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HYDRODYNAMIC LIMIT OF ONE-DIMENSIONAL EXCLUSION PROCESSES WITH SPEED CHANGE

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Hydrodynamic behavior of one-dimensional homogeneous exclusion processes with speed change on periodic lattices $\mathbb{Z}/N\mathbb{Z}$, $N = 1, 2, 3, \dots,$ is studied. For every reversible exclusion process with nearest neighbor jumps and local interactions of gradient type it is shown that under diffusion-type scaling in space and time the empirical density fields of the processes converge to a weak solution of a nonlinear diffusion equation as N goes to infinity. Two classes of examples of exclusion processes as stated are given.

1. Introduction and main result. This paper concerns the hydrodynamic behavior of exclusion processes with speed change on periodic one-dimensional lattices of large size. We derive a nonlinear diffusion equation for the macroscopic density field by passing to the hydrodynamic limit for every homogeneous exclusion process that is a reversible gradient system and involves only local interactions.

Let Γ_N be the periodic lattice $\mathbb{Z}/N\mathbb{Z}$ represented by $\{1, 2, \dots, N\}$, where $\mathbb{Z}$ is the set of integers. Let $\mathscr{X}_N = \{0, 1\}^{\Gamma_N}$, the set of configurations $\eta = \{\eta_x, x \in \Gamma_N\}$ with $\eta_x = 0$ or 1, and $\mathscr{D}_N$ be the set of functions on $\mathscr{X}_N$. Let τ_x, $x \in \mathbb{Z}$, be the shift operators acting on $\mathscr{X}_N$ by $(\tau_x \eta)_y = \eta_{y+x}$, $x \in \Gamma_N$, with addition being modulo N. They also act on $\mathscr{D}_N$ by $\tau_x f(\eta) = f(\tau_x \eta)$, $f \in \mathscr{D}_N$. For $x, y \in \Gamma_\eta$ and $\eta \in \mathscr{X}_N$, $\eta^{x,y}$ denotes an element of $\mathscr{X}_N$ obtained from η by exchanging the values of η_x and η_y; thus $(\eta^{x,y})_x = \eta_y$ and $(\eta^{x,y})_y = \eta_x$. The notations τ_x and $\eta^{x,y}$ also indicate corresponding ones for $\mathscr{X} = \{0, 1\}^{\mathbb{Z}}$, the configuration space on the whole lattice.

Let us consider a generator L_N of an exclusion process on Γ_N,

$$(1.1)\qquad L_N f(\eta) = \sum_{x=1}^{N} c_{x,x+1}(\eta)\big(f(\eta^{x,x+1}) - f(\eta)\big), \qquad f \in \mathscr{D}_N.$$

The rate function $c_{x,x+1}(\eta)$ is supposed positive and homogeneous, namely to be written in the form

$$(1.2)\qquad c_{x,x+1}(\eta) = \tau_x c(\eta), \qquad \eta \in \mathscr{X}_N,$$

with a function $c = c(\eta)$ on $\mathscr{X}$ that is *positive* if $\eta_0 \neq \eta_1$. Here $\eta \in \mathscr{X}_N$ is identified with its periodic extension to $\mathscr{X}$. Let $\eta^N(t) = \{\eta_x^N(t), x \in \Gamma_N\}$ denote the Markov chain on $\mathscr{X}_N$ governed by the infinitesimal operator $N^2 L_N$. We are interested in its macroscopic empirical-mass distribution, i.e., the measure

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valued process defined by

$$\xi^N(t, d\theta) = \frac{1}{N} \sum_{x=1}^{N} \eta_x^N(t) \delta_{x/N}(d\theta), \qquad \theta \in \mathbb{T}, \tag{1.3}$$

where $\mathbb{T}$ is the one-dimensional torus identified with $[0,1)$ and δ_θ the δ measure at θ.

We assume that the function c in (1.2) is local, i.e., depending only on $\{\eta_k, -R \le k \le R\}$ for some nonnegative integer R, and satisfies the conditions GS (gradient system) and DB (detailed balance) stated here:

(GS) There exists a local function h on $\mathscr{X}$ such that

$$c(\eta)(\eta_0 - \eta_1) = h(\eta) - \tau_1 h(\eta). \tag{1.4}$$

(DB) There exists a translation-invariant set function Φ on $\mathbb{Z}$ having a finite range R such that for every $\eta \in \mathscr{X}$,

$$c(\eta)\exp\left[-E^{\Phi}_{\{0,1\}}(\eta)\right] = c(\eta^{0,1})\exp\left[-E^{\Phi}_{\{0,1\}}(\eta^{0,1})\right]. \tag{1.5}$$

Here, for a finite subset $\Lambda \subset \mathbb{Z}$, E^{Φ}_{Λ} denotes a function on $\mathscr{X}$, defined by

$$E^{\Phi}_{\Lambda}(\eta) := \sum_{\substack{A \subset \mathbb{Z} \\ A \cap \Lambda \neq \phi}} \Phi(A) \eta^A, \qquad \eta^A = \prod_{x \in A} \eta_x. \tag{1.6}$$

The set function Φ is translation-invariant if $\Phi(A) = \Phi(A+1)$ for all $A \subset \mathbb{Z}$, where $A + 1 = \{x + 1 \colon x \in A\}$, and has a finite range R if $\Phi(A) = 0$ whenever the diameter of $A \ge R + 1$. (These two conditions on Φ necessarily follow from the locality and the homogeneity of c.) The function E^{Φ}_{Λ} is called the Hamiltonian in Λ (for Φ) and its value $E^{\Phi}_{\Lambda}(\eta)$ at η is the energy of η in Λ.

The *detailed balance* condition (DB) implies that the process $\eta^N(t)$ is reversible and the reversible measures are the Gibbs measures on $\mathscr{X}_N$ for the Hamiltonian

$$E^{\Phi}_{N}(\eta) := \sum_{A \subset \Gamma_N} \Phi(A) \eta^A, \qquad \eta \in \mathscr{X}_N, \tag{1.7}$$

provided $N > 2R$. Let $\mathscr{P}(\mathscr{X})$ be the set of probability measures on $\mathscr{X}$ where $\mathscr{X}$ is equipped with the usual product topology. It is known that to each $\rho \in [0,1]$ there uniquely corresponds $\nu_\rho \in \mathscr{P}(\mathscr{X})$, a grand canonical Gibbs measure of density ρ: $\nu_\rho(\eta_0 = 1) = \rho$, and ν_ρ, $\rho \in [0,1]$, constitute the set of extremal canonical Gibbs measures for the Hamiltonians E^{Φ}_{Λ} [8].

If the condition (GS) is satisfied, the exclusion process is called a *gradient system*. Let us write $L_N\{f(\eta)\}$ for $L_N f(\eta)$ when f is explicit. Then, taking $f(\eta) = \eta_x$ with an x fixed, the relation (1.4) implies (in fact is equivalent) that $L_N\{\eta_x\} = \tau_{x+1}h(\eta) + \tau_{x-1}h(\eta) - 2\tau_x h(\eta)$ or, in the form more relevant to our analysis, that for every function J on Γ_N,

$$L_N\left\{\sum_{x=1}^{N}\eta_x J(x)\right\} = \sum_{x=1}^{N}\tau_x h(\eta)\,\Delta J(x), \tag{1.8}$$

where $\Delta J(x) = J(x+1) + J(x-1) - 2J(x)$.

Remember that the measure valued process $\xi^N(t)$ is obtained through the diffusion-type scaling from the exclusion process generated by L_N and notice that after this scaling the relation (1.8) is transformed into a similar one with the Laplacian Δ_N on the reduced lattice Γ_N/N. Then formal arguments based on the assumption that the laws of local configurations are to be Gibbs measures suggest that limits of $\xi^N(t)$, if any, must be regulated by the nonlinear diffusion (hydrodynamic) equation

$$\frac{\partial}{\partial t}\rho(t,\theta) = \frac{\partial^2}{\partial\theta^2}P(\rho(t,\theta)), \qquad \theta\in\mathbb{T}, \tag{1.9}$$

with the function P given by

$$P(\rho) = \langle h\rangle_\rho, \tag{1.10}$$

where $\langle\,\cdot\,\rangle_\rho$, sometimes denoted by $\langle\,\cdot\,\rangle(\rho)$, stands for the integral by ν_ρ, the grand canonical Gibbs measure of a density ρ. This statement will be made precise in Theorem 1.

Throughout this paper the convergence of measures on $\mathbb{T}$ always means the usual weak convergence. We notice that any limit point of $\xi^N(t,\,\cdot\,)$ (with t fixed) necessarily has a density less than or equal to 1, as is clear from the form of $\xi^N(t)$ given in (1.3).

THEOREM 1. *Suppose that the conditions* (GS) *and* (DB) *hold. If* $\xi^N(0)$ *converges in probability to a nonrandom measure* $\rho_0(\theta)\,d\theta$ *as* $N\to\infty$, *then, for each* $t>0$, $\xi^N(t)$ *also converges in probability to* $\rho(t,\theta)\,d\theta$ *where* $\rho(t,\theta)$ *is a unique weak solution of the nonlinear diffusion equation* (1.9) *starting from the initial condition* $\rho(0,\theta)=\rho_0(\theta)$ *and satisfying* $0\le\rho(t,\theta)\le 1$.

Although the conditions (GS) and (DB) combined are rather restrictive, there are still large varieties of examples that satisfy both conditions. Two simple classes will be provided in the last section. In the multidimensional case these two conditions are very difficult to satisfy: there is an example, which we shall briefly discuss in Remark 5.3, but no other examples are known.

The proof of Theorem 1 will be carried out following the strategy developed by Guo, Papanicolaou and Varadhan [10], who treated a real valued spin system for which the Gibbs measures are product measures. For the present model the Gibbs measures are not necessarily product measures, which necessitates appropriate modifications of their proof. Except in this respect, the whole story is simplified due to the compactness of the space of spin values. The essential part of the proof consists of demonstrating two assertions, formulated as Theorems 2 and 3 (in the next section).

For Theorem 3, which is an assertion about local equilibrium, we shall prove that any limit point of the space–time average of spatially shifted law for the whole configuration $\eta^N(t)$ as $N \to \infty$ is a superposition of the grand canonical Gibbs measures ν_ρ. [An extended version of this will be stated in Proposition 4.2 and is essentially contained in relation (4.4).] This form of the local equilibrium is somewhat stronger than representing the local equilibrium states by means of the specifications of the canonical Gibbs measures, a direct analogue to the corresponding assertion in [10] (see the final step of the proof of Theorem 4.2 therein), and accordingly simplifies the rest of the proofs: For example, the uniform law of large numbers for the grand canonical Gibbs measures is easier to establish than that for the local specifications of the canonical ones [see the relation (4.16)], though in fact the required uniformity is valid also for the latter.

Theorem 2 concerns a kind of stochastic continuity of the local density. To prove it we must appropriately choose a rate function according to which particles move between two prescribed sites that are macroscopically near but microscopically far apart from each other. It turns out that Spitzer's rate function [13] is suitable for this, since by this choice a certain monotonicity requirement needed in a passage of the proof is guaranteed in general under the condition (DB).

We have assumed from the outset that our exclusion process admits only the nearest neighbor jumps, but this assumption automatically follows from reversibility combined with the gradient condition, if irreducibility is presupposed.

For physical as well as mathematical ideas on the problem of hydrodynamic limits, readers may refer to a survey paper [2], where extensive literature (up to 1982–1983) is also provided. Recent developments (mainly for the Ginzburg–Landau model) are briefly summarized in [6] (see also [7]). The method of [10] is applied in [11] to the simple exclusion process to obtain a large deviation result where the hydrodynamic limits for some perturbed processes are naturally involved.

Gibbs states and the diffusion coefficient. In the rest of this section we briefly explain how the measure ν_ρ is related to the Hamiltonians E_Λ^Φ and then introduce a Green–Kubo formula to see $P'(\rho) > 0$, which guarantees the uniqueness of the solution to the Cauchy problem for (1.9).

Let us write simply $E_x(\eta)$ for $E_{\{x\}}^\Phi(\eta)$, the energy of η at $x \in \mathbb{Z}$. A Gibbs state of a chemical potential $\lambda \in \mathbb{R}$ is a measure $\nu^{(\lambda)} \in \mathscr{P}(\mathscr{X})$ whose conditional probabilities are specified by

$$(1.11) \qquad \nu^{(\lambda)}(\eta_x = \alpha | \eta_y, y \neq x)(\omega) = Z_{x,\omega,\lambda}^{-1} \exp\{-E_x(\alpha \cdot \omega) + \lambda\alpha\}, \qquad \nu^{(\lambda)}\text{a.a. } \omega,$$

for $\alpha = 0, 1$ and $x \in \mathbb{Z}$, where $Z_{x,\omega,\lambda}$ is a normalizing constant and $\alpha \cdot \omega \in \mathscr{X}$ is a configuration such that

$$(\alpha \cdot \omega)_y = \alpha \quad \text{if } y = x \quad \text{and} \quad = \omega_y \quad \text{if } y \neq x.$$

The relation (1.11) uniquely determines $\nu^{(\lambda)}$ for every $\lambda \in \mathbb{R}$ [9]. We set $\nu^{(-\infty)} = \delta_{\mathbb{O}}$ ($\nu^{(+\infty)} = \delta_{\mathbb{1}}$), the δ-measure concentrated on the configuration $\mathbb{O} \in \mathscr{X}$ (resp., $\mathbb{1} \in \mathscr{X}$), where $\mathbb{O}_x = 0$ ($\mathbb{1}_x = 1$) for every $x \in \mathbb{Z}$. Then the mean density $\rho = \rho(\lambda) = \nu^{(\lambda)}(\eta_0 = 1)$, $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$, is a continuous and strictly increasing mapping of $\mathbb{R} \cup \{-\infty, +\infty\}$ onto $[0, 1]$ [8]. The inverse function is denoted by $\lambda = \lambda(\rho)$. The Gibbs measure ν_ρ [appearing in (1.10)] is then related to $\nu^{(\lambda)}$ by $\nu^{(\lambda)} = \nu_{\rho(\lambda)}$.

The diffusion coefficient $D(\rho) := 2P'(\rho)$ of the equation (1.9) can be expressed in the form

$$D(\rho) = \left\langle c(\eta)(\eta_0 - \eta_1)^2 \right\rangle(\rho)/\chi(\rho), \qquad 0 < \rho < 1, \tag{1.12}$$

where $\chi(\rho) := \sum_{x \in \mathbb{Z}} (\langle \eta_0 \eta_x \rangle_\rho - \rho^2)$, called the compressibility. This relation is a reduced version of the Green–Kubo formula and generally holds for reversible gradient systems. It in fact follows from the triviality

$$0 = \frac{1}{2} \sum_x x^2 \langle L\{\eta_0 \eta_x\} \rangle_\rho = \sum_x x^2 \langle \eta_0 \cdot L\{\eta_x\} \rangle_\rho - \left\langle c(\eta)(\eta_0 - \eta_1)^2 \right\rangle_\rho,$$

where $L = \lim_{N \to \infty} L_N$, together with the relation

$$\frac{d}{d\lambda} P(\rho(\lambda)) = \chi(\rho) P'(\rho) = \sum_x (\langle h \eta_x \rangle_\rho - \rho \langle h \rangle_\rho), \tag{1.13}$$

which is readily obtained from (1.11) at least at a formal level [for justification see (4.12) of this paper], since by (1.8) the rightmost member in (1.13) equals $\frac{1}{2} \sum x^2 \langle (\eta_0 - \rho) \cdot L\{\eta_x\} \rangle_\rho$. We especially see that $P(\rho)$ is increasing in ρ. The function $D(\rho)$ coincides with the bulk diffusion coefficient derived by solving the equilibrium fluctuation problem [3, 14].

2. Local equilibrium and derivation of the hydrodynamic equation. In this section we shall outline the proof of Theorem 1. Roughly speaking, it almost consists in demonstrating the claim that the equilibrium states are locally builtup in average, an average in space and time, for our system, which will be formulated, according to [10], in the form of two theorems.

Let $\mathscr{M}(\mathbb{T})$ be the compact metric space, under the weak convergence, of (nonnegative) Borel measures on the torus $\mathbb{T}$ of total mass less than or equal to 1. The integral of $J \in C(\mathbb{T})$ by a finite measure ξ is denoted by $\langle J, \xi \rangle$. When $\rho(\theta)$ is a bounded measurable function, the integral $\langle J, \xi \rangle$, $\xi = \rho(\theta)\, d\theta$, is simply written as $\langle J, \rho \rangle$. The processes $\eta^N(t)$ are supposed to be defined on some probability space $(\mathbf{P}, \mathscr{F}, \Omega)$ for all N. The expectation by $\mathbf{P}$ is denoted by $\mathbf{E}$.

As usual we decompose $\langle J, \xi^N(t) \rangle$, $J \in C^\infty(\mathbb{T})$, into

$$\left\langle J, \xi^N(t) \right\rangle = \left\langle J, \xi^N(0) \right\rangle + M_N(t) + \int_0^t b_N(s)\, ds. \tag{2.1}$$

Here $M_N(t)$ is a martingale, for which, as easily seen, $\mathbf{E}|M_N(t)|^2 \to 0$ as

$N \to \infty$. The relation (1.8) shows that the drift term can be written as

$$b_N(s) = \frac{1}{N}\sum_{x=1}^{N} J''\left(\frac{x}{N}\right)\tau_x h(\eta^N(s)) + O\left(\frac{1}{N}\right)$$

where $|O(1/N)| \le \|J'''\|_\infty \|h\|_\infty / N$.

Let Q^N be the law of $\{\xi^N(t);\ t \ge 0\}$ on $D([0,1], \mathscr{M}(\mathbb{T}))$, the space of right continuous functions $\xi\colon [0,1] \to \mathscr{M}(\mathbb{T})$ with left limits that is equipped with the Skorohod topology.

The family $\{Q^N\}$ is tight, as readily deduced from the relation (2.1) and the bounds $|b_N(s)| \le C$ and

$$\mathbf{E}\left[\{M_N(t) - M_N(s)\}^2 | \sigma\{\eta^N(s)\}\right] \le C(t-s)/N, \quad \text{for } 0 \le s \le t \le 1.$$

Under any limit point of $\{Q^N\}$ almost every sample path ξ is continuous and satisfies that for every t the measure $\xi(t, d\theta)$ has a density $\rho(t,\theta)$ such that $\rho(t,\theta) \le 1$. The sample continuity follows from the fact that the height of the jump of $\langle J, \xi^N(t)\rangle$ is at most $2\|J\|_\infty/N$ and hence vanishes in the limit.

Observe that

$$b_N(s) = N^{-1}\sum_{x=1}^{N} J''\left(\frac{x}{N}\right)\left(\frac{1}{2\varepsilon N + 1}\right)\sum_{|y-x| \le \varepsilon N} \tau_y h(\eta^N(s)) + O\left(\frac{1}{N}\right) + O(\varepsilon).$$

Then if we can prove

$$\lim_{\varepsilon \downarrow 0} \limsup_{N\to\infty} \mathbf{E}\left[\int_0^t \frac{1}{N}\sum_{x=1}^{N}\left|\frac{1}{2\varepsilon N+1}\sum_{|y-x|\le\varepsilon N}\tau_y h(\eta^N(s)) - \langle h\rangle\left(\frac{1}{2\varepsilon N+1}\sum_{|y-x|\le\varepsilon N}\eta_y^N(s)\right)\right| ds\right] = 0, \tag{2.2}$$

it is easy to see that

$$\lim_{\varepsilon\downarrow 0}\limsup_{N\to\infty}\mathbf{E}\left[\int_0^t\left|b_N(s) - \frac{1}{N}\sum_{x=1}^{N} J''\left(\frac{x}{N}\right)\langle h\rangle\left(\rho^{N,\varepsilon}\left(s, \frac{x}{N}\right)\right)\right| ds\right] = 0,$$

where $\rho^{N,\varepsilon}(s,\theta) = (2\varepsilon)^{-1}\xi^N(s, [\theta-\varepsilon, \theta+\varepsilon])$, and then that, by (2.1) and the continuity of $P = \langle h\rangle(\cdot)$, for every limit point Q of $\{Q^N\}$,

$$Q\left(\xi(\cdot, d\theta) = \rho(\cdot,\theta)\,d\theta: \quad \langle J, \rho(t)\rangle = \langle J, \rho(0)\rangle + \int_0^t \langle J'', P(\rho(s,\cdot))\rangle\,ds\right) = 1. \tag{2.3}$$

Finally, since, by the assumption on $\{\xi^N(0)\}$, $Q(\rho(0,\cdot) = \rho_0(\cdot)) = 1$, the uniqueness of weak solutions to the limit equation (Lemma 2.1) completes the proof of Theorem 1.

It remains to show (2.2). However, by noticing that an average over a long block of length $2N\varepsilon + 1$ is almost the same as an average of averages over short blocks of length $2l+1$ as argued in [10] (see the last several lines of Section 4) and by applying the continuity of $\langle h\rangle(\cdot)$, we observe with little difficulty that the verification of (2.2) (with $t=1$) is reduced to demonstrating the next two theorems, which we shall prove through the succeeding two sections.

Let μ_t^N be the distribution of $\eta^N(t)$ on $\mathscr{X}_N$ and set

$$(2.4)\qquad \bar{\mu}^N = \int_0^1 dt\,\mu_t^N \quad\text{and}\quad \tilde{\mu}^N = \frac{1}{N}\sum_{x=1}^{N} \bar{\mu}^N\circ\tau_x^{-1}.$$

THEOREM 2.

$$\lim_{\varepsilon\downarrow 0}\limsup_{l\to\infty}\lim_{L\to\infty}\limsup_{N\to\infty}\max_{L\le|z|\le\varepsilon N}\int\left|\frac{1}{2l+1}\sum_{|y|\le l}\eta_y-\frac{1}{2l+1}\sum_{|y-z|\le l}\eta_y\right|d\tilde{\mu}^N$$
$$=0.$$

THEOREM 3. *For every $f\in\mathscr{D}$,*

$$\lim_{l\to\infty}\limsup_{N\to\infty}\int\left|\frac{1}{2l+1}\sum_{|y|\le l}\tau_y f(\eta)-\langle f\rangle\left(\frac{1}{2l+1}\sum_{|y|\le l}\eta_y\right)\right|d\tilde{\mu}^N=0.$$

REMARK. In Theorems 2 and 3 we can replace $\tilde{\mu}^N$ by $\bar{\mu}^N\circ\tau_x^{-1}$ for each $x\in\mathbb{Z}$ and accordingly dispense with the operation of globally averaging in space in the relation (2.2), i.e., we have that for every $x\in\mathbb{Z}^d$ the expectation

$$\mathbf{E}\left[\int_0^t\left|\frac{1}{2\varepsilon N+1}\sum_{|y-x|\le\varepsilon N}\tau_y h(\eta^N(s))-\langle h\rangle\left(\frac{1}{2\varepsilon N+1}\sum_{|y-x|\le\varepsilon N}\eta_y^N(s)\right)\right|ds\right]$$

vanishes as $N\to\infty$ and $\varepsilon\downarrow 0$ in this order. To see this we do not have to change the proof, though some estimates should be weakened [e.g., in (4.5) ε^2 must be replaced by ε]. It is however found significant to take $\tilde{\mu}^N$ rather than $\bar{\mu}^N$ in these theorems, if one considers the multidimensional case. In fact for d-dimensional exclusion processes $I_N(\bar{\mu}^N)$ [and hence $I_N(\tilde{\mu})$] grows as $N^{d-2}=N^{d-1}\cdot N^{-1}$ and the factor N^{d-1} therein may cause difficulty in estimating the local I functions for $\bar{\mu}^N$ (cf. Lemma 4.1), while N^d (rather than N^{d-1}) is canceled for $\tilde{\mu}^N$ from the effect of the averaging in space.

LEMMA 2.1. *Let $\mathscr{H}$ be the class of $\xi(\cdot,d\theta)\in D([0,\infty),\mathscr{M}(\mathbb{T}))$ such that $\xi(t,d\theta)$ has a density $\rho(t,\theta)$ and $0\le\rho(t,\theta)\le 1$ a.e. for all $t\ge 0$. Then, given a $\rho_0(\theta)$ such that $0\le\rho_0(\theta)\le 1$ a.e., a weak solution of* (1.9) *starting from ρ_0 is unique in $\mathscr{H}$. Here by weak solution we mean that for all $J\in C^\infty(\mathbb{T})$,*

$$(2.5)\qquad \langle J,\rho(t)\rangle=\langle J,\rho_0\rangle+\int_0^t\langle J'',P(\rho(s,\cdot))\rangle\,ds,\qquad t>0.$$

PROOF. This proof is a modification of that to the corresponding uniqueness theorem in [10]. Let $\{J_\varepsilon\}_{0<\varepsilon<1/2}$ be mollifiers in $C^\infty(\mathbb{T})$: $J_\varepsilon(\theta) \geq 0$, $\langle J_\varepsilon, 1\rangle = 1$, $\operatorname{supp} J_\varepsilon \subset [-\varepsilon, \varepsilon] \subset \mathbb{T}$. Assuming that there are two solutions $\rho_i(t,\theta)$, $i = 1, 2$, in $\mathscr{H}$ with the same initial datum ρ_0, set

$$\rho_i^\varepsilon(t,\theta) = (\rho_i(t,\cdot) * J_\varepsilon)(\theta), \qquad \varepsilon > 0,$$

and $\rho_i^0(t,\theta) = \rho_i(t,\theta)$. Then further set

$$w_t^\varepsilon(\theta) = \int_0^\theta \{\rho_1^\varepsilon(t,\theta') - \rho_2^\varepsilon(t,\theta')\}\, d\theta' - \int_{\mathbb{T}} d\theta \int_0^\theta \{\rho_1^\varepsilon(t,\theta') - \rho_2^\varepsilon(t,\theta')\}\, d\theta'.$$

Since by (2.5) $\langle 1, \rho_i^\varepsilon(t)\rangle = \langle 1, \rho_0\rangle$ for all $t \geq 0$, w_t^ε is continuous on $\mathbb{T}$, i.e., $w_t^\varepsilon(0) = w_t^\varepsilon(1)$. Clearly $\langle w_t^\varepsilon, 1\rangle = 0$. We have also $\partial\rho_i^\varepsilon/\partial t = (P(\rho_i) * J_\varepsilon)''$. With the help of these relations apply the integration by parts to the integral $\langle w_t^\varepsilon, \partial w_t^\varepsilon/\partial t\rangle$ and after simple manipulations you see

$$\|w_t^\varepsilon\|_{L^2(\mathbb{T})}^2 = -\int_0^t ds \langle \rho_1^\varepsilon(s,\cdot) - \rho_2^\varepsilon(s,\cdot), \{P(\rho_1(s,\cdot)) - P(\rho_2(s,\cdot))\} * J_\varepsilon\rangle$$

for $0 < \varepsilon < \frac{1}{2}$. Letting $\varepsilon \downarrow 0$ in this relation shows $\|w_t^0\|_{L^2(\mathbb{T})}^2 \leq 0$ due to the fact that $P(\rho)$ is nondecreasing. Therefore $w_t^0(\theta) \equiv 0$, proving $\rho_1(t,\theta) = \rho_2(t,\theta)$ a.e. $(d\theta)$. □

3. Relative entropy and *I* function. According to [10], we introduce two kinds of functionals of probability measures on $\mathscr{X}_N$. Put $E_N = E_N^\Phi$ [the Hamiltonian defined in (1.7)] and let $\nu^N \in \mathscr{P}(\mathscr{X}_N)$ be the Gibbs measure corresponding to E_N, that is

$$\nu^N(\eta) = Z_N^{-1} \exp(-E_N(\eta)), \qquad \eta \in \mathscr{X}_N,$$

where Z_N is the normalizing constant. For the arguments made in this and the next section we do not make use of the assumption (GS), so all the results obtained are valid for an arbitrary rate function c that satisfies the condition (DB). In the following, as previously, we shall suppress Φ from the notations.

The relative entropy of $\mu \in \mathscr{P}(\mathscr{X}_N)$ with respect to ν^N is defined by

$$H_N(\mu) = \int_{\mathscr{X}_N} \varphi \log \varphi \, d\nu^N,$$

where $\varphi = d\mu/d\nu^N$, i.e., $\varphi(\eta) = \mu(\eta)/\nu^N(\eta)$. Observing $H_N(\mu) \leq \Sigma_\eta \mu(\eta) E_N(\eta) + \log Z_N$, we have the a priori bound

$$0 \leq H_N(\mu) \leq C_1 N, \tag{3.1}$$

with $C_1 = \log 2 + 2\sup_\eta |E_0(\eta)| < \infty$. Let I_N denote the I function associated with L_N, which we define as the Dirichlet form of $\sqrt{\varphi}$, $\varphi = d\mu/d\nu^N$,

$$I_N(\mu) = -\int_{\mathscr{X}_N} \sqrt{\varphi}\, L_N \sqrt{\varphi}\, d\nu^N.$$

Here it is recalled that the condition (DB) implies the self-adjointness of L_N in

$\mathscr{L}_2(\nu^N)$, the associated form given by

$$\int_{\mathscr{X}_N} f \cdot L_N f d\nu^N = -\tfrac{1}{2}\sum_{\eta}\sum_{x=1}^{N}\left[f(\eta^{x,x+1}) - f(\eta)\right]^2 c_{x,x+1}(\eta)\nu^N(\eta),$$

and that there is the following variational representation of $I_N(\mu)$ [4]:

$$(3.2)\quad I_N(\mu) = \sup\left\{-\int_{\mathscr{X}_N}\frac{L_N u}{u}\,d\mu\colon u \text{ is a positive function on } \mathscr{X}_N\right\}.$$

It is sometimes convenient to express $I_N(\mu)$ in the form

$$(3.3)\qquad I_N(\mu) = \sum_{x=1}^{N} I_{x,x+1}(\mu),$$

$$(3.4)\quad I_{x,y}(\mu) = \tfrac{1}{2}\sum_{\eta\in\mathscr{X}_N} K(\varphi(\eta^{x,y}),\varphi(\eta))c_{x,y}(\eta)\nu^N(\eta), \qquad x, y \in \Gamma_N,$$

where $K(a,b) = (\sqrt{a} - \sqrt{b})^2$ and $\varphi = d\mu/d\nu^N$.

PROPOSITION 3.1. *Let μ_t^N be the law of $\eta^N(t)$. Then $H_N(\mu_t^N)$ is differentiable in $t > 0$ and*

$$(3.5)\qquad \frac{d}{dt}H_N(\mu_t^N) \le -4N^2 \cdot I_N(\mu_t^N).$$

PROOF. Put $T(a,b) = (a-b)(\log a - \log b)$ and $\varphi_t = d\mu_t^N/d\nu^N$. Then

$$\frac{d}{dt}H_N(\mu_t^N) = -N^2\int L_N\varphi_t \cdot \log\varphi_t\,d\nu^N$$
$$= -\frac{N^2}{2}\sum_{x=1}^{N}\sum_{\eta\in\mathscr{X}_N} T(\varphi_t(\eta^{x,x+1}),\varphi_t(\eta))c_{x,x+1}(\eta)\nu^N(\eta)$$

and hence, using the inequality $T(a,b) \ge 4 \cdot K(a,b)$ for $a, b > 0$, we obtain (3.5). □

COROLLARY 3.1. *Set $\bar{\mu}^N = \int_0^1 dt\,\mu_t^N \in \mathscr{P}(\mathscr{X}_N)$. Then $I_N(\bar{\mu}^N) \le C_1/4N$.*

PROOF. Integrate (3.5) on $[0,1]$ and use (3.1) to first get a bound for the integral of $I_N(\mu_t^N)$ and then the desired one by applying the convexity of I_N, which follows from (3.3). □

4. Proof of Theorems 2 and 3. We shall identify a measure on $\mathscr{X}_N$ with its image measure on $\mathscr{X}$ under the mapping which extends $\eta \in \mathscr{X}_N$ periodically to an element of $\mathscr{X}$. Define a mapping $\hat{\tau}_z\colon \mathscr{X} \to \mathscr{X}^2 := \{(\eta^1,\eta^2)\colon \eta^1, \eta^2 \in \mathscr{X}\}$ by $\hat{\tau}_z\eta(\eta^1, \tau_z\eta^2)$. Then the conclusion of Theorem 2 can be rewritten as

$$\lim_{\varepsilon \downarrow 0} \limsup_{l \to \infty} \lim_{L \to \infty} \limsup_{N \to \infty} \max_{L \le |z| \le \varepsilon N} \int \left| M^1_{\Lambda_l} - M^2_{\Lambda_l} \right| d\left(\tilde{\mu}^N \circ \hat{\tau}_z^{-1} \right) = 0, \tag{4.1}$$

where $M^i_\Lambda = |\Lambda|^{-1} \Sigma_{x \in \Lambda} \eta^i_x$, $i = 1, 2$, for $\Lambda \subset \mathbb{Z}$ and $\Lambda_l = \{x \in \mathbb{Z}: |x| \le l\}$. We first focus our attention on the limit points of $\{\tilde{\mu}^N \circ \hat{\tau}_z^{-1}: L \le |z| \le \varepsilon N\}$ in $\mathscr{P}(\mathscr{X}^2)$ as $L, N \to \infty$. Note that $\mathscr{P}(\mathscr{X}^2)$ is compact. In the proof of (4.1) that follows we shall use as the property of $\tilde{\mu}^N \in \mathscr{P}(\mathscr{X})$ only the translation invariance of $\tilde{\mu}^N$ and the bound

$$I_N(\tilde{\mu}^N) \le C_1 / 4N, \tag{4.2}$$

the latter deduced from Corollary 3.1 by the convexity and the translation invariance of I_N.

Let $\mathscr{F}_\Lambda = \sigma\{\eta_y: y \in \Lambda\}$. We may assume that $c(\eta)$ is $\mathscr{F}_{\Lambda_R}$ measurable with the same R as appears in (DB). Let $L_{x, x+1}$, $x \in \mathbb{Z}$, be operators acting on $\mathscr{D}$ (and also on $\mathscr{D}_N$) defined by

$$L_{x,y} f(\eta) = c_{x,y}(\eta) \left(f(\eta^{x,y}) - f(\eta) \right). \tag{4.3}$$

Fix an arbitrary Gibbs measure $\nu \in \mathscr{P}(\mathscr{X})$ (that has no component of ν_0 or ν_1) for Hamiltonians E_Λ (e.g., take $\nu = \nu^{(0)}$). For $\mu \in \mathscr{P}(\mathscr{X}^2)$ and a bounded set Λ of $\mathbb{Z}$, define

$$I^{(1,2)}_\Lambda(\mu) = -\int_{\mathscr{X}^2} \sqrt{\varphi}\, L^{(1,2)}_\Lambda \sqrt{\varphi}\, d(\nu \times \nu), \qquad \varphi = d\mu|_{\mathscr{F}_{\bar{\Lambda}} \times \mathscr{F}_{\bar{\Lambda}}} / d(\nu \times \nu)|_{\mathscr{F}_{\bar{\Lambda}} \times \mathscr{F}_{\bar{\Lambda}}},$$

where $\bar{\Lambda} = \{x \in \mathbb{Z}: |x - y| \le R$ for some $y \in \Lambda\}$ and $L^{(1,2)}_\Lambda$ is an operator defined by

$$L^{(1,2)}_\Lambda u(\eta^1, \eta^2) = \left(L_\Lambda u(\cdot, \eta^2) \right)(\eta^1) + \left(L_\Lambda u(\eta^1, \cdot) \right)(\eta^2).$$

Here $L_\Lambda = \Sigma_{x: \{x, x+1\} \subset \Lambda} L_{x, x+1}$. For the proof of Theorem 2 we introduce an auxiliary Markov process on $\mathscr{X}^2$ in which only the spins η^1_0 and η^2_0 are exchanged. Its generator is given by

$$L^{(0)} u(\eta^1, \eta^2) = \{c^\circ(\eta^1) + c^\circ(\eta^2)\} \cdot \left\{ u\left((\eta^1, \eta^2)^0 \right) - u(\eta^1, \eta^2) \right\}.$$

Here

$$c^\circ(\eta) = \eta_0 \exp E_0(\eta)$$

and $(\eta^1, \eta^2)^0 \in \mathscr{X}^2$ denotes the configuration (η^1, η^2) with the values of η^1_0 and η^2_0 exchanged. We also define for a bounded $\Lambda \supset \Lambda_R$,

$$I^{(0)}_\Lambda(\mu) = -\int_{\mathscr{X}^2} \sqrt{\varphi}\, L^{(0)} \sqrt{\varphi}\, d(\nu \times \nu), \qquad \varphi = d\mu|_{\mathscr{F}_\Lambda \times \mathscr{F}_\Lambda} / d(\nu \times \nu)|_{\mathscr{F}_\Lambda \times \mathscr{F}_\Lambda}.$$

Clearly $I^{(1,2)}_\Lambda$ and $I^{(0)}_\Lambda$ are continuous functions of $\mu \in \mathscr{P}(\mathscr{X}^2)$.

LEMMA 4.1. *Fix* $0 < \varepsilon < \frac{1}{4}$. *Let* μ *be an arbitrary limit point in* $\mathscr{P}(\mathscr{X}^2)$ *of* $\tilde{\mu}^N \circ \hat{\tau}_z^{-1}$, $L \le |z| \le \varepsilon N$, *as* $L, N \to \infty$. *Then*

$$(4.4) \qquad I_\Lambda^{(1,2)}(\mu) = 0 \quad \textit{for every bounded interval } \Lambda$$

and

$$(4.5) \qquad I_\Lambda^{(0)}(\mu) \le C_2 \varepsilon^2 \quad \textit{for every bounded interval } \Lambda \supset \Lambda_R$$

with a constant C_2 *depending only on* Φ.

PROOF. For a given Λ, take L, N large enough that $\overline{\Lambda}$ and $\overline{\Lambda} + z$ can be regarded as disjoint subsets of Γ_N for all z satisfying $L \le |z| \le \varepsilon N$. Using the variational formula for $I_\Lambda^{(1,2)}(\mu)$, which is similar to the representation of $I_N(\mu)$ in (3.2), we obtain

$$\begin{aligned} I_\Lambda^{(1,2)}(\tilde{\mu}^N \circ \hat{\tau}_z^{-1}) &= \sup\left\{ -\int_{\mathscr{X}_N} \frac{L_{\Lambda \cup (\Lambda+z)} u}{u}\, d\tilde{\mu}^N \colon u > 0 \text{ is } \mathscr{F}_{\overline{\Lambda} \cup (\overline{\Lambda}+z)} \text{ measurable} \right\} \\ &\le \sup\left\{ -\int_{\mathscr{X}_N} \frac{L_{\Lambda \cup (\Lambda+z)} u}{u}\, d\tilde{\mu}^N \colon u \text{ is a positive function on } \mathscr{X}_N \right\} \\ &\le \sum_{x\colon \{x, x+1\} \subset \Lambda \cup (\Lambda+z)} I_{x,x+1}(\tilde{\mu}^N) \\ &\le 2|\Lambda| I_N(\tilde{\mu}^N)/N \le C_1|\Lambda|/2N^2. \end{aligned}$$

Here we applied the fact that $I_{x,x+1}(\mu)$ has the same variational representation as $I_N(\mu)$ given in (3.2) but with L_N replaced by $L_{x,x+1}$ (the supremum is attained by $d\mu/d\nu^N$ for all x). By taking the limit $N \to \infty$ in this formula we get (4.4).

For the proof of (4.5), we set

$$c_{x,y}^*(\eta) = \tau_x c^\circ(\eta) + \tau_y c^\circ(\eta).$$

Let $I_{x,y}^*$ be defined like $I_{x,y}$ by (3.4) but with $c_{x,y}^*$ in place of $c_{x,y}$. Thus

$$I_{0,z}^*(\mu) = \tfrac{1}{2}\int_{\mathscr{X}_N} c_{0,z}^*(\eta)\left\{\sqrt{\varphi}(\eta^{0,z}) - \sqrt{\varphi}(\eta)\right\}^2 d\nu^N, \qquad \varphi = d\mu/d\nu^N.$$

We claim that there exists a constant C_3 such that

$$(4.6) \qquad I_{0,z}^*(\mu) \le C_3 z \sum_{y=0}^{z-1} I_{y,y+1}(\mu)$$

for $1 \le z \le N-1$ and $\mu \in \mathscr{P}(\mathscr{X}_N)$. Note that here (and only here) $c_{x,y}^*(\eta)$ for $|x-y| > 1$ enters into our computation; it satisfies the detailed balance condition also for such x and y, namely, the function $c_{x,y}^*(\eta)\exp(-E_{\{x,y\}}(\eta))$ is invariant under the exchange of values of η_x and η_y, which guarantees the variational formula for $I_\Lambda^{(0)}$ and $I_{0,z}^*$. To see (4.6), observe that

$$\sqrt{\varphi}\,(\eta^{0,z}) - \sqrt{\varphi}\,(\eta) = \sum_{y=0}^{z-1}\Big\{\sqrt{\varphi}\Big((\bar{\sigma}_y\eta)^{y,y+1}\Big) - \sqrt{\varphi}\,(\bar{\sigma}_y\eta)\Big\}$$
$$+ \sum_{y=1}^{z-1}\Big\{\sqrt{\varphi}\Big((\bar{\bar{\sigma}}_y\eta)^{z-y-1,z-y}\Big) - \sqrt{\varphi}\,(\bar{\bar{\sigma}}_y\eta)\Big\},$$

where $\bar{\sigma}_y$ and $\bar{\bar{\sigma}}_y$ are one-to-one mappings on $\mathscr{X}$ given inductively by

$$\bar{\sigma}_y\eta = (\bar{\sigma}_{y-1}\eta)^{y-1,y} \qquad (1 \le y \le z), \quad \text{with } \bar{\sigma}_0\eta = \eta$$

and

$$\bar{\bar{\sigma}}_y\eta = \big(\bar{\bar{\sigma}}_{y-1}\eta\big)^{z-y,z-y+1} \qquad (2 \le y \le z-1), \quad \text{with } \bar{\bar{\sigma}}_1\eta = \bar{\sigma}_z\eta,$$

respectively, and moreover that $(\bar{\bar{\sigma}}_{z-1}\eta)^{0,1} = \eta^{0,z}$. Then applying Schwarz's inequality and the change of variables we see

$$I_{0,z}^*(\mu) \le \frac{2z-1}{2}\Bigg[\sum_{y=0}^{z-1}\int c_{0,z}^*\big(\bar{\sigma}_y^{-1}\eta\big)\big\{\sqrt{\varphi}\,(\eta^{y,y+1}) - \sqrt{\varphi}\,(\eta)\big\}^2 d\big(\nu^N\circ\bar{\sigma}_y^{-1}\big)$$
$$(4.7) \qquad + \sum_{y=1}^{z-1}\int c_{0,z}^*\big(\bar{\bar{\sigma}}_y^{-1}\eta\big)\big\{\sqrt{\varphi}\,(\eta^{z-y-1,z-y}) - \sqrt{\varphi}\,(\eta)\big\}^2 d\big(\nu^N\circ\bar{\bar{\sigma}}_y^{-1}\big)\Bigg].$$

Now we set (and see)

$$C_4 := \sup\{c_{0,z}^*(\eta) : \eta \in \mathscr{X}, z \in \mathbb{Z}\} < \infty,$$
$$C_5 := \inf\{c_{0,1}(\eta) : \eta \in \mathscr{X}, \eta_0 \ne \eta_1\} > 0$$

and

$$C_6 := \sup_{N,\eta,y}\left\{\frac{\big(\nu^N\circ\bar{\sigma}_y^{-1}\big)(\eta)}{\nu^N(\eta)}, \frac{\big(\nu^N\circ\bar{\bar{\sigma}}_y^{-1}\big)(\eta)}{\nu^N(\eta)}\right\} < \infty.$$

Using these bounds and (4.7), we obtain

$$I_{0,z}^*(\mu) \le \frac{2z-1}{2}\,\frac{2C_4\cdot C_6}{C_5}\sum_{y=0}^{z-1}\int c_{y,y+1}(\eta)\big\{\sqrt{\varphi}\,(\eta^{y,y+1}) - \sqrt{\varphi}\,(\eta)\big\}^2 d\nu^N$$
$$= 2(2z-1)C_4C_6C_5^{-1}\sum_{y=0}^{z-1} I_{y,y+1}(\mu)$$

and hence (4.6).

We return to the proof of (4.5). Fix $\Lambda \supset \Lambda_R$. Let $L_{x,y}^*$ be defined analogously to $L_{x,y}$ with c^* in place of c in (4.3). Then for N large enough and z

satisfying $L \le z \le \varepsilon N$,

$$\begin{aligned} I_\Lambda^{(0)}(\tilde{\mu}^N * \hat{\tau}_z^{-1}) &= \sup\left\{ -\int_{\mathscr{X}} \frac{L_{0,z}^* u}{u}\, d\tilde{\mu}^N \colon u > 0 \text{ is } \mathscr{F}_{\Lambda \cup (\Lambda + z)} \text{ measurable} \right\} \\ &\le \sup\left\{ -\int_{\mathscr{X}_N} \frac{L_{0,z}^* u}{u}\, d\tilde{\mu}^N \colon u \text{ is a positive function on } \mathscr{X}_N \right\} \\ &= I_{0,z}^*(\tilde{\mu}^N) \le C_3 z \sum_{y=0}^{z-1} I_{y,y+1}(\tilde{\mu}^N) \\ &\le C_3 z \frac{z}{N} I_N(\tilde{\mu}^N) \le \frac{C_3}{4N^2} C_1 z^2, \end{aligned}$$

where we have used (4.6) and (4.2). The same estimate is obtained for z satisfying $L \le -z \le \varepsilon N$. These prove the desired estimate (4.5). □

We can now replace (4.1) by

$$\lim_{\varepsilon \downarrow 0} \limsup_{l \to \infty} \sup_{\mu \in \mathscr{A}_\varepsilon} \int_{\mathscr{X}^2} \left| M_{\Lambda_l}^1 - M_{\Lambda_l}^2 \right| d\mu = 0, \tag{4.8}$$

where $\mathscr{A}_\varepsilon$ is the set of all $\mu \in \mathscr{P}(\mathscr{X}^2)$ that satisfy (4.4) and (4.5) and are translation-invariant: $\mu = \mu \circ (\tau_1, \tau_1)^{-1}$. For $\mu \in \mathscr{A}_\varepsilon$ we denote by $\hat{\mu}$ the distribution, under μ, of $(M_1, M_2) \equiv (\lim_{l\to\infty} M_{\Lambda_l}^1, \lim_{l\to\infty} M_{\Lambda_l}^2)$, which exists μ-a.s. owing to the individual ergodic theorem.

PROPOSITION 4.2. *If $\mu \in \mathscr{A}_\varepsilon$, then*

$$\mu = \int_{[0,1]^2} \nu_{\rho_1} \times \nu_{\rho_2} \hat{\mu}(d\rho_1\, d\rho_2). \tag{4.9}$$

PROOF. Let Λ be a bounded interval of $\mathbb{Z}$ and $\gamma_{\Lambda,n}(\cdot|\omega)$ the canonical Gibbs measure for the Hamiltonian $E_\Lambda = E_\Lambda^\Phi$ [defined by (1.6)] with particle number n and boundary condition ω, i.e.,

$$\gamma_{\Lambda,n}(\eta|\omega) = 1_{\{N_\Lambda(\eta)=n\}} \cdot Z_{\Lambda,n,\omega}^{-1} \exp\{-E_\Lambda(\eta \cdot \omega)\}, \qquad \eta \in \{0,1\}^\Lambda,$$

where $N_\Lambda(\eta) = \sum_{x\in\Lambda} \eta_x$ for $\eta \in \mathscr{X}$, $Z_{\Lambda,n,\omega}$ is the normalizing constant and $\eta \cdot \omega \in \mathscr{X}$ is a configuration that coincides with η on Λ and with ω on Λ^c.

Let $\mu_1(\eta) = \mu_1(\eta|\eta^2, \omega, n)$ be the conditional law of $\eta := \{\eta_x^1, x \in \Lambda\}$ under $\mu|_{\mathscr{F}_{\bar\Lambda} \times \mathscr{F}_{\bar\Lambda}}(d\eta^1 d\eta^2)$ given η^2, $\omega := \{\eta_x^1, x \notin \Lambda\}$ and $N_\Lambda(\eta^1) = n$; $\mu_2(\eta)$ is similarly defined. Then an application of (DB) together with the relation $K(a,b)c = K(ac, bc)$ shows that $I_\Lambda^{(1,2)}(\mu)$ can be expressed as a sum (for $i = 1,2$) of integrals (by $\mu|_{\mathscr{F}_{\bar\Lambda} \times \mathscr{F}_{\bar\Lambda}}$ over the conditioning) of

$$\sum_{\{x,x+1\}\subset\Lambda} \sum_{\eta\in\{0,1\}^\Lambda} K\big(c_{x,x+1}(\eta^{x,x+1} \cdot \omega)\mu_i(\eta^{x,x+1}), c_{x,x+1}(\eta \cdot \omega)\mu_i(\eta)\big).$$

Since $\gamma_{\Lambda,n}(\cdot|\omega)$ is a unique stationary measure for the exclusion process on Λ

generated by L_Λ with the boundary condition ω and the particle number n, the relation $I_\Lambda^{(1,2)}(\mu) = 0$ therefore implies that $\mu_1(\eta|\eta^2, \omega, n) = \mu_2(\eta|\eta^1, \omega, n) = \gamma_{\Lambda, n}(\eta|\omega)$ a.s. or, equivalently,

$$(4.10)\quad \begin{aligned}&(\mu|_{\mathscr{F}_{\bar\Lambda}\times\mathscr{F}_{\bar\Lambda}})\big([\eta^1,\eta^2]\big|(\mathscr{F}_{\bar\Lambda\setminus\Lambda}\vee\sigma\{N_\Lambda\})\times(\mathscr{F}_{\bar\Lambda\setminus\Lambda}\vee\sigma\{N_\Lambda\})\big)(\omega^1,\omega^2)\\ &\quad=\gamma_{\Lambda, N_\Lambda(\omega^1)}(\eta^1|\omega^1)\cdot\gamma_{\Lambda, N_\Lambda(\omega^2)}(\eta^2|\omega^2)\end{aligned}$$

for $\mu|_{\mathscr{F}_{\bar\Lambda}\times\mathscr{F}_{\bar\Lambda}}$-almost all (ω^1, ω^2) and all $\eta^1, \eta^2 \in \{0,1\}^\Lambda$. Here $[\eta^1, \eta^2]$ is a cylinder set in $\mathscr{X}^2$ given by

$$[\eta^1, \eta^2] = \{(\omega^1, \omega^2) \in \mathscr{X}^2\colon \omega_x^1 = \eta_x^1,\ \omega_x^2 = \eta_x^2 \text{ for every } x \in \Lambda\}.$$

From (4.10), for every $f_1, f_2 \in \mathscr{D}$ and Λ sufficiently large, we have

$$\begin{aligned}&\int_{\mathscr{X}^2} f_1(\omega^1) f_2(\omega^2)\mu(d\omega^1\, d\omega^2)\\ &\quad= \int_{\mathscr{X}^2}\langle f_1\rangle_{\Lambda, N_\Lambda(\omega^1), \omega^1}\cdot\langle f_2\rangle_{\Lambda, N_\Lambda(\omega^2), \omega^2}\mu(d\omega^1\, d\omega^2),\end{aligned}$$

where $\langle\,\cdot\,\rangle_{\Lambda, n, \omega}$ denotes the expectation with respect to $\gamma_{\Lambda, n}(\cdot|\omega)$. Since due to the equivalence of ensembles [see [8], (7.13) and (7.14)] $\nu_{\Lambda, n}(\cdot|\omega)$ weakly converges to ν_ρ for all ω as $\Lambda \uparrow \mathbb{Z}$ and $n \to \infty$ in such a way that $n/|\Lambda| \to \rho$, an application of the bounded convergence theorem proves (4.9). □

REMARK. (4.9) can be deduced only from (4.4); in other words, we can dispense with the assumption of μ being translation-invariant. Indeed (4.10) tells us that each marginal distribution of μ is a canonical Gibbs measure and by absence of phase transition it is translation-invariant. Consequently (M_1, M_2) exists almost surely and the argument in the proof of the proposition still works.

We prepare a uniform mixing property of $\{\nu_\rho\}_{\rho\in[0,1]}$.

LEMMA 4.3. *There exist positive constants C_7 and C_8 which are independent of ρ such that*

$$(4.11)\quad |\langle f_1\cdot f_2\rangle_\rho - \langle f_1\rangle_\rho\langle f_2\rangle_\rho| \le C_7\|f_1\|_\infty\|f_2\|_\infty\exp\{-C_8 d(\Lambda_1, \Lambda_2)\},$$

for every $\rho \in [0,1]$, $\Lambda_i \subset \mathbb{Z}$ and $\mathscr{F}_{\Lambda_i}$ measurable bounded functions f_i, $i = 1, 2$, where $\|f\|_\infty = \sup_\eta|f(\eta)|$ and $d(\Lambda_1, \Lambda_2) = \min\{|x_1 - x_2|\colon x_i \in \Lambda_i, i = 1, 2\}$.

PROOF. It is known that $\nu_\rho \in \mathscr{P}(\mathscr{X})$, $\rho \in [0,1]$, is the distribution of a stationary $(R-1)$-dependent Markov chain with the state space $\{0,1\}$ by regarding $x \in \mathbb{Z}$ as the time parameter; see Ellis [5, page 330]. We notice that the mapping $\rho \in [0,1] \mapsto \nu_\rho \in \mathscr{P}(\mathscr{X})$ is continuous and also that each cylinder set of $\mathscr{X}$ has positive measure under ν_ρ if $\rho \in (0,1)$ [e.g., use the relation like (4.15)]. Consequently the transition matrix p_ρ of the Markov chain corresponding to ν_ρ is continuous in $\rho \in [0,1]$ and so are its characteristic roots. Moreover, p_ρ is primitive when $\rho \in (0,1)$, so that according to the

Perron–Frobenius theorem its characteristic root 1 is simple and larger in absolute value than the others. This fact is obvious when $\rho = 0$ and 1. Therefore the desired uniform bound (4.11) can be deduced by quite standard argument. □

REMARK. The uniform bound like (4.11) holds in arbitrary dimensions and for a general potential $\Phi = \{\Phi(A),\ A \subset \mathbb{Z}^d\}$ if Φ belongs to the Dobrushin uniqueness region; cf. Künsch [12]. It proves that, for every $f \in \mathscr{D}$, the function $G(\lambda) \equiv \langle f \rangle^{(\lambda)} = \nu^{(\lambda)}(f)$ is differentiable in $\lambda \in \mathbb{R}$ and justifies a formal manipulation that leads to

$$G'(\lambda) = \sum_{x \in \mathbb{Z}} \{\langle f\eta_x \rangle^{(\lambda)} - \langle f \rangle^{(\lambda)} \langle \eta_x \rangle^{(\lambda)}\}. \tag{4.12}$$

PROOF OF THEOREM 2. Let us apply Proposition 4.2. With the help of Lemma 4.3 we have then

$$\lim_{l \to \infty} \sup_{\mu \in \mathscr{A}_\varepsilon} \int_{\mathscr{X}^2} \left| M^1_{\Lambda_l} - M^2_{\Lambda_l} \right| d\mu = \sup_{\mu \in \mathscr{A}_\varepsilon} \int |\rho_1 - \rho_2| \hat{\mu}(d\rho_1\, d\rho_2),$$

where $\hat{\mu}$ is defined just before Proposition 4.2. Therefore for the proof of (4.8) we have only to show

$$\lim_{\varepsilon \downarrow 0} \sup_{\mu \in \mathscr{A}_\varepsilon} \int |\rho_1 - \rho_2| \hat{\mu}(d\rho_1\, d\rho_2) = 0. \tag{4.13}$$

To this end we shall use (4.5). In the variational formula

$$I^{(0)}_\Lambda = \sup\left\{ -\int_{\mathscr{X}^2} \frac{L^{(0)}u}{u}\, d\mu \colon u > 0 \text{ is } \mathscr{F}_\Lambda \times \mathscr{F}_\Lambda \text{ measurable} \right\},$$

take $u = \exp[\frac{1}{2}a|\Lambda|\{(M^1_\Lambda)^2 + (M^2_\Lambda)^2\}]$ for $a > 0$. Then

$$\begin{aligned} I^{(0)}_\Lambda(\mu) &\geq -\int \frac{L^{(0)}u}{u}\, d\mu \\ &= -\int \{c^\circ(\eta^1) + c^\circ(\eta^2)\}\Big[\exp\Big(-a(\eta^1_0 - \eta^2_0)(M^1_\Lambda - M^2_\Lambda) \\ &\qquad + \frac{a}{|\Lambda|}(\eta^1_0 - \eta^2_0)^2\Big) - 1\Big]\, d\mu \\ &\geq a\int \{c^\circ(\eta^1) + c^\circ(\eta^2)\}(\eta^1_0 - \eta^2_0)(M^1_\Lambda - M^2_\Lambda)\, d\mu \\ &\qquad - a^2 e^{2a} \|c^\circ\|_\infty \int \{M^1_\Lambda - M^2_\Lambda\}^2\, d\mu + O\left(\frac{1}{|\Lambda|}\right). \end{aligned}$$

Combining this with (4.5) and letting $\Lambda \uparrow \mathbb{Z}$, we have

$$a\int_{[0,1]^2} F(\rho_1, \rho_2)\, d\hat{\mu} \le C_2\varepsilon^2 + a^2 e^{2a}\|c^\circ\|_\infty$$

for all $\mu \in \mathscr{A}_\varepsilon$ and $a > 0$, where

$$F(\rho_1, \rho_2) = \{\langle c^\circ\rangle_{\rho_1}(1-\rho_2) - \langle c^\circ\rangle_{\rho_2}(1-\rho_1)\}(\rho_1 - \rho_2), \qquad \rho_1, \rho_2 \in [0,1].$$

Denote by $\mathscr{A}^*$ the set of all limit points of $\{\hat{\mu};\ \mu \in \mathscr{A}_\varepsilon\}$ in $\mathscr{P}([0,1]^2)$ as $\varepsilon \downarrow 0$. Then, due to the continuity of F, we have

$$\sup_{\hat{\mu} \in \mathscr{A}^*} \int F(\rho_1, \rho_2)\, d\hat{\mu} \le a e^{2a}\|c^\circ\|_\infty.$$

Since $a > 0$ is arbitrary, this together with Lemma 4.4 implies $\hat{\mu}(\rho_1 = \rho_2) = 1$ for all $\hat{\mu} \in \mathscr{A}^*$, which proves (4.13).

To complete the proof of Theorem 2, we must prove

LEMMA 4.4. *Let F be the function defined above. Then $F(\rho_1, \rho_2) > 0$ if $\rho_1 \ne \rho_2$.*

PROOF. We first show the equality

$$\langle c^\circ\rangle_\rho = e^{\lambda(\rho)}(1-\rho) \quad \text{for } 0 \le \rho < 1. \tag{4.14}$$

Since $E_0(\eta)$ is $\mathscr{F}_{\Lambda_R}$ measurable,

$$\langle c^\circ\rangle_\rho = \sum_{\eta \in \mathscr{X}(\Lambda_R \setminus \{0\})} \exp\{E_0(1\cdot\eta)\}\nu_\rho([1\cdot\eta]),$$

where $[1 \cdot \eta]$ represents a cylinder set given by

$$[1\cdot\eta] = \{\omega \in \mathscr{X}\colon \omega_x = \eta_x \quad \text{for } x \in \Lambda_R \setminus \{0\} \text{ and } \omega_0 = 1\}.$$

The DLR equation (1.11) shows

$$\nu_\rho([1\cdot\eta]) = \exp(\lambda(\rho) - E_0(1\cdot\eta))\nu_\rho([0\cdot\eta]). \tag{4.15}$$

Therefore,

$$\langle c^\circ\rangle_\rho = \sum_{\eta \in \mathscr{X}(\Lambda_R \setminus \{0\})} e^{\lambda(\rho)}\nu_\rho([0\cdot\eta]) = e^{\lambda(\rho)}(1-\rho).$$

We have by (4.14),

$$F(\rho_1, \rho_2) = (e^{\lambda(\rho_1)} - e^{\lambda(\rho_2)})(1-\rho_1)(1-\rho_2)(\rho_1 - \rho_2)$$

for $0 \le \rho_1, \rho_2 < 1$. On the other hand,

$$F(1, \rho_2) = \exp\{E_0(\mathbb{1})\}(1-\rho_2)^2.$$

The lemma then follows from the fact that $\lambda(\rho)$ is strictly increasing. □

PROOF OF THEOREM 3. Let μ be an arbitrary limit point of $\{\bar{\mu}^N\}$ in $\mathscr{P}(\mathscr{X})$. Then μ is a superposition of ν_ρ as readily seen from Proposition 4.2 by considering the first marginal in the expression (4.9). Therefore, it suffices to verify that for ν_ρ's the law of large numbers holds uniformly:

$$\lim_{l\to\infty} \sup_\rho \left\langle \left| \frac{1}{2l+1} \sum_{|y|\le l} \tau_y f(\eta) - \langle f \rangle \left(\frac{1}{2l+1} \sum_{|y|\le l} \eta_y \right) \right| \right\rangle (\rho) = 0. \tag{4.16}$$

But this is an immediate consequence of Lemma 4.3 since $\langle f \rangle(\rho)$ is continuous in $\rho \in [0,1]$. □

5. Examples. We present two classes of examples for the rate function $c(\eta)$ that satisfies both (GS) and (DB). The second one is less trivial than the first. A multidimensional model to which the present method is applicable is given in Remark 5.3.

EXAMPLE 1. Let $b(k)$, $k = 0, 1, 2, \dots$, be a real sequence with $b(0) = 0$. For $x \in \mathbb{Z}$, $\eta \in \mathscr{X}$, put

$$n(x,\eta) = \max\{|A|: A \subset \mathbb{Z},\ A \text{ is connected},\ x \in A,\ \eta^A \neq 0\}$$

[i.e., $n(x,\eta)$ is the size of a component of the 1-cluster $\{y: \eta_y = 1\}$ that contains x] and

$$c(\eta) = \eta_0(1-\eta_1)e^{b(n(0,\eta))} + \eta_1(1-\eta_0)e^{b(n(1,\eta))}. \tag{5.1}$$

Then one immediately observes that the assumption (GS) is satisfied with

$$h(\eta) = \eta_0 e^{b(n(0,\eta))}.$$

The condition (DB) is also satisfied: The potential function Φ is

$$\Phi(A) = \begin{cases} u_{|A|} & \text{if } A \text{ is connected,} \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

where $u_k = b(k) - b(k-1)$ and $u_0 = 0$. In fact we have $E_x(\eta) := E_x^\Phi(\eta) = \sum_{A \ni x} u_{|A|}\eta^A$ that coincides with $b(n(x,\eta))$ if $\eta_{x+1} = 0$ or $\eta_{x-1} = 0$, so that

$$c(\eta) = |\eta_0 - \eta_1|\left(\eta_0 e^{E_0(\eta)} + \eta_1 e^{E_1(\eta)}\right),$$

showing (DB). Φ has a finite range if and only if $b(k)$ eventually become constant.

If we interpret 0 and 1 as representing, contrary to the usual, the occupancy and the vacancy of particle, respectively, then this model may be regarded as a random evolution of a system of particles whose transition rates depend on the distances between two neighboring particles.

For the potential function (5.2) the Gibbs measure ν_ρ is identified with the law of a stationary renewal process on $\mathbb{Z}$, by regarding the occurrence of 0 as renewal events. Put

$$v(k) = \sum_{j=0}^{k-1} \sum_{i=0}^{j} u_i = \sum_{j=0}^{k-1} b(j)$$

and

$$z(\gamma) = \sum_{k=1}^{\infty} e^{\gamma k - v(k)}, \qquad \gamma < \lim_{k\to\infty} b(k). \tag{5.3}$$

Then $E_A(\eta) = v(|A| + 1)$ if A is a component of the 1-cluster in the configuration η, from which one can easily deduce that the interarrival time, taking values $1, 2, \dots,$ is distributed according to $\{e^{\gamma k - v(k)}/z(\gamma)\}_{k=1}^{\infty}$ with γ determined by

$$\frac{1}{1-\rho} = \psi'(\gamma) \quad \text{where } \psi(\gamma) = \log z(\gamma). \tag{5.4}$$

The correspondence between the chemical potentials $\lambda = \lambda(\rho)$ and $\gamma = \gamma(\rho)$ is given by $e^{\lambda} = z(\gamma)$ [thus $z(\gamma)$ is nothing but the activity]. The functions $P(\rho)$, $D(\rho)$ and χ_ρ appearing in (1.10) and (1.12) also can be written explicitly in terms of γ as

$$P(\rho) = e^{\gamma}, \qquad D(\rho) = \frac{2e^{\gamma}\psi'(\gamma)^2}{\psi''(\gamma)}, \qquad \chi_\rho = \frac{\psi'''(\gamma)}{\psi'(\gamma)^3}.$$

Example 2. Let α and β be two real numbers such that $1 + \alpha > 0$, $1 + \beta > 0$ and $1 + \alpha + \beta > 0$. Put

$$\begin{aligned} c(\eta) &= \eta_0(1 - \eta_1)(1 + \alpha\eta_{-1} + \beta\eta_2) \\ &\quad + \eta_1(1 - \eta_0)(1 + \alpha\eta_2 + \beta\eta_{-1}). \end{aligned} \tag{5.5}$$

Then, as is seen in the following, $c(\eta)$ satisfies the assumption (GS) as well as (DB), with the energy being of the nearest neighbor interaction:

$$E_\Lambda(\eta) = \sum_{x\in\Lambda \text{ or } x+1\in\Lambda} (\log q)\eta_x\eta_{x+1}, \tag{5.6}$$

where

$$q = \frac{1+\alpha}{1+\beta}.$$

This example, by regarding 0 and 1 as two spin states to stress their symmetric role, may be taken as a spin-exchange model of the nearest neighbor interaction in which spins are exchanged between each couple of successive sites and the exchange rate is determined only by the spin states on the two neighboring sites to the couple. The functions c in (5.5) exhausts all spin-exchange models of this sort that satisfy both conditions (DB) and (GS). By (5.6) one sees that the same kind of spins attract (repel) each other if $0 < q < 1$ (resp., $q > 1$).

When $\beta = 0$, the formula (5.5) turns into a special case of (5.1) with $b(1) = 0$ and $b(k) = \log(1 + \alpha)$, $k \geq 2$, and similarly when $\alpha = 0$ with roles of 0 and 1 interchanged; in general, it represents a sum of these two kinds of rate functions. Therefore the corresponding process must be a gradient system. In fact, if $\alpha \neq \beta$, the decomposition into the two is provided by

$$\eta_0 c(\eta) = \eta_0(1-\eta_1)\left\{\frac{\alpha}{q-1}[(1-\eta_{-1})+q\eta_{-1}] - \frac{\beta}{q-1}[\eta_2 + q(1-\eta_2)]\right\},$$

where q is the same constant that determines E_Λ in (5.6). The two components are "reflective" of each other by the "reflection" $\eta \to \hat{\eta}$ where $\hat{\eta}_x = 1 - \eta_x$, so that we find the function $h(\eta)$ in the form

$$h(\eta) = \frac{\alpha}{q-1} h^{(q)}(\eta) + \frac{\beta}{q-1} h^{(q)}(\hat{\eta}) + \text{const.}, \tag{5.7}$$

where

$$h^{(q)}(\eta) = q\eta_0 + (1-q)(1-\eta_{-1})\eta_0(1-\eta_1).$$

If $\alpha = \beta$, we directly see that $h(\eta) = (\alpha+1)\eta_0 + \alpha(\eta_{-1} - \eta_0)(\eta_0 - \eta_1)$.

If $\alpha \neq \beta$, $c(\eta)$ can be expressed, with E_Λ given in (5.6), as

$$c(\eta) = |\eta_0 - \eta_1|\left\{p + \frac{2(1-p)}{\exp E_{\{0,1\}}(\eta) + \exp E_{\{0,1\}}(\eta^{0,1})}\right\}\exp E_{\{0,1\}}(\eta), \tag{5.8}$$

where $p = (\alpha+\beta)(1+\beta)/(\alpha-\beta)$. [Conversely $c(\eta)$ defined by (5.8) is expressed in the form (5.5), if $q \neq 1$, $q > 0$ and $c(\eta) > 0$ for all η with $\eta_0 \neq \eta_1$ or, what amounts to the same, if $q > 0$, $q \neq 1$ and $(q-1)p + 1 > 0$.] Therefore this model satisfies the condition (DB). [Notice that in the case $\alpha = \beta$, (1.5) is obviously satisfied with $\Phi \equiv 0$.]

Let us compute $P(\rho) = \langle h \rangle_\rho$ up to an additive constant, which is insignificant. The energy (5.6) is a special case of that corresponding to Φ in (5.2) with $u_1 = 0$ and $u_k = \log q$, $k \geq 2$, $z(\gamma)$ in (5.3) being given by

$$z(\gamma)e^{-\gamma} = q\left\{\frac{1}{q} - 1 + \frac{1}{1-y}\right\}, \qquad y = \frac{e^\gamma}{q}.$$

By $z'/z = 1/(1-\rho)$,

$$\left(1 - \frac{1}{q}\right)y^2 + 2\left\{\frac{1}{2\rho} - \left(1 - \frac{1}{q}\right)\right\}y - \frac{1}{q} = 0.$$

Noticing $y < 1$, we obtain

$$y = 1 - \frac{1}{2r\rho}\left\{1 - \sqrt{1 - 4r(1-\rho)\rho}\right\} \quad \text{if } q \neq 1,$$

where $r = 1 - q^{-1} = (\alpha - \beta)/(1+\alpha)$. Finally, if $\alpha \neq \beta$,

$$\begin{aligned} P(\rho) &= \frac{\alpha}{q-1}\langle h^{(q)}\rangle_\rho + \frac{\beta}{q-1}\langle h^{(q)}\rangle_{1-\rho} + \text{const.} \\ &= -\frac{1}{2}r^{-2}\left\{\frac{\alpha}{\rho} + \frac{\beta}{1-\rho}\right\}\left\{1 - \sqrt{1-4r(1-\rho)\rho}\right\} + \text{const.} \end{aligned}$$

and if $\alpha = \beta$, $P(\rho) = \rho + \alpha\rho^2$.

REMARK 5.1. For the first example the locality assumption on $c(\eta)$ can be relaxed to obtain Theorem 1 (due to the underlying renewal structure): We have only to assume that $\lim b(k) = \beta$ exists and $v(k) = b(0) + \cdots + b(k-1) - k\beta$ is bounded, though we accordingly have to alter certain statements given in Section 4 and their proofs as well (especially for Proposition 4.2 and Lemma 4.3).

REMARK 5.2. The whole class of c satisfying (GS) and (DB) would be much larger than those previously given. For example, if c is allowed to depend on $\{\eta_x, -1 \le x \le 3\}$, the essential degree of freedom for parameters determining c is 6 (in the second example it is 2). It is noted that a sum of c given in (5.1) and its reflection does not satisfy (DB) anymore.

REMARK 5.3. The special case $\alpha = \beta$ of the second example has a multidimensional analogue. Let $\Gamma_N = \mathbb{Z}^d/N\mathbb{Z}^d$ and $e(k) = (\delta_{k,j})_{j=1}^d$, $k = 1, \dots, d$ (the standard basis of $\mathbb{Z}^d$). Given real numbers α_k, $k = 1, \dots, d$, such that $1 + 2\alpha_k > 0$, the spin exchange rate between two neighboring sites x and y is defined by

$$c_{x,y}(\eta) = 1 + \alpha_k(\eta_{x-e(k)} + \eta_{y+e(k)}) \quad \text{if } y = x + e(k),$$

where $\eta \in \{0,1\}^{\Gamma_N}$ and $k = 1, \dots, d$ (otherwise no exchange). Then both conditions (GS) and (DB) are satisfied: The reversible measures are the Bernoulli measures and the function $h(\eta)$ in the gradient condition is now a vector function $h = (h_1, \dots, h_d)$, where

$$h_k(\eta) = (1 + \alpha_k)\eta_0 + \alpha_k(\eta_{-e(k)} - \eta_0)(\eta_0 - \eta_{e(k)}).$$

Precisely the same result as stated in our Theorem 1 is valid for this multidimensional model where the nonlinear diffusion equation in the limit is

$$\frac{\partial}{\partial t}\rho = \sum_{k=1}^{d} \frac{\partial^2}{\partial \theta_k^2}(\rho + \alpha_k \rho^2),$$

$\rho = \rho(t, \theta)$, $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d/\mathbb{Z}^d$. The proof is much the same as for the one-dimensional case except for the uniqueness of the solution to the limit equation, which can, however, be verified by adapting the method of [1] for the proof of its Theorem 1. [The main idea therein is to compute the derivative $(d/dt)\langle G_\alpha z(t), z(t)\rangle$ where $z(t) = z(t,\theta)$ is the difference of two solutions and G_α is the resolvent operator associated with the Brownian motion on the d-dimensional torus instead of $\langle \omega_t^\varepsilon, \partial\omega_t^\varepsilon/\partial t\rangle$ in the proof of our Lemma 2.1. After that let α go to zero, noticing that $\alpha G_\alpha z(t)$ approaches zero for every t by the ergodicity of the Brownian motion.]

REMARK 5.4. The second example also is found in the forthcoming book by Spohn [15], where the hydrodynamic limit for the special case $\alpha = \beta$ is discussed in some detail.

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