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# PERSISTENCE CRITERIA FOR A CLASS OF CRITICAL BRANCHING PARTICLE SYSTEMS IN CONTINUOUS TIME[1]

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We consider a system of particles in $\mathbb{R}^d$ performing symmetric stable motion with exponent $\alpha$, $0 < \alpha \le 2$, and branching at the end of an exponential lifetime with offspring generating function $F(s) = s + \frac{1}{2}(1 - s)^{1+\beta}$, $0 < \beta \le 1$. (This includes binary branching Brownian motion for $\alpha = 2$, $\beta = 1$.) It is shown that, for an initial Poisson population with uniform intensity, the system goes to extinction if $d \le \alpha/\beta$ and is "persistent" (i.e., preserves intensity in the large time limit) if $d > \alpha/\beta$. To this purpose a continuous-time version of Kallenberg's backward technique for computing Palm distributions of branching particle systems is developed, which permits us to adapt methods used by Dawson and Fleischmann in the study of discrete-space and discrete-time systems.

**1. Introduction.** Kallenberg (1977) presented a method of "backward trees" which allows to compute the Palm distributions of the $n$th generation of a branching particle system, and to establish criteria for "persistence," i.e., conservation of intensity in the limit $n \to \infty$. Kallenberg's method has been extended by Liemant (1981) to spatially inhomogeneous branching mechanisms in discrete time, and it has been successfully applied by Dawson and Fleischmann (1985) to establish persistence criteria for a class of branching random walks (in discrete time and random environment) on the lattice $\mathbb{Z}^d$.

Since Kallenberg's construction of backward trees relies on the generation scheme, it is not directly applicable to continuous-time branching systems, where usually at each time individuals of different generations are alive.

In the present paper, we develop the method of backward trees for a class of continuous-time branching models, whose fluctuation limits have been studied by Dawson, Fleischmann and Gorostiza (1989). These models are parametrized by $d$, $\alpha$ and $\beta$, where $d$ is the dimension of the Euclidean space in which the particles (or "individuals") live, $\alpha \in (0, 2]$ is the exponent of the symmetric stable motion which each individual performs for an exponentially distributed lifetime (at the end of which it branches) and $\beta \in (0, 1]$ is the parameter of the (critical) offspring distribution, which has moment generating function $F(s) = s + \frac{1}{2}(1 - s)^{1+\beta}$ and thus is in the domain of normal attraction of a stable law with exponent $1 + \beta$.

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Using the above-mentioned continuous-time version of the backward technique (Theorem 2.3), we then follow the program of Kallenberg (1977) and of Dawson and Fleischmann (1985), obtaining persistence criteria analogous to theirs. In particular, it turns out that the system whose initial distribution is Poisson with uniform intensity is persistent if and only if $d > \alpha/\beta$ (Theorem 2.2), a fact which in the special case ($\alpha = 2$, $\beta = 1$) of particle systems performing binary branching Brownian motion is already known: Dawson and Ivanoff (1978), Theorem 7.1, show persistence for $d \geq 3$, and nonpersistence in dimensions $d = 1, 2$ follows from Theorems 4.1 and 4.2 of Fleischman (1978), which yield, in particular, that in these cases the random numbers of particles in a bounded region are not uniformly integrable in the large time limit.

Our methods presumably can be extended without much difficulty to the case of offspring distributions in the domain of normal attraction of a stable law with exponent $1 + \beta$.

Questions of persistence also naturally arise in the study of "measure branching processes," also called "Dawson–Watanabe" processes and "superprocesses," which are scaling limits of branching particle systems of the type considered in this paper. We refer to Dawson (1977) and recent work of Dynkin (1989), in the case of offspring distributions with finite second moment. In Remark 3.2 (Theorem 3.3) we will show the relevance of our results to the "historical process" of the branching process [see Dawson and Perkins (1990)].

**2. The model and the main results.** Consider the following branching dynamics in continuous time on $\mathbb{R}^d$: An individual starting at time 0 in $x \in \mathbb{R}^d$ moves according to a symmetric stable process with exponent $\alpha \in (0, 2]$, i.e., according to the transition density $p_t(x, y) := p_t(x - y)$ whose characteristic function is given by

$$\int_{\mathbb{R}^d} e^{ix\cdot y} p_t(x)\, dx = \exp(-t|y|^\alpha), \qquad y \in \mathbb{R}^d.$$

We will refer to this process, for short, as the "basic process," and we will assume that its paths are right continuous with left limits a.s. (for $\alpha = 2$ it is Brownian motion with variance parameter 2 on $\mathbb{R}^d$). At the end of an exponentially distributed lifetime (with parameter $V$), this individual is replaced by a random number $N$ of "new" individuals, all of them then obeying (independently) the dynamics just described, starting at the parent individual's final position. The distribution of $N$ is assumed to have moment generating function

$$F(s) = \sum_{k=0}^{\infty} p_k s^k = s + \tfrac{1}{2}(1 - s)^{1+\beta} \tag{2.1}$$

for some $\beta \in (0, 1]$. Hence the branching is critical in the sense that $E[N] = \sum_{k=0}^{\infty} kp_k = 1$. With $\beta = 1$, we have $p_0 = p_2 = \frac{1}{2}$, which is the classical example of binary branching. For general $\beta$, the distribution of $N$ is in the domain of normal attraction of a stable law with exponent $1 + \beta$, and thus, for $\beta < 1$, it has finite moments only of order less than $1 + \beta$.

In the way described above, an initial individual at site $x$ gives rise to a random population at time $t > 0$, described by a random counting measure $X_t^x$. [Existence of $X_t^x$ is referred to in Dawson, Fleischmann and Gorostiza (1989), and it also follows from the construction in the proof of Proposition 3.1 below.] For any $g\colon \mathbb{R}^d \to \mathbb{R}_+$, bounded and continuous, the expected value $E\langle X_t^x, g\rangle := E[\int g(y) X_t^x(dy)]$ can be computed as follows. By a standard renewal argument (conditioning upon the first branching), we find that $E\langle X_t^x, g\rangle$ satisfies the integral equation

$$E\langle X_t^x, g\rangle = e^{-Vt} T_t g(x) + E[N] \int_0^t V e^{-Vs} T_s E\langle X_{t-s}^{\cdot}, g\rangle(x)\, ds,$$

where $T_t h(x) := \int p_t(x, y) h(y)\, dy$, and $h\colon \mathbb{R}^d \to \mathbb{R}_+$ is measurable. Since $E[N] = 1$, the (unique) solution of this equation is $E\langle X_t^x, g\rangle = T_t g(x)$. Thus the intensity measure $\pi_t(x, \cdot\,)$ of $X_t^x$ (i.e., the expectation of the random measure $X_t^x$) is given by

$$\pi_t(x, dy) = p_t(x, y)\, dy.$$

The distribution of $X_t^x$ will be denoted by $\kappa_{(x)}^t$, and the kernel $\pi_t$ will be called the intensity kernel of the branching dynamics $\kappa^t$.

For an initial population $\Phi = \sum_{i \in I} \delta_{x_i}$ of individuals (here and below, $\delta_x$ denotes Dirac measure supported by $x$), we put $X_t^\Phi := \sum_{i \in I} X_t^{x_i}$, where $(X_t^{x_i})_{i \in I}$ are independent, and we write $\kappa_{(\Phi)}^t$ for the distribution of $X_t^\Phi$. If $\Phi$ is random and has Poisson distribution with Lebesgue intensity measure (denoted by $\lambda$), the resulting random population $X_t^\Phi$ will be simply denoted by $X_t$. Note that the distribution $P_t$ of $X_t$ is infinitely divisible; its canonical measure [cf. Kallenberg (1983), page 50, or Liemant, Matthes and Wakolbinger (1988), page 18] will be denoted by $\widetilde{P}_t$.

Since $\lambda$ is invariant with respect to the kernels $\pi_t$, it follows [see Liemant, Matthes and Wakolbinger (1988), 1.5.2] that $X_t$ has intensity measure $\lambda$. Hence a general argument [cf. the proof of Proposition 2.3.1 in Liemant, Matthes and Wakolbinger (1988)] shows that for any measurable $g\colon \mathbb{R}^d \to \mathbb{R}_+$, the Laplace functionals $Ee^{-\langle X_t, g\rangle}$ are increasing in $t$, and thus, by relative compactness, $X_t$ converges towards a "steady state" $X_\infty$ (having a distribution $P_\infty$ which is invariant with respect to $\kappa_{(\Phi)}^t$ for all $t > 0$).

Convergence of $X_t$ can also be proved by the following specific calculation. The Laplace functional of $X_t$ is given by

$$Ee^{-\langle X_t, g\rangle} = \exp\{\langle \lambda, Ee^{-\langle X_t^{\cdot}, g\rangle} - 1\rangle\}. \tag{2.2}$$

A renewal argument shows that, for $g\colon \mathbb{R}^d \to \mathbb{R}_+$ continuous and with compact support, $h(x, t) := Ee^{-\langle X_t^x, g\rangle}$ satisfies the integral equation

$$h(x, t) = e^{-Vt} T_t e^{-g}(x) + \int_0^t V e^{-Vs} T_s F(h(\cdot, t - s))(x)\, ds,$$

and therefore $h$ satisfies the equation

$$\frac{\partial h}{\partial t} = (\Delta_\alpha - V)h + VF(h), \qquad h(x,0) = e^{-g(x)},$$

where $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ is the infinitesimal generator of the semigroup $(T_t)$. Since $F(s) - s = \frac{1}{2}(1-s)^{1+\beta}$, the latter equation can be rewritten as

$$\frac{\partial h}{\partial t} = \Delta_\alpha h + \frac{V}{2}(1-h)^{1+\beta}, \qquad h(x,0) = e^{-g(x)}. \tag{2.3}$$

Hence $h$ satisfies the integral equation

$$h(x,t) = T_t e^{-g}(x) + \frac{V}{2}\int_0^t T_{t-s}(1 - h(\cdot,s))^{1+\beta}(x)\,ds.$$

Substituting into (2.2), we obtain

$$Ee^{-\langle X_t, g\rangle} = \exp\left\{\langle \lambda, e^{-g} - 1\rangle + \frac{V}{2}\int_0^t \left\langle \lambda, (1 - h(\cdot,s))^{1+\beta}\right\rangle ds\right\},$$

from which the above-mentioned increase in $t$ of the Laplace functionals is immediate. Moreover, it follows that the Laplace functional of the steady state $X_\infty$ is given by

$$Ee^{-\langle X_\infty, g\rangle} = \exp\left\{\langle \lambda, e^{-g} - 1\rangle + \frac{V}{2}\int_0^\infty \left\langle \lambda, (1 - h(\cdot,s))^{1+\beta}\right\rangle ds\right\}. \tag{2.4}$$

(Note that the finiteness of the integral on the right-hand side is implied by the previous argument.)

Since the random populations $X_t$, $t \geq 0$, all have intensity measure $\lambda$, their limit in distribution $X_\infty$ has an intensity measure $\rho$ such that $\rho \leq \lambda$. $\lambda$ is called *persistent* if the intensity measure of $X_\infty$ equals $\lambda$. [We avoid in this context the term "stable" which is used in Liemant, Matthes and Wakolbinger (1988) and some earlier publications, in order to prevent confusion with the stable motion process.]

The following dichotomy holds.

$$\text{Either } \lambda \text{ is persistent or } X_\infty = \mathscr{O} \text{ a.s.}$$

(i.e., either the system preserves its intensity, or it goes to extinction in the large time limit; here and below $\mathscr{O}$ denotes the zero measure on $\mathbb{R}^d$).

Moreover, the following proposition holds.

PROPOSITION 2.1. *$\lambda$ is persistent if and only if, for some $t > 0$, the branching dynamics $\kappa^t$ has a nontrivial steady state $Y$ with locally finite intensity measure.*

(By "nontrivial" we mean that $P[Y = \mathscr{O}] < 1$.)

PROOF. One direction of this equivalence is obvious: Just take $Y := X_\infty$. To show the other direction, let $\mu$ be the intensity measure of the steady state $Y$.

Since there holds for all Borel sets $B \subseteq \mathbb{R}^d$,

$$\mu(B) = \int \mu(dx)\pi_t(x, B) = (\mu * \sigma)(B),$$

where $\sigma := \pi_t(0, \cdot)$, it follows [see Deny (1960), Section 6] that $\mu = c\lambda$ for some constant $c \geq 0$. In fact, $c$ is strictly positive since $Y$ is nontrivial. Thus, together with $\mu$, also its multiple $\lambda$ is the intensity measure of a steady state for $\kappa^t$ [see Liemant, Matthes and Wakolbinger (1988), 2.3.10], which implies that $X_\infty$ has intensity measure $\lambda$ by Liemant, Matthes and Wakolbinger (1988), Theorem 2.3.6. □

It is now a natural question to ask for which of the model parameters $d, \alpha, \beta$ the measure $\lambda$ is persistent. The same question has been studied by Dawson and Fleischmann (1985) for similar models in discrete space and discrete time, with analogous meanings of the parameters $\alpha$ and $\beta$. Their results suggest that $\lambda$ should be persistent (and hence $X_\infty$ would be non-trivial) if and only if $d > \alpha/\beta$. The qualitative content of this criterion is the following. Because of criticality of the branching, dying out of the population goes along with "large clusters of individuals" occurring in the course of time with some small probability. Now, small $\beta$ is in favor of large clusters, whereas large $d$ and small $\alpha$ are in favor of transience of the motion, a fact which tends to spread out clusters over the space and therefore counteracts the effect of large clusters.

In fact, we will prove Theorem 2.2.

THEOREM 2.2. *Lebesgue measure $\lambda$ is persistent if and only if $d > \alpha/\beta$.*

REMARK. Since $E\langle X_\infty, g\rangle = -(d/du)E\exp\{-\langle X_\infty, ug\rangle\}|_{u=0}$, denoting $h_g(x,t) := Ee^{-\langle X_t^x, g\rangle}$ we see from Theorem 2.2 and (2.4) that

$$\text{(2.5)} \quad \frac{d}{du}\int_0^\infty \left\langle \lambda, \left(1 - h_{ug}(\cdot, s)\right)^{1+\beta}\right\rangle ds\,|_{u=0} = \begin{cases} 0 & \text{if } d > \dfrac{\alpha}{\beta}, \\ \dfrac{2}{V}\langle \lambda, g\rangle & \text{if } d \leq \dfrac{\alpha}{\beta}. \end{cases}$$

On the other hand,

$$\frac{d}{du}\left(1 - h_{ug}(\cdot, s)\right)^{1+\beta}|_{u=0} = 0.$$

Hence from our results it follows that interchange of derivative and integrals in (2.5) is possible if and only if $d > \alpha/\beta$. This interchange depends on the asymptotic behavior of the solution of the nonlinear equation (2.3); however, an analytic approach to this problem seems difficult. [It can be shown that equation (2.3) has a unique global solution, and $h(x,t) \in [0,1]$ for all $x$ and $t$; see El Karoui and Roelly-Coppoletta (1989).]

The proof of Theorem 2.2 relies on a characterization of the Palm distributions of $\widetilde{P}_t$, using a continuous-time version of Kallenberg's (1977) "method of backward trees," which is established in Theorem 2.3 below. Recall that the Palm distributions of a measure $H$ on a population space $M$ (where $H$ is assumed to have locally finite intensity measure $\Lambda_H$) are a family $(H_x)_{x \in \mathbb{R}^d}$ of distributions on $M$ obeying

$$\int \langle \Phi, h \rangle g(\Phi) H(d\Phi) = \int h(x) \int g(\Phi) H_x(d\Phi) \Lambda_H(dx)$$

for all measurable $h\colon \mathbb{R}^d \to \mathbb{R}_+$ and $g\colon M \to \mathbb{R}_+$. If $H$ is concentrated on the populations with no multiple points, then $H_x$ may be thought of as the distribution of the population $\Phi$ under $H$, conditioned to contain an individual $\delta_x$ [cf. Kallenberg (1983), Section 12.4].

THEOREM 2.3. *For all $t > 0$ and all $x \in \mathbb{R}^d$, define the random population $Y_{t,x}$ by*

$$Y_{t,x} := \int_{[0,t]} \left( \sum_{i=1}^{Z_s} X_{s,i}^{W_s^x} \right) \mu(ds),$$

*where $\mu$ is a random Poisson configuration on $\mathbb{R}_+$ with intensity $V$, $(W_s^x)$ is a random path of the basic process starting in $x$, $Z_s$, $s > 0$, are random numbers with* $\operatorname{Prob}[Z_s = k] = (k+1)p_{k+1}$, $k = 0, 1, \dots$, $X_{s,i}^w$, $s > 0$, $w \in \mathbb{R}^d$, $i = 1, 2, \dots$, *have distribution $\kappa_{(w)}^s$, and all these random objects are independent.*

*Then the random population $\delta_x + Y_{t,x}$ has distribution $(\widetilde{P}_t)_x$ for $\lambda$-almost all $x \in \mathbb{R}^d$.*

Intuitively, $Y_{t,x}$ may be thought of as the population of an individual $\delta_x$'s relatives, stemming from the random ancestral line $(W_s^x)$ and the random branching times $s$ in the support of $\mu$.

**3. Proof of Theorem 2.3.** The "clustering theorem" [Matthes, Kerstan and Mecke (1978), Theorem 4.3.3] implies

$$\widetilde{P}_t = \int \kappa_{(x)}^t((\cdot) \setminus \{\mathscr{O}\}) \lambda(dx), \tag{3.1}$$

and therefore $\widetilde{P}_t$ has $\lambda$ as its intensity measure. Hence, by the definition of Palm distribution, for all $f, g\colon \mathbb{R}^d \to \mathbb{R}_+$ continuous with compact support we have

$$\begin{aligned} \int \int e^{-\langle \Phi, g \rangle} f(x) \big(\widetilde{P}_t\big)_x (d\Phi) \lambda(dx) &= \int e^{-\langle \Phi, g \rangle} \langle \Phi, f \rangle \widetilde{P}_t(d\Phi) \\ &= \langle \lambda, Ee^{-\langle X_t^{\cdot}, g \rangle} \langle X_t^{\cdot}, f \rangle \rangle. \end{aligned}$$

Thus, in order to prove Theorem 2.3, it suffices to show

$$\langle \lambda, Ee^{-\langle X_t^{\cdot}, g \rangle} \langle X_t^{\cdot}, f \rangle \rangle = \langle \lambda, Ee^{-\langle \delta_{\cdot} + Y_{t,\cdot}, g \rangle} f(\cdot) \rangle, \tag{3.2}$$

as this implies the equality of the Laplace transforms of $(\widetilde{P_t})_x$ and $\delta_x + Y_{t,x}$ for $\lambda$-almost all $x$.

We will present two proofs of (3.2). The first one relies on the "backward technique" [Kallenberg (1977), Liemant (1981) and Liemant, Matthes and Wakolbinger (1988)], which has been developed to compute the Palm distributions of random populations with branching dynamics in discrete time, and which we adapt to the continuous-time model by means of an approximation. The interest of this proof is its use of the known backward technique and its intuitive content. Since this proof is long we will only outline the main steps. The second proof deals with the continuous-time model directly, employing the Feynman–Kac formula.

Both proofs will use the basic fact

$$\lambda(dx)\pi_t(x,dz) = \lambda(dx)p_t(x,z)\lambda(dz) = \lambda(dz)\pi_t(z,dx). \tag{3.3}$$

First proof of (3.2). By the definition of Palm distribution and (3.3), we have

$$\begin{aligned}\langle \lambda, Ee^{-\langle X_t, g\rangle}\langle X_t, f\rangle\rangle &= \int\int e^{-\langle \Phi, g\rangle}\langle \Phi, f\rangle \kappa^t_{(x)}(d\Phi)\lambda(dx)\\ &= \int\int\int e^{-\langle \Phi, g\rangle} f(z)\big(\kappa^t_{(x)}\big)_z(d\Phi)\pi_t(x,dz)\lambda(dx)\\ &= \int\int\int e^{-\langle \Phi, g\rangle} f(x)\big(\kappa^t_{(z)}\big)_x(d\Phi)\pi_t(x,dz)\lambda(dx),\end{aligned}$$

and therefore (3.2) is immediate from Proposition 3.1.

Proposition 3.1. *Let $t > 0$ and $z \in \mathbb{R}^d$. Define, for $x \in \mathbb{R}^d$, the random population $Y^z_{t,x}$ as $Y_{t,x}$ in Theorem 2.3, but with $(W^x_s)$ replaced by a random path $(W^{xz}_s)$ of the basic process starting in x and conditioned to be in z at time t. Then $\delta_x + Y^z_{t,x}$ has distribution $(\kappa^t_{(z)})_x$ for $\lambda$-almost all $x \in \mathbb{R}^d$.*

Proof. 1. In order to apply the backward technique, we study the space–time evolution of generations up to time $t$, by distinguishing in each generation a motion step and a branching step. For the motion step, for each $(w,s) \in \mathbb{R}^d \times [0,t]$ we define the random element $m(w,s) := (w + W_{\eta\wedge(t-s)}, (s+\eta)\wedge t)$, where $\eta$ is exponentially distributed with parameter $V$, $(W_r)$ is a random path of the basic process starting from the origin and $\eta$, $(W_r)$ are independent, and we put $\tau_{(w,s)} :=$ distribution of $\delta_{m(w,s)}$. This means that in a motion step $\tau$, the time coordinate of an individual increases by an exponentially distributed increment, but not beyond the time horizon $t$, whereas the space coordinate changes according to the basic process. As to the branching step, we define

$$\omega_{(w,s)} := \begin{cases} \delta_{\delta_{(w,s)}} & \text{if } s = t,\\ \sum_{k=0}^{\infty} p_k \delta_{k\delta(w,s)} & \text{if } s < t.\end{cases}$$

Thus, in a branching step $\omega$, an individual with time coordinate less than $t$ produces a random number of offspring at the same time and place, whereas an individual with time coordinate $t$ remains unchanged. Further, we designate by $\gamma$ the composition of the two "clustering fields" $\tau$ and $\omega$, i.e.,

$$\gamma_{(w,s)}(\cdot) := E\omega_{m(w,s)}(\cdot).$$

For a population $\Phi = \sum_{i \in I} \delta_{y_i}$ on $\mathbb{R}^d \times [0,t]$ we define $\gamma_{(\Phi)}^{[1]} := \gamma_{(\Phi)}$ as the convolution $*_{i \in I} \gamma_{(y_i)}$, and we put, for $n = 2, 3, \ldots,$

$$\gamma_{(\Phi)}^{[n]}(\cdot) := \int \gamma_{(\Psi)}^{[n-1]}(\cdot)\gamma_{(\Phi)}(d\Psi), \qquad \gamma_{(y)}^{[n]} := \gamma_{(\delta_y)}^{[n]}.$$

The corresponding intensity kernels are defined by

$$J_\gamma^{[n]}(y,\cdot) := \int \Phi(\cdot)\gamma_{(y)}^{[n]}(d\Phi);$$

they are in fact the kernel powers of $J_\gamma := J_\gamma^{[1]}$. Due to the criticality of $\omega$, $J_\gamma((w,s),\cdot)$ is just the distribution of the random element $m(w,s)$.

2. Let $\Phi_n$ be the random population on $\mathbb{R}^d \times [0,t]$ which arises after $n$ steps of the dynamics $\gamma$, starting with $\delta_{(z,0)}$. Since each motion step increases the time coordinate by an exponential random variable until the time horizon $t$ is reached, and since any individual with time coordinate $t$ does not change anymore, it is clear that for almost all $(\Phi_n)$ there exists a (random) positive integer $N$ and a population $\Phi_\infty$ concentrated on $\mathbb{R}^d \times \{t\}$ such that $\Phi_n = \Phi_\infty$ for $n \geq N$. Writing $\Phi_n^t := \Phi_n((\cdot) \times \{t\})$, $n = 1, 2, \ldots, \infty$, we observe that $\Phi_n^t$ increases towards $\Phi_\infty^t$, and that $\Phi_\infty^t$ has distribution $\kappa_{(z)}^t$.

3. Denoting by $P_n$ the distribution of $\Phi_n$, we compute now the Palm distributions $(P_n)_{(x,t)}$, $x \in \mathbb{R}^d$. Due to the backward formula [Liemant, Matthes, and Wakolbinger (1988), 1.9.4, or Liemant (1981), Theorem 8.2], $(P_n)_{(x,t)}$ arises by coining out successively:

—the positions of the mother, grandmother, ...,
—the populations of the sisters, aunts, grandaunts, ...,
—the populations of the cousins, second-degree cousins, ...,

of the individual $\delta_{(x,t)}$. More precisely, the corresponding steps consist in forming

—a random sequence $(y_1, \ldots, y_{n-1})$ with distribution

$$\frac{1}{J_\gamma(y_0, dy_n)} J_\gamma(y_0, dy_1) J_\gamma(y_1, dy_2) \cdots J_\gamma(y_{n-1}, dy_n),$$

where $y_0 := (z, 0)$ and $y_n := (x, t)$,

—random populations $\Psi_j$ distributed according to $(\gamma_{(y_{j-1})})_{y_j}^!$, $j = 1, \ldots, n$, which are independent given $y_1, \ldots, y_{n-1}$, where $H_y^!$ stands for the reduced Palm distribution $H_y((\Phi - \delta_y) \in (\cdot))$,

—random populations $\chi_j$ with distribution $\gamma_{(\Psi_j)}^{[n-j]}$, $j = 1, \ldots, n-1$, which are independent given $\Psi_1, \ldots, \Psi_{n-1}$.

The backward formula states that $\delta_{(x,t)} + \chi_1 + \cdots + \chi_{n-1} + \Psi_n$ has distribution $(P_n)_{(x,t)}$.

4. The random sequence $(y_0, \dots, y_{n-1})$ can be realized as

$$\big((z,0), \big(W^{zx}_{t_1^n}, t_1^n\big), \dots, \big(W^{zx}_{t_\nu^n}, t_\nu^n\big), (x,t), \dots, (x,t)\big),$$

where $(W_s^{zx})$ is a random path of the basic process starting from $z$ and conditioned to be at $x$ at time $t$, $t_1^n < t_2^n < \cdots < t_\nu^n$ are the points of a Poisson process with intensity $V$ on $[0,t]$ conditioned to the event $\{\nu < n\}$, this Poisson process being independent of $(W_s^{zx})$. [The point $(x,t)$ appears $n-1-\nu$ times in the sequence.]

The random population $\Psi_j$ equals $\delta_{y_j}$ a.s. if the time coordinate of $y_j$ is $t$. If the time coordinate of $y_j$ is less than $t$, then $\Psi_j$ arises as $Z_j\delta_{y_j}$, where $Z_j$ is a random number with $\operatorname{Prob}[Z_j = k] = (k+1)p_{k+1}$, $k = 0, 1, \dots$. This is a consequence of the following fact:

Let $\pi$ be a probability distribution [in our case the distribution of $m(w,s)$ for fixed $(w,s)$], and let, for $\pi$-almost all $y$, $(p_k^y)_{k=0,1,\dots}$ be a probability on the nonnegative integers with finite mean $m_y$. Then a family of Palm distributions of

$$Q := \int \sum_k p_k^y \delta_{k\delta_y} \pi(dy)$$

is given by

$$Q_y := (m_y)^{-1} \sum_k k p_k^y \delta_{k\delta_y}.$$

Indeed,

$$\begin{aligned}\int \langle \Phi, h\rangle g(\Phi) Q(d\Phi) &= \int \sum_k p_k^y g(k\delta_y) k h(y) \pi(dy)\\ &= \int h(y)(m_y)^{-1} \sum_k k p_k^y g(k\delta_y) \Lambda_Q(dy).\end{aligned}$$

5. From the two preceding steps we infer that $(P_n)_{(x,t)}$ is represented by the distribution of

$$Y_x^{z,n} := \delta_{(x,t)} + \sum_{j=1}^{\nu} \sum_{i=1}^{Z_j} \Phi_{n-j,i}^{y_{n,j}},$$

where $y_{n,j} := (W^{zx}_{t_j^n}, t_j^n)$, $\Phi_k^y$ denotes the $k$th generation population in the system with dynamics $\gamma$ starting with the single individual $\delta_y$ and $\Phi_{k,i}^y$, $i = 1, 2, \dots$, are independent copies of $\Phi_k^y$.

The sequence of random populations $\sum_{j=1}^{\nu} \delta_{y_{n,j}}$ converges in distribution as $n \to \infty$ to $\int_{[0,t]} \delta_{(W_s^{zx}, s)} \mu(ds)$, where $\mu$ is a random Poisson configuration on $\mathbb{R}_+$ independent of $(W_s^{zx})$. Since the distribution of $\mu$ is invariant with respect to the time reversal $s \mapsto t - s$, this random population has the same distribution as $\int_{[0,t]} \delta_{(W_{t-s}^{zx}, t-s)} \mu(ds)$, which in turn equals $\int_{[0,t]} \delta_{(W_s^{xz}, t-s)} \mu(ds)$ in distribution, where $(W_s^{xz})$ denotes a random path of the basic process starting from $x$ and conditioned to lead to $z$ at time $t$, since for any $s$ the distributions of $W_s^{xz}$ and $W_{t-s}^{zx}$ coincide.

On the other hand, for any fixed $y = (w, t - s) \in \mathbb{R}^d \times [0, t]$ the sequence of random populations $\Phi_k^y((\cdot) \times \{t\})$ converges in distribution as $k \to \infty$ to the random population $X_s^w$ with distribution $\kappa_{(w)}^s$ (see the argument in step 2).

Hence we infer that the sequence of random populations $Y_{t,x}^{z,n} := Y_x^{z,n}((\cdot) \times \{t\})$ converges in distribution as $n \to \infty$ to the random population $\delta_x + Y_{t,x}^z$ defined in the proposition.

6. Finally, by using the fact that $\Phi_n^t$ increases towards $\Phi_\infty^t$ [and hence $J_y^{[n]}((0, z); dx \times \{t\})$ converges to $\pi_t(z, dx)$], the Palm distributions of $P_n$ (step 3), and dominated convergence, we have for all $g, f$: $\mathbb{R}^d \to \mathbb{R}_+$ continuous with compact support,

$$\begin{aligned} Ee^{-\langle X_t^z, g\rangle}\langle X_t^z, f\rangle &= Ee^{-\langle \Phi_\infty^t, g\rangle}\langle \Phi_\infty^t, f\rangle \\ &= \lim_{n\to\infty} E\left[e^{-\langle \Phi_n^t, g\rangle}\int\int 1_{\{t\}}(s)f(x)\Phi_n(dx, ds)\right] \\ &= \lim_{n\to\infty}\int Ee^{-\langle Y_{t,x}^{z,n}, g\rangle}f(x)J_y^{[n]}((0,z), dx \times \{t\}) \\ &= \int Ee^{-\langle \delta_x + Y_{t,x}^z, g\rangle}f(x)\pi_t(z, dx), \end{aligned}$$

which, by the definition of Palm distributions, proves that $\delta_x + Y_{t,x}^z$ has distribution $(\kappa_{(z)}^t)_x$. □

SECOND PROOF OF (3.2). Let

$$j(x,t) := Ee^{-\langle X_t^x, g\rangle}\langle X_t^x, f\rangle.$$

The renewal argument shows that $j$ satisfies the integral equation

$$j(x,t) = e^{-Vt}T_t(e^{-g}f)(x) + \int_0^t Ve^{-Vs}T_s(j(\cdot, t-s))F'(h(\cdot, t-s))(x)\,ds,$$

where $F$ is given by (2.1) and $h(x,t) := Ee^{-\langle X_t^x, g\rangle}$ satisfies (2.3). Hence $F'(s) = 1 - \frac{1}{2}(1+\beta)(1-s)^\beta$, and $j$ solves the equation

$$\begin{aligned} \frac{\partial j}{\partial t} &= (\Delta_\alpha - V)_j + V_j\left[1 - \tfrac{1}{2}(1+\beta)(1-h)^\beta\right] \\ &= \Delta_\alpha j - \tfrac{1}{2}V(1+\beta)(1-h)^\beta j, \qquad j(x,0) = e^{-g(x)}f(x). \end{aligned}$$

This is a linear equation for $j$, and its solution is given by the Feynman–Kac formula [e.g., Itô (1969)],

$$j(x,t) = E\left[\exp\left\{-\tfrac{1}{2}V(1+\beta)\int_0^t (1 - h(W_{t-s}^x, s))^\beta\,ds\right\}e^{-g(W_t^x)}f(W_t^x)\right],$$

where $W_t^x$, $t \ge 0$, denotes the basic process starting from $x$. By reversing the time and using the basic fact (3.3), we have that the left-hand side of (3.2) is given by

$$\langle \lambda, j(\cdot, t)\rangle = \left\langle \lambda, E\left[\exp\left\{-\tfrac{1}{2}V(1+\beta)\int_0^t (1 - h(W_s^{\cdot}, s))^{\beta}\, ds\right\}\right] s^{-g(\cdot)} f(\cdot)\right\rangle.$$

Therefore, to complete the proof, we must show that

$$\langle \lambda, f(\cdot) E\exp\{-\langle \delta_{\cdot} + Y_{t,\cdot}, g\rangle\}\rangle$$
$$= \left\langle \lambda, E\left[\exp\left\{-\tfrac{1}{2}V(1+\beta)\int_0^t (1 - h(W_s^{\cdot}, s))^{\beta}\, ds\right\}\right] e^{-g(\cdot)} f(\cdot)\right\rangle.$$

This amounts to computing $E\exp\{-\langle Y_{t,x}, g\rangle\}$, which is straightforward due to the explicit structure of $Y_{t,x}$. Denoting by $E'$ the conditional expectation given the path of the basic process starting from $x$, and conditioning on the number of Poisson points in $[0, t]$ (independently and uniformly distributed on $[0, t]$), we have

$$Ee^{-\langle Y_{t,x}, g\rangle} = E\exp\left\{-\int_{[0,t]} \sum_{i=1}^{Z_s} \langle X_{s,i}^{W_s^x}, g\rangle \mu(ds)\right\}$$

$$= \sum_{n=0}^{\infty} E\left[\exp\left\{-\int_{[0,t]} \sum_{i=1}^{Z_s} \langle X_{s,i}^{W_s^x}, g\rangle \mu(ds)\right\} \middle| \mu([0,t]) = n\right]$$
$$\times e^{-Vt}(Vt)^n/n!$$

$$= \sum_{n=0}^{\infty} E\left(\frac{1}{t}\int_0^t E'\exp\left\{-\sum_{i=1}^{Z_s} \langle X_{s,i}^{W_s^x}, g\rangle\right\} ds\right)^n e^{-Vt}(Vt)^n/n!$$

$$= E\exp\left\{-Vt + V\int_0^t E'\exp\left\{-\sum_{i=1}^{Z_s} \langle X_{s,i}^{W_s^x}, g\rangle\right\} ds\right\}$$

$$= E\exp\left\{-Vt + V\int_0^t \sum_{k=0}^{\infty} (k+1)p_{k+1} E'\exp\left\{-\sum_{i=1}^{k} \langle X_{s,i}^{W_s^x}, g\rangle\right\} ds\right\}$$

$$= E\exp\left\{-Vt + V\int_0^t \sum_{k=0}^{\infty} (k+1)p_{k+1}\left(E'\exp\{-\langle X_s^{W_s^x}, g\rangle\}\right)^k ds\right\}$$

$$= E\exp\left\{-Vt + V\int_0^t \left[1 - \tfrac{1}{2}(1+\beta)(1 - h(W_s^x, s))^{\beta}\right] ds\right\}$$

$$= E\exp\left\{-\frac{1}{2}V(1+\beta)\int_0^t (1 - h(W_s^x, s))^{\beta}\, ds\right\}. \qquad \square$$

REMARK 3.2. From the second proof we have

$$\left\langle \lambda, f(\cdot) \int e^{-\langle \Phi, g\rangle}(\widetilde{P}_t).\, d\Phi \right\rangle$$
$$= \left\langle \lambda, E\left[\exp\left\{-\tfrac{1}{2}V(1+\beta)\int_0^t (1 - h(W_s^{\cdot}, s))^\beta\, ds\right\}\right] e^{-g(\cdot)} f(\cdot) \right\rangle.$$

We will show that this result can be used to derive a representation similar to that of Theorem 2.3 for the (stable) measure branching process, which agrees with the corresponding result for the "historical process" obtained recently by Dawson and Perkins (1990).

The measure branching process is a scaling limit of the particle model we are studying. The appropriate scaling is the following: Each particle has mass $1/n$, the lifetime parameter is $n^\beta V$, and the initial Poisson distribution has intensity measure $n\lambda$. Then the mass process $X_t^n$ converges in law as $n \to \infty$ to the measure branching process $X_t^\infty$ for each $t$ [see Dawson and Ivanoff (1978) for $\beta = 1$ and Mélérd and Roelly-Coppoletta (1989) for $\beta < 1$].

Introducing this scaling in the formula above, we have

$$\left\langle \lambda, f(\cdot) \int e^{-\langle \Phi, g\rangle}(\widetilde{P_t^n}).\, (d\Phi) \right\rangle$$
$$(3.4) \qquad = \left\langle \lambda, E\left[\exp\left\{-\tfrac{1}{2}n^\beta V(1+\beta)\int_0^t (1 - h^n(W_s^{\cdot}, s))^\beta\, ds\right\} \times e^{-g(\cdot)/n} f(\cdot)\right] \right\rangle,$$

where $H^n(x,t) := n(1 - h^n(x,t))$ satisfies [see (2.3)]

$$\frac{\partial H^n}{\partial t} = \Delta_\alpha H^n - \frac{V}{2}(H^n)^{1+\beta}, \qquad H^n(x,0) = n(1 - e^{-g(x)/n}).$$

Since $H^n(x,0) \to g(x)$ as $n \to \infty$, then [see, e.g., Henry (1981), Theorem 3.4.1], $H^n(x,t) \to H(x,t)$ as $n \to \infty$ uniformly for $t$ in bounded intervals, where $H(x,t)$ satisfies

$$(3.5) \qquad \frac{\partial H}{\partial t} = \Delta_\alpha H - \frac{V}{2}H^{1+\beta}, \qquad H(x,0) = g(x),$$

and the right-hand side of (3.4) converges by dominated convergence:

$$\left\langle \lambda, E\left[\exp\left\{-\tfrac{1}{2}V(1+\beta)\int_0^t (n(1 - h^n(W_s^{\cdot}, s)))^\beta\, ds\right\} e^{-g(\cdot)/n} f(\cdot)\right] \right\rangle$$
$$\to \left\langle \lambda, E\exp\left\{-\tfrac{1}{2}V(1+\beta)\int_0^t (H(W_s^{\cdot}, s))^\beta\, ds\right\} f(\cdot) \right\rangle \quad \text{as } n \to \infty.$$

On the other hand, convergence in law of $X_t^n$ to $X_t^\infty$ means weak convergence of $P_t^n$ to $P_t^\infty$, the distribution of $X_t^\infty$, and since the intensity measure is preserved in the limit (direct verification), then by Kallenberg (1983), Theorem

10.4 and Lemma 10.8, $(\widetilde{P_t^n})_f$ converges weakly to $(\widetilde{P_t^\infty})_f$ as $n \to \infty$ for each $f$: $\mathbb{R}^d \to \mathbb{R}_+$ continuous with compact support, $f \not\equiv 0$, where $(\widetilde{P_t^n})_f$ and $(\widetilde{P_t^\infty})_f$ denote the randomizations of the Palm distributions by $f$ [Kallenberg (1983), Section 10.3]. This implies that the left-hand side of (3.4) converges,

$$\left\langle \lambda, f(\cdot) \int e^{-\langle \Phi, g\rangle} \left(\widetilde{P_t^n}\right)_{.} (d\Phi) \right\rangle \to \left\langle \lambda, f(\cdot) \int e^{-\langle \Phi, g\rangle} \left(\widetilde{P_t^\infty}\right)_{.} (d\Phi) \right\rangle$$

as $n \to \infty$.

Thus we have proved Theorem 3.3.

THEOREM 3.3. *For all $t > 0$, the Palm distribution $(\widetilde{P_t^\infty})_x$ of the canonical measure of the measure branching process $X_t^\infty$ has the following representation through its Laplace functional*:

$$\int e^{-\langle \Phi, g\rangle} \left(\widetilde{P_t^\infty}\right)_x (d\Phi) = E \exp\left\{ -\tfrac{1}{2} V(1+\beta) \int_0^t (H(W_s^x, s))^\beta \, ds \right\}$$

*for $\lambda$-almost all $x \in \mathbb{R}^d$, where $W_t^x$, $t \geq 0$, is the basic process starting from $x$, and $H$ is the (unique, global) solution of the nonlinear equation* (3.5).

Note that in the particle system the breakoff of mass from the trajectory of the basic process takes place at the time points of the Poisson measure $\mu$, whereas in the scaling limit the breakoff of mass goes on continuously in time.

**4. Several characterizations of persistency.** Using Theorem 2.3, we can prove the following criterion [which is an analog to Kallenberg's (1977) persistency criterion for discrete-time models; see also Liemant, Matthes and Wakolbinger (1988), Section 2.4].

PROPOSITION 4.1. *$\lambda$ is persistent if and only if for $\lambda$-almost all $x \in \mathbb{R}^d$ the random population*

$$Y_{\infty, x} := \lim_{t \to \infty} Y_{t,x} = \int_{[0,\infty)} \left( \sum_{i=1}^{Z_s} X_{s,i}^{W_s^x} \right) \mu(ds)$$

*is locally finite a.s.* [*i.e., for almost all paths* $(W_s^x)$, *Poisson configurations* $\mu$ *and random numbers* $(Z_s)$].

(The ingredients building up $Y_{\infty, x}$ have been defined in Theorem 2.3.)

PROOF. By 3.3.6 and 10.1.1 in Matthes, Kerstan and Mecke (1978), persistence of $\lambda$ is equivalent to the following uniform integrability condition on the canonical measures $\widetilde{P}_t$:

$$\lim_{m \to \infty} \sup_{t > 0} \int_{\{\Phi(B) > m\}} \Phi(B) \widetilde{P}_t(d\Phi) = 0 \tag{4.1}$$

for all bounded Borel sets $B \subset \mathbb{R}^d$.

By means of Theorem 2.3 we get

$$\int_{\{\Phi(B)>m\}} \Phi(B)\widetilde{P}_t(d\Phi) = \int 1_B(x)\big(\widetilde{P}_t\big)_x(\{\Phi(B) > m\})\lambda(dx)$$

$$= \int_B P[Y_{t,x}(B) \geq m]\lambda(dx).$$

Now suppose that $Y_{\infty,x}$ is, for $\lambda$-almost all $x$, locally finite a.s. Then we have

$$\sup_{t>0} \int_B P[Y_{t,x}(B) \geq m]\lambda(dx)$$

$$\leq \int_B P[Y_{\infty,x}(B) \geq m]\lambda(dx) \to 0 \quad \text{as } m \to \infty,$$

by dominated convergence, and hence the validity of (4.1), i.e., persistency of $\lambda$.

Conversely, assume that $\lambda$ is persistent, i.e., $P_\infty$ has intensity measure $\lambda$. Writing $H_{\kappa^t} := \int \kappa^t_{(\Phi)}(\cdot)H(d\Phi)$ for any distribution $H$, and expressing the Palm distribution $((P_\infty)_{\kappa^t})_x$ by the backward formula [Liemant, Matthes and Wakolbinger (1988), 1.9.1], we get

$$(P_\infty)_x = ((P_\infty)_{\kappa^t})_x = \int \Big(\big((P_\infty)^!_y\big)_{\kappa^t} * \big(\kappa^t_{(y)}\big)_x\Big)(\cdot)\pi_t(x, dy),$$

i.e., $(P_\infty)_x$ arises, for any $t > 0$, as the distribution of a superposition of the random population $Y_{t,x}$ and some other random population. Thus we have

$$P[Y_{t,x}(B) \geq m] \leq (P_\infty)_x(\Phi(B) \geq m)$$

for all $t, m > 0$ and bounded Borel sets $B \subset \mathbb{R}^d$.

Hence

$$P[Y_{\infty,x}(B) \geq m] = \lim_{t\to\infty} P[Y_{t,x}(B) \geq m] \leq (P_\infty)_x(\Phi(B) \geq m)$$

for all $m > 0$, which implies that $Y_{\infty,x}$ is locally finite a.s. □

The following lemma is, within the class of models considered in this paper, a continuous-time analog to Theorem 3.1 of Kallenberg (1977) and Lemma 7.4.1 of Dawson and Fleischmann (1985).

LEMMA 4.2. *$\lambda$ is persistent if and only if for all bounded Borel sets $B \subset \mathbb{R}^d$*

$$\int_{[0,\infty)} \big(\kappa^s_{(W_s)}(\Phi(B) > 0)\big)^\beta \mu(ds) < \infty \quad \textit{for } \Pi\textit{-almost all } (W_s)$$

*and almost all Poisson configurations $\mu$.*

[Here and below, $\Pi$ stands for the measure $\int \Pi_x(\cdot)\lambda(dx)$ on path space, where $\Pi_x$, $x \in \mathbb{R}^d$, denotes the distribution of the basic process $(W^x_s)$ starting in $x$.]

PROOF. 1. For any fixed realizations of $\mu$ and $(W_s)$, it follows from the Borel–Cantelli lemma that the number of summands $(\sum_{i=1}^{Z_s} X_{s,i}^{W_s})(B)$ contributing to the sum $\int_{[0,\infty)}(\sum_{i=1}^{Z_s} X_{s,i}^{W_s})(B)\mu(ds)$ is finite a.s. if and only if the sum

$$\int_{[0,\infty)} P\left[\left(\sum_{i=1}^{Z_s} X_{s,i}^{W_s}\right)(B) > 0\right]\mu(ds)$$

converges. Proposition 4.1 thus yields that $\lambda$ is persistent if and only if this sum is finite for $\Pi$-almost all $(W_s)$ and almost all $\mu$.

2. For any $s > 0$ and $w \in \mathbb{R}^d$ one has

$$\begin{aligned} P\left[\left(\sum_{i=1}^{Z_s} X_{s,i}^{w}\right)(B) > 0\right] &= \sum_{k=1}^{\infty} kp_k P\left[\left(\sum_{i=1}^{k-1} X_{s,i}^{w}\right)(B) > 0\right] \\ &= 1 - \sum_{k=1}^{\infty} kp_k(P[X_s^w(B) = 0])^{k-1} \\ &= 1 - f'(P[X_s^w(B) = 0]) \\ &= \frac{1+\beta}{2}(1 - P[X_s^w(B) = 0])^{\beta} \\ &= \frac{1+\beta}{2}\left(\kappa_{(w)}^s(\Phi(B) > 0)\right)^{\beta}. \end{aligned}$$

Combining steps 1 and 2, we arrive at our assertion. □

COROLLARY 4.3. *$\lambda$ is persistent provided that the following finiteness condition holds*:

$$(4.2)\qquad \int_{[0,\infty)} (\Pi_s(W_s, B))^{\beta}\,\mu(ds) < \infty \quad \text{for } \Pi\text{-almost all } (W_s) \text{ and almost all } \mu,$$

*for all bounded Borel sets $B \subset \mathbb{R}^d$.*

PROOF. This is immediate from Lemma 4.2, since

$$\pi_s(W_s, B) \geq \kappa_{(W_s)}^s(\Phi(B) > 0).$$ □

PROPOSITION 4.4. *Condition* (4.2) *holds if and only if $d > \alpha/\beta$.*

PROOF. 1. Let $B$ be a ball in $\mathbb{R}^d$ centered around the origin and put $M_s := \pi_s(0, B)$. By symmetry, unimodality and scaling properties of $p_t$

$$\sup_{x \in R^d} \pi_s(x, B) = M_s = \pi_1(0, s^{-1/\alpha}B).$$

By continuity of $p_1$, this implies the existence of some constants $c, C > 0$ such that

$$(4.3)\qquad cs^{-d/\alpha} \leq M_s \leq Cs^{-d/\alpha} \quad \text{for all } s \geq 1.$$

2. The condition $d > \alpha/\beta$ is equivalent to

$$\int_{[0,\infty)} s^{-\beta d/\alpha}\mu(ds) < \infty \quad \text{a.s.}$$

Indeed, writing $\mu = \sum_{i=1}^{\infty} \delta_{t_i}$, where $0 = t_0 < t_1 < t_2 < \cdots$, and putting $\tau_i := t_i - t_{i-1}$, we have

$$\sum_{i=1}^{\infty} t_i^{-\beta d/\alpha} = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{i} \tau_j \right)^{-\beta d/\alpha} = \sum_{i=1}^{\infty} \frac{1}{i^{\beta d/\alpha}} \left( \frac{1}{i} \sum_{j=1}^{i} \tau_j \right)^{-\beta d/\alpha},$$

which by the strong law of large numbers is finite if and only if $\beta d/\alpha > 1$.

3. Now assume that $d > \alpha/\beta$. Steps 1 and 2 imply, for any ball $B \subset \mathbb{R}^d$ centered around the origin,

$$\int_{[0,\infty)} (M_s)^{\beta}\mu(ds) < \infty \quad \text{a.s.},$$

which clearly yields (4.2).

4. Conversely, assume the validity of (4.2). Let $B$ be the unit ball in $\mathbb{R}^d$, and choose a starting point $x$ for the basic process such that (4.2) holds true for almost all $(W_s^x)$ starting in $x$. Moreover, we choose $x$ so close to the origin of $\mathbb{R}^d$ such that for all $s \geq 1$,

$$E[\pi_s(W_s^x, B)] = \int \pi_s(x, dy)\pi_s(y, B) = \pi_{2s}(x, B) \geq \tfrac{1}{2}M_{2s}.$$

Henceforth, we will tacitly restrict ourselves to $s \geq 1$, and will write $\xi_s := \pi_s(W_s^x, B)$.

Since, by (4.3),

$$M_{2s} \geq c2^{-d/\alpha}s^{-d/\alpha} \geq \frac{c2^{-d/\alpha}}{C}M_s = \text{const.}\, M_s,$$

there exists some $\delta \in (0, \frac{1}{2})$ such that

$$E[\xi_s] \geq \tfrac{1}{2}M_{2s} \geq 2\delta M_s.$$

By Markov's inequality,

$$\begin{aligned}
&(1-\delta)M_s P[\xi_s \leq \delta M_s] \\
&\quad = (1-\delta)M_s P[M_s - \xi_s \geq (1-\delta)M_s] \leq E[M_s - \xi_s] \\
&\quad \leq (1-2\delta)M_s.
\end{aligned}$$

Hence

$$P[\xi_s > \delta M_s] \geq 1 - \frac{1-2\delta}{1-\delta} \geq \delta.$$

From this estimate and from one more application of Markov's inequality, it follows for almost all Poisson configurations $\mu$, all $k > 0$ and $t > 1$ that

$$\begin{aligned}\delta \int_{[1,t]} (\delta M_s)^\beta \mu(ds) &\le \int_{[1,t]} P[\xi_s > \delta M_s](\delta M_s)^\beta \mu(ds)\\ &\le \int_{[1,t]} E\big[(\xi_s)^\beta\big]\mu(ds) = E\left[\int_{[1,t]} (\xi_s)^\beta \mu(ds) \middle| \mu\right]\\ &\le k + P\left[\int_{[1,t]} (\xi_s)^\beta > k \middle| \mu\right] \int_{[1,t]} (M_s)^\beta \mu(ds)\end{aligned}$$

(recall that by definition $M_s$ is an upper bound for $\xi_s$). This chain of inequalities implies

$$P\left[\int_{[1,t]} (\xi_s)^\beta \mu(ds) < k \middle| \mu\right] \ge \delta^{1+\beta} - \frac{k}{\int_{[1,t]} (M_s)^\beta \mu(ds)}.$$

Suppose now that $\int_{[1,\infty)} (M_s)^\beta \mu(ds) = \infty$ for a set $Y$ of configurations $\mu$ having nonzero probability. Then the limit $t \to \infty$ yields, for almost all $\mu \in Y$,

$$P\left[\int_{[1,\infty)} (\xi_s)^\beta \mu(ds) < k \middle| \mu\right] \ge \delta^{1+\beta},$$

and the limit $k \to \infty$ leads, for almost all $\mu \in Y$, to

$$P\left[\int_{[0,\infty)} (\xi_s)^\beta \mu(ds) = \infty \middle| \mu\right] > 0.$$

This, however, contradicts condition (4.2). Hence we have derived

$$\int_{[0,\infty)} (M_s)^\beta \mu(ds) < \infty \quad \text{a.s.,}$$

which by steps 1 and 2 implies $d > \alpha/\beta$. □

Together with Corollary 4.3, the preceding proposition shows that $\lambda$ is persistent if $d > \alpha/\beta$. The converse (and hence the complete assertion of Theorem 2.2) is established by Proposition 4.4 together with the following result:

PROPOSITION 4.5. *If $\lambda$ is persistent, then condition* (4.2) *holds.*

The following section is devoted to the proof of this proposition, thus completing the proof of Theorem 2.2.

**5. Completion of the proof of Theorem 2.2.** In the whole section, we will assume that $\lambda$ is persistent, and write $Q$ for the limiting distribution $P_\infty$.

LEMMA 5.1. *For $\lambda$-almost all $x \in \mathbb{R}^d$, the random population $Y_{\infty,x}$ (defined in Proposition* 4.1) *is distributed according to the reduced Palm distribution $\tilde{Q}_x^!$.*

PROOF. Since we have weak convergence of $P_t$ towards $Q$, as well as convergence (even coincidence) of the corresponding intensity measures, we get from Kallenberg (1983), Lemma 10.8, for any continuous $h\colon \mathbb{R}^d \to \mathbb{R}_+$ with compact support and any continuous, bounded $g\colon M \to \mathbb{R}_+$,

$$\begin{aligned}\int h(x)\int g(\Phi)\tilde{Q}_x(d\Phi)\lambda(dx) &= \int\int h(x)g(\Phi)\Phi(dx)\tilde{Q}(d\Phi)\\ &= \lim_{t\to\infty}\int\int h(x)g(\Phi)\Phi(dx)\widetilde{P_t}(d\Phi)\\ &= \lim_{t\to\infty}\int\int h(x)g(\Phi)\big(\widetilde{P_t}\big)_x(d\Phi)\lambda(dx)\\ &= \lim_{t\to\infty}\int h(x)E\big[g(\delta_x + Y_{t,x})\big]\lambda(dx)\\ &= \int h(x)E\big[g(\delta_x + Y_{\infty,x})\big]\lambda(dx),\end{aligned}$$

where we have applied Theorem 2.3 and Proposition 4.1, respectively, in the last two steps. □

The proof of Proposition 4.5 will be accomplished by considering, in addition to the clustering fields $\kappa^t_{(z)}$, other clustering fields $\tau^t_{(z)}$ which arise by displacing all individuals of the random population $X^z_t$ independently of each other with respect to some symmetric probability measure $\nu$. To give a motivation for this procedure, which is borrowed from Dawson and Fleischmann (1985), let us mention the following: A statement like that of Lemma 5.5 below, with $\kappa^t_{(z)}$ in place of $\tau^t_{(z)}$, would perfectly serve to prove Proposition 4.5. To derive this statement, one would need an analog of Lemma 5.4, again with $\kappa^t_{(z)}$ in place of $\tau^t_{(z)}$. This, however, is difficult to prove, since the backward formula yields only an expression for $(\kappa^t_{(W_t)})_x$ when $x$ is the starting point of the basic process. Hence, in order to "decouple" the "conditioning point" $x$ and the starting point of the basic process, one uses an additional randomization, which is achieved by an independent $\nu$-displacement of individuals at final time.

Thus we fix in the following a symmetric probability measure $\nu$ on $\mathbb{R}^d$, which is also supposed to have a bounded, continuous and strictly positive density with respect to $\lambda$; e.g., $\nu := \pi_1(0, \cdot\,)$.

Let $T_\nu$ denote the clustering field which corresponds to "independent $\nu$-displacement," i.e., for any measurable set $F$ of populations and $z \in \mathbb{R}^d$ one has

$$(T_\nu)_{(z)}(F) := \nu\big(\{y\colon \delta_{z+y} \in F\}\big).$$

Together with $\tilde{Q}$, also the clustered measure $\tilde{Q}_{T_\nu}$ has intensity measure $\lambda$.

We now compute the Palm distributions $(\tilde{Q}_{T_\nu})_x$ by the backward formula [Liemant, Matthes and Wakolbinger (1988), 1.9.1]. The "dual kernel" $D$ figuring there is given by $D(z, B) := \nu(B - z)$, since $\lambda(dz)\nu(dy - z) =$

$\lambda(dy)\nu(dz - y)$ due to the symmetry of $\nu$. Hence

$$\left(\tilde{Q}_{T_\nu}\right)_x = \int \nu(dy)\left(\tilde{Q}^!_{x-y}\right)_{T_\nu} * \left((T_\nu)_{(x-y)}\right)_x$$

$$= \delta_{\delta_x} * \int \nu(dy)\left(\tilde{Q}^!_{x-y}\right)_{T_\nu}.$$

Introducing, for any population $\Phi$, a random population $\Phi^\nu$ which is distributed according to $(T_\nu)_{(\Phi)}$, i.e., obtained by displacing all individuals in $\Phi$ independently of each other according to $\nu$, we thus infer Lemma 5.2 from Lemma 5.1.

LEMMA 5.2. *For $\lambda$-almost all $x \in \mathbb{R}^d$, the random population*

$$\delta_x + \left(Y_{\infty, y}\right)^\nu$$

[*where the point $y$ is random with distribution $\nu(\cdot - x)$*] *has distribution $(\tilde{Q}_{T_\nu})_x$. A fortiori, the random population $(Y_{\infty, y})^\nu$ is a.s. locally finite for $\lambda$-almost all $y \in \mathbb{R}^d$.*

Next, let us fix $z \in \mathbb{R}^d$ and $t > 0$, and calculate the Palm distributions $((\kappa^t_{(z)})_{T_\nu})_x$. Now, the dual kernel $D(x, dy)$ figuring in the backward formula [Liemant, Matthes and Wakolbinger (1988), 1.9.1] has density

$$g(x, y) := \frac{p_t(y - z)p_1(x - y)}{(p_t * p_1)(x - z)}$$

(since the density of $\nu$ with respect to $\lambda$ is $p_1$). Hence we obtain, for $\lambda$-almost all $x$,

$$\left((\kappa^t_{(z)})_{T_\nu}\right)_x = \int \lambda(dy)g(x, y)\left(\left((\kappa^t_{(z)})^!_y\right)_{T_\nu} * \left((T_\nu)_{(y)}\right)_x\right)(\cdot)$$

$$= \delta_{\delta_x} * \int \lambda(dy)g(x, y)\left((\kappa^t_{(z)})^!_y\right)_{T_\nu}.$$

Noting that, for fixed $x$, $g(x, \cdot)$ is also the conditional density of the starting point $y$ of the basic process conditioned to be at $z$ at time $t$ [where $y$ has prior distribution $\nu(\cdot - x)$], we arrive at

$$\left((\kappa^t_{(z)})_{T_\nu}\right)^!_x = P\left[(Y_{t, y})^\nu \in (\cdot) | W^y_s = z\right] \quad \lambda\text{-almost all } x, \lambda\text{-almost all } z,$$

where $(W^y_s)$ is the basic process with starting point $y$ distributed according to $\nu(\cdot - x)$.

Using the abbreviation

$$\left(\kappa^t_{(z)}\right)_{T_\nu} =: \tau^t_{(z)},$$

and exploiting the Markov property of $(W^y_s)$, from the preceding formula we obtain Lemma 5.3.

LEMMA 5.3. *For each $t > 0$ and $\lambda$-almost all $x$*

$$(5.1)\qquad \left(\tau^{t}_{(W^y_t)}\right)^!_x = P\left[(Y_{t,y})^{\nu} \in (\cdot) | (W^y_s)_{s \geq t}\right] \quad a.s.,$$

*where $(W^y_s)$ is the basic process with starting point $y$ distributed according to $\nu(\cdot - x)$.*

From this we are able to deduce Lemma 5.4.

LEMMA 5.4. *For any sequence $(t_n)$ with $t_n \to \infty$, all bounded Borel sets $B \subset \mathbb{R}^d$ and $\lambda$-almost all $x$*

$$\lim_{k\to\infty} \liminf_{n\to\infty} \left(\tau^{t_n}_{(W_{t_n})}\right)_x (\Phi(B) \leq k) = 1 \quad \textit{for } \Pi\textit{-almost all } (W_s).$$

PROOF. Lemma 5.3 guarantees that (5.1) holds for $\lambda$-almost all $x \in \mathbb{R}^d$ and all $n \geq 1$ (with $t_n$ instead of $t$). Thus we get for $\lambda$-almost all $x \in \mathbb{R}^d$ and all $k \geq 1$,

$$\begin{aligned}\left(\tau^{t_n}_{(W^y_{t_n})}\right)^!_x (\Phi(B) \leq k) &= P\left[(Y_{t,y})^{\nu}(B) \leq k | (W^y_s)_{s \geq t_n}\right] \\ &\geq P\left[(Y_{\infty,y})^{\nu}(B) \leq k | (W^y_s)_{s \geq t_n}\right],\end{aligned}$$

where $(W^y_s)$ is the basic process with starting point $y$ distributed according to $\nu(\cdot - x)$.

By the martingale convergence theorem, the right-hand side tends, for $n \to \infty$, a.s. toward

$$P\left[(Y_{\infty,y})^{\nu}(B) \leq k | F_\infty\right], \quad \text{where } F_\infty := \bigcap_{t>0} \sigma\left((W^y_s)_{s\geq t}\right).$$

Hence

$$\lim_{k\to\infty} \liminf_{n\to\infty} \left(\tau^{t_n}_{(W^y_{t_n})}\right)^!_x (\Phi(B) \leq k) \geq P\left[(Y_{\infty,x})^{\nu}(B) < \infty | F_\infty\right] \quad \text{a.s.}$$

Due to Lemma 5.2, the right-hand side equals 1 a.s. We have thus proved the statement of the lemma for almost all paths of the basic process having initial distribution $\nu(\cdot - x)$; since this initial distribution is mutually absolutely continuous with respect to $\lambda$, the proof is complete. □

LEMMA 5.5. *For all bounded Borel sets $B \subset \mathbb{R}^d$, any sequence $(t_n)$ with $t_n \to \infty$, and $\Pi$-almost all paths $(W_s)$ of the basic process, there exists a number $k = k((W_s), B)$ such that for all sufficiently large $n \in \mathbb{N}$*

$$k\tau^{t_n}_{(W_{t_n})}(\Phi(B) > 0) \geq \pi_{t_n}(W_{t_n}, B).$$

PROOF. Clearly we may assume $\lambda(B) > 0$.

1. Let $J^t_\tau(z, \cdot)$ denote the intensity measure of $\tau^t_{(z)}$. Writing $f_t(z, \cdot)$ for the density of $J^t_t(z, \cdot)/J^t_\tau(z, B)$ with respect to Lebesgue measure on $B$, we note that for $\Pi$-almost all paths $(W_s)$ there exists some $m > 0$, depending on $(W_s)$,

such that $f_t(W_t, x) \geq m$ for all $t \geq 1$ and $x \in B$. Indeed, recalling that $\nu$ has density $p_1$ and putting $g_t(z, x) := (p_t(z, \cdot) * p_1)(x) = p_{t+1}(z, x)$, for $z \in \mathbb{R}^d$ [which is the density of $J_\tau^t(z, \cdot)$], we have

$$(5.2) \qquad f_t(z, x) = \frac{p_{t+1}(z, x)}{\int_B p_{t+1}(z, y)\, dy} = \left( \int_B \frac{p_{t+1}(z, y)}{p_{t+1}(z, x)}\, dy \right)^{-1}.$$

In the case $\alpha = 2$, for $\Pi$-almost every Brownian path $(W_s)$ there exists a positive constant $c$ such that $|W_t| \leq c(1 + t^{3/4})$ for all $t \in \mathbb{R}_+$. Then the ratio

$$p_{t+1}(W_t, y)/p_{t+1}(W_t, x) = \exp\{(|W_t - y|^2 - |W_t - x|^2)/2(t+1)\}$$

is uniformly bounded for all $t \in \mathbb{R}_+$ and $x, y \in B$. The result then follows from (5.2). In the case $\alpha < 2$ there exist positive constants $K_1$ and $K_2$ such that for all $v \in \mathbb{R}^d$ with $|v| \geq 1$,

$$(5.3) \qquad K_1 |v|^{-d-\alpha} \leq p_1(v) \leq K_2 |v|^{-d-\alpha}.$$

This follows from the well-known representation of the symmetric stable process of exponent $\alpha$ on $\mathbb{R}^d$ (starting in 0) as a standard Brownian motion on $\mathbb{R}^d$ with an independent random change of time given by a one-sided stable process with exponent $\alpha/2$, and the tail behavior of the completely asymmetric stable distribution [e.g., Prokhorov and Rozanov (1969)]. The scaling property of $p_t$ together with (5.2) and (5.3) imply that $f_t(z, x)$, $t \geq 1$, $x \in B$, $z \in \mathbb{R}^d$ is uniformly bounded away from 0.

2. Next, we note that for all $z \in \mathbb{R}^d$ and $t \geq 1$,

$$\begin{aligned} J_\tau^t(z, B) &= \int \pi_t(z, dx)\nu(B - x) \geq \int_B \pi_t(z, dx)\nu(B - x) \\ &= \int_B \pi_t(z, dx) \int_B p_1(z - x)\lambda(dz) \geq C\pi_t(z, B), \end{aligned}$$

for a suitable constant $C$ not depending on $z$ and $t$ (since $p_1$ is bounded away from 0 on $B - B$).

3. For all $l \in \mathbb{N}$, $t \geq 1$ and $\Pi$-almost all paths $(W_s)$,

$$\begin{aligned} l\tau_{(W_t)}^t(\Phi(B) > 0) &\geq \int \Phi(B)\tau_{(W_t)}^t(d\Phi)1_{\{\Phi(B) \leq l\}} \\ &= \int_B J_\tau^t(W_t, dx)\left(\tau_{(W_t)}^t\right)_x(\Phi(B) \leq l) \\ &\geq mC\pi_t(W_t, B)\int_B \lambda(dx)\left(\tau_{(W_t)}^t\right)_x(\Phi(B) \leq l), \end{aligned}$$

where we used steps 1 and 2 in the last inequality.

4. Since $\lambda$-a.e. convergence on $B$ implies convergence in $\lambda$-measure on $B$, we infer, for $\Pi$-almost all paths $(W_s)$, from Lemma 5.4 the existence of a

suitable number $l = l((W_s))$ such that the set

$$B_1 := \left\{x \in B \mid \liminf_{n\to\infty}\left(\tau^{t_n}_{(W_{t_n})}\right)_x(\Phi(B) \le l) \le \tfrac{1}{2}\right\}$$

has measure $\lambda(B_1) < \lambda(B)/2$. For this number $l$, it follows by Fatou's lemma that

$$\begin{aligned}\liminf_{n\to\infty} \int_B \lambda(dx)\left(\tau^{t_n}_{(W_{t_n})}\right)_x(\Phi(B) \le l) \\ \ge \liminf_{n\to\infty} \int_{B\setminus B_1} \lambda(dx)\left(\tau^{t_n}_{(W_{t_n})}\right)_x(\Phi(B) \le l) \\ \ge \int_{B\setminus B_1} \lambda(dx) \liminf_{n\to\infty}\left(\tau^{t_n}_{(W_{t_n})}\right)_x(\Phi(B) \le l) \ge \frac{\lambda(B)}{4}.\end{aligned}$$

Together with step 3, this yields

$$l\tau^{t_n}_{(W_{t_n})}(\Phi(B) > 0) \ge mC\frac{\lambda(B)}{4}\pi_{t_n}(W_{t_n}, B)$$

for sufficiently large $n$. Hence

$$k := \frac{4l}{mC\lambda(B)}$$

is a possible choice to guarantee the assertion of the lemma. □

REMARK 5.6. The same calculation as in the proof of Lemma 4.2, step 2, shows, for any bounded Borel set $B \subset \mathbb{R}^d$, any $s > 0$ and $w \in \mathbb{R}^d$,

$$P\left[\left(\sum_{i=1}^{Z_s} (X^w_{s,i})^\nu\right)(B) > 0\right] = \frac{1+\beta}{2}\left(\tau^s_{(w)}(\Phi(B) > 0)\right)^\beta.$$

Since we know from Lemma 5.2 that

$$(Y_{\infty, W_0})^\nu(B) = \int_{[0,\infty)} \sum_{i=1}^{Z_s} (X^{W_s}_{s,i})^\nu(B)\mu(ds) < \infty$$

for $\Pi$-almost all $(W_s)$ and almost all $\mu$ and $(Z_s)$, we infer from the Borel–Cantelli lemma that

$$\int_{[0,\infty)} \left(\tau^s_{(W_s)}(\Phi(B) > 0)\right)^\beta \mu(ds) < \infty$$

for $\Pi$-almost all $(W_s)$ and almost all $\mu$. Due to Lemma 5.5, this implies

$$\int_{[0,\infty)} (\pi_s(W_s, B))^\beta \mu(ds) < \infty$$

for $\Pi$-almost all $(W_s)$ and almost all $\mu$, which is condition (4.2).

Having shown that condition (4.2) is a consequence of the persistency of $\lambda$, we have thus proved Proposition 4.5, and also completed the proof of Theorem 2.2.

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