The Annals of Probability
1991, Vol. 19, No. 1, 289–311

CONDITIONAL LIMIT DISTRIBUTIONS OF CRITICAL BRANCHING BROWNIAN MOTIONS

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A critical branching Brownian motion in R^d is studied where the initial state is either a single particle or a homogeneous field with finite or infinite density. Conditioned on survival in a bounded subset B of R^d at a large time t, some normalized limits of the number of particles in a bounded subset A are obtained. When the initial state is a single particle, the normalization factor is a power of t in low dimensions, a power of $\log t$ in the critical dimension and a constant in high dimensions. Extensions to the other initial states and/or more general critical offspring distributions are discussed. Both factors affect the critical dimension. The results are motivated by probabilistic consideration and are proved with the aid of analytic technique of differential equations.

0. Introduction. This paper deals with some conditional limits of critical branching Brownian motions (CBBM) in R^d, initially either a single particle or a homogeneous random field. For simplicity of presentation we assume a particle to follow a Brownian motion (BM) in R^d, obeying (H1)–(H3):

(H1) The particle lives an exponentially distributed lifetime of mean $\frac{1}{2}$.

(H2) The particle is replaced at death by either no descendant or two descendants at the same position with equal probability.

(H3) Each descendant follows a BM and obeys (H1) and (H2). All Brownian motions, lifetimes and decisions on death or birth are independent of one another.

Several initial states will be considered. We denote by P_x the probability distribution of our CBBM, initially a single particle at x, by Q_a that initially a random field μ_a with exponent $0 < a \leq 1$, characterized by

$$E^{\mu_a}\left\{\prod_{x_j \in X}\left[1 - f(x_j)\right]\right\} = \exp\left[-\int_{R^d} f(x)^a\, dx\right], \tag{0.1}$$

for $0 \leq f(x) \leq 1$, where $X = \{x_j;\ j = 1, 2, \dots\}$ denotes the position of particles. Throughout this paper we use E_x (E^a, resp.) for the expectation associated with P_x (Q_a, resp.) and call a subset of R^d proper when it is bounded with positive Lebesgue measure. The present article deals with $N_A(t)$, the number of particles in a proper set A at time t.

Received November 1988; revised October 1989.
AMS 1980 *subject classifications*. 60F05, 60J65, 60J80, 35K55.
Key words and phrases. Branching Brownian motion, critical branching, semilinear parabolic equation, survival probability, conditional limit distributions.

The following quantities and related differential equations are important for our discussion. Let $\{x_j(t)\}$ be the position of particles survival at time t. For $h, \eta \in L^1(R^d)$ and $\eta(x) \in [0,1]$ for $x \in R^d$, define

$$m(t,x;h) \equiv E_x\left\{\sum_{j=1}^{N(t)} h(x_j(t))\right\},$$

$$u(t,x;h) \equiv 1 - E_x\left\{\prod_{j=1}^{N(t)} \left[1 - \eta(x_j(t))\right]\right\},$$

where $N(t)$ is the total number of particles existent at time t. They satisfy the following:

(A) $$m_t - \frac{\Delta}{2}m = 0 \quad \text{in } R^+ \times R^d \quad \text{and} \quad m(0,x;h) = h(x),$$

(B) $$Lu \equiv u_t - \frac{\Delta}{2}u + u^2 = 0 \quad \text{in } R^+ \times R^d \quad \text{and} \quad u(0,x;h) = h(x).$$

See [10] and references therein for the derivation.

Let 1_A denote the index function associated with A. It is easy to see

(0.2) $$E_x\{N_A(t)\} = m(t,x;1_A),$$

(0.3) $$P_x\{e^{-\theta N_A(t)}\} = 1 - u(t,x,1-e^{-\theta 1_A}) \quad \text{for } \theta \geq 0,$$

(0.4) $$P_x\{N_A(t) > 0\} = \lim_{\theta\to\infty} u(t,x;1-e^{-\theta 1_A}) = u(t,x;1_A).$$

From a probabilistic viewpoint it is natural to look into $E_{t^{1/2}y}\{N_A(t)\}$ and $P_{t^{1/2}y}\{N_A(t) > 0\}$ as $t \to \infty$, where the scaling $t^{1/2}y$ is suggested by BM. For fixed $y \in R^d$, define

(0.5) $$c(t) \equiv E_{t^{1/2}y}\{N_A(t) | N_A(t) > 0\} = \frac{m(t, t^{1/2}y; 1_A)}{u(t, t^{1/2}y; 1_A)}.$$

We anticipate the following type of limit theorem:

(0.6) Conditioned on $\{N_A(t) > 0\}$ and w.r.t. $P_{t^{1/2}y}$, $N_A(t)/c(t)$ converges in distribution as $t \to \infty$.

An elementary computation using (0.3) and (0.4) gives the generating function of $N_A(t)/c(t)$,

(0.7) $$E_{t^{1/2}y}\left\{\exp\left[-\theta c(t)^{-1}N_A(t)\right] | N_A(t) > 0\right\} = 1 - \left[\frac{u(t, t^{1/2}y; \theta c(t)^{-1}1_A)}{u(t, t^{1/2}y; 1_A)}\right]$$

for $\theta \geq 0$. (0.6) is therefore equivalent to:

(0.8) $$\frac{u(t, t^{1/2}y; \theta c(t)^{-1}1_A)}{u(t, t^{1/2}y; 1_A)} \quad \text{converges as } t \to \infty \text{ for } \theta \geq 0.$$

Our goal is to prove various limit theorems like (0.6) and (0.8). Regarding (0.6), our results show three regimes of different limiting behaviors of $N_A(t)$: $c(t)$ is of the order of $t^{1/2}$ when $d = 1$, of $\log t$ when $d = 2$ and of a constant when $d \geq 3$. The existence of the critical dimension $d = 2$ can be seen from the picture provided by CBBM as follows. Assuming the survival of the process at a large time t, the number of particles is, roughly speaking, of the order of t and they spread in a region of area of the order of $t^{d/2}$. Let A be a proper set in R^d. This suggests that $c(t)$, i.e., $N_A(t)$ conditioned on survival, is of the order of $t^{1/2}$ when $d = 1$ and is finite when $d \geq 3$ as $t \to \infty$. A closer look, as in the Introduction of [9], at CBBM actually anticipates $t^{1/2}$ for $d = 1$, $\log t$ for $d = 2$ and finite for $d \geq 3$. Some discussions from different points of view can be found in [4, 6] and references therein.

From (0.5), $c(t)$, hence the critical dimension, can also be determined by the behavior of $u(t, t^{1/2}y; 1_A)$ and $m(t, t^{1/2}y; 1_A)$ as $t \to \infty$. The latter quantity is of the order of $t^{-d/2}$. From (0.8) we see that $u(t, t^{1/2}y; \theta c(t)^{-1} 1_A)$ as well as $u(t, t^{1/2}y; 1_A)$ should be investigated. This is undertaken by a comparison principle which we now illustrate to estimate the large time behaviors of $u(t, t^{1/2}y; 1_A)$ and $c(t)$. Using functions of the form $\zeta(t)g(t, x)$, $g(t, x) \equiv (2\pi t)^{-d/2} \exp[-|x|^2/2t]$, the heat kernel, a simple computation yields that the following $\zeta(t)$ makes $\zeta(t)g(t, x)$ subsolutions of (B) [i.e., "$\leq$" is satisfied instead of "$=$" in (B)] for $t \geq 1$:

$$(0.9)\quad \zeta(t) = \begin{cases} \left[\dfrac{1}{\zeta(1)} + (2\pi)^{-1/2} 2(t^{1/2} - 1)\right]^{-1} \\ \quad \sim (2\pi)^{1/2} 2^{-1} t^{-1/2} \quad \text{when } d = 1, \\ \left[\dfrac{1}{\zeta(1)} + (2\pi)^{-1} \log t\right]^{-1} \sim (2\pi)(\log t)^{-1} \quad \text{when } d = 2, \\ \left[\dfrac{1}{\zeta(1)} + (2\pi)^{-d/2}\left(\dfrac{d}{2} - 1\right)^{-1}(1 - t^{1-d/2})\right]^{-1} \\ \quad \sim \left[\dfrac{1}{\zeta(1)} + (2\pi)^{-d/2}\left(\dfrac{d}{2} - 1\right)^{-1}\right]^{-1} \quad \text{when } d \geq 3, \end{cases}$$

as $t \to \infty$. We thus have some subsolutions with the large time asymptotics being

$$\frac{1}{2} t^{-1} \exp\left[-\frac{x^2}{2t}\right] \quad \text{for } d = 1, \qquad (t \log t)^{-1} \exp\left[-\frac{|x|^2}{2t}\right] \quad \text{for } d = 2,$$

and $kt^{-d/2}$, $k > 0$, for $d \geq 3$. Supersolutions with the same magnitude of asymptotics will be given in the proofs of the theorems. We conclude, using (0.5), the preceding discussion with the suggested formula,

$$(0.10)\qquad c(t) \sim \begin{cases} O(t^{1/2}) & \text{when } d = 1, \\ O(\log t) & \text{when } d = 2, \\ O(1) & \text{when } d \geq 3. \end{cases}$$

The assertion (0.6) with these $c(t)$ will be verified in Theorems 1.4, 2.4 and 3.2, respectively.

The remainder of this section contains the motivation and statements of our results in the order of their logical dependence and also some accounts for generalizations. The proofs are placed in the subsequent sections (for example, the proof of Lemma 1.1 is in Section 1).

First of all, there are two fundamental facts, a scaling structure and a comparison principle. Since both are intuitive and easy to prove, we omit their proofs.

SCALING LEMMA. *Let* $L_p u \equiv u_t - (\Delta/2)u + u^p$ *for* $p > 1$, $(J_\lambda u)(t,x) \equiv \lambda^{1/(p-1)}u(\lambda t, \lambda^{1/2}x)$ *and* $J_\lambda \Omega \equiv \{(t,x)|(\lambda t, \lambda^{1/2}x) \in \Omega\}$ *for a domain* $\Omega \subseteq R^+ \times R^d$. *Then* $(L_p u)(t,x) = 0$ *in* Ω *if and only if* $(L_p J_\lambda u)(t,x) = 0$ *in* $J_\lambda \Omega$.

COMPARISON LEMMA. *Suppose* $f \in C^1(R)$ *and* $\overline{u}(t,x)$ *and* $\underline{u}(t,x)$ *satisfy* (0.11)–(0.14):

(0.11) $\overline{u}$ *is a supersolution and* $\underline{u}$ *is a subsolution, i.e.,*

$$(\overline{u})_t - \frac{\Delta}{2}\overline{u} + f(\overline{u}) \geq 0 \quad and \quad (\underline{u})_t - \frac{\Delta}{2}\underline{u} + f(\underline{u}) \leq 0$$

$$in\ (0,T) \times R^d;$$

(0.12)
$$\sup_{\substack{t \leq s \\ x \in R^d}} |\overline{u}(t,x)| + |\underline{u}(t,x)| < \infty \quad for\ every\ s < T;$$

(0.13)
$$\overline{u}(0,x) \geq \underline{u}(0,x) \quad for\ x \in R^d;$$

(0.14)
$$\liminf_{r \to \infty} \inf_{t \leq s,\, x = r} [\overline{u}(t,x) - \underline{u}(t,x)] \geq 0 \quad for\ every\ s < T.$$

Then $\overline{u}(t,x) \geq \underline{u}(t,x)$ *in* $(0,T) \times R^d$.

REMARK. It is not difficult to remove condition (0.14) and cover more general nonlinearity, e.g., $f(t,x,u)$. The present form serves our purpose well.

Now we begin with the low dimension $d = 1$. Some special solutions of (B) play important roles in this case: $U(t,x) = t^{-1}F(t^{-1/2}x)$, the unique positive self-similar solution of (B) in $R^+ \times R$ (leaving out the initial condition) with $F'(0) = 0$ and $\lim_{|y| \to \infty} y^2 F(y) = 0$; $V(t,x;a)$, $a > 0$, the unique solution of (B) with initial value $a\delta_0$, where δ_0 is the Dirac measure at the origin. See [1, 2] for the existence, uniqueness and the following two properties of U and V:

$F(y) \sim O(|y|e^{-y^2/2})$ as $|y| \to \infty$.

For $a > 0$ there exists $c > 0$ such that $a(2\pi t)^{-1/2}\exp[-x^2/2t] \geq V(t,x;a) \geq [a(2\pi t)^{-1/2} - c]\exp[-x^2/2t]$ for all $t > 0$ and $x \in R$. Let $M_{\exp}$ denote the totality of nonnegative measurable functions (from R^d to R, $d \in \mathbf{N}$) smaller than $k\exp[-|x|^2/2m]$ for some positive k and m but not identically zero and set $\overline{h} \equiv \int h(x)\,dx$. Lemma 1.1 is a simple consequence of [5] (see [4]).

LEMMA 1.1. *If* $d = 1$ *and* $h \in M_{\exp}$, *then* $\lim_{t\to\infty} tu(t, t^{1/2}y, h) = F(y)$ *for all* $y \in R$.

Lemma 1.1 enables us to estimate $P_{t^{1/2}y}\{N_A(t) > 0\}$ and $Q_a\{N_A(t) > 0\}$ in Theorem 1.2.

THEOREM 1.2. *Suppose* $d = 1$ *and* A *and* B *are proper. Then the following hold*:

(0.15) $\lim_{t\to\infty} tP_{t^{1/2}y}\{N_A(t) > 0\} = F(y)$ *for* $y \in R$ (*independent of* A);

(0.16) *as* $t \to \infty$, $Q_a\{N_A(t) > 0\}$ *tends to* 0, $1 - \exp[-\int F(y)^{1/2}\,dy]$ *or* 1 *in accordance with* $1 \ge a > \frac{1}{2}$, $a = \frac{1}{2}$ *or* $\frac{1}{2} > a > 0$; *moreover, we have* $\lim_{t\to\infty} t^{a-1/2}Q_a\{N_A(t) > 0\} = \int F(y)^a\,dy$ *for* $a > \frac{1}{2}$ *and* $\lim_{t\to\infty} t^{a-1/2}\log Q_a\{N_A(t) = 0\} = -\int F(y)^a\,dy$ *for* $\frac{1}{2} > a$;

(0.17)
$$\lim_{t\to\infty} P_{t^{1/2}y}\{N_A(t) > 0 | N_B(t) > 0\} = 1 \quad \textit{for all } y \in R,$$
$$\lim_{t\to\infty} Q_a\{N_A(t) > 0 | N_B(t) > 0\} = 1 \quad \textit{for } a \in (0,1].$$

REMARK. From the fact $P_{t^{1/2}y}\{N(t) > 0\} = (1 + t)^{-1}$, it follows that
$$\lim_{t\to\infty} P_{t^{1/2}y}\{N_A(t) > 0 | N(t) > 0\} = F(y).$$
Applying the comparison lemma to
$$\bar{u}(t,x) \equiv u(t,x;h),$$
$$h \equiv (2\pi)^{-1/2}\exp\left[-\frac{|x|^2}{2}\right]$$
and
$$\underline{u}(t,x) \equiv \left[1 + (2\pi)^{-1/2}2\left((t+1)^{1/2} - 1\right)\right]^{-1} \times [2\pi(t+1)]^{-1/2}\exp\left[-\frac{|x|^2}{2(t+1)}\right]$$
[see (0.9)], Lemma 1.1 yields $F(y) \ge \frac{1}{2}\exp[-|y|^2/2]$ for all $y \in R$, in particular, $F(0) \ge \frac{1}{2} > 0$. We suspect $\lim_{t\to\infty} P_0\{N_A(t) > 0 | N(t) > 0\} = F(0)$ to be strictly less than 1 but do not have a proof.

Theorem 1.3 is an analytic result whose probabilistic implication, stated in Theorem 1.4 is our main result for $d = 1$.

THEOREM 1.3. *Suppose* $d = 1$ *and* $h \in M_{\exp}$. *Then*
$$\lim_{t\to\infty} tu(t - s, t^{1/2}y; t^{-1/2}h) = V(1, y; \bar{h}) \quad \textit{for all } s \in R \textit{ and } y \in R.$$

THEOREM 1.4. *Let B and A_j, $1 \le j \le k$, be proper sets in R and $\theta \ge 0$. Then*

$$\lim_{t\to\infty} E_{t^{1/2}y}\left\{\exp\left[-t^{-1/2}\sum_{j=1}^{k}\theta_j N_{A_j}(t)\right]\middle| N_B(t) > 0\right\} = 1 - \frac{V(1,y;\Sigma\theta_j|A_j|)}{F(y)} \quad \text{for all } y \in R, \tag{0.18}$$

where $|A|$ denotes the Lebesgue measure of A, i.e., conditioned on $\{N_B(t) > 0\}$ and w.r.t. $P_{t^{1/2}y}$, $(t^{-1/2}N_{A_j}(t);\ 1 \le j \le k)$ converges in distribution to $(|A_j|Z;\ 1 \le j \le k)$ as $t \to \infty$, where Z is characterized by $E\{\exp[\theta Z]\} = 1 - V(1,y;\theta)/F(y)$ for $\theta \ge 0$.

For $1 \ge a \ge \frac{1}{2}$,

$$\lim_{t\to\infty} E^a\left\{\exp\left[-t^{-1/2}\Sigma\theta_j N_{A_j}(t)\right]\middle| N_B(t) > 0\right\} = \begin{cases} 1 - \dfrac{\int V(1,x;\Sigma\theta_j|A|)^a\,dx}{\int F(x)^a\,dx} & \text{for } 1 \ge a > \frac{1}{2}, \\[2ex] 1 - \dfrac{\left\{1 - \exp\left[-\int V(1,x;\Sigma\theta_j|A_j|)^{1/2}\,dx\right]\right\}}{\left\{1 - \exp\left[-\int F(x)^{1/2}\,dx\right]\right\}} & \text{for } a = \frac{1}{2}. \end{cases} \tag{0.19}$$

This is a weak-convergence result similar to (0.18) w.r.t. Q^a, $1 \ge a \ge \frac{1}{2}$.

The case $d = 2$ is unlike $d = 1$ at least in that there is no self-similar solution of (B) such as U. Practically, what we need in order to establish similar results is some supersolutions and subsolutions with a matching leading term. This is roughly the idea of Theorem 2.1. Our comparison techniques should be also useful in asymptotic analysis of many nonlinear differential equations other than (B).

THEOREM 2.1. *For $d = 2$ and any small positive ε there exist $\bar{u}(t,x)$, $\underline{u}(t,x)$ and t_0 which satisfy (0.20) and (0.21) for all $t \ge t_0$ and $x \in R^2$:*

$\underline{u} \ge 0, \quad L\underline{u} \le 0$ *and*

$$(t\log t)^{-1}2\exp\left[-\frac{|x|^2}{[2(1+\varepsilon)t]}\right] \ge \underline{u}(t,x) \ge (t\log t)^{-1}2(1-\varepsilon)\exp\left[-\frac{|x|^2}{2t}\right], \tag{0.20}$$

$\bar{u} \ge 0, \quad L\bar{u} \ge 0$ *and*

$$(t\log t)^{-1}2(1+\varepsilon)\exp\left[-\frac{|x|^2}{[2(1+\varepsilon)t]}\right] \ge \bar{u}(t,x) \ge (t\log t)^{-1}2\exp\left[-\frac{|x|^2}{2t}\right]. \tag{0.21}$$

We also have:

(0.22) *If* $d = 2$ *and* $h \in M_{\exp}$, *then* $\lim_{t\to\infty}(t \log t)u(t, t^{1/2}y; h) = 2\exp[-|y|^2/2]$ *for all* $y \in R^2$.

By Theorem 2.1 we derive the asymptotic behavior of some survival probabilities.

THEOREM 2.2. *Let* $d = 2$ *and* A *and* B *be proper sets in* R^2. *Then*

$$(0.23)\quad \lim_{t\to\infty}(t\log t)P_{t^{1/2}y}\{N_A(t) > 0\} = 2\exp\left[-\frac{|y|^2}{2}\right] \quad \textit{for all } y \in R^2,$$

$$(0.24)\quad \begin{aligned} &\lim_{t\to\infty}(\log t)Q_1\{N_A(t) = 0\} = 4\pi \quad \textit{and} \\ &\lim_{t\to\infty}\frac{\log Q_a\{N_A(t) = 0\}}{\left[t^{1-a}(\log t)^{-a}\right]} = -2^a a^{-1/2}(2\pi), \end{aligned}$$

in particular;

$$\lim_{t\to\infty} Q_a\{N_A(t) = 0\} = 1, \quad \textit{for } 1 > a > 0,$$

$$(0.25)\quad \begin{aligned} &\lim_{t\to\infty} P_{t^{1/2}y}\{N_A(t) > 0|N_B(t) > 0\} = 1 \quad \textit{and} \\ &\lim_{t\to\infty} Q_1\{N_A(t) > 0|N_B(t) > 0\} = 1. \end{aligned}$$

REMARK. (0.24) is proved by showing that

$$\lim_{t\to\infty}(\log t)\int u(t, x; h)\, dx = 4\pi$$

for $h \in M_{\exp}$. This specific results gives a positive answer to an open problem [7]. Moreover, it holds that $\lim_{t\to\infty}$ $(\log t)\int u(t, x; (\log t)^{-1}h)\, dx = 4\pi\bar{h}/(4\pi + \bar{h})$ (a consequence of Theorem 2.3).

Theorem 2.3 establishes an analytic result crucial to Theorem 2.4, our main result for $d = 2$.

THEOREM 2.3. *Suppose* $d = 2$ *and* $h(x) \in M_{\exp}$. *Then*

$$\lim_{t\to\infty}(t\log t)u\left(t - s, t^{1/2}y; (\log t)^{-1}h\right) = \left(1 + 4\pi(\bar{h})^{-1}\right)2\exp\left[-\frac{|y|^2}{2}\right]$$

for all $s \in R$ *and* $y \in R^2$.

THEOREM 2.4. *Let* $d = 2$ *and let* B *and* A_j, $1 \le j \le k$, *be proper sets in* R^2. *Then*

$$(0.26)\quad \begin{aligned} &\lim_{t\to\infty} E_{t^{1/2}y}\left\{\exp\left[-(\log t)^{-1}\Sigma\theta_j N_{A_j}(t)\right]\Big|N_B(t) > 0\right\} \\ &= \left[1 + 4\pi\left(\Sigma\theta_j|A_j|\right)^{-1}\right]^{-1} \end{aligned}$$

for all $y \in R^2$ and $\theta_j \geq 0$, $1 \leq j \leq k$;

$$\lim_{t\to\infty} E^1\left\{\exp\left[-(\log t)^{-1}\sum_{j=1}^{k}\theta_j N_{A_j}(t)\right]\middle| N_B(t) > 0\right\} = \left[1 + 4\pi\left(\sum_{j=1}^{k}\theta_j|A_j|\right)^{-1}\right]^{-1}. \tag{0.27}$$

For $d \geq 3$ we take the following lemma from Theorem 3.1 of [8] [there, $p \geq (N+2)/N$ should be $p > (N+2)/N$].

LEMMA 3.1. *If $d \geq 3$, $h(x) \geq 0$ and $h \in L^1(R^d)$, then*

$$\lim_{t\to\infty} t^{d/2}u(t, t^{1/2}y; h) = \alpha(y, h)$$

exists and is positive for $h \not\equiv 0$.

The limit distributions for $d \geq 3$ depend not only on $|A_j|$, $1 \leq j \leq k$, but also on their shape and relative position. This is already suggested in (0.9) by the dependence of the leading term of $\zeta(t)$ on $\zeta(1)$ and is made clear in Theorem 3.2.

THEOREM 3.2. *Let $d \geq 3$ and B and A_j, $1 \leq j \leq k$, be proper sets in R^d and $A \equiv \bigcup_{j=1}^{k} A_j$. Then*

$$\lim_{t\to\infty} t^{d/2}P_{t^{1/2}y}\{N_B(t) > 0\} = \alpha(y; 1_B), \tag{0.28}$$

$$\lim_{t\to\infty} E_{t^{1/2}y}\left\{\exp\left[-\sum_{j=1}^{k}\theta_j N_{A_j}(t)\right]\middle| N_A(t) > 0\right\} = 1 - \frac{\alpha\left(y; 1 - \exp\left[-\Sigma\theta_j 1_{A_j}\right]\right)}{\alpha(y; 1_A)}, \tag{0.29}$$

$$\lim_{t\to\infty} E^1\left\{\exp\left[-\Sigma\theta_j N_{A_j}(t)\right]\right\} = \exp\left[-\int\alpha\left(y; 1 - \exp\left[-\Sigma\theta_j 1_{A_j}\right]\right)dy\right]. \tag{0.30}$$

(0.31)
With respect to Q_a, $a < 1$, $N_B(t)$ tends to ∞ in distribution.

We now remark on some generalizations of the previously mentioned results.

1. Consider, for $1 < p < \infty$ and $d \in R$,

$$u_t - \frac{1}{2}\left(D^2 + \frac{d-1}{x}D\right)u + u^p = 0 \quad \text{for } t > 0,\ x > 0, \tag{C}$$

and let $u(t, x; h)$ be the solution with the Neumann boundary condition at

$x = 0$ and initial value $h(x)$. The symmetric extension of h is assumed to belong to $M_{\exp}$. The critical dimension of (C), in the sense of the behavior of $c(t)$ in (0.5) and (0.8), is $2/(p-1)$. Analytic results parallel to those in this article (Lemma 1.1, Theorems 1.3, 2.1 and 2.3 and Lemma 3.1) can be obtained with obvious modification of the arguments adopted here and in the quoted references. The computation leading to (0.9), for instance, yields

$$\zeta(t) \sim \begin{cases} O\left(t^{(d/2)-1/(p-1)}\right) & \text{for } d < 2/(p-1), \\ O\left((\log t)^{-1/(p-1)}\right) & \text{for } d = 2/(p-1), \\ \text{constant depending on } \zeta(1) & \text{for } d > 2/(p-1). \end{cases}$$

2. For $1 < p \leq 2$ and $d \in R$, (C) corresponds to a branching process with Bessel process of index d, instead of BM, as its spatial motion and with its critical offspring distribution belonging to the normal domain of attraction of stable distribution with exponent p, $1 < p \leq 2$. One can therefore derive conditional limit theorems parallel to the case when $p = 2$ (Theorems 1.2, 1.4, 2.2, 2.4 and 3.2). Although our analytic results extend to $1 < p < \infty$ with no difficulty, the probabilistic meaning for $p > 2$ is unknown and is interesting to seek. Note that the critical dimension depends on p as well as α [see (0.1)]. Extensions to critical measure-valued processes require little modification besides interpreting $N_A(t)$ as the mass occupying A at time t. See [3] and references therein for the construction of measure-valued processes. Partial results for general p and for measure-valued processes were obtained in [4, 7].

3. Since only large time behavior of solutions of (C), which tend to zero, are essential in this study, extensions to the following equation (D) are not difficult (see [9] for some details of analysis required).

$$\text{(D)} \qquad u_t - \frac{1}{2}\left(D^2 + \frac{d-1}{x}D\right)u + f(t,x,u)u^p = 0,$$

where $f(t,x,u)$ tends to a positive constant in an appropriate sense as $t \to \infty$, $x \to \infty$ and $u \to 0$. (D) corresponds to processes with waiting times and critical offspring distributions depending on t and x.

1.

Proof of Lemma 1.1. It was already proved in [5] (see [4]) that

$$\lim_{t\to\infty} \sup_{x\in R} |tu(t,x;h) - F(t^{-1/2}x)| = 0 \quad \text{for } h \in M_{\exp} \cap C^2(R).$$

For $h \in M_{\exp} \cap C^2(R)$, substituting x by $t^{1/2}y$ yields Lemma 1.1. In view of the facts that $u(t,x;h)$ is increasing in h by the comparison lemma and that for $h \in M_{\exp}$ there exist h_+ and $h_- \in M_{\exp} \cap C^2(R)$ such that $h_+ \geq h \geq h_-$, the differentiability condition is unnecessary and the proof is complete. □

PROOF OF THEOREM 1.2. (0.15) follows immediately from (0.4) and Lemma 1.1. In order to prove (0.16), we note for $a \in (0,1]$ and $d \in \mathbf{N}$ that

$$
\begin{aligned}
Q_a\{N_A(t) = 0\} &= E^{\mu_a}\left\{\prod_{j=1}^{\infty} P_{x_j}\{N_A(t) = 0\}\right\} \\
&= E^{\mu_a}\left\{\sum_{j=1}^{\infty}\left[1 - u(t, x_j; 1_A)\right]\right\} \qquad (1.1)\\
&= \exp\left[-\int_{R^d} u(t, x; 1_A)^a\, dx\right].
\end{aligned}
$$

The first task is to determine for $d = 1$ when $\int_R u(t, x; 1_A)^a\, dx$ tends to zero, a constant or infinity. Since there exists some $c > 1$ such that $1_A \le cF(|x|)$ and thus,

$$
\begin{aligned}
tu(t, t^{1/2}y; 1_A) &\le tu(t, t^{1/2}y; cF) \le ctu(t, t^{1/2}y; F) \\
&= ctu(t, t^{1/2}y; U(1, x)) = ctu(t + 1, t^{1/2}y) \qquad (1.2)\\
&= [ct/(t+1)]F([t/(t+1)]^{1/2}y),
\end{aligned}
$$

where the second inequality is due to the comparison lemma using $cu(t, x; F)$ as a supersolution and $u(t, x, cF)$ as a subsolution. By Lemma 1.1, (1.2) and the property of F in the paragraph preceding Lemma 1.1, the dominated convergence theorem implies

$$
\begin{aligned}
\lim_{t\to\infty} t^{a-1/2}\int_R u(t, x; 1_A)^a\, dx &= \lim_{t\to\infty}\int_R \left[tu(t, t^{1/2}y; 1_A)\right]^a dy \qquad (1.3)\\
&= \int F(y)^a\, dy.
\end{aligned}
$$

In view of (1.1) this proves (0.16) for the case $a \le \frac{1}{2}$ immediately. For $a > \frac{1}{2}$, note from (1.3) that $\lim_{t\to\infty}\int u(t, x; 1_A)^a\, dx = 0$. Thus,

$$
\begin{aligned}
\lim_{t\to\infty} t^{a-1/2} Q_a\{N_A(t) > 0\} &= \lim_{t\to\infty} t^{a-1/2}\left\{1 - \exp\left[-\int u(t, x; 1_A)^a\, dx\right]\right\} \\
&= \lim_{t\to\infty} t^{a-1/2}\int u(t, x; 1_A)^a\, dx \\
&= \lim_{t\to\infty}\int tu(t, t^{1/2}y; 1_A)^a\, dy \\
&= \int F(y)^a\, dy.
\end{aligned}
$$

Next, we turn to (0.17). An elementary computation gives

$$P_{t^{1/2}y}\{N_A(t) > 0 | N_B(t) > 0\}$$

$$(1.4) \qquad = \frac{\left[P_{t^{1/2}y}\{N_A(t) > 0\} + P_{t^{1/2}y}\{N_B(t) > 0\} - P_{t^{1/2}y}\{N_{A\cup B}(t) > 0\}\right]}{P_{t^{1/2}y}\{N_B(t) > 0\}}.$$

From (1.4) and Lemma 1.1 the first assertion of (0.17) follows. A similar computation with $P_{t^{1/2}y}$ replaced by Q_a, $1 \geq a > 0$, proves the second assertion of (0.17) and ends the proof. □

Proof of Theorem 1.3. Define

$$(1.5) \qquad \begin{aligned} b(y;h) &= \inf_{s\in R} \liminf_{t\to\infty} tu(t-s, t^{1/2}y; t^{-1/2}h), \\ B(y;h) &\equiv \sup_{s\in R} \limsup_{t\to\infty} tu(t-s, t^{1/2}y; t^{-1/2}h). \end{aligned}$$

Theorem 1.3 is equivalent to

$$(1.6) \qquad b(y,h) \geq V(1,y;\bar{h}) \quad \text{and} \quad B(y,h) \leq V(1,y;\bar{h}).$$

The proof for $b(y;h)$ is broken up into four steps and a similar proof for $B(y;h)$ will be omitted.

Step 1.

$$b(y,h) = b(y;G_\lambda h), \quad \text{where } (G_\lambda h)(x) \equiv \lambda^{1/2}h(\lambda^{1/2}x).$$

By the scaling lemma, $\lambda u(\lambda(t-s),\ \lambda^{1/2}x;\ (\lambda T)^{-1/2}h) = u(t-s,\ x;\ T^{-1/2}G_\lambda h)$ for $t \geq s$ and $T > 0$, which implies

$$b(y;G_\lambda h) = \inf_{\lambda s\in R} \liminf_{\lambda T\to\infty} (\lambda T)u\big(\lambda T - \lambda s, (\lambda T)^{1/2}y; (\lambda T)^{-1/2}h\big) = b(y;h).$$

Step 2.

$$b(y;h) = b(y;H_\lambda h),$$

where

$$(H_\lambda h)(x) = \int (2\pi\lambda)^{-1/2} \exp\left[-\frac{|x-y|^2}{2\lambda}\right] h(y)\,dy.$$

A simple computation shows that $(t + \theta^{-1}\|h\|_\infty^{-1})^{-1}$ is a solution of (B) with a constant initial value $\theta\|h\|_\infty$ for any dimension and $\theta > 0$. It then follows easily that $u(t,x;\theta h)$ satisfies $u_t - (\Delta/2)u + (t + \theta^{-1}\|h\|_\infty^{-1})^{-1}u \geq 0$ and from the maximum principle that

$$(1.7) \qquad u(t,x;\theta h) \geq \|h\|_\infty^{-1}\big(t + \theta^{-1}\|h\|_\infty^{-1}\big)^{-1}(H_t h)(x),$$

where the r.h.s. is the exact solution of $v_t - (\Delta/2)v + (t + \theta^{-1}\|h\|_\infty^{-1})^{-1}v = 0$ and $v(0,x) = \theta h$. Finally, letting $t = \lambda$ and $\theta = T^{-1/2}$, (1.7) yields for $d = 1$

that

$$\begin{aligned} b(y,h) &\equiv \inf_{s\in R} \liminf_{T\to\infty} Tu(T-s, T^{1/2}y; T^{-1/2}h) \\ &= \inf_{s\in R} \liminf_{T\to\infty} Tu(T-(s+\lambda), T^{1/2}y; u(\lambda, \cdot\,; T^{-1/2}h)) \\ &= \inf_{t\in R} \liminf_{T\to\infty} Tu\Big(T-t, T^{1/2}y; \|h\|_\infty^{-1}\big(\lambda + T^{1/2}\|h\|_\infty^{-1}\big)^{-1} H_\lambda h\Big) \\ &\ge \inf_{t\in R} \liminf_{T\to\infty} g(T) Tu(T-t, T^{1/2}y; T^{-1/2}H_\lambda h) \\ &= b(y; H_\lambda h) \quad \text{for any } \lambda \ge 0, \end{aligned}$$

where $g(T) \equiv T^{1/2}\|h\|_\infty^{-1}(\lambda + T^{1/2}\|h\|_\infty^{-1})^{-1} \uparrow 1$ as $T \to \infty$.

The last inequality holds because $g(T)u$ is a subsolution of (B). The inequality in the other direction follows easily from $u(\lambda, x; t^{-1/2}h) \le t^{-1/2}H_\lambda h$ (by the maximum principle).

STEP 3. Define

$$g_c(x) \equiv (2\pi c)^{-1/2} \exp\left[-\frac{|x|^2}{2c}\right], \qquad c > 0.$$

We prove $b(y;h) = b(y;\bar{h}g_1)$ for all $h \in M_{\exp}$. For $h \in M_{\exp}$ we have $a(\lambda)\downarrow 1$ and $b(\lambda)\uparrow 1$ as $\lambda \to \infty$ such that

(1.8) $a(\lambda)^{3/2}\bar{h}g_{a(\lambda)}(x) \ge (H_1 G_\lambda h)(x) \ge b(\lambda)^{3/2}\bar{h}g_{b(\lambda)}(x)$ for $x \in R$.

The first inequality together with the results of Steps 1 and 2 yields

$$\begin{aligned} b(y;h) &= b(y; H_1 G_\lambda h) \le b\big(y; a(\lambda)^{3/2}\bar{h}g_{a(\lambda)}\big) \\ &= b\big(y; a(\lambda)^{3/2}\bar{h}g_1\big) \le a(\lambda)^{3/2} b(y;\bar{h}g_1). \end{aligned}$$

The last inequality is obtained by the comparison lemma. Letting λ tend to ∞ yields $b(y;h) \le b(y;\bar{h}g_1)$. The opposite inequality is obtained similarly, using the second inequality of (1.8) and the results of Steps 1 and 2.

STEP 4. $b(y;h) \ge V(1,y;\bar{h})$ for all $h \in M_{\exp}$.

$$\begin{aligned} b(y,h) &= b(y;\bar{h}g_1) \\ &\equiv \inf_{s\in R} \liminf_{\varepsilon\to 0} \varepsilon^{-1}u\big(\varepsilon^{-1} - s; \varepsilon^{-1/2}y; \varepsilon^{1/2}g_1\big) \\ &= \inf_{s} \liminf_{\varepsilon\to 0} u\big(1-\varepsilon s, y; \bar{h}g_\varepsilon\big) \\ &\ge \inf_{s} \liminf_{\varepsilon\to 0} u(1-\varepsilon s, y; V(\varepsilon, x; \bar{h})) \\ &\qquad \text{(see the paragraph preceding Lemma 1.1)} \\ &= \inf_{s} \liminf_{\varepsilon\to 0} V(1-\varepsilon s+\varepsilon, y; \bar{h}) = V(1,y;\bar{h}). \end{aligned}$$

The opposite inequality is obtained using $V(t, x; a) \geq [a(2\pi t)^{-1/2} - c]\exp[-x^2/2t]$ for $t > 0$ and $x \in R$ for some constant c.

We omit the similar proof of $B(y, h) \leq V(1, y; \bar{h})$, which requires only minor modification. The proof of Theorem 1.3 is complete. □

PROOF OF THEOREM 1.4. In view of (0.17) we need only consider the case $B \equiv \bigcup_{j=1}^{k} A_j$ when proving (0.18) and (0.19). A straightforward computation yields

$$\lim_{t\to\infty} E_{t^{1/2}y}\left\{\exp\left[-t^{-1/2}\sum_{j=1}^{k}\theta_j N_{A_j}(t)\right]\middle| N_B(t) > 0\right\}$$

$$= \lim_{t\to\infty} \frac{\left[E_{t^{1/2}y}\left\{\exp\left[-t^{-1/2}\Sigma\theta_j N_{A_j}(t)\right]\right\} - P_{t^{1/2}y}\{N_B(t) = 0\}\right]}{P_{t^{1/2}y}\{N_B(t) > 0\}}$$

$$(1.9)\qquad = 1 - \lim_{t\to\infty}\frac{u\left(t, t^{1/2}y; 1 - \exp\left[-t^{-1/2}\Sigma\theta_j 1_{A_j}\right]\right)}{u\left(t, t^{1/2}y; 1_B\right)}$$

$$= 1 - \frac{\lim_{t\to\infty} tu\left(t, t^{1/2}y; t^{-1/2}\Sigma\theta_j 1_{A_j}\right)}{\lim_{t\to\infty} tu\left(t, t^{1/2}y; 1_B\right)}$$

$$= 1 - \frac{V\left(1, y; \Sigma\theta_j |A_j|\right)}{F(y)},$$

where the third equality is by the comparison lemma and the fact that $\{1 - \exp[-t^{1/2}\Sigma\theta_j 1_{A_j}]\}/(t^{-1/2}\Sigma\theta_j 1_{A_j})$ tends to 1 as $t \to \infty$ uniformly for $x \in \bigcup_{j=1}^{k} A_j$ and equals 0 for $x \notin \bigcup_{j=1}^{k} A_j$. The last equality is due to Lemma 1.1 and Theorem 1.3. This proves (0.18).

(0.19) is proved by a computation similar to (1.9) and (1.1); an exchange of limit and integration is needed and can be easily justified by the dominated convergence theorem as in (1.2) and (1.3). □

2.

PROOF OF THEOREM 2.1. Recall from (0.9) that there is a subsolution $\zeta(t)g(t, x) \sim (t \log t)^{-1}\exp[-|x|^2/2t]$ as $t \to \infty$. The present Theorem 2.1 asserts that $2(t \log t)^{-1}\exp[-|x|^2/2t]$ is the universal large time asymptotic behavior obeyed by all positive solutions of (B) with $M_{\exp}$ initial value. The idea which leads us to find this factor 2, roughly, is the following. Let $w \equiv (t \log t)^{-1}\exp[-|x|^2/2t]$ and let $Lu \equiv u_t - (\Delta/2)u + u^2$. A straightforward computation yields that

$$L(kw) = (t\log t)^{-2}\left(k^2\exp\left[-\frac{|x|^2}{t}\right] - k\exp\left[-\frac{|x|^2}{2t}\right]\right).$$

This immediately shows that kw, $0 < k \leq 1$, are subsolutions of (B). From this we also expect the first two leading terms of solutions $u(t, x)$ of (B) to be

$$u(t,x) = kw + t^{-1}(\log t)^{-2} g(t^{-1/2}x) + o\big(t^{-1}(\log t)^{-2}\big), \tag{2.1}$$

for some smooth function g because this leads to

$$Lu \equiv (t \log t)^{-2}\left\{k^2 \exp\left[-\frac{|x|^2}{t}\right] - k \exp\left[-\frac{|x|^2}{2t}\right] - \left[\left(\frac{\Delta}{2} + \frac{x}{2}\cdot\nabla + I\right)g\right](t^{-1/2}x)\right\} + o(t\log t)^{-2}.$$

In order to make the leading term of Lu vanish, k must be such that

$$\left[\left(\frac{\Delta}{2} + \frac{x}{2}\cdot\nabla + I\right)g\right](x) = k^2 \exp\left[-|x|^2\right] - k \exp\left[-\frac{|x|^2}{2}\right]$$

has a solution g in a reasonable class of functions. We need a fact concerning $H \equiv -(\Delta/2 + (x/2)\cdot\nabla + I)$: H is "self-adjoint" with respect to the measure $\exp[|x|^2/2]\,dx$, $x \in R^2$, and $c\exp[-|x|^2/2]$, $c \neq 0$, is the principal eigenfunction associated with the maximal eigenvalue zero. Fredholm's theorem then implies that there exists g if and only if $(k^2\exp[-|x|^2] - k\exp[-|x|^2/2]) \perp \exp[-|x|^2/2]$ w.r.t. $\exp[|x|^2/2]\,dx$, this is equivalent to $k = 2$. We shall use this idea of formal expansion to construct $\underline{u}$ and $\bar{u}$ with the desired properties (0.20) and (0.21) and to prove (0.22).

To prove (0.20), let ε be a small positive number and define

$$G(s,y,z) \equiv s^{-1}(2\pi)^{-1}(1-s)^{-1}\exp\left[-\frac{|y - s^{1/2}z|^2}{2(1-s)}\right],$$

$$H \equiv -\left(\frac{\Delta}{2} + \frac{y}{2}\cdot\nabla + I\right)$$

and

$$f_-(y) \equiv \left[\frac{(2-\varepsilon)^2}{2}\right]\int_0^1\int_{R^2} G(s,y,z)\left(\exp\left[-\frac{|z|^2}{2}\right] - 2\exp\left[-|z|^2\right]\right)dz\,ds.$$

In order not to interrupt the argument we postpone the proof of the following facts to the Appendix.

(2.2) $(Hf_-)(y) = [(2-\varepsilon)^2/2)](\exp[-|y|^2/2] - 2\exp[-|y|^2])$ and there exist positive constants K and M such that $K\exp\{-|y|^2/[2(1+\varepsilon)]\} \geq f_-(y) \geq -M\exp[-|y|^2/2]$.

Consider $\underline{u}(t,x) \equiv (2-\varepsilon)w(t,x) + t^{-1}(\log t)^{-2} f_-(t^{-1/2}x)$. We need only to show $\underline{u}(t,x) \geq 0$ and $L\underline{u} \leq 0$ for large t. It follows from (2.2) and a straightforward computation that

$$\underline{u}(t,x) \geq (t\log t)^{-1}\exp\left[-|x|^2/2t\right]\left[2-\varepsilon-M(\log t)^{-1}\right] > 0$$

for large t and

$$L\underline{u} \equiv (t\log t)^{-2}\exp\left[-\frac{|x|^2}{2t}\right]\left\{-\frac{\varepsilon(2-\varepsilon)}{2} + (\log t)^{-1}2[M+(2-\varepsilon)K]\cdot + (\log t)^{-2}(K^2+M^2)\right\} \leq 0$$

for large t. It is readily verified that $\underline{u}$ satisfies (0.20).

To prove (0.21) we also use a similar construction. Define

$$f_+(y) \equiv \left[\frac{(2+\varepsilon)^2}{2+\varepsilon/2}\right]\int_0^1\int_{R^2} G(s,y,z) \times \left[\exp\left[-\frac{|y|^2}{2+\varepsilon/2}\right] - \left(2+\frac{\varepsilon}{2}\right)\exp\left[-|z|^2\right]\right] dz\,ds.$$

A fact, proved in the Appendix, is needed:

$$(2.3)\quad (Hf_+)(y) = \left[\frac{(2+\varepsilon)^2}{2+\varepsilon/2}\right]\exp\left[-\frac{|y|^2}{2+\varepsilon/2}\right] - (2+\varepsilon)^2\exp\left[-|z|^2\right]$$

and there exist positive constants K and M such that

$$K\exp\left[-\frac{|y|^2}{2+\varepsilon/2}\right] \geq f_+(y) \geq -M\exp\left[-\frac{|y|^2}{2}\right].$$

Consider $\bar{u}(t,x) \equiv (2+\varepsilon)w + t^{-1}(\log t)^{-2}f_+(t^{-1/2}x)$. From (2.3) and a simple computation it follows that $\bar{u}(t,x) \geq 0$ and

$$L\bar{u} \geq (t\log t)^{-2}\left\{\left[\frac{\varepsilon(2+\varepsilon)}{4+\varepsilon}\right]\exp\left[-\frac{|x|^2}{(2+\varepsilon/2)t}\right] - (\log t)^{-1}\left[2K\exp\left(-\frac{|x|^2}{(2+\varepsilon/2)t}\right) + 2(2+\varepsilon)M\exp\left(-\frac{|x|^2}{t}\right)\right]\right\} \geq 0$$

for large t. It is easy to check that $\bar{u}$ satisfies (0.21).

(0.22) will be proved in two steps.

STEP 1. Suppose $h(x)$ is nonnegative and not identically zero. Then

$$\liminf_{t\to\infty}(t\log t)u(t,t^{1/2}y;h)\geq 2\exp\left[-\frac{|y|^2}{2}\right].$$

For small positive ε let $\underline{u}$ be as described in (0.20). Consider the subsolution $\lambda\underline{u}(\lambda(t+1),\lambda^{1/2}x)$, the initial value is $\lambda\underline{u}(\lambda,\lambda^{1/2}x)$. Assuming that $h(x)$ is greater than a multiple of $\exp\{-|x|^2/[2(1+\varepsilon)]\}$ and using (0.20), we can choose λ so large that

$$h\geq(\log\lambda)^{-1}2\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]\geq\lambda\underline{u}(\lambda,\lambda^{1/2}x).$$

The comparison lemma and (0.20) then imply that

$$\begin{aligned}u(t,t^{1/2}y;h)&\geq\lambda\underline{u}\big(\lambda(t+1),(\lambda t)^{1/2}y\big)\\&\geq(t+1)^{-1}(\log\lambda(t+1))^{-1}2(1-\varepsilon)\exp\left[-\frac{t|y|^2}{2(1+t)}\right].\end{aligned}$$

Letting t tend to ∞ followed by letting ε tend to 0, Step 1 is completed for $h(x)$ satisfying the previously mentioned assumption, which we now remove. If $h(x)$ is nonnegative and not identically zero, by (1.7) there exists a positive $c(t)$ such that

$$u(t,x;h)\geq c(t)\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]\quad\text{for } t>1+\varepsilon.$$

Step 1 can be easily extended to cover general h.

STEP 2. $\limsup_{t\to\infty}(t\log t)u(t,t^{1/2}y;h)\leq 2\exp[-|y|^2/2]$ for all $h\in M_{\exp}$. For $\varepsilon>0$ let t_0 and $\bar{u}(t,x)$ be as described in (0.21). Without loss of generality, assume $t_0>1$ and $h(x)\leq b\exp[-|x|^2/2m]$ for some $b>0$ and $m>t_0$; (0.21) guarantees the existence of $c>1$ such that $h(x)\leq c\bar{u}(m,x)$. The comparison lemma then implies

$$u(t,x;h)\leq u(t,x;c\bar{u}(m,\cdot))\leq c\bar{u}(t+m,x)\qquad\text{for } t\geq 0\text{ and } x\in R^2. \tag{2.4}$$

Next, we consider a solution $\lambda u(\lambda(t+t_0),\lambda^{1/2}x;h)$. By (2.4) its initial value is not greater than $\lambda c\bar{u}(\lambda t_0+m,\lambda^{1/2}x)$. It then follows from (0.21) that

$$\begin{aligned}\lambda u(\lambda t_0,\lambda^{1/2}x;h)&\leq\lambda c(\lambda t_0+m)^{-1}[\log(\lambda t_0+m)]^{-1}\\&\quad\times 2(1+\varepsilon)\exp\left[-\frac{\lambda|x|^2}{2(1+\varepsilon)(\lambda t_0+m)}\right].\end{aligned}\tag{2.5}$$

We can choose λ so large that $\lambda/[1+\varepsilon)(\lambda t_0+m)] \geq 1/[(1+2\varepsilon)t_0]$ and

$$\lambda c(\lambda t_0+m)^{-1}[\log(\lambda t_0+m)]^{-1}(1+\varepsilon)$$
$$\leq [t_0(1+2\varepsilon)]^{-1}\{\log[t_0(1+2\varepsilon)]\}^{-1}.$$

Using such a λ, (2.5) together with (0.21) yields

$$\lambda u(\lambda t_0, \lambda^{1/2}x; h) \leq [t_0(1+2\varepsilon)]^{-1}\{\log[t_0(1+2\varepsilon)]\}^{-1} 2\exp\left[-\frac{|x|^2}{2(1+2\varepsilon)t_0}\right]$$
$$\leq \bar{u}(t_0(1+2\varepsilon), x).$$

The comparison lemma then implies

$$\lambda u(\lambda(t_0+t), \lambda^{1/2}x; h) \leq \bar{u}(t+t_0(1+2\varepsilon), x). \tag{2.6}$$

Multiplying (2.6) by $(t_0+t)^{-1}[\log\lambda(t_0+t)]^{-1}$ and letting x be $(t_0+t)^{1/2}y$, (0.21) implies

$$\limsup_{t\to\infty}(t\log t)u(t, t^{1/2}y; h) \leq 2(1+\varepsilon)\exp\left[-\frac{|y|^2}{2(1+\varepsilon)}\right].$$

Since ε can be arbitrarily small, this completes Step 2 and ends the proof. □

PROOF OF THEOREM 2.2. Using Theorem 2.1 in the place of Lemma 1.1, the proof requires an obvious modification of the proof of Theorem 1.2. □

PROOF OF THEOREM 2.3. Define

$$B(y,h) \equiv \sup_{s\in R}\limsup_{T\to\infty}(T\log T)u(T-s, T^{1/2}y, (\log T)^{-1}h)$$

and

$$b(y;h) \equiv \inf_{s\in R}\liminf_{T\to\infty}(T\log T)u(T-s, T^{1/2}y, (\log T)^{-1}h).$$

It is easy to prove by the same kind of argument as in Steps 1–3 of the proof of Theorem 1.3 that

$$B(y;h) = B\left(y; \bar{h}(2\pi)^{-1}\exp\left[-\frac{|x|^2}{2}\right]\right) \tag{2.7}$$

and the same conclusion for $b(y;h)$. Due to (2.7), the proof can be broken up into two steps.

STEP 1. $B(y; a\exp[-|x|^2/2]) \le (1 + 2a^{-1})^{-1}2\exp[-|y|^2/2]$.

Let ε be a small positive number and recall from (0.21) that $(\log\lambda)^{-1}2\exp[-|x|^2/2] \le 2\exp[-|x|^2/2] \le \lambda\bar{u}(\lambda, \lambda^{1/2}x)$ for large λ. Choosing $\lambda = \lambda(T) \equiv T^{2/a}$, we conclude that

$$\begin{aligned} u\left(t-s, x; (\log T)^{-1}a\exp\left[-\frac{|x|^2}{2}\right]\right) &= u\left(t-s, x; (\log\lambda)^{-1}2\exp\left[-\frac{|x|^2}{2}\right]\right) \\ &\le u\big(t-s, x; \lambda\bar{u}(\lambda, \lambda^{1/2}\cdot)\big) \\ &= \lambda u\big(\lambda(t-s), \lambda^{1/2}x; \bar{u}(\lambda, \cdot)\big) \\ &\le \lambda\bar{u}\big(\lambda(t-s+1), \lambda^{1/2}x\big), \end{aligned}$$

where the last equality is due to the scaling lemma and the last inequality is due to the comparison lemma. It then follows from (0.21) that

$$\begin{aligned} &\sup_{s\in R}\limsup_{T\to\infty}(T\log T)u\left(T-s, T^{1/2}y, (\log T)^{-1}a\exp\left[-\frac{|x|^2}{2}\right]\right) \\ &\quad\le \sup_{s}\limsup_{T\to\infty}(T\log T)\lambda\bar{u}\big(\lambda(T-s+1), \lambda^{1/2}T^{1/2}y\big) \\ &\quad\le \sup_{s}\limsup_{T\to\infty}(T\log T)(T-s+1)^{-1}[\log\lambda(T-s+1)]^{-1} \\ &\qquad\times 2(1+\varepsilon)\exp\left[-\frac{T|y|^2}{2(1+\varepsilon)(T+s+1)}\right] \\ &\quad= (1+2a^{-1})^{-1}2(1+\varepsilon)\exp\left[-\frac{|y|^2}{2(1+\varepsilon)}\right]. \end{aligned}$$

Since ε is arbitrarily small, Step 1 is complete.

STEP 2. $b(y; a\exp[-|x|^2/2]) \ge (1 + 2a^{-1})^{-1}2\exp[-|y|^2/2]$.

We need only show that

$$b\left(y; a(1+\varepsilon)^{-1}\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]\right) \ge (1+2a^{-1})^{-1}2\exp\left[-\frac{|y|^2}{2}\right]$$

because $b(y; h)$ depends on $\bar{h}$ only. Recall from (0.20) that

$$\lambda\underline{u}(\lambda, \lambda^{1/2}x) \le (\log\lambda)^{-1}2\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]$$

for large λ. Choosing $\lambda = \lambda(T) \equiv T^{-2(1+\varepsilon)/a}$ and using an argument similar to

Step 1, we see that

$$u\left(t-s,x;(\log T)^{-1}a(1+\varepsilon)^{-1}\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]\right)$$
$$=u\left(t-s,x;(\log\lambda)^{-1}2\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]\right)$$
$$\geq u\big(t-s,x;\lambda\underline{u}(\lambda,\lambda^{1/2}\cdot)\big)$$
$$=\lambda u\big(\lambda(t-s),\lambda^{1/2}x;\underline{u}(\lambda,\cdot)\big)$$
$$\geq \lambda\underline{u}\big(\lambda(t-s+1),\lambda^{1/2}x\big).$$

(0.20) then implies that

$$b\left(y;a(1+\varepsilon)^{-1}\exp\left[-\frac{|x|^2}{2(1+\varepsilon)}\right]\right)$$
$$\geq \inf_{s\in R}\liminf_{T\to\infty}(T\log T)\lambda\underline{u}\big(\lambda(T-s+1),\lambda^{1/2}T^{1/2}y\big)$$
$$\geq \inf_{s\in R}\liminf_{T\to\infty}(T\log T)\lambda[\lambda(T-s+1)]^{-1}$$
$$\times[\log\lambda(T-s+1)]^{-1}2(1-\varepsilon)\exp\left[-\frac{T|y|^2}{2(T-s+1)}\right]$$
$$=\left[1+2(1-\varepsilon)a^{-1}\right]^{-1}(1-\varepsilon)2\exp\left[-\frac{|y|^2}{2}\right].$$

Since ε is arbitrarily small, this completes Step 2 and ends the proof. □

PROOF OF THEOREM 2.4. The proof is very similar to that of Theorem 1.4. We use Theorem 2.3 in the place of Theorem 1.3. □

3.

PROOF OF THEOREM 3.2. This result is a simple consequence of Lemma 3.1 and a computation as in the proof of Theorem 1.4. □

APPENDIX

Define

$$G(s,y,z)\equiv s^{-1}(2\pi)^{-1}(1-s)^{-1}\exp\left[\frac{|y-s^{1/2}z|^2}{2(1-s)}\right],$$
$$H\equiv-\left(\frac{\Delta}{2}+\frac{y}{2}\cdot\nabla+I\right)$$

and

$$(Gq)(y) \equiv \int_0^1 \int_{R^2} G(s,y,z) q(z)\, dz\, ds,$$

which is, in general, a divergent improper integral and will be proved to converge when $|q| \in M_{\exp}$ with $\bar{q} \equiv \int q(x)\, dx = 0$.

LEMMA.

(3.1) *If* $|q| \in M_{\exp}$ *and* $\bar{q} \equiv 0$, *then* $H(Gq) = q$.

For $\varepsilon > 0$, *there exist positive* K *and* M *(depending on* ε*) such that*

$$\begin{aligned} (3.2)\qquad & K \exp\left[-\frac{|y|^2}{2+\varepsilon} \right] \\ & \geq \int_0^1 \int_{R^2} G(s,y,z) \left(\exp\left[-\frac{|z|^2}{2} \right] - 2\exp\left[-|z|^2 \right] \right) dz\, ds \\ & \geq -M \exp\left[-\frac{|y|^2}{2} \right], \\ & K \exp\left[-\frac{|y|^2}{2+\varepsilon} \right] \\ & \geq \int_0^1 \int_{R^2} G(s,y,z) \left(\exp\left[-\frac{|z|^2}{2+\varepsilon} \right] - (2+\varepsilon)\exp\left[-|z|^2 \right] \right) dz\, ds \\ & \geq -M \exp\left[-\frac{|y|^2}{2} \right]. \end{aligned}$$

PROOF. We shall first prove (3.2) because it handles the q's which we actually use to prove Theorem 2.1 and also because its proof indicates why $H(Gq) = q$ holds for more general function q, such as $|q| \in M_{\exp}$ and $q = 0$.
A simple heat-semigroup computation shows

$$\begin{aligned} (3.3)\qquad & \int_0^1 \int_{R^2} G(s,y,z) \left(\exp\left[-\frac{|z|^2}{2} \right] - 2\exp\left[-|z|^2 \right] \right) dz\, ds \\ & = \int_0^1 s^{-1} \left[\exp\left[-\frac{|y|^2}{2} \right] - \left(1 - \frac{s}{2}\right)^{-1} \exp\left(-\frac{|y|^2}{2(1-s/2)} \right) \right] ds. \end{aligned}$$

Convergence of this improper integral is easily obtained by L'Hôpital's rule or by integration by parts (integrate s^{-1} and differentiate the rest).

One side of the desired inequality follows immediately from (3.3):

$$\begin{aligned}\int_0^1 s^{-1}\left[\exp\left(-\frac{|y|^2}{2}\right) - \left(1-\frac{s}{2}\right)^{-1}\exp\left(-\frac{|y|^2}{2(1-s/2)}\right)\right] ds\\ \geq \int_0^1 s^{-1}\left[1-\left(1-\frac{s}{2}\right)^{-1}\right] ds \exp\left(-\frac{|y|^2}{2}\right)\\ = -\left[\int_0^1 \frac{ds}{(2-s)}\right]\exp\left(-\frac{|y|^2}{2}\right).\end{aligned}$$

The other side of the desired inequality is derived by integration by parts:

$$\int_0^1 s^{-1}\left[\exp\left(-\frac{|y|^2}{2}\right) - \left(1-\frac{s}{2}\right)^{-1}\exp\left(-\frac{|y|^2}{2(1-s/2)}\right)\right] ds$$

$$(3.4)\quad \begin{aligned}&= \int_0^1 (-\log s)\left[\frac{|y|^2}{4}\left(1-\frac{s}{2}\right)^{-3} - \frac{1}{2}\left(1-\frac{s}{2}\right)^{-2}\right]\exp\left(-\frac{|y|^2}{2(1-s/2)}\right) ds\\ &\leq \frac{|y|^2}{4}\left[\int_0^1 (-\log s)\left(1-\frac{s}{2}\right)^{-3}\right]\exp\left(-\frac{|y|^2}{2}\right)\\ &\leq K\exp\left(-\frac{|y|^2}{2+\varepsilon}\right)\quad \text{for some } K = K(\varepsilon) > 0.\end{aligned}$$

Next we turn to the case $q(z) = \exp[-|z|^2/(2+\varepsilon)] - (2+\varepsilon)\exp(-|z|^2)$. Similarly to (3.3), we have

$$\begin{aligned}\int_0^1\int_{R^2} G(s,y,z)\left(\exp\left[-\frac{|z|^2}{2+\varepsilon}\right] - (2+\varepsilon)\exp\left[-|z|^2\right]\right) dz\, ds\\ = (1+\varepsilon)\int_0^1 s^{-1}\left[\left(1+\frac{\varepsilon s}{2}\right)^{-1}\exp\left(-\frac{|y|^2}{2(1+\varepsilon s/2)}\right)\right.\\ \left. -\left(1-\frac{s}{2}\right)^{-1}\exp\left(-\frac{|y|^2}{2(1-s/2)}\right)\right] ds.\end{aligned}$$

The greater than $-M\exp[-|y|^2/2]$ part is as easy to obtain as in the preceding case. We then prove the other side of the inequality: Choosing any number a, $0 < a < 1$, and breaking

$$\begin{aligned}\int_0^1 s^{-1}\left[\left(1+\frac{\varepsilon s}{2}\right)^{-1}\exp\left(-\frac{|y|^2}{2(1+\varepsilon s/2)}\right)\right.\\ \left. -\left(1-\frac{s}{2}\right)^{-1}\exp\left(-\frac{|y|^2}{2(1-s/2)}\right)\right] ds\end{aligned}$$

into two parts, $\int_0^a$ and $\int_a^1$, it is easy to see that the part $\int_a^1$ is no more than $(\int_0^1 s^{-1}(1+\varepsilon s/2)^{-1}\,ds)\exp[-|y|^2/(2+\varepsilon)]$ and, by the same integration-by-parts argument as in the preceding case, that the part $\int_0^a$ is no more than $c\exp[-|y|^2/(2+\varepsilon)]$, where c is a positive constant.

The proof of (3.2) is now complete.

To prove $H(Gq)=q$, note for $v(t,x)=t^{-1}f(xt^{-1/2})$ that $(Hf)(y)=q(y)$ if and only if $(\partial_t-\Delta/2)v=t^{-2}q(xt^{-1/2})$. Now, if we consider

$$v(t,x)=\int_0^t\int_{R^2}(2\pi)^{-1}(t-s)^{-1}\exp\left[-\frac{|x-w|^2}{2(t-s)}\right]s^{-2}q(s^{-1/2}w)\,dw\,ds,$$

which is a convergent integral and $(\partial_t-\Delta/2)v=t^{-2}q(xt^{-1/2})$ when, for example,

$$q(z)=a^{-1}\exp\left[-\frac{|z-a_1|^2}{2a}\right]-b^{-1}\exp\left[-\frac{|z-b_1|^2}{2b}\right]$$

for some $a,b>0$ and $a_1,b_1\in R^2$.

Also, $v(t,x)$, when exists, is automatically self-similar, i.e., $v(t,x)=t^{-1}v(1,t^{-1/2}x)$. Thus, $Hv(1,y)=q(y)$. Finally, note that

$$v(1,y)=\int_0^1\int_{R^2}(2\pi)^{-1}(1-s)^{-1}\exp\left[-\frac{|y-w|^2}{2(1-s)}\right]s^{-2}q(s^{-1/2}w)\,dw\,ds,$$

by the change of variable $w=s^{1/2}z$, is exactly $(Gq)(y)$.

We have just proved (3.1) for

$$q(z)=a^{-1}\exp\left[-\frac{|z-a_1|^2}{2a}\right]-b^{-1}\exp\left[-\frac{|z-b_1|^2}{2b}\right].$$

Since H is a linear operator, (3.1) holds for all linear combinations of these special q's and conceivably for those q with $|q|\in M_{\exp}$ with $\bar{q}=0$. The details are omitted. The proof is complete. □

Acknowledgment. The author wishes to thank the referee for many valuable comments.

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