The Annals of Probability
1991, Vol. 19, No. 1, 221–225

THE RANGE OF A LÉVY PROCESS

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It is shown that all Lévy processes on the line whose paths are of bounded variation have a closed range over any finite time interval that is nowhere dense except for those processes having positive (negative) drift with Lévy measure finite on $(0, \infty)$ [finite on $(-\infty, 0)$].

We will consider a Lévy process $X(t)$ on the line with initial point being the origin 0. A problem of some interest is to determine the structure of the closure R_t of the range of the process over the interval $[0, t]$, $t > 0$. An important tool in the investigation of this problem are two results due to Barlow [1]. The first, known as Barlow's 0–1 law, asserts that for every $t > 0$, either R_t is nowhere dense with probability 1 or with probability 1 R_t contains an interval. The second asserts that if R_t contains an interval with probability 1 and 0 is regular for $(0, \infty)$ [respectively, for $(-\infty, 0)$], then R_t contains an interval of the form $(0, \delta]$ (respectively, $[-\delta, 0]$) with probability 1.

In this paper we will consider processes $X(t)$ having paths of bounded variation. We will show all such processes have R_t nowhere dense with probability 1 except those having either drift $b > 0$ and Lévy measure μ such that $\mu(0, \infty) < \infty$ or $b < 0$ and $\mu(-\infty, 0) < \infty$.

Investigations of the structure of R_t for certain processes $X(t)$ having paths of nonbounded variation were carried out by Kesten [6], Pruitt and Taylor [7] and Barlow [2]. These authors investigate processes having the property that 0 is regular for $\{0\}$. Such processes have at each point x a local time $l_x(t)$ with $t \to l_x(t)$ continuous. Kesten [6] shows that if $l_x(t)$ is jointly continuous in (x, t), then R_t contains an interval about 0 with probability 1. He also investigates certain processes close to asymmetric Cauchy processes and shows that these have R_t nowhere dense with probability 1. Pruitt and Taylor [7] show that the asymmetric Cauchy processes have nowhere dense range provided their Lévy measure is not one sided. In [3], the problem is posed of determining if all Lévy processes having 0 regular for $\{0\}$ with Lévy measure of both $(-\infty, 0)$ and $(0, \infty)$ infinite have R_t nowhere dense iff there is no version of $l_x(t)$ that is jointly continuous.

Let $X(t)$ be a Lévy process. Then the paths are of bounded variation iff

$$-\log Ee^{i\theta X(1)} = i\theta b + \int (e^{i\theta x} - 1)\mu(dx),$$

where $\int_{|x| \le 1} |x| \mu(dx) < \infty$. (See [5], page 279.)

Received October 1989; revised February 1990.
[1]Partially supported by NSF Grant DMS-86-01800.
AMS 1980 *subject classifications*. Primary 60J30, 60G17.
Key words and phrases. Lévy process, range.

THEOREM. *For every* $t > 0$, $X(t)$ *has* R_t *nowhere dense with probability* 1 *except if* (i) $b > 0$ *and* $\mu(0, \infty) < \infty$ *or* (ii) $b < 0$ *and* $\mu(-\infty, 0) < \infty$. *In these cases* R_t *contains an interval with probability* 1.

The proof will be carried out via a sequence of lemmas. For any x, let P_x be the law of the process starting at x. Set $P = P_0$. For a Borel set A, let $T_A = \inf\{t > 0: X(t) \in A\}$ ($= \infty$ if there is no such t).

LEMMA 1. *If either* (i) *or* (ii) *holds, then* R_t *is not nowhere dense.*

PROOF. Assume (i) holds. The time τ to the first positive jump is exponentially distributed with mean $\mu(0, \infty)^{-1}$. Rogazin's theorem (see [5], page 316, Theorem 1) shows that $X_t/t \to b$ with probability 1 as $t \downarrow 0$. We can thus find t such that $P(\tau \le t) \le \frac{1}{4}$ and $P(X_t > (b/2)t) > \frac{3}{4}$: Hence $P(R_t \supseteq (0, (b/2)t)) \ge P(X_t > (b/2)t) - P(\tau \le t) \ge \frac{1}{2}$. Barlow's 0–1 law now shows R_t cannot be nowhere dense. A similar argument shows that when (ii) holds, R_t is not nowhere dense. □

It is a known fact that if a process has paths that are cadilag and if it fails to hit points, then its closed range over any finite time interval is nowhere dense. However, we have not been able to locate this fact in the literature. Since we need this fact we will take this opportunity to place it in the literature. This will be done in the Appendix.

LEMMA 2. *If* $b = 0$, *then* R_t *is nowhere dense.*

PROOF. If $b = 0$ and $\mu(-\infty, \infty) < \infty$, the process is a compound Poisson process. Its range R_t is then a finite set. If $\mu(-\infty, \infty) = \infty$ the process fails to hit points (see [4]). In that case the Lebesgue measure of its range is 0. Since the paths are cadilag it follows that R_t must be nowhere dense. (See Proposition 3 of the Appendix.) □

Henceforth we consider processes with $b > 0$ and $\mu(0, \infty) = \infty$. The same arguments applied to $-X(t)$ will establish the theorem for $b < 0$ and $\mu(-\infty, 0) = \infty$.

Let $\gamma > 0$. The process $X(t)$ is distributed like the sum of two independent processes $Y(t)$ and $V(t)$, where

$$-\log Ee^{i\theta Y(1)} = i\theta b + \int_{-\infty}^{\gamma} (e^{i\theta x} - 1)\mu(dx)$$

and

$$-\log Ee^{i\theta V(1)} = \int_{\gamma}^{\infty} (e^{i\theta x} - 1)\mu(dx).$$

Let $\tau_\gamma = \inf\{t: X(t) - X(t-) > \gamma\}$. Then τ_γ and Y are independent and τ_γ is exponentially distributed with mean $\mu(\gamma, \infty)^{-1}$. Hence $P(\tau_\gamma < \infty) = 1$ for all γ

sufficiently small, say $\gamma \le \gamma_0$. For such γ let

$$Z(t) = Y(\tau_\gamma - t) - Y(\tau_{\gamma-}) \quad \text{on } 0 < t < \tau_\gamma.$$

LEMMA 3. *The process $Z(t)$ is distributed like the process $-Y(t)$ killed at an independent exponential time T distributed like τ_γ.*

PROOF. Straightforward computations show the finite marginals of the Z process are the same as those for the Y process killed at time T from which the result follows. □

LEMMA 4. *Let γ_0 be such that $P(\tau_\gamma < \infty) = 1$ for $\gamma \le \gamma_0$ and let $\varepsilon > 0$. There are then a $t_0 > 0$ and a $\gamma_1 > 0$, $\gamma_1 \le \gamma_0$ such that for $\gamma \le \gamma_1$,*

$$P\Big(R_{t_0} \cap \big[(X(\tau_{\gamma-}), X(\tau_\gamma))\big] = \varnothing, \tau_\gamma \le t_0\Big) \ge 1 - \varepsilon. \tag{1}$$

PROOF. Let $A_\gamma = [R_{\tau_\gamma} \cap [(X(\tau_\gamma -), X(\tau_\gamma)) = \varnothing], \tau_\gamma \le t_0]$ and let $B_\gamma = [R(\tau_\gamma, t_0) \cap [(X(\tau_\gamma -), X(\tau_\gamma)) = \varnothing]]$, where $R(\tau_\gamma, t_0)$ is the range of $X(t)$ over the interval (τ_γ, t_0). Then

$$\begin{aligned} P\big(B_\gamma | \mathscr{F}_{\tau_\gamma}\big) &= P_{X_{\tau_\gamma}}\Big(R_{t_0 - \tau_\gamma} \cap \big(X(\tau_{\gamma-}), X(\tau_\gamma)\big) = \varnothing\Big) 1_{[\tau_\gamma \le t_0]} + 1\big[\tau_\gamma > t_0\big] \\ &\ge P_{X_{\tau_\gamma}}\Big(R_{t_0} \cap \big(-\infty, X(\tau_\gamma)\big) = \varnothing\Big) = P\big(T_{(-\infty,0)} > t_0\big). \end{aligned} \tag{2}$$

Thus

$$P\big(B_\gamma | \mathscr{F}_{\tau_\gamma}\big) \ge P\big(T_{(-\infty,0)} > t_0\big). \tag{3}$$

Since $A_\gamma \in \mathscr{F}_{\tau_\gamma}$, it follows from (3) that

$$P\Big(R_{t_0} \cap \big(X(\tau_{\gamma-}), X(\tau_\gamma)\big) = \varnothing, \tau_\gamma \le t_0\Big) \ge P(A_\gamma) P\big(T_{(-\infty,0)} > t_0\big). \tag{4}$$

Now the range of X over $(0, \tau_\gamma)$ is the same as the range of Y over $(0, \tau_\gamma)$. Let $\tilde{T}_{(0,\infty)}$ be $T_{(0,\infty)}$ for the process $-X(t)$ and let $\tilde{T}^\gamma_{(0,\infty)} = \inf\{t > 0: Z(t) \in (0,\infty)\}$. Then using Lemma 3, we find

$$\begin{aligned} P(A_\gamma) &= P\Big(\big[\text{Range } Z \text{ over } (0,\tau_\gamma) \cap \big(X(\tau_{\gamma-}), X(\tau_\gamma)\big) = \varnothing\big], \tau_\gamma \le t_0\Big) \\ &\ge P\big(\tilde{T}^\gamma_{(0,\infty)} > \tau_\gamma, \tau_\gamma \le t_0\big) \\ &= \int_0^{t_0} P\big(\tilde{T}^\gamma_{(0,\infty)} > t\big) \lambda(\gamma) e^{\lambda(\gamma)t}\, dt \\ &\ge P\big(\tilde{T}^\gamma_{(0,\infty)} > t_0\big)\big[1 - e^{-\lambda(\gamma)t_0}\big], \end{aligned}$$

where $\lambda(\gamma) = \mu(\gamma, \infty)$. Since $P(\tilde{T}^\gamma_{(0,\infty)} > t_0) \ge P(\tilde{T}_{(0,\infty)} > t_0)$ we see that

$$P(A_\gamma) \ge P\big(\tilde{T}_{(0,\infty)} > t_0\big)\big[1 - e^{-\lambda(\gamma)t_0}\big]. \tag{5}$$

Since $b > 0$, 0 is irregular for $(-\infty, 0)$ for the X process and 0 is irregular for $(0,\infty)$ for the $-X$ process. Hence we can choose t_0 such that $P(\tilde{T}_{(\infty,0)} > t_0) \ge (1 - \varepsilon)^{1/3}$ and $P(T_{(-\infty,0)} > t_0) > (1 - \varepsilon)^{1/3}$. Since $\lambda(\gamma) \uparrow \infty$ as $\gamma \downarrow 0$, we can then choose $\gamma_1 \le \gamma_0$ such that $1 - e^{-\lambda(\gamma_1)t_0} > (1-\varepsilon)^{1/3}$. Using (4) and (5) we find that (1) holds. □

PROOF OF THE THEOREM. Fix $\delta > 0$. Then

$$\begin{aligned} P\big(R_{t_0} \supsetneqq (0,\delta)\big) &\ge P\big(R_{t_0} \cap \big(X(\tau_{\gamma-}), X(\tau_\gamma)\big) = \varnothing, \\ &\qquad X(\tau_\gamma) < \delta, \tau_\gamma \le t_0, T_{(-\infty,0)} > t_0\big) \\ &\ge P\big(R_{t_0} \cap \big(X(\tau_{\gamma-}), X(\tau_\gamma)\big) = \varnothing\big) \\ &\qquad - P\big(X(\tau_\gamma) \ge \delta\big) - P\big(T_{(-\infty,0)} \le t_0\big). \end{aligned} \tag{6}$$

By Lemma 4 we can find t_0 and $\gamma_1 \le \gamma_0$ such that $P(R_{t_0} \cap [X(\tau_{\gamma-}), X(\tau_\gamma)] = \varnothing, \tau_\gamma \le t_0) \ge 1 - \frac{1}{6}$. Since $P(T_{(-\infty,0)} \le t) \to 0$ as $t \downarrow 0$, we can assume that t_0 is also such that $P(T(-\infty,0) \le t_0) \le \frac{1}{6}$. Now $\tau_\gamma \to_P 0$ as $\gamma \to 0$. Hence there is a subsequence $\gamma_n \downarrow 0$ such that $P(\tau_{\gamma_n} \to 0) = 1$. By right continuity of $X(t)$, $P(X(\tau_{\gamma_n}) \to 0) = 1$. Therefore we can find $0 < \gamma \le \gamma_1$ such that $P(X(\tau_\gamma) \ge \delta) < \frac{1}{6}$. Using (6) we find that $P(R_{t_0} \supsetneqq (0,\delta)) \ge 1 - \frac{3}{6}$. Thus for any $\delta > 0$, $P(R_{t_0} \supsetneqq (0,\delta)) \ge \frac{1}{2}$. Hence $P(R_{t_0}$ does not contain an interval $(0,\delta)) = \lim_{n\to\infty} P(R_{t_0} \supsetneqq (0,(1/n))) \ge \frac{1}{2}$. Barlow's two results now show that $P(R_{t_0}$ is nowhere dense$) = 1$. □

APPENDIX

PROPOSITION 1. *Suppose x_t is defined on $[0,\infty]$ and is right continuous with left limits at each point. For $t > 0$, let $R_t' = \{x\colon x_s = x, 0 \le s \le t\}$ be the range of the function x_t on $[0,t]$. Then the closure R_t of R_t' is $S = \{x\colon x_s$ or $x_{s-} = x, 0 < s < t\} \cup \{x_0, x_t\}$.*

PROOF. Suppose $x_n \in R_t'$ and $x_n \to x$. Since $[0,t]$ is compact we can find a sequence $t_n \in [0,t]$ that converges to some $\tau \in [0,t]$ and $x(t_n) = x_n$. Since either $t_n \le \tau$ i.o. or $t_n \ge \tau$ i.o., we can assume that either $t_n < \tau$ for all n or $t_n \ge \tau$ for all n. In the first case $x = x_{\tau-}$ and in the second case $x = x_\tau$; thus $R_t \supseteq S$. Since x_t has left and right limits for all t, $S \subseteq R_t$. □

PROPOSITION 2. *Let x_t be a function as in Proposition 1. If R_t' has 0 Lebesgue measure, then R_t is nowhere dense.*

PROOF. Proposition 1 shows that R_t is obtained by adding all the left limit points of the function on $(0,t]$ to R_t. Since x_t has no discontinuities of the second kind, it can have for any finite t at most countably many jump discontinuities. Thus R_t must also have 0 Lebesgue measure. But then R_t can contain no open interval. Thus R_t is nowhere dense. □

PROPOSITION 3. (i) *If X_t is a standard stochastic process, $X_0 = 0$ and $P(X_t = x$ for some $t < \infty) = 0$ for all x, then the range of X_t on $[0,t]$ is nowhere dense with probability one.*

PROOF. (i) Let R be the entire range of X_t and let $|R|$ be its Lebesgue measure. Then $E|R| = \int_0^\infty P_0(X_t = x \text{ for some } t < \infty)\, dx = 0$. Hence $P(|R| = 0) = 1$. Proposition 2 now shows $P(R_t$ is nowhere dense$) = 1$. □

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