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MICROSCOPIC STRUCTURE OF TRAVELLING WAVES IN THE ASYMMETRIC SIMPLE EXCLUSION PROCESS

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The one-dimensional nearest neighbor asymmetric simple exclusion process has been used as a microscopic approximation for the Burgers equation. This equation has travelling wave solutions. In this paper we show that those solutions have a microscopic structure. More precisely, we consider the simple exclusion process with rate p (respectively, $q = 1 - p$) for jumps to the right (left), $\frac{1}{2} < p \leq 1$, and we prove the following results: There exists a measure μ on the space of configurations approaching asymptotically the product measure with densities ρ and λ to the left and right of the origin, respectively, $\rho < \lambda$, and there exists a random position $X(t) \in \mathbb{Z}$, such that, at time t, the system "as seen from $X(t)$," remains distributed according to μ, for all $t \geq 0$. The hydrodynamical limit for the simple exclusion process with initial measure μ converges to the travelling wave solution of the inviscid Burgers equation. The random position $X(t)/t$ converges strongly to the speed $v = (1 - \lambda - \rho)(p - q)$ of the travelling wave. Finally, in the weakly asymmetric hydrodynamical limit, the stationary density profile converges to the travelling wave solution of the Burgers equation.

1. Introduction. The inviscid Burgers equation is the first order quasi-linear hyperbolic equation given by

$$\frac{\partial u}{\partial t} + \theta \frac{\partial[u(1-u)]}{\partial r} = 0, \tag{1.1}$$

where $\theta > 0$ is a fixed parameter. For $\rho < \lambda$ this equation admits travelling wave (weak) solutions $\bar{u}(r,t) = u_0(r - vt)$, where $v = (1 - \rho - \lambda)\theta$ and

$$u_0(r) = \begin{cases} \rho & \text{if } r \leq 0, \\ \lambda & \text{if } r > 0. \end{cases} \tag{1.2}$$

On the other hand, the asymmetric simple exclusion process is one of the simplest infinite particle systems which can be informally described as follows: Each site of $\mathbb{Z}$ can be occupied by one particle at most and each particle jumps to its nearest right neighbor (respectively, left) with rate p (respectively, $q = 1 - p$) unless the site is occupied. The product Bernoulli measures ν_ρ $(0 \leq \rho \leq 1)$ of density ρ are invariant for this process. See [16] for a complete description.

When $p > \frac{1}{2}$, it has been proved by Benassi and Fouque [5] and Andjel and Vares [3] that if the initial state is the product measure $\nu_{\rho,\lambda}$ with density ρ to

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the left and λ to the right of the origin (with $\rho < \lambda$), the hydrodynamical limit is given by the travelling wave solution of the inviscid Burgers equation [where in (1.1), $\theta = p - q$]. See the introduction of [2] or [8] for a description of the hydrodynamical limit.

However, it was not clear whether the sharpness of the shock was due to the scaling and, in particular, if by going to an intermediate scale, one would not find a smooth profile. This question was considered important in order to explain why the system chose by itself the entropic solution which can be obtained as the limit of the Burgers equation. See [5] for a discussion on the entropic solution. It was proven that the shock is sharp in various special cases (all assuming $\rho = 0$) by Kesten (quoted in [17], and also [16]) and for related models by Wick [18], in the case $p = 1$, and by De Masi, Kipnis, Presutti and Saada [9] for any $p > \frac{1}{2}$. Boldrighini, Cosimi, Frigio and Grasso Nunes [6] simulated the shock for $p = 1$ and $\rho > 0$ and obtained evidence that it was sharp.

One major problem for studying the microscopic structure at the shock is that there is no obvious microscopic definition of the location of the shock. In the case when $\rho = 0$ a natural candidate is the position of the leftmost particle. This fact was used in the previously mentioned results combined with the explicit knowledge of the invariant measures as seen from the leftmost particle ([11]).

When the left density is not zero, one naturally tries a coupling between two copies of the simple exclusion process: one with initial measure ν_ρ and the other with initial measure $\nu_{\rho,\lambda}$. The coupling, called the basic coupling [16], is constructed in such a way that the first marginal is always ν_ρ. We call second class particles, those particles of the second marginal that do not have a correspondent in the first one. By analogy with the case $\rho = 0$, one would like to look at the leftmost second class particle and see if there exists an invariant measure μ for the process as seen from its position. If the microscopic profile is sharp, the asymptotic density of particles of the second marginal of μ should be λ to the right and ρ to the left. One also would like to know enough about the measure μ to be able to prove laws of large numbers and central limit theorems for the position of the leftmost second class particle properly rescaled. Our attempts to find a more or less explicit form of this measure failed for reasons that will appear later. However, a useful observation is that, provided that we have all the needed ergodic theorems, the rigidity of the shock implies that the speed of the leftmost second class particle is given by the speed of the shock. This speed is explicitly known and coincides with the speed of a tagged second class particle in a translation invariant and stationary system that would couple ν_λ and ν_ρ.

In this paper we give a definition of what we call *microscopic interface*. Actually, in order to define this interface it is necessary to couple three processes: the two described previously and one with initial density λ. In this coupling, by identifying particles of types 1 and 2, we recover the process that we want to study. The position of the interface is defined as a process depending on the whole trajectory of the coupled process with three marginals.

We do not want to enter into details here but we can say to those that are familiar with couplings that the microscopic interface coincides with the position of a tagged second class particle in the translation invariant two-particle system obtained when identifying particles of types 2 and 3.

We prove that there exists an invariant measure for the three-particle coupled process, as seen from the interface, under which the distribution of the second marginal approaches asymptotically ν_ρ and ν_λ at infinity. Under that invariant measure, the distance between the leftmost particle and the microscopic interface remains tight. Initially, the number of type 2 (respectively, 3) particles to the left (right) of the interface is random but the difference between the number of type 3 particles to the right and the number of type 2 particles to the left of the interface is constant in time.

Our analysis gives no direct information about the speed at which the measure relaxes to ν_λ and ν_ρ at infinity. However, by looking at the weakly asymmetric hydrodynamical limit [10, 12], the density profile of the measure that we found converges to the travelling wave solution of the Burgers equation (with viscosity 1).

It remains an open problem to prove that at the shock, the system goes to a superposition of ν_ρ and ν_λ [14, 15]. This is proved by [1] when $\lambda + \rho = 1$ and [18] and [9] for $\rho = 0$. For the general case one should study the fluctuations of the microscopic interface around its mean vt. The result of [1] when the velocity of the shock is zero indicates that this fluctuation is not tight.

2. Results. Let η_t be the asymmetric simple exclusion process. This is the Markov process on the state space $\mathbf{X} := \{0,1\}^{\mathbb{Z}}$ with generator (applied to cylindrical functions f) defined by

$$(2.1)\quad Lf(\eta) = \sum_{x\in\mathbb{Z}} \sum_{y=x\pm 1} \eta(x)(1-\eta(y))p(x,y)[f(\eta^{x,y}) - f(\eta)],$$

where $p(x, x+1) = p$, $p(x, x-1) = 1 - p := q$, $p(x,y) = 0$ if $|x-y| > 1$, and

$$(2.2)\quad \eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x. \end{cases}$$

We refer to Liggett [16] for the construction and the ergodic properties of this process. Let ν_ρ be the product measure on $\mathbf{X}$ with density ρ. The family $\{\nu_\rho: 0 \le \rho \le 1\}$ contains all the extremal translation invariant and invariant measures for the process [16]. Let $\nu_{\rho,\lambda}$ be the product measure on $\mathbf{X}$ with marginals

$$\nu_{\rho,\lambda}\{\eta(x) = 1\} = \begin{cases} \rho & \text{if } x \le 0, \\ \lambda & \text{if } x > 0. \end{cases}$$

From now on we assume $\rho < \lambda$ and $p > q$. In what follows we use the notation $\mu f := \int f d\mu$. Define the configuration $\tau_x\eta$ by $(\tau_x\eta)(z) = \eta(z+x)$, the function $\tau_x f$ by $\tau_x f(\eta) := f(\tau_x\eta)$ and the measure $(\tau_x\mu)$ by $(\tau_x\mu)f := \mu(\tau_x f)$.

THEOREM 2.3. *There exist a version of the asymmetric simple exclusion process η_t in a probability space $(\Omega, \mathscr{F}, P)$, an $\mathscr{F}$-measurable stochastic process $X(t) \in \mathbb{Z}$ and a measure μ on* $\mathbf{X}$ *satisfying*

$$\lim_{x \to -\infty} \tau_x \mu = \nu_\rho, \qquad \lim_{x \to \infty} \tau_x \mu = \nu_\lambda \quad \textit{weakly}, \tag{2.4}$$

such that the process $\tau_{X(t)}\eta_t$ is distributed according to μ, for all $t \geq 0$.

REMARK 2.5. Neither the process $X(t)$ nor $\tau_{X(t)}\eta_t$ are Markovian. Moreover, the asymptotic profile of the measure μ is established in Theorem 2.14.

THEOREM 2.6 (Law of large numbers). *The process $X(t)/t$ converges P-almost surely to $v := (p - q)(1 - \rho - \lambda)$.*

In the next theorem we prove the hydrodynamical limit for the simple exclusion process starting with the initial measure μ of Theorem 2.3. The Proof of Theorem 2.7 applies to a family of initial measures that includes the product measure $\nu_{\rho,\lambda}$. In this case the Theorem was proved in [5] and [3]. In the proof we use a different approach that exploits the existence of the microscopic interface of Theorem 2.3. On the other hand, we believe that the invariant measure μ is absolutely continuous with respect to $\nu_{\rho,\lambda}$ but we do not know how to prove it. In the case $\mu \ll \nu_{\rho,\lambda}$, Theorem 2.7 would be just a corollary of [5] and [3].

THEOREM 2.7. *Let μ be the measure introduced in Theorem 2.3, $S(t)$ be the semigroup corresponding to the simple exclusion process, $t \geq 0$, and r be real numbers. Then*

$$\lim_{\varepsilon \to 0} \tau_{\varepsilon^{-1}r} \mu S(\varepsilon^{-1}t) = \begin{cases} \nu_\rho & \text{if } r < vt \\ \nu_\lambda & \text{if } r > vt \end{cases} \tag{2.8}$$

weakly.

Our next result is a weak law of large numbers for the density profile of the measure μ of Theorem 2.3, in the "weakly asymmetric" [10, 12] hydrodynamical limit. Let μ_ε be the measure introduced in Theorem 2.3 when $p = 1/2 + \varepsilon(\theta/2)$, where $\theta \geq 0$ is the parameter of (1.1). Let Y_ε be the density field, given by

$$Y_\varepsilon(\phi, \eta) := \varepsilon \sum_{x \in \mathbb{Z}} \phi(\varepsilon x)\eta(x), \tag{2.9}$$

where ϕ is a test function, $\phi \in \mathscr{S}(\mathbb{R})$, the Schwartz space of rapidly decaying functions on $\mathbb{R}$. Let

$$Y^1(\phi) := \int \phi(r)u(r)\,dr, \qquad Y^0(\phi) := \int \phi(r)u_0(r)\,dr, \tag{2.10}$$

where

$$(2.11)\quad u(r) := \rho + \frac{\lambda - \rho}{1 + e^{-2\theta r(\lambda - \rho)}}, \qquad u_0(r) := \rho 1\{r \le 0\} + \lambda 1\{r > 0\}.$$

REMARK 2.12. The function u defined in (2.11) is the travelling wave solution with asymptotic densities λ and ρ of the Burgers equation with viscosity 1,

$$(2.13)\qquad \frac{\partial u}{\partial t} + \theta \frac{\partial[u(1-u)]}{\partial r} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2}.$$

That means that $\bar{u}(r,t) := u(r - vt)$ is the stationary solution of (2.13), where $v = (1 - \lambda - \rho)\theta$.

THEOREM 2.14. *For all $\phi \in \mathscr{S}(\mathbb{R})$ the following hold:*

$$(2.15)\qquad \lim_{\varepsilon \to 0} \int \mu(d\eta)|Y_\varepsilon(\phi, \eta) - Y^0(\phi)| = 0,$$

$$(2.16)\qquad \lim_{\varepsilon \to 0} \int \mu_\varepsilon(d\eta)|Y_\varepsilon(\phi, \eta) - Y^1(\phi)| = 0.$$

In particular, under μ (respectively, μ_ε), as $\varepsilon \to 0$, Y_ε converges in probability to the deterministic profile u_0 (respectively, u). Moreover, for $r \in \mathbb{R}$,

$$(2.17)\qquad \lim_{\varepsilon \to 0} \tau_{\varepsilon^{-1}r}\mu_\varepsilon = \nu_{u(r)}$$

weakly.

A related result is given by [7]. In that paper the invariant measures in boxes of length ε^{-1} are studied for a fixed total density and the analogue of (2.17) is found.

THEOREM 2.18 (Weakly asymmetric hydrodynamics). *Let u be the travelling wave solution* (2.11) *of the Burgers equation*, (2.13). *Let μ_ε be the measure of Theorem* 2.3, *when $p = 1/2 + (\theta/2)\varepsilon$. Then*

$$(2.19)\qquad \lim_{\varepsilon \to 0} \tau_{\varepsilon^{-1}r}\mu_\varepsilon S(\varepsilon^{-2}t) = \nu_{u(r-vt)}$$

weakly.

3. Microscopic interface. Define the process $(\sigma_t, \gamma_t, \zeta_t) \in \mathbf{X}^3 \cap \{\sigma(x) + \gamma(x) + \zeta(x) \le 1,\ x \in \mathbb{Z}\} := \mathbf{X}_3$ with generator

$$(3.1)\qquad \begin{aligned} L_3 f(\sigma, \gamma, \zeta) = \sum_x \sum_{y = x \pm 1} \big(&\sigma(x)(1 - \sigma(y))p(x,y) \\ &\times[f(\sigma^{x,y}, \gamma^{x,y}, \zeta^{x,y}) - f(\sigma, \gamma, \zeta)] \\ &+\gamma(x)(1 - \sigma(y) - \gamma(y))p(x,y) \\ &\times[f(\sigma, \gamma^{x,y}, \zeta^{x,y}) - f(\sigma, \gamma, \zeta)] \\ &+\zeta(x)(1 - \sigma(y) - \gamma(y) - \zeta(y))p(x,y) \\ &\times[f(\sigma, \gamma, \zeta^{x,y}) - f(\sigma, \gamma, \zeta)]\big). \end{aligned}$$

In words, the generator L_3 describes the behavior of particles of three different types, with priorities. In the language of Andjel and Kipnis [2], the first coordinate has priority over the second, which has priority over the third.

Let $(\Omega, \mathscr{F}, P)$ be the canonical probability space associated with this process. Let E be the expectation with respect to P. All the processes defined later are realized in $(\Omega, \mathscr{F}, P)$. We use the notations P_μ and E_μ for the probability and expectation with respect to the process with initial measure μ.

The relation of the process $(\sigma_t, \gamma_t, \zeta_t)$ with the simple exclusion process is given in the following lemma. Call $S_3(t)$ the Markov semigroup generated by L_3. Denote by $\sigma + \gamma \in \mathbf{X}$ the configuration defined by $(\sigma + \gamma)(x) = \sigma(x) + \gamma(x)$.

LEMMA 3.2. *The processes σ_t, $\sigma_t + \gamma_t$ and $\sigma_t + \gamma_t + \zeta_t$ are, in distribution, the asymmetric simple exclusion process.*

PROOF. Follows from the definition of L_3 by taking functions depending only on σ, $\sigma + \gamma$ and $\sigma + \gamma + \zeta$, respectively. □

It is convenient to consider the process (σ_t, ξ_t) on $\mathbf{X}_2 := \mathbf{X}^2 \cap \{(\sigma, \xi): \sigma(x) + \xi(x) \le 1,\ x \in \mathbb{Z}\}$, where $\xi_t := \gamma_t + \zeta_t$, obtained by making particles γ and ζ indistinguishable. We call L_2 and $S_2(t)$ the generator and the corresponding semigroup of this process. We say that sites such that $\sigma(x) = 1$ are occupied by a first class particle, while sites such that $\xi(x) = 1$ are occupied by a second class particle. This is a way to look at the basic coupling $(\eta_t^\sigma, \eta_t^{\sigma+\xi})$, where the upper index indicates the initial configuration [16]. At time t, the configuration σ_t is the set of sites where the two marginals coincide and ξ_t is the set of sites where they do not.

LEMMA 3.3. *There exists an unique invariant and translation invariant measure μ_2 on $\mathbf{X}_2$ for the process (σ_t, ξ_t) such that the first marginal of μ_2 is ν_ρ and $\int d\mu_2\, f(\sigma + \xi) = \nu_\lambda f$.*

PROOF. We couple initially ν_ρ and ν_λ in the following natural way: To each site $x \in \mathbb{Z}$ associate a random variable U_x uniformly distributed on $[0, 1]$, $\{U_x\}$ mutually independent. Given a realization $\{u_x\}$ of those random variables, $\sigma(x) = 1$ if $u_x < \rho$ [$\sigma(x) = 0$ otherwise] and $\xi(x) = 1$ if $\rho \le u_x < \lambda$ [$\xi(x) = 0$ otherwise]. Call ν_2 the resulting distribution of (σ, ξ). Let μ_2 be any weak limit point of $(1/T)\int_0^T \nu_2 S_2(t)\, dt$. By Proposition 2.14 of Chapter 8 of [16], μ_2 is invariant for $S_2(t)$ and by Lemma 3.2, the marginals satisfy the required property.

To prove uniqueness, consider two measures satisfying the conditions of Lemma 3.3. Now couple these two measures in such a way that their first marginals coincide. Construct a basic coupling between the four processes. The first marginals are always the same. The second marginals will converge in measure to the same distribution, as follows from Lemma 3.2 of Chapter 8 of [16]. □

DEFINITION 3.4 (The process as seen from a second class particle). Let (σ_s', ξ_s') on $\mathbf{X}_2' := \mathbf{X}_2 \cap \{(\sigma, \xi): \xi(0) = 1\}$ be the process with generator

$$
\begin{aligned}
L_2' f(\sigma, \xi) = \sum_{x \neq 0} \sum_{y = x \pm 1 \neq 0} &\Big(\sigma(x)(1 - \sigma(y))p(x,y)[f(\sigma^{x,y}, \xi^{x,y}) - f(\sigma, \xi)] \\
&+ \xi(x)(1 - \sigma(y) - \xi(y))p(x,y) \\
&\times [f(\sigma, \xi^{x,y}) - f(\sigma, \xi)]\Big) \qquad (3.5)\\
+ \sum_{y = \pm 1} &\Big(\sigma(y)p(y,0)\big[f(\tau_y \sigma^{0,y}, \tau_y \xi^{0,y}) - f(\sigma, \xi)\big] \\
&+ (1 - \sigma(y) - \xi(y))p(0,y)\big[f(\tau_y \sigma, \tau_y \xi^{0,y}) - f(\sigma, \xi)\big]\Big).
\end{aligned}
$$

Let $X(t)$ be the algebraic number of shifts of (σ_s', ξ_s') in the time interval $[0, t]$.

LEMMA 3.6. *The measure $\mu_2' = \mu_2(\cdot | \xi(0) = 1)$ is invariant for the process with generator L_2'.*

PROOF. Here we use the argument of Section 2 of [11]. Call $S_2'(t)$ the semigroup generated by L_2' and recall that $S_2(t)$ is the semigroup of the process (σ_t, ξ_t). Then, we have that

$$\mu_2' S_2'(t) = \mu_2 S_2(t)\{\cdot | \xi(0) = 1\}. \qquad (3.7)$$

Indeed, for every continuous function f on $\mathbf{X}_2$, an easy computation shows that

$$\int d\mu_2(\sigma, \xi)\xi(0) L_2' f(\sigma, \xi) = \int d\mu_2(\sigma, \xi) L_2[\xi(0) f(\sigma, \xi)], \qquad (3.8)$$

which is the main step in the proof of (the analogue of) (3.7) in Theorem 2.3 of [11]. Therefore, the invariance of μ_2 and (3.7) imply that μ_2' is invariant. □

DEFINITION 3.9. Let $A, B, C \subset \mathbb{Z}$ be finite and disjoint sets. Define

$$f_{A,B,C}(\sigma, \gamma, \zeta) = \prod_{x \in A} \sigma(x) \prod_{y \in B} \gamma(y) \prod_{z \in C} \zeta(z).$$

Let μ_3' be the measure on $\mathbf{X}_3' := \mathbf{X}_3 \cap \{\gamma(0) + \zeta(0) = 1\}$ given by

$$
\begin{aligned}
\mu_3' f_{A,B,C} = \int_{\mathbf{X}_2'} d\mu_2'(\sigma, \zeta) \prod_{x \in A} \sigma(x) \prod_{y \in B} \xi(y) \frac{(p/q)^{n(y,\xi)}}{1 + (p/q)^{n(y,\xi)}} \qquad (3.10)\\
\times \prod_{z \in C} \xi(z) \frac{(p/q)^{-n(z,\xi)}}{1 + (p/q)^{-n(z,\xi)}},
\end{aligned}
$$

where

$$n(x,\xi) = \begin{cases} \sum_{j=1}^{x} \xi(j) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\sum_{j=x}^{-1} \xi(j) & \text{if } x < 0. \end{cases} \tag{3.11}$$

When $p = 1$ we interpret $(p/q)^n/(1 + (p/q)^n)$ as 1 if $n > 0$, as $\frac{1}{2}$ if $n = 0$ and as 0 if $n < 0$. In this way μ'_3 is well defined for all p. The same interpretation is used in Remark 3.12.

REMARK 3.12. An equivalent way to define μ'_3 is to introduce a family of independent Bernoulli random variables $\{\kappa(x)\}_{x \in \mathbb{Z}}$ with distribution α such that $\alpha(\kappa(x) = 1) = (1 + (p/q)^{-x})^{-1}$ and write

$$\begin{aligned} \mu'_3 f_{A,B,C} = \int_{\mathbf{X}_2 \times \mathbf{X}} & d(\mu'_2 \times \alpha)(\sigma, \xi, \kappa) \prod_{x \in A} \sigma(x) \\ & \times \prod_{y \in B} \xi(y)\kappa(n(y,\xi)) \prod_{z \in C} \xi(z)\kappa(-n(z,\xi)). \end{aligned} \tag{3.13}$$

This means that in order to construct the three-particle measure μ'_3 one can put a configuration with two types of particles distributed according to μ'_2 and then decide which ξ particles will be γ particles independently according to the random variables κ.

The fact that μ'_3 would be invariant for the process was suggested to the authors by the "blocking" invariant measures of Liggett (Example 2.8 of Chapter 8 in [16]). The Liggett measures are product and reversible for the process η_t in the state space $\{\eta: \sum_{x \le 0} \eta(x) + \sum_{x > 0}(1 - \eta(x)) < \infty\}$. The reader can check that when $\rho = 0$ and $\lambda = 1$ the γ marginal of μ'_3 coincides with one of those. Let's call it ν. When η is initially distributed according to ν, the rescaled density is an exact stationary solution of the microscopic approximation of the Burgers equation for all p and all ε. We just realized that in the macroscopic limit $\varepsilon \to 0$, the marginal $\sigma + \gamma$ of μ'_3 has the same property.

PROPOSITION 3.14. *The measure μ'_3 is invariant for the process*

$$(\sigma'_t, \gamma'_t, \zeta'_t) := (\tau_{X(t)}\sigma_t, \tau_{X(t)}\gamma_t, \tau_{X(t)}\zeta_t).$$

REMARK 3.15. Even if $X(t)$ is not explicitly defined in terms of the process $(\sigma'_t, \gamma'_t, \zeta'_t)$, it is measurable w.r.t. the σ-fields generated by $\{(\sigma'_s, \gamma'_s, \zeta'_s): 0 \le s \le t\}$. Actually, $X(t)$ is the position of a tagged second class particle in the (σ_t, ξ_t)-process. Hence, $X(t)$ is measurable w.r.t. the σ-fields generated by $\{(\sigma'_s, \gamma'_s + \zeta'_s): 0 \le s \le t\}$. This means that, in the three-particle representation, interchanges of particles γ and ζ do not contribute to the motion of $X(t)$. As a consequence of that, $\sum_{x \le 0} \gamma'_t(x) - \sum_{x > 0} \zeta'_t(x)$ is constant in time.

PROOF. The generator of the process $(\sigma_t', \gamma_t', \zeta_t')$ is given by $L_3' = L_2'' + L_3''$, where

$$
\begin{aligned}
L_2'' f(\sigma,\gamma,\zeta) &= \sum_{x\neq 0}\sum_{y=x\pm 1\neq 0}\big(\sigma(x)(1-\sigma(y))p(x,y)\\
&\qquad\times[f(\sigma^{x,y},\gamma^{x,y},\zeta^{x,y})-f(\sigma,\gamma,\zeta)]\\
&\qquad+(\gamma(x)+\zeta(x))(1-\sigma(y)-\gamma(y)-\zeta(y))p(x,y)\\
&\qquad\times[f(\sigma,\gamma^{x,y},\zeta^{x,y})-f(\sigma,\gamma,\zeta)]\big)\\
&\quad+\sum_{y=\pm 1}\Big(\sigma(y)p(y,0)\big[f\big(\tau_y\sigma^{0,y},\tau_y\gamma^{0,y},\tau_y\zeta^{0,y}\big)-f(\sigma,\gamma,\zeta)\big]\\
&\qquad+(1-\sigma(y)-\gamma(y)-\zeta(y))p(0,y)\\
&\qquad\times\big[f\big(\tau_y\sigma,\tau_y\gamma^{0,y},\tau_y\zeta^{0,y}\big)-f(\sigma,\gamma,\zeta)\big]\Big)
\end{aligned}
\tag{3.16}
$$

and

$$
L_3'' f(\sigma,\gamma,\zeta) = \sum_x\sum_{y=x\pm 1}\gamma(x)\zeta(y)p(x,y)[f(\sigma,\gamma^{x,y},\zeta^{x,y})-f(\sigma,\gamma,\zeta)]. \tag{3.17}
$$

Notice that L_2'', when applied to functions depending only on $(\sigma, \gamma+\zeta)$, is the generator of (3.5). Moreover, defining

$$
\begin{aligned}
\bar f_{A,B,C}(\sigma,\xi) &= \prod_{x\in A}\sigma(x)\prod_{x\in B}\xi(x)\frac{(p/q)^{n(x,\xi)}}{1+(p/q)^{n(x,\xi)}}\\
&\quad\times\prod_{x\in C}\xi(x)\frac{(p/q)^{-n(x,\xi)}}{1+(p/q)^{-n(x,\xi)}},
\end{aligned}
\tag{3.18}
$$

we have that

$$
\mu_3' L_2'' f_{A,B,C} = \mu_2' L_2' \bar f_{A,B,C} = 0, \tag{3.19}
$$

where, since $\bar f$ is cylindrical, the second identity follows from Lemma 3.6.

On the other hand,

$$
\begin{aligned}
&\int_{\mathbf{X}_3'} d\mu_3'(\sigma,\gamma,\zeta)p(x,y)\gamma(x)\zeta(y)f_{A,B,C}(\sigma,\gamma,\zeta)\\
&\quad=\int_{\mathbf{X}_2'} d\mu_2'(\sigma,\xi)\prod_{j\in A}\sigma(j)\prod_{k\in B\setminus\{x\}}\xi(k)\frac{(p/q)^{n(k,\xi)}}{1+(p/q)^{n(k,\xi)}}\\
&\qquad\times\prod_{l\in C\setminus\{y\}}\xi(l)\frac{(p/q)^{-n(l,\xi)}}{1+(p/q)^{-n(l,\xi)}}\\
&\qquad\times\xi(x)\xi(y)p(x,y)\frac{(p/q)^{n(x,\xi)}}{1+(p/q)^{n(x,\xi)}}\frac{(p/q)^{-n(y,\xi)}}{1+(p/q)^{-n(y,\xi)}}\\
&\quad=\int_{\mathbf{X}_3'} d\mu_3'(\sigma,\gamma,\zeta)p(y,x)\gamma(y)\zeta(x)f_{A,B,C}(\sigma,\gamma^{x,y},\zeta^{x,y}),
\end{aligned}
\tag{3.20}
$$

where the last identity follows from

$$\begin{aligned}(3.21)\qquad &\xi(x)\xi(x+1)p\frac{(p/q)^{n(x,\xi)}}{1+(p/q)^{n(x,\xi)}}\frac{(p/q)^{-n(x+1,\xi)}}{1+(p/q)^{-n(x+1,\xi)}}\\ &=\xi(x)\xi(x+1)q\frac{(p/q)^{-n(x,\xi)}}{1+(p/q)^{-n(x,\xi)}}\frac{(p/q)^{n(x+1,\xi)}}{1+(p/q)^{n(x+1,\xi)}}\end{aligned}$$

for $y = x + 1$ and similarly for $y = x - 1$. For $y \neq x \pm 1$, $p(x,y) = 0$ and (3.20) is automatic.

Equation (3.20) implies that μ'_3 is reversible for the generator L''_3 and, in particular, that

$$(3.22)\qquad \mu'_3 L''_3 f_{A,B,C} = 0.$$

Finally, (3.22) and (3.19) imply the proposition. □

LEMMA 3.23. *Let ν_2 be a measure on $\mathbf{X}_2$ such that the first marginal of ν_2 is ν_ρ and, for $f \in C(\mathbf{X})$, $\int d\nu_2(\sigma,\xi) f(\sigma + \xi) = \nu_\lambda f$. Let $n(x,\xi)$ be the signed number of ξ particles between* 0 *and x, as defined in* (3.11). *Then*

$$(3.24)\qquad \lim_{x\to\pm\infty}\frac{n(x,\xi)}{x} = \lambda - \rho \qquad \nu_2\text{-}a.s.$$

PROOF. Since, under ν_2, σ is distributed according to ν_ρ and $\sigma + \xi$ is distributed according to ν_λ, we have that

$$\begin{aligned}(3.25)\qquad &\lim_{x\to\pm\infty}\frac{n(x,\sigma)}{x} = \rho \qquad \nu_2\text{-a.s.},\\ &\lim_{x\to\pm\infty}\frac{n(x,\sigma+\xi)}{x} = \lambda \qquad \nu_2\text{-a.s.}\end{aligned}$$

Calling $E = \{\lim n(x,\xi)/x = \lambda - \rho\}$, (3.25) implies $\nu_2(E) = 1$. □

LEMMA 3.26. *Let ν_2 be a measure on $\mathbf{X}_2$ such that the first marginal of ν_2 is ν_ρ and, for $f \in C(\mathbf{X})$, $\int d\nu_2(\sigma,\xi) f(\sigma+\xi) = \nu_\lambda f$. Let f_A and g_B be cylindrical functions on $\mathbf{X}_2 = \{(\sigma,\xi)\}$ depending on the finite sets of coordinates A and B, respectively, such that f_A depends only on σ, while g_B depends only on $\sigma + \xi$. Then,*

$$(3.27)\qquad \lim_{x,y\to\infty}\nu_2(\tau_{-x} f_A \tau_y g_B) = \nu_\rho f_A \nu_\lambda g_B.$$

PROOF. Fix x and consider a subsequence $\{y_{(l)}\}$ such that the limit on the left-hand side of (3.27) exists and $\tau_{y(l)}B \cap \tau_{y(l+1)}B = \varnothing$. Calls this limit a_x. The Cesaro limit in l converges also to a_x,

$$(3.28)\qquad a_x = \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\nu_2(\tau_{-x} f_A \tau_{y(l)} g_B) = \nu_2(\tau_{-x} f_A)\nu_\lambda g_B,$$

where the second identity follows from the following: (a) under ν_2, $\sigma + \xi$ is distributed according to ν_λ; (b) g_B depends only on $(\sigma + \xi)$; (c) the law of large numbers for ν_λ. This implies that the limit in y exists and equals the right-hand side of (3.28), uniformly in x. The same argument for the limit in x shows the lemma. □

COROLLARY 3.29. *Lemmas* 3.23 *and* 3.26 *hold for any measure absolutely continuous w.r.t.* ν_2.

PROOF. The proofs of Lemmas 3.23 and 3.26 use only a law of large numbers, that also holds for the measures of the corollary. □

PROOF OF THEOREM 2.3. Define the measure μ on $\mathbf{X}$ by

$$\int_{\mathbf{X}} d\mu(\eta)\, f(\eta) = \int_{\mathbf{X}'_3} d\mu'_3(\sigma,\gamma,\zeta)\, f(\sigma+\gamma). \tag{3.30}$$

By Proposition 3.14, if $(\sigma_0, \gamma_0, \zeta_0)$ has distribution μ'_3, then $\tau_{X(t)}\eta_t = \sigma'_t + \gamma'_t$ has distribution μ for all $t \geq 0$.

Now we only need to prove (2.4). Assume $f_A(\eta) = \prod_{y\in A}\eta(y)$, for finite $A \subset \mathbb{Z}$. Use Remark 3.12 to compute

$$\begin{aligned}\int_{\mathbf{X}} d\mu(\eta)\, f_A(\tau_x\eta) &= \int_{\mathbf{X}'_3} d\mu'_3(\sigma,\gamma,\zeta) \prod_{y\in A}\left(\sigma(y+x) + \gamma(y+x)\right)\\ &= \int_{\mathbf{X}'_2\times\mathbf{X}} d(\mu'_2\times\alpha)(\sigma,\xi,\kappa)\\ &\qquad \times \prod_{y\in A}\left(\sigma(y+x) + \xi(y+x)\kappa(y+x,\xi)\right),\end{aligned}$$

which as $x \to \infty$ converges to $\nu_\lambda f_A$ by Corollary 3.29, the fact that $\lim_{x\to\infty}\kappa(x,\xi) = 1$ α-a.s. and dominated convergence. The same proof works for negative x because $\lim_{x\to-\infty}\kappa(x,\xi) = 0$ α-a.s. □

In the following lemma we prove that the distance between the interface and the leftmost γ-particle has finite expectation. This implies that the process as seen from the leftmost second class particle admits an invariant measure with the asymptotic properties of μ'_3.

LEMMA 3.31. *Let* $Z(\sigma,\gamma,\zeta)$ [*respectively*, $G(\sigma,\gamma,\zeta)$] *be the position of the rightmost* (*leftmost*) ζ *particle* (γ *particle*) *in the configuration* (σ,γ,ζ). *Then* $\mu'_3(Z) = \mu'_3(-G) < \infty$.

PROOF. The distributions of Z and $-G$ coincide as a consequence of the translation invariance of μ_2 and Remark 3.12. Let K be the label of the rightmost ξ particle that is chosen to be a ζ particle. Then, from Remark 3.12,

$$\mu'_3(K=k) = \frac{1}{1+(p/q)^k}\prod_{i>k}\frac{1}{1+(p/q)^{-i}},$$

from where $\mu'_3 K < \infty$. On the other hand, letting A_i be the distance between the $(i-1)$-th and the i-th ξ particles to the right of the origin,

$$\mu'_3 Z = \mu'_3 \sum_{i=1}^{K} A_i = \mu'_3 K \mu'_3 A_1 = \mu'_3 K \frac{1}{(\lambda - \rho)},$$

where the second equality follows from the independence between A_i and K and the third one from the translation invariance of μ_2. □

4. Law of large numbers. First we prove a law of large numbers for the flux of particles through the origin. In this section we work in the space $\mathbf{X}_2 = \{(\sigma, \xi)\}$. It is convenient to use a notation for the holes (the sites where there are no particles): Let $\delta(x) := 1 - \xi(x) - \sigma(x)$. Notice that the δ particles evolve according to a simple exclusion process with rate q to jump to the right and p to jump to the left. Define now the total flux of holes, first class particles and second class particles:

$$\begin{aligned} &F_t^0 := \text{algebraic number of } \delta\text{-particles that crossed } \tfrac{1}{2} \text{ in the interval } [0,t]; \\ (4.1)\quad &F_t^1 := \text{algebraic number of } \sigma\text{-particles that crossed } \tfrac{1}{2} \text{ in the interval } [0,t]; \\ &F_t^2 := \text{algebraic number of } \xi\text{-particles that crossed } \tfrac{1}{2} \text{ in the interval } [0,t]. \end{aligned}$$

LEMMA 4.2. *Let ν_2 be a measure on $\mathbf{X}_2$ such that the first marginal of ν_2 is ν_ρ and, for $f \in C(\mathbf{X})$, $\int d\nu_2(\sigma, \xi) f(\sigma + \xi) = \nu_\lambda f$. Then,*

$$\begin{aligned} &\lim_{t\to\infty} \frac{F_t^0}{t} = -\lambda(1-\lambda)(p-q), \\ (4.3)\quad &\lim_{t\to\infty} \frac{F_t^1}{t} = \rho(1-\rho)(p-q), \\ &\lim_{t\to\infty} \frac{F_t^2}{t} = (\lambda-\rho)(1-\lambda-\rho)(p-q), \end{aligned}$$

P_{ν_2}-almost surely.

PROOF. Since the total flux must be null, we have that

$$F_t^0 + F_t^1 + F_t^2 = 0. \tag{4.4}$$

Hence, the third line in (4.3) follows from the first two lines in the same equation. Notice that the limit in the second line of (4.3) depends on the first marginal of the process that has distribution P_{ν_ρ}. The law of large numbers for F_t^1/t can be proven using the ergodic theorem because ν_ρ is extremal invariant (see Theorem 4 of [13] or [9]). For the first line, it suffices to observe that the initial distribution of the holes is $\nu_{(1-\lambda)}$ and they evolve as an asymmetric exclusion process constructed with q and p instead of p and q. The conclusion follows from the same argument. □

THEOREM 4.5. *Weak law of large numbers. Let ν_2 be a measure on $\mathbf{X}_2$ such that the first marginal of ν_2 is ν_ρ and, for $f \in C(\mathbf{X})$, $\int d\nu_2(\sigma, \xi) f(\sigma + \xi) = \nu_\lambda f$. Let $X(t)$ be the position at time t of a second class particle that at time $t = 0$ is at the origin. Then,*

$$\lim_{t \to \infty} \frac{X(t)}{t} = v$$

in $P_{\nu_2'}$-probability, where $\nu_2' := \nu_2(\cdot \mid \xi(0) = 1)$ and $v = (1 - \rho - \lambda)(p - q)$.

PROOF. Since particles cannot be overcome, we have the identity

$$F_t^2 = n(X(t), \xi_t), \tag{4.6}$$

where F_t^2 is the flux of ξ particles, while $n(X(t), \xi_t)$ is the number of ξ particles between 0 and $X(t)$, as defined in (4.1) and (3.11), respectively. Now, by Lemma 4.2, F_t^2/t converges a.s. to $(p - q)(\lambda - \rho)(1 - \lambda - \rho)$. Since the laws of σ_t and δ_t do not depend on t, we have for any positive c,

$$\lim_{a \to \infty} P_{\nu_2}\left(\sup_{x:\, |x - b| \geq a} \left| \frac{n(x, \xi_t) - n(b, \xi_t)}{x - b} - (\lambda - \rho) \right| \geq c \right) = 0, \tag{4.7}$$

uniformly on b and t.

Hence, if $A_t := \{|X(t)/t - v| \geq c\}$, we also have

$$\frac{F_t^2 - n(vt, \xi)}{t} = \frac{n(X(t), \xi_t) - n(vt, \xi_t)}{t},$$

so that the left-hand side converges in probability to zero whereas by Lemmas 3.23 and 4.2 on A_t the right-hand side is larger than $c(\lambda - \rho)$. So,

$$\lim_{t \to \infty} P_{\nu_2}(A_t) = 0 = \lim_{t \to \infty} P_{\nu_2'}(A_t),$$

where the second identity follows by absolute continuity. This proves the theorem. □

In order to prove Theorem 2.6, we let $\mu_2' = \int_I \mu_\beta' m(d\beta)$, where I is a set of labels, m is a measure on I and the μ_β' are the extremal invariant measures for the process with generator L_2'. Next, we prove a strong law for each extremal invariant measure.

PROPOSITION 4.8. *The process $X(t)/t$ converges $P_{\mu_\beta'}$-a.s. to a constant v_β.*

PROOF. Write

$$X(t) = N_t(\sigma, -1) + N_t(\sigma, 1) + N_t(\xi, 1) + N_t(\xi, -1),$$

where $N_t(\cdot, \cdot)$ are measurable with respect to the natural sigma fields associated to the process with generator L'_2 and defined by

$$N_t(\sigma, j) := \text{number of } \sigma \text{ particles that jumped from } j \text{ to } 0 \text{ in } [0, t],$$

$$N_t(\xi, j) := \text{number of jumps of the tagged particle from } 0 \text{ to } j \text{ in } [0, t].$$

The fact that μ'_β is extremal leads to the $(P_{\mu'_\beta})$ strong law of large numbers for $N_t(\cdot, j)$ (see [13] or [9]). □

PROOF OF THEOREM 2.6. Let $A_n = \{\beta: |v_\beta - v| > 2/n\}$ and write

$$P_{\mu'_2}\left(\left|\frac{X(t)}{t} - v\right| > \frac{1}{n}\right) \geq \int_{A_n} P_{\mu'_\beta}\left(\left|\frac{X(t)}{t} - v\right| > \frac{1}{n}\right) m(d\beta),$$

and, taking $t \to \infty$, Theorem 4.5 implies that $m(A_n) = 0$ for all n. Hence, $m\{\beta: v_\beta \neq v\} = 0$. Finally,

$$\begin{aligned} P_{\mu'_2}\left(\lim_{t\to\infty}\left|\frac{X(t)}{t} - v\right| \neq 0\right) &= \int_I P_{\mu'_\beta}\left(\lim_{t\to\infty}\left|\frac{X(t)}{t} - v\right| \neq 0\right) m(d\beta) \\ &= m\{\beta: v_\beta \neq v\} = 0, \end{aligned}$$

which concludes the proof. □

5. Hydrodynamics.

PROOF OF THEOREM 2.7. Let $A \subset \mathbb{Z}$ be a finite set and let $f_A(\eta) := \prod_{x \in A} \eta(x)$.

$$\begin{aligned} &\int_{\mathbf{X}} d\mu(\eta) E_\eta \tau_{\varepsilon^{-1}r} f_A(\eta_{\varepsilon^{-1}t}) \\ &\quad = \int_{\mathbf{X}'_3} d\mu'_3(\sigma, \gamma, \zeta) E_{(\sigma,\gamma,\zeta)} \tau_{\varepsilon^{-1}r} f_A(\sigma_{\varepsilon^{-1}t} + \gamma_{\varepsilon^{-1}t}) \\ (5.1) &\quad = \int_{\mathbf{X}'_3} d\mu'_3(\sigma, \gamma, \zeta) E_{(\sigma,\gamma,\zeta)} \tau_{\varepsilon^{-1}r - X(\varepsilon^{-1}t)} f_A(\sigma'_{\varepsilon^{-1}t} + \gamma'_{\varepsilon^{-1}t}) \\ &\quad = \int_{\mathbf{X}'_3} d\mu'_3(\sigma, \gamma, \zeta) E_{(\sigma,\gamma,\zeta)} \sum_x \tau_{\varepsilon^{-1}r - x} f_A(\sigma'_{\varepsilon^{-1}t} + \gamma'_{\varepsilon^{-1}t}) 1\{X(\varepsilon^{-1}t) = x\}, \end{aligned}$$

where E_η is the expected value of the process with initial configuration η, etc. Now, consider a number a such that $|r - vt| > a$, and decompose the sum in the last line of (5.1) in three parts: $\{x: |\varepsilon^{-1}r - x| \leq \varepsilon^{-1}a\}$, $\{x: \varepsilon^{-1}r - x > \varepsilon^{-1}a\}$ and $\{x: \varepsilon^{-1}r - x < -\varepsilon^{-1}a\}$. The integral of the first of those three sums goes to zero by Theorem 4.5 and dominated convergence. As in the proof of Theorem 2.3, by Corollary 3.29 and the fact that as $x \to \infty$, $\kappa(x, \xi) \to 1$, the second part converges to $\nu_\lambda f_A$ if $r > vt$ and to zero otherwise. In the same way, using the fact that as $x \to -\infty$, $\kappa(x, \xi) \to 0$, the third part converges to $\nu_\rho f_A$ if $r < vt$ and to zero otherwise. □

REMARK 5.2. The proof of Theorem 2.7 also works for initial measures other than μ. In fact, the proof is based on Theorem 4.5 and Corollary 3.29, which only use the marginal distributions of σ and $\sigma + \xi$ for the system of two particles and the asymptotic behavior of $\kappa(x, \xi)$. The initial measure for the three-particle process can be any measure ν_3 on $\mathbf{X}_3$ defined from a measure ν_2 on $\mathbf{X}_2$ with the good marginals [i.e., $\int d\nu_2(\sigma, \xi) f(\sigma) = \nu_\rho f$ and $\int d\nu_2(\sigma, \xi) f(\sigma + \xi) = \nu_\lambda f$] by the same transformation that constructed μ'_3 from μ'_2 (see Remark 3.12). Given a configuration (σ, ξ), the distribution of the γ and ζ particles is reversible with respect to the generator L''_3 of (3.17). This implies that this distribution is independent of time. This fact allows us to consider as initial measure ν any measure that comes from a ν_3 as μ comes from μ'_3. In particular, let ν_2 be the product measure on $\mathbf{X}_2$ defined as in the proof of Lemma 3.3, $\nu'_2 := \nu_2(\cdot \mid \xi(0) = 1)$, let ν'_3 on $\mathbf{X}_3$ be obtained from ν'_2 as in Remark 3.12 and let ν be the measure defined from ν'_3 as μ is defined from μ'_3. It is easy to see that ν is equivalent to $\nu_{\lambda,\rho}$ ($\nu_{\nu,\rho} \ll \nu$ and $\nu \ll \nu_{\lambda,\rho}$). In this way we can recover the hydrodynamics proven by [5] for the case $\rho < \lambda$.

6. Weakly asymmetric asymptotic profile. The measures considered below were defined in Section 3. We add a subscript ε indicating that they are constructed with $p = 1/2 + \varepsilon(\theta/2)$.

PROOF OF THEOREM 2.14. We consider ϕ as the indicator function of the box $[a, b]$, the extension to other functions being immediate. Let $\mathbf{Y} \subset \mathbf{X}_2$ be the set ($[\cdot]$ is the integer part)

$$\begin{aligned}
\mathbf{Y} := &\left\{(\sigma, \xi):\ \lim_{\varepsilon \to 0} \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \sigma(x) = \rho(b-a)\right\} \\
&\cap \left\{(\sigma, \xi):\ \lim_{\varepsilon \to 0} \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \xi(x) = (\lambda - \rho)(b-a)\right\} \\
&\cap \bigcap_{r \in [a,b] \cap \mathbb{Q}} \left\{(\sigma, \xi):\ \lim_{\varepsilon \to 0} \varepsilon n(\varepsilon^{-1} r, \xi) = (\lambda - \rho) r\right\} \\
:= &\ \mathbf{Y}_1 \cap \mathbf{Y}_2 \cap \mathbf{Y}_3.
\end{aligned}$$

Observe that, by Lemma 3.3,

$$\mu_{2,\varepsilon}(\mathbf{Y}_1) = \nu_\rho \left\{ \lim_{\varepsilon \to 0} \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \sigma(x) = \rho(b-a)\right\} = 1. \tag{6.1}$$

On the other hand,

$$\mathbf{Y}_2 \supset \mathbf{Y}_1 \cap \mathbf{Y}_4, \tag{6.2}$$

where

$$\mathbf{Y}_4 := \left\{ (\sigma, \xi) \colon \lim_{\varepsilon \to 0} \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} (\sigma(x) + \xi(x)) = \lambda(b-a) \right\}.$$

This, Lemma 3.3 and (6.1) imply that

$$\mu_{2,\varepsilon}(\mathbf{Y}_2) = \mu_{2,\varepsilon}(\mathbf{Y}_4) = \nu_\lambda \left\{ \eta \colon \lim_{\varepsilon \to 0} \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \eta(x) = \lambda(b-a) \right\} = 1. \tag{6.3}$$

Finally, from Lemma 3.23, it follows that

$$\mu_{2,\varepsilon}(\mathbf{Y}_3) = 1. \tag{6.4}$$

Using (3.30) and Remark 3.12, compute

$$\begin{aligned} &\mu_\varepsilon |Y_\varepsilon(\phi, \eta) - Y(\phi)| \\ &\quad = \mu'_{3,\varepsilon} |Y_\varepsilon(\phi, \sigma + \gamma) - Y(\phi)| \\ &\quad = (\mu'_{2,\varepsilon} \times \alpha_\varepsilon) \Bigg| \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} (\sigma(x) + \xi(x)\kappa_\varepsilon(n(x,\xi))) \\ &\qquad\qquad - \left((b-a)\rho + \int_a^b \frac{\lambda - \rho}{1 + e^{2\theta r\lambda - \rho}}\, dr \right) \Bigg|, \end{aligned} \tag{6.5a}$$

where $\kappa_\varepsilon(y)$ is the family of independent Bernoulli random variables with distribution α_ε, such that

$$\alpha_\varepsilon(\kappa_\varepsilon(y) = 1) = \left(1 + \left(\frac{1 + \varepsilon\theta}{1 - \varepsilon\theta} \right)^{-y} \right)^{-1},$$

defined in Remark 3.12, for $p = 1/2 + (\theta/2)\varepsilon$. The last member of (6.5a) is bounded by

$$\begin{aligned} &\mu'_{2,\varepsilon} \left| \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \sigma(x) - (b-a)\rho \right| \\ &\quad + (\mu'_{2,\varepsilon} \times \alpha_\varepsilon) \left| \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} (\xi(x) - (\lambda - \rho))\kappa_\varepsilon(n(x,\xi)) \right| \\ &\quad + (\mu'_{2,\varepsilon} \times \alpha_\varepsilon) \Bigg| (\lambda - \rho) \Bigg(\varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \kappa_\varepsilon(n(\dot{x},\xi)) \\ &\qquad\qquad - \int_a^b \frac{1}{1 + e^{2\theta r(\lambda - \rho)}}\, dr \Bigg) \Bigg|. \end{aligned} \tag{6.5b}$$

Since $\mu'_{2,\varepsilon} = \mu_{2,\varepsilon}(\cdot\,|\xi(0) = 1)$ and $\mu_{2,\varepsilon}(\xi(0) = 1) > 0$, we can consider $\mu_{2,\varepsilon}$ instead of $\mu'_{2,\varepsilon}$. As $\varepsilon \to 0$, the first line in (6.5b) vanishes by equation (6.1) and dominated convergence. In the second line, since $\kappa_\varepsilon(y) \le 1$, use (6.3) to prove that also this line goes to zero. The third line is bounded by

$$\begin{aligned} &\varepsilon^{-1}(\lambda - \rho) \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} |(\mu'_{2,\varepsilon} \times \alpha_\varepsilon)(\kappa_\varepsilon(n(x,\xi)) - \kappa_\varepsilon([x(\lambda - \rho)]))| \\ &\quad + (\lambda - \rho)(\mu'_{2,\varepsilon} \times \alpha_\varepsilon)\left| \varepsilon^{-1} \sum_{x=[a\varepsilon^{-1}]}^{[b\varepsilon^{-1}]} \kappa_\varepsilon([x(\lambda - \rho)]) - \int_a^b \frac{1}{1 + e^{2\theta r(\lambda - \rho)}}\, dr \right|. \end{aligned} \tag{6.6}$$

The second line in (6.6) goes to zero by the law of large numbers for independent random variables. Since $\kappa_\varepsilon(\cdot)$ are zero–one random variables, each term in the sum of the first line of (6.6) is bounded by

$$\left| \left(1 + \left(\frac{1 + \varepsilon\theta}{1 - \varepsilon\theta}\right)^{-n(x,\xi)}\right)^{-1} - \left(1 + \left(\frac{1 + \varepsilon\theta}{1 - \varepsilon\theta}\right)^{-x(\lambda - \rho)}\right)^{-1} \right|,$$

which goes to zero by (6.4). The sum then goes to zero by (6.4) and dominated convergence. This proves (2.16). The proof of (2.15) is easier.

For (2.17), compute, using (3.30) $[f_A = \prod_{x \in A} \eta(x)]$,

$$\begin{aligned} \tau_{\varepsilon^{-1}r}\mu_\varepsilon f_A &= \mu'_{3,\varepsilon} \prod_{y \in A} \left(\sigma([y + \varepsilon^{-1}r]) + \gamma([y + \varepsilon^{-1}r])\right) \\ &= (\mu'_{2,\varepsilon} \times \alpha_\varepsilon) \prod_{y \in A} \Big(\sigma([y + \varepsilon^{-1}r]) \\ &\qquad + \xi([y + \varepsilon^{-1}r])\kappa_\varepsilon(n([y + \varepsilon^{-1}r], \xi))\Big). \end{aligned} \tag{6.7}$$

Now, the law of large numbers for $n(x, \xi)$ of Lemma 3.23, Corollary 3.29 and the absolute continuity of $\mu'_{2,\varepsilon}$ w.r.t. $\mu_{2,\varepsilon}$, imply that $\mu'_{2,\varepsilon}(\sigma([\varepsilon^{-1}r])) \to \rho$, $\mu'_{2,\varepsilon}(\xi([\varepsilon^{-1}r])) \to \lambda - \rho$ and $n([\varepsilon^{-1}r], \xi)\varepsilon \to r(\lambda - \rho)$. Hence, for $p = 1/2 + \varepsilon(\theta/2)$, as $\varepsilon \to 0$, $(p/q)^{n([\varepsilon^{-1}r],\xi)} \to e^{2\theta r(\lambda-\rho)}$, $\mu'_{2,\varepsilon}$-almost surely and the last line of (6.7) converges to the solution u of Theorem 2.14 by dominated convergence. □

7. Weakly asymmetric hydrodynamics.

The proof of Theorem 2.18 is very similar to that of Theorem 2.7. The important observation is that in order to prove Theorem 4.5 it was only necessary to have a *weak* law of large numbers for the flux of ξ particles. But this was a consequence of the laws of large numbers for the fluxes of σ and δ particles, which move according to simple asymmetric exclusion process. Let Q^ε be the measure for the weakly asymmetric simple exclusion process [i.e., a process with generator as in (2.1) constructed with $p = 1/2 + \varepsilon(\theta/2)$] with initial measure ν_ρ, and let F_t be the algebraic number of particles that crossed the point $\frac{1}{2}$ during the time interval $[0, t]$.

THEOREM 7.1. *For all positive c, the following holds*:

$$\lim_{\varepsilon \to 0} Q^{\varepsilon}\big(|\varepsilon F_{t\varepsilon^{-2}} - \theta\rho(1-\rho)t| > c\big) = 0.$$

PROOF. Adapting to our case the proof of Theorem 2 of [13], it is not hard to prove that $\epsilon F_{(t+s)\varepsilon^{-2}} - \varepsilon F_{t\varepsilon^{-2}}$ and $\varepsilon F_{t\varepsilon^{-2}}$ are negatively associated random variables. This implies that $V^{\varepsilon}F_t/t$ is decreasing in t, where V^{ε} denotes the variance w.r.t. Q^{ε}. So that, if $w < t\varepsilon^{-2}$,

$$\frac{V^{\varepsilon}F_{\varepsilon^{-2}t}}{t\varepsilon^{-2}} \le \frac{V^{\varepsilon}F_w}{w}. \tag{7.2}$$

Besides, Q^{ε} converges weakly, as $\varepsilon \to 0$, to Q^0, the law of the symmetric simple exclusion process. On the other hand, $Q^{\varepsilon}(F_w)^4$ is bounded uniformly in ε, by comparing to a Poisson process with rate 2. So we have that, as $\varepsilon \to 0$, $V^{\varepsilon}F_w$ converges to V^0F_w, which, by Arratia [4], behaves like $\sqrt{w}$. (Here we used that the flux is bounded in absolute value by the position of the tagged particle.) The theorem is now a consequence of (7.2), by letting first $\varepsilon \to 0$ and then $w \to \infty$. □

The proof of the following corollary is the same as the proof of Theorem 4.5.

COROLLARY 7.3. *For all positive c, the following holds*:

$$\lim_{\varepsilon \to 0} Q^{\varepsilon}\big(|\varepsilon X(t\varepsilon^{-2}) - \theta(1-\lambda-\rho)| > c\big) = 0.$$

PROOF OF THEOREM 2.18. The proof follows from the weak law of large numbers of Corollary 7.3 and from the fact that

$$\alpha_{\varepsilon}\big(\kappa_{\varepsilon}([\varepsilon^{-1}r]) = 1\big) = \left(1 + \left(\frac{1+\varepsilon\theta}{1-\varepsilon\theta}\right)^{-[\varepsilon^{-1}r]}\right)^{-1}$$

converges uniformly in ε to $1/(1+e^{2\theta r(\lambda-\rho)})$. □

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