The Annals of Probability
1991, Vol. 19, No. 1, 329–341

THE ASYMPTOTIC BEHAVIOR OF THE REWARD SEQUENCE IN THE OPTIMAL STOPPING OF I.I.D. RANDOM VARIABLES

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Let $X_1, X_2, \dots$ be integrable, i.i.d. r.v.'s with common distribution function F and let $\{v_n\}_{n \geq 1}$ be the sequence of optimal rewards or values in the associated optimal stopping problem, i.e., $v_n = \sup\{E(X_T)\colon T$ is a stopping time for $\{X_m\}_{m \geq 1}$ and $T \leq n\}$ for $n \geq 1$. For distribution functions F in the domain of attraction of one of the three classical extreme-value laws G_{I}, G_{II}^{α} or $G_{\mathrm{III}}^{\alpha}$, it is shown that $\lim_n n(1 - F(v_n)) = 1, 1 - \alpha^{-1}$, or $1 + \alpha^{-1}$ if $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, respectively. From this result, the growth rate of $\{v_n\}_{n \geq 1}$ is obtained and compared to the growth rate of the expected maximum sequence. Also, the limit distribution of the optimal reward r.v.'s $\{X_{T_n^*}\}_{n \geq 1}$ is derived, where $\{T_n^*\}_{n \geq 1}$ are the optimal stopping times defined by $T_n^* \equiv 1$ if $n = 1$ and, for $n = 2, 3, \dots$, by $T_n^* = \min\{1 \leq k < n\colon X_k > v_{n-k}\}$ if this set is not equal to $\varnothing$ and equal to n otherwise. This tail-distribution growth rate is shown to be sufficient for *any* threshold sequence to be asymptotically optimal.

1. Introduction and statement of results. Let $X_1, X_2, \dots$ and X be integrable, i.i.d. random variables with common distribution function F. The solution of the optimal stopping problem associated with these random variables, as described by Chow, Robbins and Siegmund [2], centers around the sequence of optimal rewards or values

$$v_n = \sup\{E(X_T)\colon T \text{ is a stopping time for } \{X_m\}_{m \geq 1} \text{ and } T \leq n\}, \quad n = 1, 2, \dots. \tag{1.1}$$

The recursive representation of this value sequence $\{v_n\}_{n \geq 1}$, given by

$$v_1 = EX \quad \text{and} \quad v_{n+1} = E(X \vee v_n) \quad \text{for } n = 1, 2, \dots, \tag{1.2}$$

identifies these numbers as thresholds which target when to stop using optimal stopping times $\{T_n^*\}_{n \geq 1}$ for (1.1); that is, $v_n = E(X_{T_n^*})$ for $n \geq 1$, with the stopping times defined by $T_1^* \equiv 1$ and by

$$T_n^* = \min\{1 \leq k < n\colon X_k > v_{n-k}\} \quad \text{if this set is} \neq \varnothing, \text{ and } = n \text{ otherwise,} \tag{1.3}$$

for $n = 2, 3, \dots$. In this paper, the limit behavior of the value sequence

Received July 1988; revised August 1989.

[1]This author is grateful to the School of Mathematics, Georgia Institute of Technology, for support during the academic year 1987–1988.

[2]Supported in part by NSF grants.

AMS 1980 *subject classifications*. Primary 60G40; secondary 62L15, 60F05.

Key words and phrases. Optimal stopping, extreme-value theory, maxima of i.i.d. r.v.'s, regular variation, domains of attraction.

$\{v_n\}_{n\geq 1}$ and the optimal reward r.v.'s $\{X_{T_n^*}\}_{n\geq 1}$ are determined, for a large class of distribution functions.

This limit behavior clearly depends on the distribution function F and, in particular, on the form of the upper tail of F. In this paper, the distribution functions considered are those with (extended) regularly varying upper tails; that is, the distribution functions of extreme-value theory (cf. e.g., the books by Galambos [3], Leadbetter, Lindgren and Rootzén [10] and Resnick [13]). If we denote $x_F = \sup\{x\colon F(x) < 1\}$ and $M_n = \max_{1\leq i\leq n} X_i$ for $n \geq 1$, then we will assume that the distribution function F satisfies the following property:

(1.4) There exist sequences of constants $\{a_n\}_{n\geq 1}$ with $a_n > 0$ and $\{b_n\}_{n\geq 1}$ for which $a_n(M_n - b_n)$ converges in distribution to some nondegenerate distribution function G as $n \to \infty$,

which implies that for each $0 < \eta < \infty$, there exists a sequence of numbers $\{u_n\}_{n\geq 1}$ for which $\lim_n n(1 - F(u_n)) = \eta$.

A distribution function F which satisfies property (1.4) with distribution function G is said to be in the domain of attraction of G for maxima, and we write $F \in \mathscr{D}(G)$ in this case. The extremal types theorem ([10], Theorem 1.4.2) states that any limiting d.f. G of (1.4) must be of the same type as one of the extreme-value distributions G_{I}, G_{II}^{α} or $G_{\mathrm{III}}^{\alpha}$ for some $\alpha > 0$, where

$$
\begin{aligned}
G_{\mathrm{I}}(x) &= \exp(-e^{-x}) \quad \text{for } -\infty < x < \infty,\\
G_{\mathrm{II}}^{\alpha}(x) &= 0 \quad \text{for } x \leq 0,\\
&= \exp(-x^{-\alpha}) \quad \text{for } 0 \leq x,\\
G_{\mathrm{III}}^{\alpha}(x) &= \exp\left(-(-x)^{\alpha}\right) \quad \text{for } x \leq 0,\\
&= 1 \quad \text{for } 0 \leq x.
\end{aligned} \tag{1.5}
$$

Appropriate constants which yield convergence in (1.4) to these three limiting distributions are

$$
\begin{aligned}
a_n &= (g(\gamma_n))^{-1} && \text{and} \quad b_n = \gamma_n \quad \text{if } F \in \mathscr{D}(G_{\mathrm{I}}),\\
a_n &= \gamma_n^{-1} && \text{and} \quad b_n = 0 \quad \text{if } F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha}),\\
a_n &= (x_F - \gamma_n)^{-1} && \text{and} \quad b_n = x_F \quad \text{if } F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha}),
\end{aligned} \tag{1.6}
$$

where $\gamma_n = \inf\{x\colon F(x) \geq 1 - n^{-1}\}$ and $g(t) = \int_t^{x_F}(1 - F(u))\,du/(1 - F(t))$ for $t < x_F$. In this paper, the norming constants of (1.6) are used throughout (by Khintchine's theorem [10], Theorem 1.2.3, there is no loss of generality).

It is immediate that the value sequence $\{v_n\}_{n\geq 1}$ increases to x_F as $n \to \infty$. How does $\{v_n\}_{n\geq 1}$ grow? What is the rate at which $\{v_n\}_{n\geq 1}$ converges to x_F? A key result toward answering these questions is given in the following main theorem of this paper.

THEOREM 1.1. *If* $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ *and* $\alpha > 1$, *or* $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ *and* $\alpha > 0$, *then, respectively,*

$$\lim_n n(1 - F(v_n)) = 1,\, 1 - \alpha^{-1} \text{ or } 1 + \alpha^{-1}. \tag{1.7}$$

The proof of Theorem 1.1 is given in Section 2. Examples and numerical calculations for specific distributions F, giving special cases of (1.7), are found in Gilbert and Mosteller [4], Kennedy and Kertz [7] and Petrucelli [11]. For classifications of specific distributions by their domain of attraction for maxima, see [3], [10] and [13].

A direct description of the growth of $\{v_n\}_{n \ge 1}$ follows from Theorem 1.1. For this description, denote the rate functions and inverse rate functions associated with the extreme-value distributions G_{I}, G_{II}^{α} and $G_{\mathrm{III}}^{\alpha}$, respectively, by

$$\begin{aligned} R_{\mathrm{I}}(x) &= e^{-x} \quad \text{for } -\infty < x < \infty, \\ R_{\mathrm{II}}^{\alpha}(x) &= x^{-\alpha} \quad \text{for } 0 \le x, \\ R_{\mathrm{III}}^{\alpha}(x) &= (-x)^{\alpha} \quad \text{for } x \le 0; \\ \text{for } x > 0, \quad R_{\mathrm{I}}^{-1}(x) &= -\log x, \\ (R_{\mathrm{II}}^{\alpha})^{-1}(x) &= x^{-1/\alpha}, \\ (R_{\mathrm{III}}^{\alpha})^{-1}(x) &= (-1)x^{1/\alpha} \end{aligned} \tag{1.8}$$

and write G_F, R_F and R_F^{-1} for the extreme-value d.f., associated rate function and inverse rate function, linked to F [i.e., $F \in \mathscr{D}(G_F)$]. To find the growth rate of $\{v_n\}_{n \ge 1}$, use Theorem 1.1 and the result that

$$\begin{aligned} &\text{for each } 0 < x < \infty, \quad \lim_n n(1 - F(u_n)) = x \quad \text{if and only if} \\ &\lim_n a_n(u_n - b_n) = R_F^{-1}(x). \end{aligned} \tag{1.9}$$

[To see (1.9), use, for example that $\lim_n n(1 - F(a_n^{-1}y + b_n)) = R_F(y)$, which is part of Theorem 1.1 of [7].] This yields the following theorem.

THEOREM 1.2. *If* $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ *and* $\alpha > 1$, *or* $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ *and* $\alpha > 0$, *then, respectively,*

$$\lim_n a_n(v_n - b_n) = 0,\, (1 - \alpha^{-1})^{-1/\alpha} \text{ or } (-1)(1 + \alpha^{-1})^{1/\alpha}, \tag{1.10}$$

for the sequences $\{a_n\}_{n \ge 1}$ *and* $\{b_n\}_{n \ge 1}$ *of* (1.6).

The limit of (1.7) draws a comparison between optimal stopping and extreme-value probabilities, through the representation

$$\lim_n n(1 - F(v_n)) = \lim_n P(X > v_n)/P(X > M_n). \tag{1.11}$$

For comparisons between the value sequence $\{v_n\}_{n \geq 1}$ and the expected partial maxima sequences $\{m_n\}_{n \geq 1}$, with $m_n = E(M_n)$, we recall growth properties of the sequence $\{m_n\}_{n \geq 1}$ from Pickands [12], Section 2.1 of Resnick [13] and page 946 of Gradshteyn and Ryzhik [5] and use (1.9) to obtain,

if $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, then, respectively,

$$\begin{aligned} &\lim_n n(1 - F(m_n)) = e^{-\gamma}, \left(\Gamma(1 - \alpha^{-1})\right)^{-\alpha}, \text{or } \left(\Gamma(1 + \alpha^{-1})\right)^{\alpha}, \\ &\lim_n a_n(m_n - b_n) = \gamma, \Gamma(1 - \alpha^{-1}), \text{or } (-1)\Gamma(1 + \alpha^{-1}), \end{aligned} \tag{1.12}$$

where γ = Euler's constant = 0.5772... and Γ is the gamma function. Ratio and difference growth comparisons between $\{v_n\}_{n \geq 1}$ and $\{m_n\}_{n \geq 1}$, which follow from Theorem 1.2 and (1.12), are given in Theorem 1.3.

THEOREM 1.3. (i) *For* $F \in \mathscr{D}(G_{\mathrm{I}})$, $\lim_n m_n/v_n = 1$,

$$\lim_n (m_n - \gamma_n)/(v_n - \gamma_n) = +\infty$$

and

$$\lim_n (m_n - v_n)/g(\gamma_n) = \lim_n (m_n - v_n)/(n(\gamma_{n+1} - \gamma_n)) = \gamma.$$

(ii) *For* $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ *and* $\alpha > 1$, $\lim_n m_n/v_n = (1 - \alpha^{-1})^{1/\alpha}\Gamma(1 - \alpha^{-1})$ *and*

$$\lim_n (m_n - v_n)/\gamma_n = \Gamma(1 - \alpha^{-1}) - (1 - \alpha^{-1})^{-1/\alpha}.$$

(iii) *For* $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ *and* $\alpha > 0$, $\lim_n m_n/v_n = 1$,

$$\lim_n (x_F - m_n)/(x_F - v_n) = (1 + \alpha^{-1})^{-1/\alpha}\Gamma(1 + \alpha^{-1}),$$

$$\lim_n (m_n - v_n)/(x_F - \gamma_n) = (1 + \alpha^{-1})^{1/\alpha} - \Gamma(1 + \alpha^{-1}).$$

Comparisons of value and expected maximum over a given class of stochastic processes, taking the form of sharp inequalities or regions, can be found in the literature on prophet problems; for the class of i.i.d. r.v.'s, see [6] and [8]; for a survey on prophet problems, see [9]. We emphasize that in this paper the sequence of integrable, i.i.d. r.v.'s $\{X_n\}_{n \geq 1}$ and d.f. F is fixed.

As another application of Theorem 1.1, use the representation

$$P(X_{T_n^*} \le x_n) = \sum_{k=1}^{n-1} \left(\prod_{j=1}^{k-1} F(v_{n-j}) \right) (F(x_n) - F(v_{n-k}))_+ + \left(\prod_{j=1}^{n-1} F(v_{n-j}) \right) F(x_n)$$

for $n > 1$ and a straightforward limiting argument to identify the distributional convergence of the normalized optimal reward r.v.'s as follows.

THEOREM 1.4. *If $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, then the sequence $\{a_n(X_{T_n^*} - b_n)\}_{n \ge 1}$ converges in distribution, with the limiting distribution given, respectively, by*

$$\begin{aligned} H_{\mathrm{I}}(x) &= (\tfrac{1}{2})e^{x} \quad \text{if } x \le 0,\\ &= 1 - (\tfrac{1}{2})e^{-x} \quad \text{if } 0 \le x;\\ H_{\mathrm{II}}^{\alpha}(x) &= 0 \quad \text{if } x \le 0,\\ &= (2 - \alpha^{-1})^{-1}[(1 - \alpha^{-1})x^{\alpha}]^{1-(1/\alpha)} \quad \text{if } 0 \le x \le (1 - \alpha^{-1})^{-1/\alpha},\\ &= 1 - (2 - \alpha^{-1})^{-1}x^{-\alpha} \quad \text{if } (1 - \alpha^{-1})^{-1/\alpha} \le x;\\ \text{or}\quad H_{\mathrm{III}}^{\alpha}(x) &= (2 + \alpha^{-1})^{-1}[(1 + \alpha^{-1})(-x)^{-\alpha}]^{1+(1/\alpha)} \quad \text{if } x \le -(1 + \alpha^{-1})^{1/\alpha},\\ &= 1 - (2 + \alpha^{-1})^{-1}(-x)^{\alpha} \quad \text{if } -(1 + \alpha^{-1})^{1/\alpha} \le x \le 0,\\ &= 1 \quad \text{if } 0 \le x. \end{aligned} \tag{1.13}$$

Theorem 1.1 may also be combined with Theorem 3.3 of [7] to obtain the limiting joint distribution of $\{(X_{T_n^*}, M_n)\}_{n \ge 1}$.

In Section 3, we show that *any* sequence of thresholds with the same tail-distribution rates of growth as those of (1.7) will be "asymptotically optimal." Specifically, for any sequence of real numbers $\{u_n\}_{n \ge 1}$, define *threshold stopping times* $\{T_n\}_{n \ge 1}$ by $T_1 \equiv 1$ and for $n = 2, 3, \ldots,$

$$T_n = \min\{1 \le k < n \colon X_k > u_{n-k}\} \quad \text{if this set} \ne \varnothing \text{ and } = n \text{ otherwise.} \tag{1.14}$$

THEOREM 1.5. *Let $\{u_n\}_{n \ge 1}$ be a sequence of real numbers and F be a distribution function satisfying $\lim_n n(1 - F((u_n)) = 0$, $1 - \alpha^{-1}$ or $1 + \alpha^{-1}$ for $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, respectively. Let $\{T_n\}_{n \ge 1}$ be the threshold stopping times associated with $\{X_n\}_{n \ge 1}$*

and $\{u_n\}_{n\geq 1}$. *Then*

(1.15) $$\lim_n a_n E(X_{T_n^*} - X_{T_n}) = 0.$$

Indeed, a more general result is proved in Section 3, which also shows that the tail-distribution rate of growth of (1.7) is in a sense necessary for a threshold sequence to be asymptotically optimal.

2. Proof of Theorem 1.1. In this section, Theorem 1.1 is proved. First, assume $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$. In this case $x_F < \infty$ and $1 - F(x_F - x^{-1}) = x^{-\alpha}L(x)$, where $L(x)$ is slowly varying at infinity. Denote $G(x) = \int_x^{x_F}(1 - F(y))\,dy$ and obtain $v_n = v_{n-1} + G(v_{n-1})$ for $n \geq 2$ from (1.2).

The proof rests on properties of the auxiliary function $H(x) = (x_F - x)/G(x)$ for $x < x_F$. First, we claim that

(2.1) $$\lim_{x \uparrow x_F} H(x)(1 - F(x)) = \alpha + 1.$$

This follows from the representation

$$H(x)(1 - F(x)) = y^{-(\alpha+1)}L(y) \Big/ \int_y^{\infty} z^{-(\alpha+2)}L(z)\,dz,$$

where $y = (x_F - x)^{-1}$, letting $x \to x_F$ (and $y \to \infty$) and applying Proposition 1.5.10 of [1]. Next, we claim that

(2.2) $$\lim_n (H(v_n) - H(v_{n-1})) = \alpha.$$

This follows from the representation

$$H(v_n) - H(v_{n-1}) = H(v_n)(G(v_{n-1}))^{-1}(G(v_{n-1}) - G(v_n)) - 1,$$

from the inequality

$$\begin{aligned} H(v_n)(1 - F(v_n)) &\leq H(v_n)(G(v_{n-1}))^{-1}(G(v_{n-1}) - G(v_n)) \\ &\leq H(v_n)(1 - F(v_{n-1})) \end{aligned}$$

and the limits (2.1) and $\lim_n(1 - F(v_{n-1}))/(1 - F(v_n)) = 1$. Now, the conclusion follows in this case by observing that $\lim_n n^{-1}H(v_n) = \alpha$ from (2.2) and using this and (2.1) to obtain $\lim_n n(1 - F(v_n)) = 1 + \alpha^{-1}$.

In the case of $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, define $H(x) = x/G(x)$ for $x < x_F = \infty$, observe that $\lim_{x \to \infty} H(x)(1 - F(x)) = \alpha - 1$ and proceed as above. Finally, in the case $F \in \mathscr{D}(G_{\mathrm{I}})$, define $H(x) = \int_x^{x_F} G(y)\,dy/(G(x))^2$ for $x < x_F$ and observe that $H(x) < \infty$ and $\lim_{x \uparrow x_F} H(x)(1 - F(x)) = 1$ from Lemma 1.8 and Proposition 1.9 of [13]. The argument is then analogous to the above one.

3. Expectations of threshold-stopped random variables. The main result of this section describes the rate of growth of expectations of threshold-stopped random variables. Theorem 1.5 follows as an immediate consequence of this result and Theorem 1.2.

THEOREM 3.1. *Let $\{u_n\}_{n\ge 1}$ be a sequence of numbers satisfying*

$$\lim_n n(1 - F(u_n)) = \eta$$

for some $0 < \eta < \infty$ and let $\{T_n\}_{n\ge 1}$ be the threshold stopping times associated with $\{X_n\}_{n\ge 1}$ and $\{u_n\}_{n\ge 1}$. If $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, then $\lim_n E\{a_n(X_{T_n} - b_n)\} = f_{\mathrm{I}}(\eta)$, $f_{\mathrm{II}}(\eta;\alpha)$ or $f_{\mathrm{III}}(\eta;\alpha)$, respectively, where

$$f_{\mathrm{I}}(x) = 1 - x^{-1} - \log x \quad \text{for } 0 < x,$$

$$(3.1)\qquad \begin{aligned} f_{\mathrm{II}}(x;\alpha) &= (1-\alpha^{-1})^{-1} x^{1-(1/\alpha)} (x+\alpha^{-1})^{-1} \qquad \text{for } 0 < x \text{ and } 1 < \alpha; \\ \text{for } \alpha > 0, \quad f_{\mathrm{III}}(x,\alpha) &= -\infty \quad \text{for } 0 < x \le \alpha^{-1}, \\ &= (-1)(1+\alpha^{-1})^{-1} x^{1+(1/\alpha)} (x-\alpha^{-1})^{-1} \qquad \text{for } \alpha^{-1} < x. \end{aligned}$$

PROOF. Let $\{u_n\}_{n\ge 1}$, η and $\{T_n\}_{n\ge 1}$ be numbers and r.v.'s satisfying the hypotheses. Theorem 3.1 is an immediate consequence of results (3.2) and (3.3).

For each $0 < \varepsilon < 1$,

$$(3.2)\qquad \begin{aligned} &\lim_n E\{a_n(X_{T_n} - b_n) I_{(T_n \le n - [n\varepsilon])}\} \\ &= f_{\mathrm{I}}(\eta)(1-\varepsilon^{\eta}) + (-\log\varepsilon)\varepsilon^{\eta} \qquad \text{if } F \in \mathscr{D}(G_{\mathrm{I}}), \\ &= f_{\mathrm{II}}(\eta;\alpha)(1-\varepsilon^{\eta+(1/\alpha)}) \qquad \text{if } F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha}) \text{ and } \alpha > 1, \\ &= f_{\mathrm{III}}^{0}(\eta;\alpha)\{(1-\varepsilon^{\eta-(1/\alpha)}) I_{(\eta\ne\alpha^{-1})} + (-\log\varepsilon) I_{(\eta=\alpha^{-1})}\} \\ &\qquad \text{if } F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha}) \text{ and } \alpha > 0, \end{aligned}$$

where

$$\begin{aligned} f_{\mathrm{III}}^{0}(x;\alpha) &= (-1)(1+\alpha^{-1})^{-1} x^{1+(1/\alpha)} (x-\alpha^{-1})^{-1} \qquad \text{for } x \ne \alpha^{-1}, \\ &= (-1)(1+\alpha^{-1})^{-1} x^{1+(1/\alpha)} \qquad \text{for } x = \alpha^{-1}. \end{aligned}$$

(3.3)(i) Let $F \in \mathscr{D}(G_{\mathrm{I}})$. For any $0 < \delta < \eta$, there is a sufficiently large integer M for which

$$\limsup_n \left|E\{a_n(X_{T_n} - b_n) I_{(n-[n\varepsilon] < T_n \le n-M)}\}\right|$$
$$= O(\varepsilon^{\eta-\delta} \log \varepsilon^{-1}) \quad \text{as } \varepsilon \downarrow 0,$$
$$\text{and} \quad \lim_n E\{a_n(X_{T_n} - b_n) I_{(n-M < T_n)}\} = 0.$$

(3.3)(ii) Let $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$. For any $0 < \delta < \eta + \alpha^{-1}$, there is a sufficiently large integer M for which

$$\limsup_n \left|E\{a_n(X_{T_n} - b_n) I_{(n-[n\varepsilon] < T_n \le n-M)}\}\right|$$
$$= O(\varepsilon^{\eta+(1/\alpha)-\delta}) \quad \text{as } \varepsilon \downarrow 0,$$
$$\text{and} \quad \lim_n E\{a_n(X_{T_n} - b_n) I_{(n-M < T_n)}\} = 0.$$

(3.3)(iii) Let $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$, $0 < \alpha$ and $\alpha^{-1} < \eta$. For any $0 < \delta < \eta - \alpha^{-1}$, there is a sufficiently large integer M for which

$$\limsup_n \left|E\{a_n(X_{T_n} - b_n) I_{(n-[n\varepsilon] < T_n \le n-M)}\}\right|$$
$$= O(\varepsilon^{\eta-\delta} \log \varepsilon^{-1}) \quad \text{as } \varepsilon \downarrow 0,$$
$$\text{and} \quad \lim_n E\{a_n(X_{T_n} - b_n) I_{(n-M < T_n)}\} = 0.$$

In the three cases of $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, and $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, the proofs of both (3.2) and (3.3) are similar. We give here the proofs for $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$. In this case, $x_F < \infty$ and $1 - F(x_F - x^{-1}) = x^{-\alpha}L(x)$, where $L(x)$ is slowly varying at infinity. The norming constants are $a_n = (x_F - \gamma_n)^{-1}$ and $b_n = x_F$ for $n \ge 1$. The sequence $\{a_n\}_{n \ge 1}$ satisfies $\lim_n a_n(u_{[ns]} - b_n) = (-1)(\eta/s)^{1/\alpha}$ for each $s > 0$ and is regularly varying with exponent α^{-1} with representation $a_n = n^{1/\alpha}c_n \exp(\sum_{k=1}^{n}\delta_k/k)$, where $\{c_n\}_{n \ge 1}$ are numbers satisfying $\lim_n c_n = c_0$ for some $c_0 \in (0, \infty)$ and $\lim_n \delta_n = 0$. (For reference, see Chapter 1 of [13] and Sections 1.5, 1.9 and 8.13 of [1].) Also, let $\{\beta_n\}_{n \ge 1}$ be numbers satisfying $\log F(u_n) = (-\eta + \beta_n)/n$ and $\lim_n \beta_n = 0$.

To prove (3.2) in this case, first use the representation

$$E\{(x_F - X) I_{\{X > u\}}\}$$
$$= (x_F - u)(1 - F(u)) - (x_F - u)\int_0^1 \{1 - F(x_F - x(x_F - u))\}\, dx$$

and the dominated convergence theorem to obtain

$$\lim_n nE\{a_n(X-b_n)I_{\{X>u_n\}}\} = -(1+\alpha^{-1})^{-1}\eta^{1+(1/\alpha)}. \tag{3.4}$$

Next, let $\varepsilon > 0$, denote $s_n(u) = E\{a_n(X-b_n)I_{\{X>u\}}\}$ and calculate as follows: For all n,

$$\begin{aligned}
&E\{a_n(X_{T_n}-b_n)I_{(T_n\le n-[n\varepsilon])}\}\\
&\quad=\sum_{k=1}^{n-[n\varepsilon]} E\{a_n(X_k-b_n)I_{(T_n=k)}\}\\
&\quad=\sum_{k=1}^{n-[n\varepsilon]} a_nE\{(X-x_F)I_{(X>u_{n-k})}\}\prod_{r=n-k+1}^{n-1}F(u_r)\\
&\quad=n^{-1}\sum_{k=1}^{n-[n\varepsilon]}\left(\frac{na_n}{(n-k)a_{n-k}}\right)((n-k)s_{n-k}(u_{n-k}))\prod_{r=n-k+1}^{n-1}F(u_r) \qquad (3.5)\\
&\quad=n^{-1}\sum_{k=1}^{n-[n\varepsilon]}\left(\frac{n}{n-k}\right)^{1+(1/\alpha)}\left(\frac{c_n}{c_{n-k}}\right)((n-k)s_{n-k}(u_{n-k}))\\
&\qquad\times\exp\left\{\sum_{r=n-k+1}^{n-1}\left(\frac{-\eta+\beta_r+\delta_r}{r}\right)+\frac{\delta_n}{n}\right\},
\end{aligned}$$

where $\lim_n c_n = c_0 \in (0,\infty)$ and $\lim_n \delta_n = 0 = \lim_n \beta_n$. Now let $n\to\infty$ in (3.5) and use (3.4) to obtain

$$\begin{aligned}
&\lim_n E\{a_n(X_{T_n}-b_n)I_{(T_n\le n-[n\varepsilon])}\}\\
&\qquad=\int_0^{1-\varepsilon}(1-s)^{-(1+(1/\alpha))}(-1)(1+\alpha^{-1})^{-1}\eta^{1+(1/\alpha)}\\
&\qquad\qquad\times\exp\left\{-\eta\int_{1-s}^{1}x^{-1}\,dx\right\}ds,
\end{aligned}$$

and (3.2) is proved for this case.

Next, (3.3)(iii) is proved. Let $0<\varepsilon<1$ and $0<\delta<\eta-\alpha^{-1}$. Choose an integer M sufficiently large so that $\delta_m+\beta_m<\delta$ and $c_m>0$ for all $m\ge M$. Note that M depends on δ but not on ε. Also let $K>1$ be a constant for which $|ms_m(u_m)|<K^{1/2}$ for all $m\ge 1$ and $(c_n/c_m)<K^{1/2}$ for all $n\ge m\ge M$. For the first part of (3.3)(iii), obtain the following inequality for all n

sufficiently large:

$$\begin{aligned}
&\left|E\{a_n(X_{T_n} - b_n)I_{(n-[n\varepsilon]<T_n\le n-M)}\}\right| \\
&= \left| n^{-1} \sum_{k=n-[n\varepsilon]+1}^{n-M} (na_n/((n-k)a_{n-k}))((n-k)s_{n-k}(u_{n-k})) \times \prod_{r=n-k+1}^{n-1} F(u_r) \right| \\
&\le Kn^{-1} \sum_{k=n-[n\varepsilon]+1}^{n-M} (1-(k/n))^{-1-(1/\alpha)} \times \exp\left\{ n^{-1} \sum_{r=n-k+1}^{n-1} (-\eta+\delta)(r/n)^{-1} \right\}.
\end{aligned} \tag{3.6}$$

Then take a limit in (3.6) along the appropriate subsequence to obtain

$$\begin{aligned}
&\limsup_n \left|E\{a_n(X_{T_n} - b_n)I_{(n-[n\varepsilon]<T_n\le n-M)}\}\right| \\
&\le K\int_{1-\varepsilon}^{1} (1-s)^{-1-(1/\alpha)} \exp\left\{(-\eta+\delta)\int_{1-s}^{1} x^{-1}\,dx\right\} ds \\
&= K(\eta-\delta-(1/\alpha))^{-1}\varepsilon^{\eta-\delta-(1/\alpha)}.
\end{aligned}$$

For the second part of (3.3)(iii), obtain the following inequality for all n sufficiently large:

$$\begin{aligned}
&\left|E\{a_n(X_{T_n} - b_n)I_{(n-M<T_n)}\}\right| \\
&\le a_n E(x_F - X) \sum_{k=n-M+1}^{n} P(T_n > k-1) \\
&\le Ma_n E(x_F - X)P(T_n > n-M) \\
&\le Ca_n \prod_{r=1}^{n} F(u_r) \\
&= Cn^{1/\alpha}c_n \exp\left\{\sum_{r=1}^{n}(-\eta/r)\right\}\exp\left\{\sum_{r=1}^{n}(\delta_r+\beta_r)/r\right\} \\
&= Cn^{1/\alpha}c_n \exp\{-\eta\log n - \eta(\gamma+\rho_n)\}\exp\left\{\sum_{r=1}^{n}(\delta_r+\beta_r)/r\right\} \\
&\le Dn^{-(\eta-(1/\alpha))}\exp\left\{\sum_{r=1}^{n}(\delta_r+\beta_r)/r\right\},
\end{aligned} \tag{3.7}$$

where C and D are positive constants, $\lim_n c_n = c_0 \in (0, \infty)$, $0 = \lim_n \delta_n = \lim_n \beta_n = \lim_n \rho_n$ and γ is Euler's constant. The second part of (3.3)(iii) now follows by letting $n \to \infty$ in (3.7). □

REMARK. We note that it is possible to provide an alternative proof of Theorem 3.1 (and indeed give a slightly stronger result). First, one establishes the analogue of Theorem 1.4 for general threshold sequences $\{u_n\}$ and their associated threshold stopping times $\{T_n\}$. Second, by verifying a further condition of uniform-integrability type, viz.

$$\lim_{L \to \infty} \limsup_{n \to \infty} E\Big(\big|a_n(X_{T_n} - b_n)\big| I\Big(\big|a_n(X_{T_n} - b_n)\big| > L\Big)\Big) = 0, \tag{3.8}$$

one shows that $a_n(EX_{T_n} - b_n)$ converges to the appropriate limit. The verification of condition (3.8) may be modelled on arguments within the proof of Proposition 2.1 in [13]. The argument for type II follows easily [because in this case $a_n(X_{T_n} - b_n)$ is dominated by $a_n(M_n - b_n)$ to give a right-tail estimate and the left-tail essentially plays no role]. The arguments for types I and III, however, require estimates analogous to those in (3.6) and (3.7); thus no substantial shortening of the above arguments results.

Theorem 3.1 can be used to show that the tail-distribution rate of growth of (1.7) is in a sense necessary for a threshold sequence to be asymptotically optimal.

COROLLARY 3.2. *Let $\{u_n\}_{n \geq 1}$ be any sequence of numbers satisfying $\lim_n n(1 - F(u_n)) = \eta$ for some $0 < \eta < \infty$ and let $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$. Let $\{T_n\}_{n \geq 1}$ be the threshold stopping times of* (1.14) *associated with $\{X_n\}_{n \geq 1}$ and $\{u_n\}_{n \geq 1}$. Then if* (1.15) *holds, it must follow that $\eta = 1$, $1 - \alpha^{-1}$ or $1 + \alpha^{-1}$ for $F \in \mathscr{D}(G_{\mathrm{I}})$, $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$ and $\alpha > 1$, or $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$ and $\alpha > 0$, respectively.*

PROOF. The result is an immediate consequence of Theorems 1.2 and 3.1 and the following easily verifiable facts about the functions f_{I}, f_{II} and f_{III} of (3.1):

(3.9)(i) $\max_{x > 0} f_{\mathrm{I}}(x) = f_{\mathrm{I}}(1) = 0$ and $x = 1$ is the only maximizer.

(3.9)(ii) For each $\alpha > 1$,

$$\max_{x > 0} f_{\mathrm{II}}(x; \alpha) = f_{\mathrm{II}}(1 - \alpha^{-1}; \alpha) = (1 - \alpha^{-1})^{-1/\alpha}$$

and $x = 1 - \alpha^{-1}$ is the only maximizer.

(3.9)(iii) For each $\alpha > 0$,

$$\max_{x > 0} f_{\mathrm{III}}(x; \alpha) = f_{\mathrm{III}}(1 + \alpha^{-1}; \alpha) = -(1 + \alpha^{-1})^{1/\alpha}$$

and $x = 1 + \alpha^{-1}$ is the only maximizer. □

REMARK 3.3. For simplicity in statement, convenience in proof and analogy to theorems in extreme-value theory, the statement of Theorem 3.1 is the appropriate one. However, Theorem 3.1 can be restated in terms of a rate of convergence on thresholds implying a rate of convergence on expectations of threshold-stopped r.v.'s as follows:

(3.10)(i) If $F \in \mathscr{D}(G_{\mathrm{I}})$ and $\lim_n a_n(u_n - b_n) = \nu$ with $-\infty < \nu < \infty$, then $\lim_n a_n(E(X_{T_n}) - b_n) = f_{\mathrm{I}}((R_{\mathrm{I}}^{-1}(\nu))$.
(3.10)(ii) If $F \in \mathscr{D}(G_{\mathrm{II}}^{\alpha})$, $\alpha > 1$ and $\lim_n a_n(u_n - b_n) = \nu$ with $0 < \nu < \infty$, then $\lim_n a_n(E(X_{T_n}) - b_n) = f_{\mathrm{II}}((R_{\mathrm{II}}^{\alpha})^{-1}(\nu); \alpha)$.
(3.10)(iii) If $F \in \mathscr{D}(G_{\mathrm{III}}^{\alpha})$, $\alpha > 0$ and $\lim_n a_n(u_n - b_n) = \nu$ with $-\infty < \nu < 0$, then $\lim_n a_n(E(X_{T_n}) - b_n) = f_{\mathrm{III}}((R_{\mathrm{III}}^{\alpha})^{-1}(\nu); \alpha)$.

The constants $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ are those in (1.6), the inverse rate functions R_{I}^{-1}, $(R_{\mathrm{II}}^{\alpha})^{-1}$ and $(R_{\mathrm{III}}^{\alpha})^{-1}$ are defined in (1.8) and the functions f_{I}, f_{II} and f_{III} are defined in (3.1). For the value sequence $\{v_n\}_{n \geq 1}$, the number $\lim_n a_n(v_n - b_n) = \nu^*$ of (1.10) is a fixed point of $g_i(x) = f_i(R_i^{-1}(x))$, and the unique maximizer and maximum of g_i, for each $i = \mathrm{I}$, II and III. Uniqueness and identification of the maximizer follow from (3.9).

Acknowledgment. The authors thank a referee for the main ideas in the proof of Theorem 1.1 given here; the original proof was much longer and relied on use of Theorem 3.1.

REFERENCES

[1] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). Regular variation. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press.
[2] CHOW, Y. S., ROBBINS, H. and SIEGMUND, D. (1971). *Great Expectations: The Theory of Optimal Stopping*. Houghton-Mifflin, New York.
[3] GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics*. Wiley, New York.
[4] GILBERT, J. and MOSTELLER, F. (1966). Recognizing the maximum of a sequence. *J. Amer. Stat. Assoc.* **61** 35–73.
[5] GRADSHTEYN, I. S. and RYZHIK, I. M. (1980). *Tables of Integrals, Series and Products*. Academic, New York.
[6] HILL, T. P. and KERTZ, R. P. (1982). Comparisons of stop rule and supremum expectations of i.i.d. random variables. *Ann. Probab.* **10** 336–344.
[7] KENNEDY, D. P. and KERTZ, R. P. (1990). Limit theorems for threshold-stopped random variables with applications to optimal stopping. *Adv. in Appl. Probab.* **22** 396–411.
[8] KERTZ, R. P. (1986). Stop rule and supremum expectations of i.i.d. random variables: A complete comparison by conjugate duality. *J. Multivariate Anal.* **19** 88–112.
[9] KERTZ, R. P. (1986). Prophet problems in optimal stopping: Results, techniques, and variations. Technical Report, School of Mathematics, Georgia Inst. Tech.
[10] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York.
[11] PETRUCCELLI, J. D. (1985). Maximin optimal stopping for normally distributed random variables. *Sankhyā Ser. A* **47** 36–46.

[12] PICKANDS, J. (1968). Moment convergence of sample extremes. *Ann. Math. Statist.* **39** 881–889.

[13] RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

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