\left(E\left|\sum a_k\theta_k\right|^q\right)^{1/q}.$$

(iii) *For each* $s > 0$ *there exists* $K > 0$ *such that*

$$E(|t\theta|^p - 1)^+ \le KE(|t\theta|^2 \wedge 1),$$

for all $t \le s$.

THEOREM 2.2. *Let* $p > 1$ *and* θ *be a symmetric random variable. Then the following conditions are equivalent*:

(i) $\theta \in \mathrm{HC}(p, q, F)$ *for some (equivalently, for all)* $q \in (1, p)$ *and for every Banach space* F.

(ii) *For each (equivalently, for some)* $q \in (0, p)$ *there is a constant* $C > 0$ *such that for every sequence* (x_k) *of vectors in an arbitrary Banach space* F,

$$\left(E\left\|\sum x_k\theta_k\right\|^p\right)^{1/p} \le C\left(E\left\|\sum x_k\theta_k\right\|^q\right)^{1/q}.$$

(iii) *For each* $s > 0$ *there exists* $K > 0$ *such that*

$$E(|t\theta|^p - 1)^+ \le KP(|t\theta| > 1),$$

for all $t \le s$.

PROOF OF THEOREM 2.2. The equivalence of (ii) and (iii) is known and can be found, for example, in Vakhania, Tarieladze and Chobanyan (1987), Theorem 5.1 and Remark 1. Implication (i) $\Rightarrow$ (ii) is explained in the Introduction. Thus, it is enough to prove (iii) $\Rightarrow$ (i), i.e., that (1.2) holds. Without loss of generality we may assume that $\|x\| = 1$. Put $t = \|y\|$ and let $\sigma < \frac{1}{2}(E|\theta|^q)^{1/q}/(E|\theta|^p)^{1/p}$. Since

$$(E\|x + \sigma y\theta\|^p)^{1/p} \le 1 + \sigma t(E|\theta|^p)^{1/p},$$
$$(E\|x + y\theta\|^q)^{1/q} \ge t(E|\theta|^q)^{1/q} - 1,$$

the inequality (1.2) is fulfilled if $t > 4/(E|\theta|^q)^{1/q} =: C$. So, assume that $t < C$ and let (iii) hold for all $t \le C/2$ with a constant K. Since $s \mapsto E\|x + sy\theta\|^p - 1$ is a convex symmetric function vanishing at 0, we have, for $4\sigma < 1$,

$$E\|x + \sigma y\theta\|^p - 1 \le 4\sigma(E\|x + y\theta/4\|^p - 1) \le 4\sigma(A + B), \tag{2.1}$$

where

$A = E(\|x + y\theta/4\|^p - 1)\mathbb{1}_{\{\|y\theta\| \le 4\}}$ and $B = E(\|x + y\theta/4\|^p - 1)\mathbb{1}_{\{\|y\theta\| > 4\}}$.

Since

$$\frac{a^p + b^p}{2} - 1 \le C_{p,q}\left(\frac{a^q + b^q}{2} - 1\right),$$

for $0 \le a, b \le 2$, $a + b \ge 2$, where $C_{p,q} = [p(p-1)2^p - 2]/[q(q-1)2^q - 2]$, we infer that

$$\begin{aligned} A &= E\left(\frac{\|x + y\theta/4\|^p + \|x - y\theta/4\|^p}{2} - 1\right)\mathbb{1}_{\{\|y\theta\| \le 4\}} \\ &\le C_{p,q}E\left(\frac{\|x + y\theta/4\|^q + \|x - y\theta/4\|^q}{2} - 1\right)\mathbb{1}_{\{\|y\theta\| \le 4\}} \\ &\le C_{p,q}E(\|x + y\theta\|^q - 1). \end{aligned} \tag{2.2}$$

Since $(1 + x/2)^p \le x^p$ for $x > 2$, then (ii) yields

$$\begin{aligned} B &\le E\big((1 + |t\theta/4|)^p - 1\big)\mathbb{1}_{\{|t\theta| > 4\}} \\ &\le E\big(|t\theta/2|^p - 1\big)^+ \le KP(|t\theta/2| > 1) \\ &= KP(\|y\theta\| > 2) \le KE(\|x + y\theta\|^q - 1) \end{aligned} \tag{2.3}$$

because

$$\mathbb{1}_{\{\|y\theta\| > 2\}} \le \frac{\|x + y\theta\|^q + \|x - y\theta\|^q}{2} - 1 \quad \text{a.s.}$$

Combining (2.1), (2.2) and (2.3), we obtain the estimate

$$E\|x + \sigma y\theta\|^p - 1 \le 4\sigma(K + C_{p,q})E(\|x + y\theta\|^q - 1).$$

Therefore, for $\sigma \le (4(K + C_{p,q}))^{-1}$, we get

$$\begin{aligned} (E\|x + \sigma y\theta\|^p)^{1/p} &= (E\|x + \sigma y\theta\|^p - 1 + 1)^{1/p} \\ &\le (E\|x + y\theta\|^q)^{1/p} \le (E\|x + y\theta\|^q)^{1/q}, \end{aligned}$$

because $E\|x + y\theta\|^q > 1$. This completes the proof of Theorem 2.2. □

REMARK. The implication (iii) ⇒ (ii) follows also from Krakowiak and Szulga (1988), Theorem 3.12.

PROOF OF THEOREM 2.1. We proceed almost identically as in the proof of Theorem 2.2, except for the following differences. Equivalence of (ii) and (iii) is known, and belongs to Dacunha-Castelle (1970).

To show (iii) ⇒ (i), we may assume that $x = 1$ and $y = t > 0$. The only modification is to use condition (ii) of Theorem 2.1 (instead of that of Theorem 2.2) in the proof of a counterpart of (2.3). If K is such that (ii) of Theorem 2.1 holds for $t < C/2$, then

$$E(|t\theta/2|^p - 1)^+ \le KE\big(|t\theta/2|^2 \wedge 1\big) \le C_q KE(|1 + t\theta|^q - 1),$$

because $1 \wedge (u/2)^2 \le C_q((|1 + u|^q + |1 - u|^q)/2 - 1)$ for every $u \in \mathbb{R}$, where C_q is a positive constant $[C_q < (4q(q - 1))^{-1}]$.

We repeat the final argument with $\sigma \le (4(KC_q + C_{p,q}))^{-1}$. □

If θ is such that $N(x) = P(|\theta| > x)$ is a regularly varying function with an exponent $p_\theta > p$, then θ fulfills conditions (iii) of Theorems 2.1 and 2.2 and thus $\theta \in \mathrm{HC}(p, q, F)$ for each $q \in (1, p)$ and every Banach space F. If $p > 2$ and $E|\theta|^p < \infty$, then $\theta \in \mathrm{HC}(p, q, \mathbb{R})$ for each $q \in (1, p)$. For a random variable θ with the tail distribution $N(x) = P(|\theta| > x) \approx (x \log x)^{-2}$, we check that $\theta \in \mathrm{HC}(2, q, \mathbb{R})$ and $\theta \notin \mathrm{HC}(2, q, c_0)$ for all $q \in (1, 2)$, where c_0 is the space of all real null sequences with the supremum norm.

The question, whether, in a fixed normed space, hypercontractivity of a random variable with parameters p and q, $1 < q < p < \infty$, and comparability of pth and qth moments of linear combinations of independent copies of θ are equivalent, is open. It seems plausible that hypercontractivity of a random variable would depend on its distribution and, as well, on a geometry of the underlying normed space. Some information may be found in Krakowiak and Szulga (1988). Namely, if $1 < q < p$ and F is a two-convex normed space, then a random variable $\theta \in \mathrm{HC}(p, q, F)$ if and only if $\theta \in \mathrm{HC}(p, q, \mathbb{R})$. Therefore, in this case, we can use Theorem 2.1, so the answer to the problem is positive.

The obstacle to carry over the results on hypercontractivity to the case when $q < 1$, is that it can be $E\|x + y\theta\|^q < \|x\|^q$. For example, if $E\theta^2 < \infty$ and $q < 1$ then $\theta \notin \mathrm{HC}(p, q, \mathbb{R})$. Examples of hypercontractive random variables with parameter $q < 1$ refer to α-stable distributions [see Szulga (1990)]. We shall show that if $q \le 1$ then hypercontractivity is an absolute feature of a random variable regardless of a norm.

PROPOSITION 2.3. *Let $0 < q \le 1$, $q < p$ and θ be a random variable. If $\theta \in \mathrm{HC}(p, q, \mathbb{R}, \sigma)$ for some $\sigma > 0$, then $\theta \in \mathrm{HC}(p, q, F, \sigma)$ for every normed space F.*

PROOF. Ferguson (1962) [independently, Herz (1963)] proved that any two-dimensional normed space can be embedded isometrically into L_1, and hence, by Lévy integration, it can be embedded isometrically into L_q for $q \le 1$. Therefore, for fixed $x, y \in F$, and $p > 0$, there is a probability measure μ on $\mathbb{R}^2$ such that

$$\|ux + vy\| = \left(\int_{\mathbb{R}^2} |ua + vb|^q \mu(da, db) \right)^{1/q} \quad \text{for all } u, v \in \mathbb{R}.$$

Assume that $\theta \in \mathrm{HC}(p, q, \mathbb{R}, \sigma)$ with $\sigma > 0$. Then we have

$$\begin{aligned} (E\|x + \sigma\theta y\|^p)^{1/p} &= \left(E\left(\int_{\mathbb{R}^2} |a + \sigma\theta b|^q \mu(da, db) \right)^{p/q} \right)^{1/p} \\ &\le \left(\int_{\mathbb{R}^2} (E|a + \sigma\theta b|^p)^{q/p} \mu(da, db) \right)^{1/q} \\ &\le \left(\int_{\mathbb{R}^2} E|a + \theta b|^q \mu(da, db) \right)^{1/q} \\ &= (E\|x + \theta y\|^q)^{1/q}, \end{aligned}$$

where we use the fact that $(E(\int f^r)^{1/r}) \le (\int (Ef)^r)^{1/r}$ for $r \le 1$ and $f \ge 0$ [cf. (1.1)]. This completes the proof. □

Since a Hilbert space can be embedded isometrically into each L_q, the same argument yields the following proposition.

PROPOSITION 2.4. *Let $0 < q < p < \infty$ and θ be a random variable. Let H be a Hilbert space. Then $\theta \in \mathrm{HC}(p, q, H, \sigma)$ provided $\theta \in \mathrm{HC}(p, q, \mathbb{R}, \sigma)$.*

3. Hypercontraction and Talagrand's inequality. The result below is crucial for what follows.

LEMMA 3.1. *Let $p \geq 4$. If σ is such that*

$$(1 + \sigma t)^p - pt\sigma \leq 1 + \left(\frac{p}{2} - 1\right)t^2 + t^p, \qquad t > 0, \tag{3.1}$$

then for every $s > 0$ and every random variable θ such that $E\theta \geq 0$ we have

$$\left(E\left((\sigma\theta + s)^2 + 1\right)^{p/2}\right)^{2/p} \leq E(\theta + s)^2 + (E1 \vee |\theta|^p)^{2/p}. \tag{3.2}$$

PROOF. Fix $p \geq 4$ and a random variable θ with $c = E\theta > 0$. Put $a = E|\theta|^p$, $b = E\theta^2$. Observe that since $1 \vee |x|^p > 1 + |x|^p - x^2 \geq 0$,

$$E1 \vee |\theta|^p \geq 1 \vee (1 + a - b). \tag{3.3}$$

We shall show that

$$\begin{aligned}\left(\frac{\sigma^2x^2}{s^2+1} + \frac{2\sigma xs}{s^2+1} + 1\right)^{p/2} &\leq \left(\frac{\sigma^2x^2}{s^2+1} + \frac{2\sigma|x|}{\sqrt{s^2+1}} + 1\right)^{p/2} \\ &\quad - \frac{p\sigma|x|}{\sqrt{s^2+1}} + \frac{p\sigma xs}{s^2+1},\end{aligned} \tag{3.4}$$

for every real x and $s > 0$. Indeed, for $A \geq 1$, $A + u \geq 0$ and $v \geq |u|$, we have

$$(A + u)^{p/2} - \frac{pu}{2} \leq (A + v)^{p/2} - \frac{pv}{2} \tag{3.5}$$

[in our case $A = 1 + \sigma^2x^2/(s^2 + 1)$, $u = 2\sigma xs/(s^2 + 1)$], and $v = 2\sigma|x|/\sqrt{s^2 + 1}$]. (3.5) can be easily derived since $u \mapsto (A + u)^{p/2} - pu/2$ is an increasing function for $u \geq 0$, and since, for $0 < u < A$,

$$g(u) = (A + u)^{p/2} - (A - u)^{p/2} \geq pu.$$

The latter inequality follows since $g(0) = 0$, and because

$$g'(u) = \frac{p}{2}\left((A + u)^{p/2-1} + (A - u)^{p/2-1}\right) \geq pA^{p/2-1} \geq p.$$

Therefore, using (3.3) and (3.4), it is enough to prove that

$$\begin{aligned}(s^2 + 1)\left(\left(E\left(\left(\frac{\sigma|\theta|}{\sqrt{s^2+1}} + 1\right)^p - \frac{p\sigma|\theta|}{\sqrt{s^2+1}}\right) + \frac{p\sigma cs}{s^2+1}\right)^{2/p} - 1\right) \\ \leq 2sc + b - 1 + 1 \vee (1 + a - b)^{2/p},\end{aligned}$$

or, denoting $t = 1/(s^2 + 1)$ and using assumption (3.1), to check that

$$\begin{aligned}&\left(1 + (p/2 - 1)bt + at^{p/2} + cp\sigma st\right)^{2/p} - 1 \\ &\quad \leq t\left(2sc + b - 1 + 1 \vee (1 + a - b)^{2/p}\right).\end{aligned}$$

In general, to prove an inequality of type $(A + Dc)^{2/p} \le Rc + V$, where all entries are positive numbers, it suffices to check that $(2/p)DA^{2/p-1} \le R$, and to prove the inequality for $c = 0$, i.e., $A^{2/p} \le V$.

Applying this argument to our case, we first examine the inequality

$$(2/p) \cdot p\sigma st\big(1 + (p/2 - 1)bt + at^{p/2}\big)^{2/p-1} \le 2st,$$

and find it obvious even for $\sigma \le 1$, and then we handle the case when $c = 0$. In other words, it suffices now to prove that, for $0 \le t \le 1$,

$$\begin{aligned} g(t) &= \big(1 + (p/2 - 1)bt + at^{p/2}\big)^{2/p} \\ &\le 1 + \big(b - 1 + 1 \vee (1 + a - b)^{2/p}\big)t =: 1 + Vt. \end{aligned}$$

It is easily seen that $g''(t)$ increases on $(0, \infty)$ which, in turns, means that $g(t)$ is at first concave and then eventually becomes convex. Thus, to prove that $g(t) \le 1 + Vt$, $0 \le t \le 1$, it suffices to check that

$$1^\circ \quad g(0) \le 1, \qquad 2^\circ \quad g'(0) \le V, \qquad 3^\circ \quad g(1) \le 1 + V.$$

1° and 2° are satisfied since $g(0) = 1$, and $g'(0) = 2/p(p/2 - 1)b \le b \le V$. Condition 3° reads

$$(bp/2 + A)^{2/p} \le b + 1 \vee A^{2/p}, \qquad b, A \ge 0$$

(here $A = 1 + a - b$), and the latter estimate follows from Bernoulli's inequality. This completes the proof of Lemma 3.1. □

The constant σ is of order $\ln p/p$ when $p \to \infty$. Indeed, the inequality (3.1) holds trivially for $t \ge 1/(1 - \sigma)$. Consider (3.1) for $t < 1/(1 - \sigma)$. Using the fact that $(1 + a)^p \le e^{ap}$ and that $a \mapsto (e^{ap} - pa - 1)/a^2$ is an increasing function for $a > 0$, it is enough to assume that $\exp\{p\sigma/(1 - \sigma)\} \le p/2$ which is satisfied if

$$\sigma \le \frac{\ln(p/2)/p}{1 + \ln(p/2)/p}.$$

COROLLARY 3.2. *Let p and σ be as above. Then for any zero-mean random variable X taking values in a normed space F, the following estimate holds*:

$$\Big(E\big(\|x + \sigma X\|^2 + t^2\big)^{p/2}\Big)^{1/p} \le \Big(E\|x + X\|^2 + \big(Et^p \vee \|X\|^p\big)^{2/p}\Big)^{1/2},$$

for every $x \in F$ and $t > 0$.

PROOF. By homogeneity, we may consider only the value $t = 1$. Since

$$\|x + \sigma X\| \le \sigma(\|x + X\| - \|x\|) + \|x\|,$$

it is enough to apply Corollary 3.2 with $\theta = \|x + X\| - \|x\|$, $s = \|x\|$. □

THEOREM 3.3. *Let $p > 4$ and $C_p = 1 + p/\ln(p/2)$. Then for every sequence $(X_i,\ i = 1, \dots, n)$ of independent zero-mean random variables taking values in* *a normed space F the following inequality holds*:

$$\left(E\left\|\sum_i X_i\right\|^p\right)^{1/p} \le C_p\left(\left(E\left\|\sum_i X_i\right\|^2\right)^{1/2} + \left(E\left(\|X_n\|^p \vee \left(E\|X_{n-1}\|^p \vee \left(\cdots \vee E\|X_1\|^p\right)\cdots\right)\right)\right)^{1/p}\right).$$

PROOF. Define inductively functions

$$h_0(t) = t, \qquad h_n(t) = \left(E|h_{n-1}(t)|^p \vee \|X_n\|^p\right)^{1/p}, \qquad n \ge 1.$$

Denote $S_n = \sum_{i=1}^n$. We shall prove that

$$\left(E\left(\|x + \sigma S_n\|^2 + t^2\right)^{p/2}\right)^{1/p} \le \left(E\|x + S_n\|^2 + h_n^2(t)\right)^{1/2}, \tag{3.6}$$

from which the theorem follows by setting $t = 0$ and $x = 0$.

The case $n = 1$ is contained in Corollary 3.2. Assuming that (3.6) holds for n, we shall show (3.6) for $n + 1$. Denote $E' = E[\cdot|X_{n+1}]$ and $E'' = E[\cdot|X_1, \dots, X_n]$. Using Fubini's theorem and (3.6), we have

$$\begin{aligned}
&\left(E\left(\|x + \sigma S_n + \sigma X_{n+1}\|^2 + t^2\right)^{p/2}\right)^{1/p} \\
&\quad = \left(E''E'\left(\|x + \sigma S_n + \sigma X_{n+1}\|^2 + t^2\right)^{p/2}\right)^{1/p} \\
&\quad \le \left(E''\left(E'\|x + S_n + \sigma X_{n+1}\|^2 + h_n^2(t)\right)^{p/2}\right)^{1/p}.
\end{aligned}$$

We estimate the latter quantity from above by

$$\begin{aligned}
&\left(E'\left(E''\left(\|x + S_n + \sigma X_{n+1}\|^2 + h_n^2(t)\right)^{p/2}\right)^{2/p}\right)^{1/2} \\
&\quad \le \left(E'E''\|x + S_n + X_{n+1}\|^2 + \left(E''h_n(t)^p \vee \|X_{n+1}\|^p\right)^{2/p}\right)^{1/2} \\
&\quad = \left(E\|x + S_{n+1}\|^2 + h_{n+1}^2(t)\right)^{1/2},
\end{aligned}$$

using (1.1) in the form $(E''(E'|Z|^2)^{p/2})^{1/p} \le (E'(E''|Z|^p)^{2/p})^{1/2}$, where $Z = (\|x + S_n + \sigma X_{n+1}\|^2 + h_n^2(t))^{1/2}$, and then applying Corollary 3.3. This completes the proof. □

REMARK 3.4. The bound $h_n(0) \le (E \sup\|X_i\|^p)^{1/p}$ yields (1.5). Talagrand's result involving the L_1-norm,

$$\left(E\left\|\sum_i X_i\right\|^p\right)^{1/p} \le KC_p\left(E\left\|\sum_i X_i\right\| + \left(E \sup_i\|X_i\|^p\right)^{1/p}\right),$$

follows from Hoffmann–Jørgensen's inequalities (1974).

Let S be a sum of independent uniformly bounded random variables. We conclude this section by proving existence of "gamma-moments," $E\Gamma(1 + \alpha\|S\|)$. The ideas and methods belong to de Acosta (1980).

By Stirling's formula, $\Gamma(1 + x) \approx (2\pi)^{-1/2} \exp\{x \ln x - x - 1/2 \ln x\}$, it is readily seen that $E\Gamma(1 + \beta W) < \infty$ for all $0 < \beta < \alpha$ if and only if $Ee^{\beta W \ln_+ W} < \infty$ for all $\beta < \alpha$. Also, any of the above conditions are equivalent to

$$\int_1^\infty \ln t \exp\{\beta t \ln t\} P(W > t)\, dt < \infty, \qquad \beta < \alpha. \tag{3.7}$$

PROPOSITION 3.5. *Let (X_i) be a sequence of independent zero-mean random variables such that $\|X_i\| \le M$. Suppose that $S = \sum_i X_i$ converges a.s. and denote $W = \sup_n \|\sum_{i=1}^n X_i\|$. Then $E\Gamma(1 + \alpha W) < \infty$ for every $\alpha < 1/M$.*

PROOF. Let $S = \sum_i X_i$ converge a.s. Fix $\varepsilon > 0$. Since the series S converges also in L_2 [Hoffmann–Jørgensen (1974)], then we can write $X_n = X_n' + X_n''$, where $\|X_n'\| \le \varepsilon/4$, $\|\sum_i X_i'\|_2 \le \varepsilon/4$ and $\sum_i P(X_i'' \ne 0) < \infty$. Denote the corresponding series and suprema by S', S'', W' and W'', respectively. For large p, the inequality (1.5) implies an estimate for $\|W\|_p$,

$$\|W'\|_p \le 2p/\ln p\Big(\|S'\|_2 + \sup_i \|X_i'\|_\infty\Big) \le \varepsilon p/\ln p.$$

Then, by the Chebyshev inequality, $P(W' > t) \le (\varepsilon p/(t \ln p))^p$. This, after changing the variable $t = e\varepsilon p/\ln p$ in (3.7), yields $E\Gamma(1 + \alpha' W') < \infty$ for $\alpha' < 1/e\varepsilon$.

On the other hand, $E\Gamma(1 + \alpha'' W'') < \infty$ if $\alpha'' M < 1$. To see this, put $p_i = P(X_i'' \ne 0)$, $m = \sum_i p_i$, and check the bounds

$$\begin{aligned} E\Gamma(1 + \alpha'' W'') &\le 1 + \int_1^\infty E t^{\alpha'' W''} e^{-t}\, dt \\ &\le 1 + \int_1^\infty \prod_i \big(1 + p_i(t^{\alpha'' M} - 1)\big) e^{-t}\, dt \\ &\le 1 + e^{-m} \int_1^\infty \exp\{m t^{\alpha'' M} - t\}\, dt < \infty. \end{aligned}$$

Since the function $\Gamma(1 + x)$ is convex then for $\alpha = \alpha'\alpha''/(\alpha' + \alpha'')$ we have

$$\begin{aligned} E\Gamma(1 + \alpha(W' + W'')) &\le \frac{\alpha''}{\alpha' + \alpha''} E\Gamma(1 + \alpha' W') \\ &\quad + \frac{\alpha'}{\alpha' + \alpha''} E\Gamma(1 + \alpha'' W'') < \infty. \end{aligned}$$

To complete the proof, it suffices to take $\alpha < 1/(M + e\varepsilon)$ and then choose ε small enough. □

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