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INTERSECTION LOCAL TIMES FOR INFINITE SYSTEMS OF BROWNIAN MOTIONS AND FOR THE BROWNIAN DENSITY PROCESS

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The Brownian density process is a distribution-valued process that arises either via a limiting operation on an infinite collection of Brownian motions or as the solution of a stochastic partial differential equation. It has a (self-) intersection local time, that is formally defined through an operation involving delta functions, much akin to the better studied intersection local time of measure-valued ("super") processes. Our main aim is to show that this formal definition not only makes sense mathematically, but can also be understood, at least in two and three dimensions, via the intersection local times of simple Brownian motions. To show how useful this way of looking at the Brownian density intersection local time can be, we also derive a Tanaka-like evolution equation for it in the two-dimensional case.

1. Introduction. Let Π^λ be a Poisson point process on $\mathbb{R}^d$ of intensity λ, i.e., the number of points of Π^λ in a Borel set $A \subset \mathbb{R}^d$ is a Poisson random variable with parameter $\lambda|A|$, ($|\ |$ denotes Lebesgue measure), and the numbers in disjoint sets are independent. Since the probability that any two points of Π^λ lie exactly the same distance from the origin is 0, we can order them by magnitude, and shall denote them by $X_0^1, X_0^2, \dots$.

Let $X_t^1, X_t^2, \dots,\ t \geq 0$, be a sequence of independent, $\mathbb{R}^d$-valued Brownian motions, with initial values given by $X_0^1, X_0^2, \dots,$ and let $\sigma^1, \sigma^2, \dots$ denote a sequence of independent Rademacher random variables. ($P\{\sigma^i = +1\} = P\{\sigma^i = -1\} = \frac{1}{2}$.) The two sequences and Π^λ are assumed independent of one another except for the fact that Π^λ determines the initial values of the X^i.

For $\phi \in \mathscr{S}_d = \mathscr{S}(\mathbb{R}^d)$, the Schwartz space of infinitely differentiable functions on $\mathbb{R}^d$ decreasing rapidly at ∞, let η_t^λ be the $\mathscr{S}_d'$-valued random process defined by

$$\eta_t^\lambda(\phi) = \lambda^{-1/2} \sum_{i=1}^{\infty} \sigma^i \phi(X_t^i). \tag{1.1}$$

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[If one were to think of the random signs as signed particles moving throughout space according to independent Brownian motions, then $\eta_t^\lambda(1_A)$ would describe the average net charge in the set A at time t, if only it were true that indicator functions belonged to $\mathscr{S}_d$.]

The $\lambda \to \infty$ behavior of η_t^λ has been a subject of some considerable interest, and the most complete results can be found in Walsh (1986), where it is shown that η_t^λ converges in distribution in the Skorohod space $D([0,1], \mathscr{S}_d')$ to the solution of the stochastic partial differential equation (SPDE)

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{1}{2} \Delta \eta + \nabla \cdot W, \\ \eta_0 &= \tilde{\Pi}, \end{aligned} \tag{1.2}$$

where $\tilde{\Pi}$, a Gaussian white noise on $\mathbb{R}^d$, is the weak limit in $\mathscr{S}_d'$ of $\lambda^{-1/2}\sum_i \sigma^i \phi(X_0^i)$, and W is an $\mathbb{R}^d$-valued Gaussian white noise on $\mathbb{R}^d \times \mathbb{R}_+$. Equation (1.2) should be understood in the weak form developed in Walsh (1986): i.e., for every $\phi \in \mathscr{S}_d$,

$$\eta_t(\phi) = \tfrac{1}{2}\int_0^t \eta_s(\Delta\phi)\, ds + \int_0^t \int_{\mathbb{R}^d} \langle \nabla\phi(x), W(dx, ds)\rangle.$$

The solution η_t of (1.2) is called the *Brownian density process*. The SPDE is not quite as forbidding as it at first seems. It is easy to see that η_t^λ satisfies a similar equation by simply applying Itô's formula to (1.1), and then to obtain (1.2) by a formal passage to the limit. The Brownian density process seems to have been originally introduced into the probability literature by Martin-Löf (1976), and studied in depth, for the case $d = 1$, by Itô (1983). By choosing different values of the initial state η_0, the resulting Brownian density process corresponds to different centerings in functional central limit results of the kind we have just described.

There is another, perhaps mathematically less demanding, definition of the Brownian density process, based on the fact that it is Gaussian. We shall give this in the following section, and rely on it heavily for moment calculations.

We have two main aims in this paper. The first is to establish that the Brownian density process has an intersection local time up to dimension 3. The second, and more important, will be to try to understand what this local time represents, and how it is related to the individual Brownian motions that go toward making up (1.1). As a consequence of the latter, we shall obtain a Tanaka-like formula that describes the temporal evolution of the intersection local time of η_t. While this result is of considerable intrinsic interest, it is mainly included to show how much one can do, easily, from the understanding we shall develop.

To describe our results, we start with the intersection local time of η_t. Note first, however, that for a test function $\phi \in \mathscr{S}_d$ we shall take the liberty of writing

$$\begin{aligned} \eta_t(\phi) &= \eta_t(\phi(x)) \\ &= \langle \phi, \eta \rangle \\ &= \int_{\mathbb{R}^d} \phi(x)\eta_t(x)\, dx = \int \phi(x)\eta_t(x)\, dx. \end{aligned} \tag{1.3}$$

Let $\phi = \phi(x, y)$ be a test function in $\mathscr{S}_{2d}$. Our first task will be to show that

$$(1.4) \qquad (\eta_s \times \eta_t)(\phi(x,y)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,y)\eta_s(x)\eta_t(y)\,dx\,dy$$

makes sense, and yields a $(2d)$-dimensional tempered distribution. This is reasonably straightforward.

Now let $\delta(x) = \delta_d(x)$ denote the Dirac delta function on $\mathbb{R}^d$. We then have to show how to make mathematics out of the following expression, which we shall call the *intersection local time process* corresponding to the Brownian density process, and show that it exists as a $\mathscr{S}_d'$-valued stochastic process on [0, 1]:

$$(1.5) \qquad \Psi_t(\phi) := \int_0^t du \int_0^u dv \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\eta_u \otimes \eta_v)(\delta(x-y))\phi(x)\,dx\,dy.$$

The switch from the regular product of distributions in (1.4) to the $\otimes$ product in (1.5) is not accidental. Making mathematics out of (1.5) with the regular product requires a certain renormalization, which we indicate by a change of notation. Full details are given in Sections 2 and 4.

There are essentially three different cases to be considered in constructing the intersection local time of the Brownian density process. When $d = 1$, then Ψ_t exists, for each $t \in [0, 1]$, as an ordinary function, and there is no need to treat it as a distribution. When $d = 2, 3$ it exists as a random distribution, and for $d \geq 4$ there is no renormalization that allows a straightforward analog of (1.5) to be defined. Higher dimensions, as well as higher orders of intersection, require a more delicate analysis, details of which can be found in Adler and Rosen (1990). This is consistent with well-known results on the intersection local time of d-dimensional Brownian motions, and this connection will become clearer later.

A general word of warning to the reader, regarding the issue of dimension. The only results in this paper that always hold are the planar ones, and, in many ways, the easiest way to read this paper is to imagine that $d = 2$ throughout. Very often we have written results also for $d = 3$ or for general d. The rule we have followed has been to state each result at the highest level of generality for which the proof given works *without adjustment*. When extra work is required for a more general, or slightly different result, we shall say so explicitly.

Intersection local times of the above form, for measure-valued process, have been the subject of intense recent interest, commencing with the work of Perkins (1988) and followed by Dynkin (1988a, b), Dawson, Iscoe and Perkins (1989) and others. The emphasis in Perkins' work was on the dimensionality of the support of the intersection local time, while Dynkin was primarily concerned with integral representations of the intersection local time itself. (In fact, in the planar case, we shall also obtain a representation for Ψ_t as a *process* in t, as a multiple stochastic integral, but of a somewhat different form.) In both of these works a structure akin to (1.5) was taken at face value as representing a generalization of the intersection local time concept to distribution or measure-valued processes, with no real justification other than "things seemed to work." Our aim is to show why this was the right thing to do.

To see how to do this, we return to the sum (1.1). We shall require another process of considerable independent interest, also defined in the same setup. Set

$$\mu_t^\lambda(\phi) = \lambda^{-1/2}\sum_{i=1}^{\infty}\sigma^i\int_0^t \phi(X_s^i)\,ds = \int_0^t \eta_s^\lambda(\phi)\,ds \tag{1.6}$$

for $\phi \in \mathscr{S}_d$. By taking $\phi = 1_A$, where $A \subset \mathbb{R}^2$ (and ignoring the fact that 1_A is not an element of $\mathscr{S}_d$), we see that $\mu_t^\lambda(1_A)$ describes the average "net charge" of the X^i in A until time t, and so we shall refer to μ_t^λ, and its $\lambda \to \infty$ limit, as *Brownian occupation processes*.

Since integration is a continuous functional in $D([0,1],\mathscr{S}_d')$ the limiting distribution of $\mu_t^\lambda(\phi)$, as a process in t, is clearly that of $\int_0^t\eta_s(\phi)\,ds$. It then follows (as, in fact, it does from a simple central limit theorem) that the limiting marginal distribution of μ_t^λ is that of a centered $\mathscr{S}_d'$-valued Gaussian random variable, which we shall denote by μ_t. Limit theorems of this form (modulo some minor technical differences—see the comments at the end of this section) were discussed in detail in Adler and Epstein (1987), where we also discussed more complicated, and more interesting, limit theorems for sums of additive functionals of quite general symmetric Markov processes. A particular, and important, special case was a limit theorem for sums of intersection local times of *planar* Brownian motions. The intersection local time of two such processes X and Y is defined formally as the continuous $\mathscr{S}_2'$-valued process

$$L_t(\phi; X, Y) = \int_0^t du\int_0^u \phi(Y_v)\delta(X_u - Y_v)\,dv, \tag{1.7}$$

where δ is the Dirac delta function, or, rigorously, as the $\mathscr{L}^2(P)$ limit, as $\varepsilon \to 0$, of

$$L_t^\varepsilon(\phi; X, Y) = \int_0^t du\int_0^u \phi(Y_v)e^{-\varepsilon/2}p_\varepsilon(X_u - Y_v)\,dv, \tag{1.8}$$

where p_t is the transition density of planar Brownian motion. (Recall that the transition density of d-dimensional Brownian motion is given by

$$p_t(x,y) = p_t(x-y) = \frac{1}{(2\pi t)^{d/2}}e^{-\|x-y\|^2/2t}, \tag{1.9}$$

where, hopefully, our use of the same function p_t to denote both a function on $\mathbb{R}^d \times \mathbb{R}^d$ and $\mathbb{R}^d$ will not lead to too much confusion.) The limit theorem studied, for fixed t, sums of the form

$$\Psi_t^\lambda(\phi) = \lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j L_t(\phi; X^i, X^j), \tag{1.10}$$

which were shown to converge in distribution, as $\lambda \to \infty$, to certain $\mathscr{S}_2'$-valued random variables which could be represented as double Wiener–Itô integrals of the Gaussian process μ_t.

The main result of this paper is that the weak, $\lambda \to \infty$, limit of Ψ_t^λ is the intersection local time Ψ_t. For us, this is what justifies calling (1.5) an intersection local time.

However, there are a number of other applications of this weak convergence. For example, we can derive without much pain a result somewhat akin to Tanaka's formula for the temporal development of the local time of a single Brownian motion, so that we shall be able to write Ψ_t in terms of an evolution equation driven by Gaussian white noises.

The evolution equation itself is of considerable interest. The process Ψ_t appears naturally in a model of interacting, signed particles [cf. Adler (1990)] and the evolution equation formulation adds insight into that model. Since the infinite particle limit in that model is a Euclidean field theory, the added insight extends beyond the specific prescriptions of that paper.

In the following two sections we shall present all our major results, with some discussion but without proofs. These follow in the remaining sections. The main technical tool used there is the weak convergence theory developed in Walsh (1986), and we are grateful to John Walsh for long ago providing us with a prepublication copy of his excellent set of notes.

The link between Brownian motion intersection local times and the intersection local time that we shall present is restricted to the two-dimensional case. It is not too hard to see that most of our arguments also extend to three dimensions, although there are some nontrivial technical problems to overcome on the way. The important fact to note, however, is that although some details change, the same intuition developed for the intersection local time of the planar Brownian density process carries over qualitatively to the three-dimensional one as well.

We close the Introduction with some technical notes, explaining why the processes considered here are directly comparable neither with those treated in Adler and Epstein (1987) nor with the measure-valued processes referred to above. The reader not interested in this can skip immediately to the following section without loss of continuity.

TECHNICAL ASIDES. First, we note that our use of the term "Brownian density process" is somewhat more restrictive than that of Walsh (1986) mentioned above. Walsh's version of the SPDE (1.2) has an extra term on the right-hand side, that comes from a certain branching mechanism that we have chosen not to include at this stage.

In Adler and Epstein (1987) and Adler (1990) our basic processes were general Markov processes with symmetric transition densities. Here we treat only Brownian motion, and so are, in essence, obtaining much more detail for a much smaller class of processes. In Adler and Epstein (1987) our processes were not started according to the points of a Poisson process, but either according to a (nonprobability but σ-finite) uniform measure on $\mathbb{R}^2$, or according to a rather awkward probabilistic way of spreading points out through a partition of $\mathbb{R}^2$ [also used in Adler (1989)]. We could have saved ourselves a substantial amount of trouble had we used the Poissonization trick, for no lack of realism in the model. Thus we do so here.

Furthermore, objects like $L_t(\phi; X, Y)$ were replaced by intersection local times of the form

$$L^*(\phi; X, Y) = \int_0^\infty \int_0^\infty e^{-u} e^{-v} \phi(Y_v) \delta(X_u - Y_v)\, du\, dv, \tag{1.11}$$

so that the corresponding limit theorems for objects like Ψ_t^λ are somewhat different. Because of the exponential weighting in (1.11), and the fact that the integration on u and v is over the entire real line, the results of this and the previous paper are not strictly comparable. Both, however, show similar phenomena. We emphasize again, however, that the results of the current paper, in so far as planar Brownian motions are concerned, give substantially more detailed information.

We also note that our processes and those of Perkins and Dynkin mentioned above are not strictly comparable, and, despite the fact that they are distributions rather than measures, are actually somewhat simpler to work with. (In the measure-valued setting, each one of our Brownian motions must be replaced with a branching Brownian motion with a branching rate that goes to ∞ as $\lambda \to \infty$.) Nevertheless, we thought we might start with the simpler case, both because of our original interest in it resulting from the interacting particle results described above and because, as will become clear in future sections, even this case is not all that easy.

2. The Brownian density process and its intersection local time. In this section we shall give a more self-contained definition of the Brownian density process than that of the Introduction, and also carefully formulate its intersection local time.

Rather than work with the SPDE (1.2), we shall now give an alternative definition of the Brownian density process η_t. This will be in a Gaussian rather than stochastic analysis framework.

All our calculations take place on a complete probability space $(\Omega, \mathscr{F}, P)$. Recall that p_t is the standard Brownian transition density (1.9).

Definition 2.1. The d-dimensional Brownian density process on $[0, 1]$ is the centered $\mathscr{S}'(\mathbb{R}^d)$ valued Gaussian process with covariances given by

$$E\{\eta_t(\phi_1) \cdot \eta_s(\phi_2)\} = \iint \phi_1(x) p_{t-s}(x, y) \phi_2(y)\, dx\, dy, \tag{2.1}$$

for $\phi_1, \phi_2 \in \mathscr{S}_d$ and $t > s$, and for $t = s$ by

$$E\{\eta_t(\phi_1) \cdot \eta_t(\phi_2)\} = \int \phi_1(x)\phi_2(x)\,dx. \tag{2.2}$$

Our first requirement toward defining an intersection local time for η_t is to properly define the product distribution $\eta_s \times \eta_t$ as a random distribution in $\mathscr{S}'_{2d}$. For fixed ω this is standard, since for $\phi \in \mathscr{S}_{2d}$ we can set

$$(\eta_s \times \eta_t)(\phi(x,y)) = \eta_t(\eta_s(\phi(x,y))), \tag{2.3}$$

where the inner operation $\eta_s(\phi(x,y))$ should be thought of as η_s operating on the function $\phi(\cdot, y)$ for fixed y [cf. Gelfand and Shilov (1964)]. There is no difficulty in showing that, again for fixed ω, this yields a continuous linear functional on $\mathscr{S}_{2d}$. However, since η_t in general operates on the random test function $\psi(x) = \eta_s(\phi(x,y))$ on the right-hand side of (2.3), we shall need to define $\eta_s \times \eta_t$ by a somewhat more circuitous route if we want to establish joint measurability in (ω, s, t), something that will be important to us later.

In order to do this, equip $\mathscr{S}_d$ with the usual topology defined by either of the equivalent sequences of seminorms

$$\sup_{\substack{x \\ |p|, |q| \le n}} |x^p D^q \phi(x)|, \qquad \sup_{|p|, |q| \le n} \left[\int (x^p D^q \phi(x))^2\,dx\right]^{1/2}, \qquad n = 0, 1, \dots, \tag{2.4}$$

where p and q are multi-indices, and

$$x^p = x_1^{p_1} \cdots x_d^{p_d}, \qquad D^q = \left(\frac{\partial}{\partial x_1}\right)^{q_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{q_d}, \qquad |p| = \sum_{i=1}^{d} p_i.$$

Consider $\mathscr{A}_{2d}$, the dense subset of $\mathscr{S}_{2d}$ made up of functions of the form

$$\phi_N(x,y) = \sum_{k=1}^{N} \phi_k(x) \cdot \psi_k(y), \tag{2.5}$$

where $\phi_k(x), \psi_k(y) \in \mathscr{S}_d$. We define the product distribution $\eta_s \times \eta_t$ on $\mathscr{S}_d \times \mathscr{S}_d$ by setting

$$(\eta_s \times \eta_t)(\phi(x) \times \psi(y)) = \eta_s(\phi(x)) \cdot \eta_t(\psi(y)), \tag{2.6}$$

and extend it to functions of the form (2.5) in $\mathscr{A}_{2d}$ by setting

$$(\eta_s \times \eta_t)(\phi_N(x,y)) = \sum_{k=1}^{N} \eta_s(\phi_k(x)) \cdot \eta_t(\psi_k(y)). \tag{2.7}$$

The fact that this construction can be extended to all of $\mathscr{S}_{2d}$ to define a $\mathscr{S}'_{2d}$-valued stochastic process on $[0,1] \times [0,1]$ that is jointly measurable in (ω, s, t) is our first result.

THEOREM 2.1. *There exists a $\mathscr{S}'_{2d}$-valued stochastic process $\eta_s \times \eta_t$ on $[0,1] \times [0,1]$, jointly measurable in (ω, s, t), such that*:

(i) *If $\phi \in \mathscr{A}_{2d}$ then $(\eta_s \times \eta_t)\phi$ is given by* (2.7).

(ii) *For each $\phi(x,y) \in \mathscr{S}_{2d}$, and distinct $s,t,u,v \in [0,1]$,*

$$E\{(\eta_s \times \eta_t)(\phi(x,y))\} = \iint p_{|t-s|}(x,y)\phi(x,y)\,dx\,dy, \tag{2.8}$$

$$\begin{aligned} &E\{(\eta_s \times \eta_t)(\phi(x,y)) \cdot (\eta_u \times \eta_v)(\psi(x,y))\} \\ &\quad = \iint p_{|t-s|}(x,y)\phi(x,y)\,dx\,dy \cdot \iint p_{|v-u|}(x,y)\psi(x,y)\,dx\,dy \\ &\qquad + \iiiint p_{|s-u|}(x,z)p_{|t-v|}(y,w)\phi(x,y)\psi(z,w)\,dx\,dy\,dz\,dw \\ &\qquad + \iiiint p_{|s-v|}(x,w)p_{|t-u|}(y,z)\phi(x,y)\psi(z,w)\,dx\,dy\,dz\,dw. \end{aligned} \tag{2.9}$$

In the cases $s = t$ and/or $u = v$, (2.8) and (2.9) make sense if at each point we formally replace $p_0(x,y)$ by a delta function. That this is permissible will be a consequence of the proof of the theorem, given in detail in Section 4 below. Thus, for example, we have that

$$\begin{aligned} &E\{[(\eta_t \times \eta_t)(\phi(x,y))]^2\} \\ &\quad = \iint (\phi(x,x)\phi(y,y) + \phi(x,y)\phi(x,y) + \phi(x,y)\phi(y,x))\,dx\,dy. \end{aligned} \tag{2.10}$$

We are now ready to define the intersection local time of η_t. As we saw in the previous section, this involves an integral of the product $\eta_s \times \eta_t$ applied to a delta function. Unfortunately, without appropriate renormalization, this leads to a plethora of infinities. Thus, for $\phi \in \mathscr{S}_{2d}$ define the renormalized, or centered, product

$$(\eta_s \otimes \eta_t)(\phi) := (\eta_s \times \eta_t)(\phi) - E\{(\eta_s \times \eta_t)(\phi)\}. \tag{2.11}$$

The following theorem is the main result of this section.

THEOREM 2.2. *Let $t \in [0,1]$, $\phi \in \mathscr{S}_d$ and $d = 1, 2$ or 3. Then the following expression has a limit in $\mathscr{L}^2(P)$ as $\varepsilon \to 0$*:

$$\Psi_t^\varepsilon(\phi) := \iint_{0 \le u \le v \le t} (\eta_u \otimes \eta_v)\big(e^{-\varepsilon/2} p_\varepsilon(x,y)\phi(x)\big)\,du\,dv. \tag{2.12}$$

The limit is denoted by $\Psi_t(\phi) = \Psi_t(\phi(x))$ and is called the (self)intersection local time process corresponding to the Brownian density process. As t varies, Ψ_t yields a $\mathscr{S}'_d$-valued stochastic process.

REMARK 1. The need for the renormalization in (2.12) comes from the fact that when $d = 2, 3$, $\lim_{\varepsilon \to 0} E\{(\eta_s \times \eta_t)(p_\varepsilon)\} = \infty$, which implies that none of the $\mathscr{L}^2$ calculations performed below on $\eta_s \otimes \eta_t$ are correct if performed on $\eta_s \times \eta_t$. The divergence of this limit is reasonably easy to check, along the lines of the calculations involved in the proof. We shall leave verification, however, to the reader.

REMARK 2. The factor $e^{-\varepsilon/2}$ in (2.12) has, of course, no real effect in the limit as $\varepsilon \to 0$, and could well have been left out of the definition. We include it only for "semantic" consistency with the Brownian motion intersection local time of the following section, where it will have the affect of making some of the proofs algebraically simpler.

As we remarked in the Introduction, definitions akin to (2.12) have been used by a number of authors in the past to define intersection local time for measure-valued processes. Although there are some intrinsic reasons for doing so, we believe that the results of the following section are what justify this choice fully.

3. The particle picture. Our aim in this section is to show how the intersection local times of the individual processes X_t^i appearing in (1.1) can be used to understand the intersection local time of the Brownian density process. To do this, we shall present a series of results, commencing with a fuller formulation of the weak convergence of η_t^λ to η_t than that given in the Introduction, followed by weak convergence results for intersection local times, and concluding with an application of the theory to the development of a Tanaka-like formula for the intersection local time of η_t.

We retain the notation and general setting of the Introduction, so that $X_t^1, X_t^2, \dots$ is a sequence of $\mathbb{R}^d$-valued Brownian motions, started at the points of a homogeneous Poisson process Π^λ of intensity λ. As before, η_t^λ is given by

$$\eta_t^\lambda(\phi) = \lambda^{-1/2} \sum_{i=1}^{\infty} \sigma^i \phi(X_t^i). \tag{3.1}$$

The σ^i are, of course, the random Rademacher signs of the Introduction.

Our first result incorporates the central limit theorem for η_t^λ described in the Introduction, but also includes information on two (orthogonal) martingale (signed) measures. The first is defined for each $\lambda > 0$ as

$$W^\lambda(A,t) = \lambda^{-1/2} \sum_{i=1}^{\infty} \sigma^i \int_0^t 1_A(X_u^i)\, dX_u^i, \tag{3.2}$$

where $A \subset \mathbb{R}^d$ and we restrict t to the interval [0, 1]. The second measure is the $\mathbb{R}^d$-valued Gaussian white noise W on $\mathbb{R}^d \times \mathbb{R}_+$, defined by the requirement that the components $W_i(A,t)$, $i = 1, \dots, d$, of $W(A,t)$ be independent, zero mean Gaussian random variables, and for all $A, B \subset \mathbb{R}^d$ and $s < t$, $u < v$,

$$\begin{aligned} &E\{[W_i(A,t) - W_i(A,s)] \cdot [W_j(B,v) - W_j(B,u)]\} \\ &\quad = \delta_{ij}|A \cap B| \cdot |[s,t] \cap [u,v]|, \end{aligned} \tag{3.3}$$

where $|\cdot|$ is Lebesgue measure in the appropriate dimension.

The following result is a consequence of Proposition 8.16 of Walsh (1986). We associate with the Poisson point process Π^λ giving the initial points of the

Brownian motions a signed version defined, in distribution form, by

$$\tilde{\Pi}^\lambda(\phi) = \sum_{i=1}^{\infty} \sigma^i \phi(X_0^i),$$

so that $\lambda^{-1/2}\tilde{\Pi}^\lambda(\phi) = \eta_0^\lambda(\phi)$. (Recall that the points of Π^λ are the X_0^i.) Weak convergence is denoted by $\Rightarrow$

THEOREM 3.1. *As* $\lambda \to \infty$,

$$(\lambda^{-1/2}\tilde{\Pi}^\lambda, W^\lambda, \eta^\lambda) \Rightarrow (\tilde{\Pi}, W, \eta),$$

where $\tilde{\Pi}$ *is Gaussian white noise on* $\mathbb{R}^d$, W *is defined above*, η *is defined as either the solution of the SPDE* (1.2) *with initial condition* $\eta_0 = \tilde{\Pi}$ *or as in Definition* 2.1 *and the weak convergence is on the Skorohod space* $D([0,1], \mathscr{S}_d' \times \mathscr{S}_d' \times \mathscr{S}_d')$.

Since Theorem 3.1 shows one how to think of the Brownian density process as a limit of a large number of individual Brownian motions—themselves particularly simple processes—we shall call any proof or description based on this approach as being based on a "particle picture" of η_t.

The next step is to set up a central-limit-like result for sums of intersection local times.

Let X and Y be two generic processes from our collection of Brownian motions. Then, as noted in the Introduction, the intersection local time, $L_t(\phi; X, Y)$ between X and Y, up to time t, and weighted by the test function $\phi \in \mathscr{S}_d$, is defined as the $\mathscr{L}^2$ limit, as $\varepsilon \to 0$, of

$$L_t^\varepsilon(\phi; X, Y) = \int_0^t du \int_0^u e^{-\varepsilon/2} p_\varepsilon(X_u - Y_v)\phi(Y_v)\, dv. \tag{3.4}$$

That the limit exists when $d = 2, 3$ follows from results in Dynkin (1981), and it is this limit that gives a precise meaning to the formal expression (1.7).

Defining, as before,

$$\Psi_t^\lambda(\phi) = \lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j L_t(\phi; X^i, X^j), \tag{3.5}$$

and Ψ_t as the intersection local time of Theorem 2.2, we can formulate the main result of this section, and, indeed, of the paper as a whole.

THEOREM 3.2. *Let* $d = 2$ *or* 3. *Then* $\Psi^\lambda \Rightarrow \Psi$ *as* $\lambda \to \infty$ *on the Skorohod space* $D([0,1], \mathscr{S}_d')$.

NOTE. One immediate consequence of Theorem 3.2 is the fact that Ψ takes values in a Skorohod space. While the direct approach of the previous section establishes the existence of Ψ_t as an $\mathscr{L}^2$ limit, the extra cadlag property implicit in Theorem 3.2 is not available there.

The reason why this result provides the justification for the definition of Brownian density intersection local time we claimed above comes from the fact that we know that L_t is precisely what we want to serve as the intersection local time of two Brownian motions, and Ψ_t^λ, as a sum of such local times, is well understood, and has support exactly on the intersections of the paths of the individual Brownian motions. It seems reasonable, therefore, that the $\lambda \to \infty$ limit of Ψ_t^λ should be an appropriate candidate for the intersection local time of the Brownian density process. Since Theorem 3.2 gives us that this limit is the Ψ_t of Theorem 2.2, the particle picture helps understand the more formal, direct definition of the previous section.

At first inspection, Theorem 3.2 should be a "straightforward" consequence of Theorem 3.1, via an appropriate version of the continuous mapping theorem, since we know by Theorem 3.1 that $\eta^\lambda \Rightarrow \eta$, that Ψ_t^λ is a functional defined on η_t^λ and that Ψ_t is a similar functional on η. The difficulty with this line of argument, however, is that functionals based on intersection local times, whether they be of Brownian motions or the Brownian density process, are generally not smooth enough to apply continuity arguments of this kind. [A related, but somewhat different problem, was studied by Dynkin (1988b).]

As a consequence of this, the proof of Theorem 3.2 is unfortunately somewhat circuitous, and will rely on results (to be detailed in Section 5) which are more involved and less interesting. Fortunately, however, another consequence of our hard work will be the Tanaka-like representation of the intersection local time process Ψ_t in two dimensions.

To state this result, we need one more piece of notation. Let g be the Green's function of X, given by either

$$g(x) = \int_0^\infty e^{-t/2} p_t(x)\,dt \quad \text{or} \quad g(x,y) = \int_0^\infty e^{-t/2} p_t(x-y)\,dt. \tag{3.6}$$

Also, to save on notation, we shall write A_{st} to denote the centered product $\eta_s \otimes \eta_t$. Equality in law is denoted by $=_{\mathscr{L}}$. We can now state Theorem 3.3.

THEOREM 3.3. *If $d = 2$ the following equality holds for all $\phi \in \mathscr{S}_2$:*

$$\begin{aligned}\Psi_t(\phi) =_{\mathscr{L}} &\tfrac{1}{2}\int_0^t du \int_0^u dv\, A_{uv}(g(x-y)\phi(y)) \\ &+ \int_0^t du\, A_{uu}(g(x-y)\phi(y)) \\ &- \int_0^t du\, A_{tu}(g(x-y)\phi(y)) \\ &+ \int_0^t \int_{\mathbb{R}^2} \mu_u(\nabla g(x-\cdot)\phi(\cdot))W(dx,du),\end{aligned} \tag{3.7}$$

where the stochastic integral is of the type studied by Walsh (1986).

We shall conclude this section by indicating the main results that go toward establishing Theorem 3.3, and, in particular, emphasizing why the particle picture is so useful in establishing (3.7).

Our first task is to look a little more carefully at the *intersection local time* between two generic Brownian motions X and Y, and to represent it via a Tanaka-like formula. This is given in the following result.

THEOREM 3.4. *If $d = 2$ then the following equality holds for all $\phi \in \mathscr{S}_2$:*

$$L_t(\phi; X, Y) = \tfrac{1}{2}\int_0^t du \int_0^u g(X_u - Y_v)\phi(Y_v)\, dv + \int_0^t g(X_u - Y_u)\phi(Y_u)\, du$$

$$- \int_0^t g(X_t - Y_u)\phi(Y_u)\, du + \int_0^t dX_u \int_0^u \nabla g(X_u - Y_v)\phi(Y_v)\, dv.$$

This result is basically due to Rosen (1986). Details of the proof appear in the following section.

IMPORTANT REMARK ON NOTATION. Note that since $X_t \in \mathbb{R}^2$ we should really write the last integrand above as the inner product $\langle \phi(Y_v)\nabla g(X_u - Y_v), dX_u\rangle\, dv$, using the second option in (3.6) to define g. To keep our formulas reasonably neat, however, we shall use the more ambiguous formulation above throughout the paper, and the reader will do well to keep this in mind later on.

The Tanaka formula for the intersection local times of the individual Brownian processes lifts, in a reasonably straightforward fashion, to the process Ψ_t^λ, which is the centered sum of these local times, taken over all pairs. To see how this works, set, for $\phi \in \mathscr{S}_4$,

$$A_{st}^\lambda(\phi) = \lambda^{-1}\sum_{i \neq j} \sigma^i\sigma^j\phi\big(X_s^i, X_t^j\big). \tag{3.8}$$

Note that the sum here does *not* include the diagonal $i = j$.

To help out in the following, if $\Phi \in \mathscr{S}_d'$ is a distribution, and $\phi \in \mathscr{S}_d$ a test function, we shall often write $\Phi(\phi(x))$ to denote $\Phi(\phi)$.

LEMMA 3.5. *Let $d = 2$. Then, for every $t \in [0,1]$, $\lambda > 0$, $\phi \in \mathscr{S}_2$,*

$$\begin{aligned}\Psi_t^\lambda(\phi) &= \tfrac{1}{2}\int_0^t du \int_0^u dv\, A_{uv}^\lambda(g(x-y)\phi(y)) \\ &\quad + \int_0^t du\, A_{uu}^\lambda(g(x-y)\phi(y)) - \int_0^t du\, A_{tu}^\lambda(g(x-y)\phi(y)) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} \mu_u^\lambda(\nabla g(x - \cdot)\phi(\cdot)) W^\lambda(dx, du) - R_t^\lambda(\phi),\end{aligned} \tag{3.9}$$

where

$$R_t^\lambda(\phi) := \lambda^{-1} \sum_i \int_0^t \int_0^u \nabla g(X_u^i - X_v^i)\phi(X_v^i)\,dv\,dX_u^i,$$

and $R_t^\lambda(\phi) \to_{\mathscr{P}} 0$ *as* $\lambda \to \infty$, *for all* $t \in [0,1]$ *and all* $\phi \in \mathscr{S}_2$.

Once again, it rather looks as if any "easy" proof of Theorem 3.3 would be to apply the continuous mapping theorem via Theorem 3.1 and Lemma 3.5. In particular, for the reader familiar with the weak convergence theorems of Walsh (1986), which we shall rely on heavily in the following section, it would seem that virtually all the work has already been done in that paper. The difficulty in following this direct route, however, lies in the fact that the functions g and ∇g appearing above are not always the best behaved [e.g., $g(x,x) \equiv \infty$, $\nabla g(x-y)\phi(y) \notin \mathscr{L}^2(\mathbb{R}^2 \times \mathbb{R}^2)$] and so substantial technical difficulties arise.

The key to proving both Theorems 3.2 and 3.3 is the following result, based on the random distributions

$$\Psi_t^{\lambda\varepsilon}(\phi) := \lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j L_t^\varepsilon(\phi; X^i, X^j), \tag{3.10}$$

where L_t^ε was defined at (3.4). Let Ψ_t^ε be the approximate intersection local time of η_t defined at (2.12). [In the terminology of Dynkin (1988), $\Psi_t^{\lambda\varepsilon}$ and Ψ_t^ε provide "links" between Ψ_t^λ and Ψ_t.]

THEOREM 3.6. *Let* $d = 2$ *or* 3. *Then* $\Psi_t^{\lambda\varepsilon} \Rightarrow \Psi_t^\varepsilon$ *as* $\lambda \to \infty$, *for every* $\varepsilon > 0$, *on the Skorohod space* $D([0,1], \mathscr{S}_d')$.

Since it is the main result of the preceding section that $\Psi_t^\varepsilon \to_{\mathscr{L}^2} \Psi_t$ as $\varepsilon \to 0$, and it follows from the definition of $\Psi_t^{\lambda\varepsilon}$ and the $\mathscr{L}^2$ convergence of L_t^ε to L_t that $\Psi_t^{\lambda\varepsilon} \to_{\mathscr{L}^2} \Psi^\lambda$ as $\varepsilon \to 0$ for every $\lambda > 0$, it is now easy to see how to prove Theorem 3.3 from Lemma 3.5. We shall give the details in Section 5.

In closing this section, however, we note that a result similar to Theorem 3.3 holds also for the intersection local time of the Brownian density process on $\mathbb{R}^3$. The technical differences in this case, referred to briefly in the Introduction, arise primarily from the fact that the corresponding summands in the remainder term $R_t^\lambda(\phi)$ of (3.9) do not exist in this case. Nevertheless, approximations to these, analogous to the summands of $R_t^{\lambda\varepsilon}(\phi)$ of (5.12) in Section 5 do exist, and judicious handling of the $\lambda \to \infty$ and $\varepsilon \to 0$ limits overcomes the difficulties. We shall not go into the details, however.

4. Proofs for Section 2.

PROOF OF THEOREM 2.1. We commence by noting that the measurability referred to in the statement of the theorem holds automatically if the space of test functions is restricted to $\mathscr{A}_{2d}$. [This is an immediate consequence of the definition (2.7), the measurability of η_t in ω and its continuity in t.] The next step is to establish (2.9) for

$$\Phi_N(x,y) = \sum_{k=1}^{N} \phi_k(x)\cdot\hat{\phi}_k(y), \qquad \Psi_N(x,y) = \sum_{k=1}^{N} \psi_k(x)\cdot\hat{\psi}_k(y),$$

elements of $\mathscr{A}_{2d}$. Equation (2.8) is easier, and follows from similar calculations.

Recall [e.g., Adler (1981), page 108] that if $Y_1,\dots,Y_{2m}$ is any set of $2m$ centered Gaussian variables, then

$$E\{Y_1 \cdots Y_{2m}\} = \sum E\{Y_{i_1}Y_{i_2}\} \cdots E\{Y_{i_{2m-1}}Y_{i_{2m}}\}, \tag{4.1}$$

where the sum is over the $(2m)!/m!2^m$ different ways of grouping $Y_1,\dots,Y_{2m}$ into pairs. For ease of notation, we denote the right-hand integral in (2.1) by $\langle\phi_1,\phi_2\rangle_{t-s}$. It then follows from (4.1), with $m=2$, the form of Φ_N and Ψ_N, (2.7) and (2.1) and (2.2) that

$$\begin{aligned}
&E\{(\eta_s\times\eta_t)(\Phi_N(x,y))\cdot(\eta_u\times\eta_v)(\Psi_N(x,y))\}\\
&\quad= \sum_{k=1}^{N}\sum_{k'=1}^{N}\big\{\langle\phi_k,\hat{\phi}_k\rangle_{|t-s|}\cdot\langle\hat{\psi}_{k'},\psi_{k'}\rangle_{|v-u|} + \langle\phi_k,\hat{\psi}_{k'}\rangle_{|v-s|}\cdot\langle\hat{\phi}_k,\psi_{k'}\rangle_{|t-u|}\\
&\qquad\qquad + \langle\phi_k,\psi_{k'}\rangle_{|u-s|}\cdot\langle\hat{\phi}_k,\hat{\psi}_{k'}\rangle_{|v-t|}\big\},
\end{aligned}$$

where we have considered u, v, s and t, distinct, and leave it as an exercise to the reader to handle the other, easier, cases along the lines noted in Section 2.

Consider, for example, the first summand above. Interchanging the order of integration and summation gives

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{k'=1}^{N}\langle\phi_k,\hat{\phi}_k\rangle_{|t-s|}\cdot\langle\hat{\psi}_{k'},\psi_{k'}\rangle_{|v-u|}\\
&\quad= \iint\sum_{k=1}^{N}\phi_k(x)\hat{\phi}_k(y)p_{|t-s|}(x,y)\,dx\,dy\\
&\qquad\times\iint\sum_{k'=1}^{N}\phi_{k'}(z)\hat{\phi}_{k'}(w)p_{|u-v|}(z,w)\,dz\,dw\\
&\quad= \iint p_{|t-s|}(x,y)\Phi_N(x,y)\,dx\,dy\cdot\iint p_{|v-u|}(x,y)\Psi_N(x,y)\,dx\,dy,
\end{aligned}$$

which is the first term of (2.9) for the functions Φ_N and Ψ_N. The remaining terms are handled similarly, and this establishes the theorem for test functions in $\mathscr{A}_{2d}$.

Now take $\Phi\in\mathscr{S}_{2d}$ and let $\{\Phi_N\}_N$ be a sequence of $\mathscr{A}_{2d}$ functions converging pointwise to Φ. We shall show that $\{(\eta_s\times\eta_t)(\Phi_N)\}_N$ is Cauchy in $\mathscr{L}^2(P)$. This is enough to establish the theorem in full generality.

By (2.9), with $s = u \neq t = v$, we have

$$E\{[(\eta_s \times \eta_t)(\Phi_N - \Phi_M)]^2\}$$

$$\begin{aligned} &= \left| \iint p_{|t-s|}(x,y)[\Phi_N(x,y) - \Phi_M(x,y)]\,dx\,dy \right|^2 \\ &\quad + \iint [\Phi_N(x,y) - \Phi_M(x,y)]^2\,dx\,dy \\ &\quad + \iiiint p_{|t-s|}(x,w)p_{|t-s|}(y,z) \\ &\qquad \times [\Phi_N(x,y) - \Phi_M(x,y)][\Phi_N(z,w) - \Phi_M(z,w)]\,dx\,dy\,dz\,dw. \end{aligned} \tag{4.2}$$

Setting $\varepsilon_{NM} = \sup_{xy} |\Phi_N(x,y) - \Phi_M(x,y)|$ and using the fact that, for $t \neq s$, $p_{|t-s|}$ is a probability density, we easily have that the first and last terms on the right-hand side of (4.2) are each bounded above by ε_{NM}^2, which goes to 0 as $N, M \to \infty$ by virtue of the equivalence of norms on $\mathscr{S}_d$, noted above when we set up the topology. The second term also goes to 0, by a similar argument.

A similar argument also works for the case $s = t$ [in which case all the integrals corresponding to (4.2) appear without the factor $p_{|t-s|}$]. This establishes the existence of $(\eta_s \times \eta_t)(\phi)$ as an $\mathscr{L}^2$ limit for each $\phi \in \mathscr{S}_{2d}$, as well as the moment formulae (2.8) and (2.9). It follows now from (2.9) that for $\phi \in \mathscr{S}_{2d}$,

$$\int_0^1 \int_0^1 E|(\eta_s \times \eta_t)(\phi)|^2\,ds\,dt \leq 3 \sup_{x,y} |\phi(x,y)|^2, \tag{4.3}$$

so that the joint measurability of $\eta_s \times \eta_t$ in (ω, s, t) is a consequence of Fubini's theorem.

To complete the proof of Theorem 2.1, we still need to establish that the process $\eta_s \times \eta_t$, defined as a $\mathscr{L}^2$ limit, for each fixed $\phi \in \mathscr{S}_{2d}$ is, in fact, a $\mathscr{S}'_{2d}$-valued random variable. In order to establish this we require some notation and a lemma. We follow Martin-Löf (1976).

Introduce a sequence of norms on $\mathscr{S}_d$ defined by $\|\phi\|_n = [\int (H^n\phi(x))^2\,dx]^{1/2}$, $n \geq 0$, where $\eta \in \mathscr{S}_d$ and H is the operator defined by $H\phi = |x|^2\phi - \Delta\phi$. This sequence generates an equivalent topology on $\mathscr{S}_d$ to that generated by the seminorms (2.4). Define a topology on $\mathscr{S}'_d$ via the dual norms

$$\|\eta\|_{-n} = \sup_{\|\phi\|_n \leq 1} |\eta(\phi)|. \tag{4.4}$$

The norms $\|\phi\|_n$ increase in $n \geq 0$, and so $\|\eta\|_{-n} \geq \|\eta\|_{-n-1}$ for $n \geq 0$.

The importance of these norms lies in the fact, among other things, that Martin-Löf showed that there exists a p such that

$$P\{\|\eta_t\|_{-p} < \infty\} = 1 \quad \text{for all } t \in [0,1]. \tag{4.5}$$

Recall that a linear operator T with domain $D(T) \subset X$ into Y is said to be *closable* or *pre-closed*, if the closure on $X \times Y$ of the graph $G(T) = \{(x, Tx)$: $x \in D(T)\}$ is the graph of a linear operator S from $D(S)$ into Y. The following result is standard [e.g., Yoshida (1980)].

LEMMA 4.1. *If X and Y are quasi-normed linear spaces, then T is closable if and only if*

$$\{x_n\} \subseteq D(T), \ \lim_{n\to\infty} x_n = 0, \ \lim_{n\to\infty} = y \text{ implies } y = 0, \tag{4.6}$$

where the limits are taken in the strong topologies.

We can now return to the proof of Theorem 2.1. Choose p so that (4.5) holds. It is immediate, from the continuity and linearity of η_t, that $\eta_s \times \eta_t$ restricted to $\mathscr{A}_{2d}$ is a continuous linear functional. Write $\mathscr{S}_{2d,p}$ to denote $\mathscr{S}_{2d}$ with the topology generated by $\|\cdot\|_p$, and $\mathscr{S}'_{2d,p}$ to denote its topological dual. Note that $\mathscr{S}_{2d,p}$ is a Hilbert space, and, in particular, quasi-normed. Also, $\mathscr{A}_{2d}$ is dense in $\mathscr{S}_{2d,p}$. Using now the $\mathscr{L}^2$ convergence of $(\eta_s \times \eta_t)(\Phi_N)$ and (4.3) it is easy to check that (4.6) is satisfied, with probability 1. Applying Lemma 4.1, the Banach closed graph theorem and the joint measurability of $\eta_s \times \eta_t$, it follows that $\eta_s \times \eta_t$ can be extended to a random, continuous, linear mapping from $\mathscr{S}_{2d,p}$ to $\mathbb{R}$. That is, $\eta_s \times \eta_t \in S'_{2d,p}$ for all $s, t \in [0,1]$. Since the topology we have placed on S'_{2d} makes $\mathscr{S}'_{2d} = \bigcup_p \mathscr{S}'_{2d,p}$, this completes the proof of the theorem. □

To prepare ourselves for the next proof, note that (4.3) also implies that the integral $\iint_{0\le s\le t\le T}(\eta_s \times \eta_t)(\phi(x,y))\,ds\,dt$ is well defined as a $\mathscr{S}'_{2d}$ distribution.

PROOF OF THEOREM 2.2. Fix $t \in (0,1]$ and denote the integral in (2.12) by $I_\varepsilon(\phi)$. To prove that it has an $\mathscr{L}^2$ limit, it suffices to show that

$$\lim_{\varepsilon\to 0,\,\delta\to 0} E\{I_\varepsilon(\phi)\cdot I_\delta(\phi)\} = \text{constant}.$$

From Theorem 2.1 and the form of $\eta_s \otimes \eta_t$ it follows that, up to a factor of $\exp(-(\varepsilon+\delta)/2)$, which we shall neglect, $E\{I_\varepsilon(\phi)\cdot I_\delta(\phi)\}$ is given by

$$\begin{aligned}&\int_D\int_{\mathbb{R}^{4d}} p_{|s-u|}(x,z)\,p_{|t-v|}(y,w)\,p_\varepsilon(x,y)\,p_\delta(z,w)\,\phi(x)\phi(z)\\ &\quad+\int_D\int_{\mathbb{R}^{4d}} p_{|s-v|}(x,w)\,p_{|t-u|}(y,z)\,p_\varepsilon(x,y)\,p_\delta(z,w)\,\phi(x)\phi(z),\end{aligned}\tag{4.7}$$

where the inner integrals are over $(x,y,z,w) \in \mathbb{R}^{4d}$, the outer integrals over $D := \{0 \le s \le t \le T,\ 0 \le u \le v \le T\}$, and we have left out the eight differentials to save space. (Again, we are implicitly assuming that s, t, u and v are distinct, and are leaving the other cases to the reader.) We shall show that the first integral in (4.7) has a limit as $\varepsilon, \delta \to 0$. The second is handled similarly.

Let T_ε be the contraction semigroup given by

$$(T_\varepsilon F)(x) = \int_{\mathbb{R}^d} p_\varepsilon(x,y)F(y)\,dy,$$

so that recalling the symmetry of $p_t(x,y)$ we can write the first integral in (4.7) as

$$(4.8)\quad \int_D \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(T_\varepsilon(\phi(\cdot)p_{|s-u|}(\cdot, z))\big)(y) \cdot \big(T_\delta(p_{|t-v|}(\cdot, y))\big)(z)\phi(z)\, dy\, dz.$$

By contractivity we can interchange the order of integration and take the $\varepsilon, \delta \to 0$ limit, as long as the limit we thus obtain is finite. Modulo this technicality, the limit of (4.8) is thus given by

$$(4.9)\quad \int_D \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{|s-u|}(y,z) p_{|t-v|}(y,z)\phi(y)\phi(z)\, dy\, dz\, ds\, dt\, du\, dv,$$

and this is enough to establish the $\mathscr{L}^2$ convergence claimed in the statement of the theorem.

It remains to establish, however, that (4.9) is finite, which we shall now show is the case for $d = 1, 2, 3$. In fact, since test functions in $\mathscr{S}_d$ are bounded, it suffices to establish that the following integral is finite:

$$\int_D \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{|s-u|}(y,z) p_{|t-v|}(y,z)|\phi(z)|\, dy\, dz\, ds\, dt\, du\, dv$$

$$(4.10)\quad = \int_D \int_{\mathbb{R}^d} p_{(|s-u|+|t-v|)}(z,z)|\phi(z)|\, dz\, ds\, dt\, du\, dv$$

$$= C \cdot \int_D [|s-u| + |t-v|]^{-d/2}\, ds\, dt\, du\, dv,$$

where the first equality is a consequence of the Chapman–Kolmogorov equation, the second of the fact that $p_t(z,z) = [2\pi]^{-d/2}$, and $C = [2\pi]^{-d/2}\int|\phi(z)|\, dz < \infty$.

Since for all $x, y \geq 0$, $\sqrt{2xy} \leq x + y$, the last term in (4.10) is bounded above by

$$C' \cdot \int_D \frac{1}{|s-u|^{d/4}|t-v|^{d/4}}\, ds\, dt\, du\, dv.$$

Standard estimates now show that this is finite for $d \leq 3$, and this completes the proof of the $\mathscr{L}^2$ convergence part of the theorem. It remains to establish, as in the previous section, that the $\mathscr{L}^2$ limit, established independently for each test function ϕ, can be extended to a proper distribution.

In order to do this we need the following result. The proof is almost identical to that of a stronger result of Martin-Löf (1976), Lemma 4, for Gaussian distributions. All that needs to be changed is to replace the parts of his argument that require specific bounds related to Gaussian distributions with Chebyshev's inequality. The notation is as at the end of the previous section.

LEMMA 4.2. *Let η be a $\mathscr{S}_d'$-valued random variable, with covariances $E\{\eta(\phi)\eta(\psi)\} = R(\phi, \psi)$. If $\sum_i R(\phi_i, \phi_i) = D < \infty$ for some p-orthonormal sequence* [*i.e.*, $(\phi_i, \phi_j)_p = \int H^p\phi_i(x)H^p\phi_j(x)\,dx = \delta_{ij}$] *then*

$$P\{\|\eta\|_{-p} \geq \lambda\} \leq D/\lambda^2. \tag{4.11}$$

We can now return to the proof of Theorem 2.2. The easiest approach is to think of the $I_\varepsilon(\phi)$ as a sequence of continuous linear functionals applied to the test function $\phi \in \mathscr{S}_d$, an approach that is permitted by Theorem 2.1.

Since, for each $\phi \in \mathscr{S}_d$, $I_\varepsilon(\phi) \to_{\mathscr{L}^2} \Psi_t(\phi)$, as $\varepsilon \to 0$, there is a subsequence $\{\varepsilon_n\}$ along which the convergence occurs with probability 1. Since $\mathscr{S}_{2d}$ is separable we can assume that this holds simultaneously for all ϕ in a dense subset. Furthermore, for each ε_n, I_{ε_n} is an element of $\mathscr{S}_d'$. Thus, since $\mathscr{S}_d'$ is weakly complete [Gelfand and Vilenkin (1966)], it will follow that the limit is also in $\mathscr{S}_d'$ if we can show that for some p, $\|\sup_n I_{\varepsilon_n}\|_{-p} < \infty$, with probability 1. This, however, follows from the nuclearity of $\mathscr{S}_d$, the $\mathscr{L}^2$ calculations above, Lemma 4.2 and the Borel–Cantelli lemma. □

5. Proofs for Section 3. We shall start by proving the central Theorem 3.2. In fact, since we shall require something a little stronger in order to establish the Tanaka formula of Theorem 3.3, we shall prove a little more than necessary at this stage. For this we shall require a little extra notation.

For $\phi \in \mathscr{S}_{2d}$ let $U_t^\lambda(\phi(x,y))$ be given by

$$U_t^\lambda(\phi) := \int_0^t \int_{\mathbb{R}^d} \mu_u^\lambda(\phi(x,\cdot))W^\lambda(dx,du), \tag{5.1}$$

and, similarly, define

$$U_t(\phi) := \int_0^t \int_{\mathbb{R}^d} \mu_u(\phi(x,\cdot))W(dx,du). \tag{5.2}$$

The next result is the key step in establishing all of our main results.

THEOREM 5.1. *Let $\lambda \to \infty$ along a countable sequence.* (*We should therefore really replace λ by λ_n in what follows, but there is a limit to how many subscripts and superscripts the human mind can absorb.*) *Then, for all $\phi_1, \phi_2, \phi_3 \in \mathscr{S}_d$, and $\phi_4, \phi_5 \in \mathscr{S}_{2d}$,*

$$\begin{aligned}&\left\langle \lambda^{-1/2}\tilde{\Pi}(\phi_1), W^\lambda(\phi_2), \eta^\lambda(\phi_3), U^\lambda(\phi_4), A^\lambda(\phi_5)\right\rangle \\ &\quad\Rightarrow \left\langle \tilde{\Pi}(\phi_1), W(\phi_2), \eta(\phi_3), U(\phi_4), A(\phi_5)\right\rangle,\end{aligned} \tag{5.3}$$

as $\lambda \to \infty$ along this sequence.

PROOF. The general theory of weak convergence that we require to prove this result is developed and expounded in detail in Walsh (1986). To give a fully detailed, self-contained version of the proof of (5.3), we would have to copy two pages of definitions from Walsh's notes, and then go through a number of pages of rather detailed, and essentially uninteresting, calculations.

Since this seems to be a somewhat unjustified addition to what is already an overlong paper, we shall assume that the reader is familiar with Walsh's notes, and merely point out how Theorem 5.1 follows from the results and techniques developed there.

Note first that the weak convergence of the triple $\langle \lambda^{-1}\tilde{\Pi}, W^{\lambda}, \eta^{\lambda}\rangle$ is a special case of Theorem 3.1 (itself a direct consequence of Theorem 8.16 of Walsh), since here we are only taking $\lambda \to \infty$ through a countable sequence. Appending the convergence of A^{λ}_{st} and U^{λ}_t follows as in the proof of Proposition 8.17 of Walsh's notes, once we have checked that the individual limits are as claimed. We shall check this only for A_{st}, this being the harder of the two, and somewhat different to the example treated by Walsh.

Consider first $\phi \in \mathscr{S}_{2d}$ of the form $\phi(x, y) = \psi(x)\hat{\psi}(y)$, $\psi, \hat{\psi} \in \mathscr{S}_d$. Then

$$A^{\lambda}_{ts}(\phi) = \eta^{\lambda}_t(\psi)\eta^{\lambda}_s(\hat{\psi}) - \lambda^{-1}\sum_i \psi(X^i_t)\hat{\psi}(X^i_s). \tag{5.4}$$

Consider the second term here, noting that if X denotes a generic term of our sequence $X_1, X_2, \dots, B$ a standard Brownian motion on $\mathbb{R}^d$ starting at the origin, and F a functional on $C([0,1], \mathbb{R}^d)$, then

$$E\{F(X) \mid X_0 = x\} = E\{F(B + x)\}.$$

Consequently, using the fact that the sequence of Brownian motions $X^1, X^2, \dots$ commence at the points of a Poisson process of intensity λ, it follows that

$$E\left\{\sum_i F(X^i)\right\} = \lambda\int_{\mathbb{R}^d} E\{F(B + x)\}\, dx. \tag{5.5}$$

It is now a simple moment computation to check that the second term in (5.4) converges in probability to the deterministic expression

$$E\{\psi(X^i_t)\hat{\psi}(X^i_s)\} = \begin{cases} \int \psi(x)\, dx \int \hat{\psi}(y)\, dy\, p_{|t-s|}(x - y) & \text{if } t \neq s, \\ \int \psi(x)\hat{\psi}(x)\, dx & \text{if } t = s. \end{cases}$$

Since this expression is equivalent to $E\{(\eta_t \times \eta_s)(\phi)\}$ (cf. Theorem 2.1), it follows from (5.4), the convergence of η^{λ} to η, and the continuous mapping theorem that for ϕ of product form

$$A^{\lambda}_{ts}(\phi) \Rightarrow A_{ts}(\phi) = (\eta_t \otimes \eta_s)(\phi) = (\eta_t \times \eta_s)(\phi) - E\{(\eta_t \times \eta_s)(\phi)\}. \tag{5.6}$$

Using now the fact that sums of the form $\sum_{k=1}^{N}\psi_k(x)\hat{\psi}_k(y)$ are $\mathscr{L}^2$ dense in $\mathscr{S}_{2d}$, the extension of (5.6) to all $\phi \in \mathscr{S}_{2d}$ is standard.

This fact, together with the comments made above, completes the proof of the theorem. □

REMARK. It is important to note that since the proof of Theorem 5.1 relies on results proved by Walsh, the weak convergence in (5.3) can only be shown, at this stage, to hold for nice functions ϕ_k, and not for functions like g and ∇g, which is what we need. To handle these functions, we need the extra work of the following arguments.

PROOF OF THEOREM 3.6. We need to show that $\Psi_t^{\lambda\varepsilon} \Rightarrow \Psi_t^{\varepsilon}$ as $\lambda \to \infty$, for every $\varepsilon > 0$, on $D([0,1], \mathscr{S}_d')$. Recall that

$$\Psi_t^{\lambda\varepsilon}(\phi) = \int_0^t \int_0^u A_{uv}^{\lambda}\big(e^{-\varepsilon/2} p_\varepsilon(x-y)\phi(y)\big)\, dv\, du, \tag{5.7}$$

and Ψ_t^{ε} is given by an identical expression with A_{uv} replacing A_{uv}^{λ}.

Note first that since $e^{-\varepsilon/2} p_\varepsilon(x-y)\phi(y) \in \mathscr{S}_{2d}$ for every $\varepsilon > 0$, it follows that for every $\phi \in \mathscr{S}_d$ we have $A_{uv}^{\lambda}(e^{-\varepsilon/2} p_\varepsilon(x-y)\phi(y)) \Rightarrow A_{uv}(e^{-\varepsilon/2} p_\varepsilon(x-y)\phi(y))$. Since integration with respect to the two time parameters is a continuous functional on $D([0,1]\times[0,1], \mathscr{S}_{2d}')$, it follows that $\Psi_t^{\lambda\varepsilon}(\phi) \Rightarrow \Psi_t^{\varepsilon}(\phi)$ for every $\phi \in \mathscr{S}_d$ and each fixed $t \in [0,1]$. By the Cramér–Wold device (to handle the t parameter) and linearity (to handle the ϕ parameter) this convergence can be lifted to that of finite-dimensional distributions. The problem now is to establish tightness in t for fixed ϕ. Theorem 6.15 of Walsh (1986) then gives us that $\Psi_t^{\lambda\varepsilon} \Rightarrow \Psi_t^{\varepsilon}$ on $D([0,1], \mathscr{S}_d')$, which completes the proof of the theorem.

For fixed ϕ, however, tightness in t follows easily from standard moment conditions, using the integral form of (2.12) and the fact that $A_{uv}^{\lambda}(\phi)$ has moments of all orders [cf. Rosen (1986)]. □

PROOF OF THEOREM 3.2. We want to prove that $\Psi^{\lambda} \Rightarrow \Psi$ on the appropriate Skorohod space. Now, however, we must restrict ourselves to the cases $d = 2$ and $d = 3$. Note first that

$$\Psi_t^{\lambda}(\phi) - \Psi_t^{\lambda\varepsilon}(\phi) = \lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j \big[L_t(\phi; X^i, X^j) - L_t^{\varepsilon}(\phi; X^i, X^j)\big],$$

and so

$$E\Big\{\big|\Psi_t^{\lambda\varepsilon}(\phi) - \Psi_t^{\lambda}(\phi)\big|^2\Big\} = E\Big\{\big|(L_t - L_t^{\varepsilon})(\phi, X^1, X^2)\big|^2\Big\} = C_\varepsilon(\phi),$$

where $C_\varepsilon(\phi) \to 0$ as $\varepsilon \to 0$, by the very definition of L_t as the $\mathscr{L}^2$ limit of L_t^{ε}. Consequently, by Chebyshev's inequality,

$$P\Big\{\big|\Psi_t^{\lambda\varepsilon}(\phi) - \Psi_t^{\lambda}(\phi)\big| \geq \delta\Big\} \leq \frac{C_\varepsilon(\phi)}{\delta} \quad \text{for every } \phi \in \mathscr{S}_d \tag{5.8}$$

where $C_\varepsilon(\phi) \to 0$ as $\varepsilon \to 0$, for every $\delta > 0$.

We now want to show that for fixed $t \in [0,1]$ and $\phi \in \mathscr{S}_d$ that $\Psi_t^\lambda(\phi) \to_{\mathscr{L}} \Psi_t(\phi)$ as $\lambda \to \infty$. For $x \in \mathbb{R}$ and $\delta > 0$,

$$\begin{aligned}
\lim_{\lambda\to\infty} P\{\Psi_t^\lambda(\phi) \le x\} &= \lim_{\varepsilon\to 0}\lim_{\lambda\to\infty} P\{\Psi_t^\lambda(\phi) \le x\} \\
&= \lim_{\varepsilon\to 0}\lim_{\lambda\to\infty} P\{[\Psi_t^\lambda(\phi) - \Psi_t^{\lambda\varepsilon}(\phi)] + \Psi_t^{\lambda\varepsilon}(\phi) \le x\} \\
&\le \lim_{\varepsilon\to 0}\lim_{\lambda\to\infty} \Big[P\{|(\Psi_t^\lambda(\phi) - \Psi_t^{\lambda\varepsilon})(\phi)| \ge \delta\} \\
&\qquad + P\{\Psi_t^{\lambda\varepsilon}(\phi) \le x + \delta\}\Big] \\
&\le \lim_{\varepsilon\to 0}\left[\frac{C_\varepsilon(\phi)}{\delta} + P\{\Psi_t^\varepsilon(\phi) \le x+\delta\}\right] \\
&= P\{\Psi_t(\phi) \le x + \delta\},
\end{aligned}$$

where the first line follows from the independence of both the right- and left-hand sides of ε, the second and third are trivial, the fourth is a consequence of Theorem 3.6 and the last from the fact that $\Psi_t^\varepsilon \to_{\mathscr{L}^2} \Psi_t$ (Theorem 2.2).

A similar argument shows that $\lim_{\lambda\to\infty} P\{\Psi_t^\lambda(\phi) \le x\} \ge P\{\Psi_t(\phi) \le x - \delta\}$, and since δ was arbitrary we have that $\Psi_t^\lambda(\phi) \to_{\mathscr{D}} \Psi_t(\phi)$ for fixed t and ϕ. Arguing as in the proof of Theorem 3.6, we can also establish full weak convergence on $D([0,1], \mathscr{S}_d)$, and so complete the proof of the theorem. □

The remainder of this section is devoted to the proof of the Tanaka formula for the Brownian density intersection local time, so we now restrict ourselves to the case $d = 2$.

We commence by trying to get a result like Theorem 3.3, but for the process $\Psi_t^{\lambda\varepsilon}$, i.e., an evolution equation representation for $\Psi_t^{\lambda\varepsilon}$. Recall that $\Psi_t^{\lambda\varepsilon}$ was defined in (3.10), as a sum of "approximate" intersection local times. For $\varepsilon > 0$ set

$$K^\varepsilon(x) = \int_\varepsilon^\infty e^{-t/2} p_t(x)\, dt. \tag{5.9}$$

Note that $K^\varepsilon \to_{\mathscr{L}^2} g$ as $\varepsilon \to 0$, where g is the Green's function (3.6). Unlike g, however, K^ε is well behaved in that it is C^∞, everywhere finite and $\nabla K^\varepsilon \in \mathscr{L}^2(\mathbb{R}^2)$. Furthermore,

$$\tfrac{1}{2}(-\Delta + 1)K^\varepsilon(x) = e^{-\varepsilon/2} p_\varepsilon(x), \tag{5.10}$$

as is easily checked by direct differentiation. This leads us to the following.

LEMMA 5.2. *For all* $\lambda, \varepsilon > 0$, $t \in [0,1]$ *and* $\phi \in \mathscr{S}_2$,

$$\begin{aligned} \Psi_t^{\lambda\varepsilon}(\phi) = {} & \tfrac{1}{2}\lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j\int_0^t du\int_0^u dv\,K^\varepsilon(X_u^i - X_v^j)\phi(X_v^j) \\ & + \lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j\int_0^t du\,K^\varepsilon(X_u^i - X_u^j)\phi(X_u^j) \\ & - \lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j\int_0^t du\,K^\varepsilon(X_t^i - X_u^j)\phi(X_u^j) \\ & + \lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j\int_0^t\int_0^u \nabla K^\varepsilon(X_u^i - X_v^j)\phi(X_v^j)\,dv\,dX_u^i. \end{aligned} \tag{5.11}$$

PROOF. Apply Itô's formula [using (5.10)] to the C^∞ function

$$f(t,x) = \int_0^t K^\varepsilon(x - X_v^j)\phi(X_v^j)\,dv,$$

replace x by X_u^i, multiply by $\sigma^i\sigma^j$ and sum over $i \neq j$. [A similar argument, used in establishing the original Tanaka formula for Brownian motion intersection local time, can be found in Rosen (1986).] □

PROOF OF THEOREM 3.4. In the proof of the above lemma we actually established, *en passant*, a version of Theorem 3.4 with L_t^ε replacing L_t and K^ε replacing g. Sending $\varepsilon \to 0$ to obtain $\mathscr{L}^2$ limits on both sides of the equation is not trivial, but, fortunately, has already been done for us in Rosen (1986). (Rosen actually treats *self*-intersections of Brownian motions, so his proof is a little harder, and the precise terms in the Tanaka formula slightly different. Nevertheless, the proofs carry over almost verbatim.)

To convert (5.11) to a form more reminiscent of an evolution equation, we need to make optimal use of the notation set up in the previous section. We need, furthermore, one more piece of notation, and so for $\phi \in \mathscr{S}_2$ we set

$$R_t^{\lambda\varepsilon}(\phi) := \lambda^{-1}\sum_i \int_0^t\int_0^u \nabla K^\varepsilon(X_u^i - X_v^i)\phi(X_v^i)\,dv\,dX_u^i. \tag{5.12}$$

LEMMA 5.3.

$$\begin{aligned} \Psi_t^{\lambda\varepsilon}(\phi) = {} & \tfrac{1}{2}\int_0^t du\int_0^u dv\,A_{uv}^\lambda(K^\varepsilon(x-y)\phi(y)) \\ & + \int_0^t du\,A_{uu}^\lambda(K^\varepsilon(x-y)\phi(y)) - \int_0^t du\,A_{tu}^\lambda(K^\varepsilon(x-y)\phi(y)) \\ & + \int_0^t\int_{\mathbb{R}^2}\mu_u^\lambda(\nabla K^\varepsilon(x-\cdot)\phi(\cdot))W^\lambda(dx,du) - R_t^{\lambda\varepsilon}(\phi). \end{aligned}$$

PROOF. The above is basically a rewrite of (5.11). The first three terms are easily seen to be equivalent to the first three terms of (5.11) on applying the definition (3.8) of A^λ_{st}. To obtain the last two terms, we write the last expression in (5.11) as

$$\lambda^{-1} \sum_i \sum_j \sigma^i \sigma^j \int_0^t \int_0^u \nabla K^\varepsilon(X^i_u - X^j_v) \phi(X^j_v)\, dv\, dX^i_u$$

$$- \lambda^{-1} \sum_i \int_0^t \int_0^u \nabla K^\varepsilon(X^i_u - X^i_v) \phi(X^i_v)\, dv\, dX^i_u.$$

Consider the first term here. (The second is much easier, and it is easy to see that it is equal to $R_t^{\lambda\varepsilon}$.) By Fubini's theorem for stochastic integrals [e.g., Revuz and Yor (1987), Section 6, Lemma 1.4], this is equal to

$$\lambda^{-1/2} \sum_j \sigma^j \int_0^t dv \left[\lambda^{-1/2} \sum_i \sigma^i \int_v^t \nabla K^\varepsilon(X^i_u - X^j_v) \phi(X^j_v)\, dX^i_u \right].$$

From the definition of the measure W^λ, it follows that this is equal to

$$\lambda^{-1/2} \sum_j \sigma^j \int_0^t dv \left[\int_v^t \int_{\mathbb{R}^2} \nabla K^\varepsilon(x - X^j_v) \phi(X^j_v) W^\lambda(dx, du) \right].$$

[This follows from Proposition 8.3 of Walsh (1986). The proof there proceeds by establishing a result like the above first for indicator functions ϕ, and then via passage to the limit for general $\phi \in \mathscr{L}^2$. It is precisely at this point that we need $\nabla K^\varepsilon \in \mathscr{L}^2(\mathbb{R}^2)$.]

A stochastic Fubini theorem for worthy martingale measures [Walsh (1986), Theorem 2.6] implies that the above equals

$$\lambda^{-1/2} \sum_j \sigma^j \int_0^t \int_{\mathbb{R}^2} \left[\int_0^u \nabla K^\varepsilon(x - X^j_v) \phi(X^j_v)\, dv \right] W^\lambda(dx, du).$$

Interchanging summation and integration yields

$$\int_0^t \int_{\mathbb{R}^2} \left\{ \int_0^u \lambda^{-1/2} \sum_j \sigma^j \left[\nabla K^\varepsilon(x - X^j_v) \phi(X^j_v)\, dv \right] \right\} W^\lambda(dx, du)$$

$$= \int_0^t \int_{\mathbb{R}^2} \mu^\lambda_u (\nabla K^\varepsilon(x - \cdot) \phi(\cdot)) W^\lambda(dx, du),$$

where the last line follows from the definition of μ^λ_t at (1.6).

This completes the proof of the lemma. □

The next step is to show that $R_t^{\lambda\varepsilon}$, the remainder term (5.12), goes to 0 as $\lambda \to \infty$.

LEMMA 5.4. *$R_t^{\lambda\varepsilon}(\phi) \to_{\mathscr{P}} 0$ as $\lambda \to \infty$, for each $t \in [0,1]$, $\varepsilon > 0$ and $\phi \in \mathscr{S}_2$.*

PROOF. Define

$$r_t(X) = r_t(X;\phi,\varepsilon) = \int_0^t \int_0^u \nabla K^\varepsilon(X_u - X_v)\phi(X_v)\,dv\,dX_u,$$

so that $R_t^{\lambda\varepsilon}(\phi) = \lambda^{-1}\Sigma_i r_t(X^i)$. Thus, calculating moments as in the proof of Theorem 5.1,

$$E\Big\{\big[R_t^{\lambda\varepsilon}(\phi)\big]^2\Big\} = \lambda^{-2}E\Big\{\sum_i \big[r_t(X^i)\big]^2\Big\}$$

$$= \lambda^{-1}\int_{\mathbb{R}^2} E\bigg[\int_0^t\int_0^u \nabla K^\varepsilon(B_u - B_v)\phi(B_v + x)\,dv\,dB_u\bigg]^2\,dx,$$

the last line following from (5.5). Standard inequalities for the moments of stochastic integrals [e.g., Ikeda and Watanabe (1981), page 110], combined with the fact that ϕ decays rapidly at ∞ give us that the triple integral above is finite, and so $R_t^{\lambda\varepsilon} \to_{\mathscr{P}} 0$ as $\lambda \to \infty$, and the lemma is established. □

PROOF OF THEOREM 3.3. It follows from Theorems 3.6 and 5.1 and Lemmas 5.3 and 5.4, that for all $\varepsilon > 0$, $t \in [0,1]$ and $\phi \in \mathscr{S}_2$,

$$\begin{aligned}\Psi_t^\varepsilon(\phi) =_{\mathscr{L}} {} & \frac{1}{2}\int_0^t du \int_0^u dv\, A_{uv}(K^\varepsilon(x-y)\phi(y)) \\ & + \int_0^t du\, A_{uu}(K^\varepsilon(x-y)\phi(y)) - \int_0^t du\, A_{tu}(K^\varepsilon(x-y)\phi(y)) \\ & + \int_0^t\int_{\mathbb{R}^2} \mu_u(\nabla K^\varepsilon(x-\cdot)\phi(\cdot))W(dx,du).\end{aligned} \tag{5.13}$$

All we need to do to prove Theorem 3.3 is to show that each term in (5.13) has a well-defined limit as $\varepsilon \to 0$. The $\mathscr{L}^2$ convergence of Ψ_t^ε to Ψ_t is Theorem 2.2. Thus we need only work on the four terms on the right-hand side of (5.13). Consider the first of these, which, modulo a factor of $\frac{1}{2}$, equals

$$I_\varepsilon(\phi) = \int_0^t du \int_0^u dv\, A_{uv}(K^\varepsilon(x-y)\phi(y)).$$

In order to show that I_ε converges in $\mathscr{L}^2$, it suffices to show that $E\{I_\varepsilon(\phi)I_\delta(\phi)\}$ tends to a limit as $\varepsilon, \delta \to 0$. By Theorem 2.1

$$\begin{aligned}E\{I_\varepsilon(\phi)I_\delta(\phi)\} = {} & \int_D\int_{\mathbb{R}^8} p_{|u-u'|}(x,z)p_{|v-v'|}(y,w)K^\varepsilon(x-y) \\ & \quad\times\phi(y)K^\delta(z-w)\phi(w) \\ & + \int_D\int_{\mathbb{R}^8} p_{|u-v'|}(x,w)p_{|v-u'|}(y,z)K^\varepsilon(x-y) \\ & \quad\times\phi(y)K^\delta(z-w)\phi(w),\end{aligned} \tag{5.14}$$

where $D = \{0 \le v \le u \le t;\ 0 \le v' \le u' \le t\}$ and we have neglected to write the eight differentials in each of the multiple integrals.

Consider the first term in (5.14). (The second is handled in an almost identical fashion.) By the definition of K^ε this is equal to

$$\begin{aligned}
&\int_D \int_{\mathbb{R}^8} p_{|u-u'|}(x,z)\, p_{|v-v'|}(y,w) \int_\varepsilon^\infty e^{-\alpha/2} p_\alpha(x,y)\phi(y)\, d\alpha \\
&\quad \times \int_\delta^\infty e^{-\beta/2} p_\beta(z,w)\phi(w)\, d\beta \\
&\quad = \int_\varepsilon^\infty e^{-\alpha/2}\, d\alpha \int_\delta^\infty e^{-\beta/2}\, d\beta \int_D \int_{\mathbb{R}^4} dy\, dw\, \phi(y)\phi(w) \\
&\quad \times \int_{\mathbb{R}^2} dz\, p_\beta(z,w) \int_{\mathbb{R}^2} dx\, p_{|u-u'|}(x,z)\, p_\alpha(x,y)\, p_{|v-v'|}(y,w).
\end{aligned}$$

Integrating over x and then z, by applying the Chapman–Kolmogorov equation twice, we obtain that this is equal to

$$\int_\varepsilon^\infty e^{-\alpha/2}\, d\alpha \int_\delta^\infty e^{-\beta/2}\, d\beta \int_D \int_{\mathbb{R}^4} dy\, dw\, \phi(y)\phi(w)\, p_{|v-v'|}(y,w)\, p_{(|u-u'|+\alpha+\beta)}(y,w).$$

By the Lebesgue dominated convergence theorem this will converge, as $\varepsilon, \delta \to 0$ to the finite constant

$$4 \int_D \int_{\mathbb{R}^4} dy\, dw\, \phi(y)\phi(w)\, p_{|v-v'|}(y,w)\, p_{|u-u'|}(y,w),$$

as long as $\int_D \int_{\mathbb{R}^4} dy\, dw\, \phi(y)\phi(w) p_{|v-v'|}(y,w) p_{|u-u'|+\alpha+\beta}(y,w)$ is bounded uniformly in ε and δ for each $\phi \in \mathscr{S}_2$. Calculations similar to those made in the proof of Theorem 2.2 easily show this to be the case.

This establishes the required convergence for the first term on the right-hand side of (5.13). The next two terms are handled similarly, and we leave the details to the reader. The last term is somewhat different, however, so we treat it in detail. Set

$$J_\varepsilon(\phi) = \int_0^t \int_{\mathbb{R}^2} \mu_u(\nabla K^\varepsilon(x - \cdot)\phi(\cdot))\, W(dx, du).$$

Then

$$\begin{aligned}
E\{[(J_\varepsilon - J_\delta)(\phi)]^2\} &= E\left\{\left[\int_0^t \int_{\mathbb{R}^2} \mu_u(\nabla(K^\varepsilon - K^\delta)(x - \cdot)\phi(\cdot))\, W(dx, du)\right]^2\right\} \\
&\le E\left\{\int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\mu_u(\nabla(K^\varepsilon - K^\delta)(x - \cdot)\phi(\cdot))| \right. \\
&\qquad \times |\mu_u(\nabla(K^\varepsilon - K^\delta)(y - \cdot)\phi(\cdot))| \\
&\qquad \left. \times \delta(x-y)\, dx\, dy\, du \right\},
\end{aligned}$$

where the last line follows from Theorem 2.5 of Walsh (1986), and is, in turn, equal to

$$E\left\{\int_0^t\int_{\mathbb{R}^2}\left|\mu_u(\nabla(K^\varepsilon - K^\delta)(x-\cdot)\phi(\cdot))\right|^2 dx\,du\right\}$$

$$= \int_0^t\int_{\mathbb{R}^2} dx\,du\,E\left|\mu_u(\nabla(K^\varepsilon - K^\delta)(x-\cdot)\phi(\cdot))\right|^2$$

$$= \int_0^t\int_{\mathbb{R}^2} dx\,du\,E\left|\int_0^u \eta_v(\nabla(K^\varepsilon - K^\delta)(x-\cdot)\phi(\cdot))\,dv\right|^2$$

$$= \int_0^t\int_{\mathbb{R}^2} dx\,du\int_0^u\int_0^u E\left[\eta_{v_1}(\nabla(K^\varepsilon - K^\delta)(x-\cdot)\phi(\cdot))\right.$$
$$\left.\times\,\eta_{v_2}(\nabla(K^\varepsilon - K^\delta)(x-\cdot)\phi(\cdot))\right]dv_1\,dv_2$$

$$= \int_0^t\int_{\mathbb{R}^2} dx\,du\int_0^u\int_0^u E\left[\left(\nabla(K^\varepsilon - K^\delta)(x - X_{v_1})\phi(X_{v_1})\right)\right.$$
$$\left.\times\left(\nabla(K^\varepsilon - K^\delta)(x - X_{v_2})\phi(X_{v_2})\right)\right]dv_1\,dv_2,$$

the last line following from Walsh (1986), page 389. Note that for each $\varepsilon > 0$,

$$K^\varepsilon(x) = \frac{2}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ip\cdot x}\frac{e^{-\varepsilon(1+\|p\|^2)/2}}{1+\|p\|^2}\,dp.$$

Substitute this into the above to obtain

$$E\{[(J_\varepsilon - J_\delta)(\phi)]^2\} \le Ct\int_0^t dv_1\int_0^{v_1} dv_2\int_{\mathbb{R}^2} dx\int_{\mathbb{R}^2} dp_1\int_{\mathbb{R}^2} dp_2\prod_{k=1}^2\frac{\|p_k\|}{1+\|p_k\|^2}$$
$$\tag{5.15} \times\left[e^{-\varepsilon(1+\|p_k\|^2)/2} - e^{-\delta(1+\|p_k\|^2)/2}\right]$$
$$\times E\left\{e^{ip_1(x-X_{v_1})}\phi(X_{v_1})e^{ip_2(x-X_{v_2})}\phi(X_{v_2})\right\}.$$

We now proceed much as Rosen (1986) argued when dealing with similar expressions that arose in studying Brownian motion self-intersections. The expectation in (5.15) is a simple Gaussian calculation, and is easily seen to be equal to

$$\tag{5.16} e^{i(p_1+p_2)x}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\phi(y)\phi(z)e^{-ip_1y-ip_2z}\,p_{|v_1-v_2|}(y-z)\,dy\,dz.$$

Note the elementary inequality $|e^{-\varepsilon a^2} - e^{-\delta a^2}| \le C_\alpha(a^2|\varepsilon-\delta|)^\alpha$, for every $\alpha < 1$, and the fact that $\int e^{i(p_1+p_2)x}\,dx = \delta(p_1+p_2)$, where δ here represents the Dirac delta function. Thus, substituting (5.16) into (5.15), integrating out x

and applying these two facts we obtain that (5.15) is bounded above by

$$C_\alpha|\varepsilon - \delta|^{2\alpha} t \int_0^t dv_1 \int_0^{v_1} dv_2 \int_{\mathbb{R}^2} dp \frac{\|p\|^2}{(1+\|p\|^2)^2}(1+\|p\|^2)^{2\alpha}$$

$$\times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(y)\phi(z) p_{|v_1 - v_2|}(y-z) e^{ip(y-z)}\, dy\, dz$$

$$= C_\alpha|\varepsilon - \delta|^{2\alpha} t \int_0^t dv \int_v^t dw \int_{\mathbb{R}^2} dp \frac{\|p\|^2}{(1+\|p\|^2)^{2-2\alpha}} \tag{5.17}$$

$$\times \int_{\mathbb{R}^2} dz\, p_v(z) e^{ipz} \int_{\mathbb{R}^2} dy\, \phi(y)\phi(y-z)$$

$$\le C_\alpha|\varepsilon - \delta|^{2\alpha} t^2 \int_{\mathbb{R}^2} dp \frac{\|p\|^2}{(1+\|p\|^2)^{2-2\alpha}} \int_{\mathbb{R}^2} dz\, e^{ipz} f(z) g_t(z),$$

where C_α may change from line to line, and the functions f and g are given by

$$f(z) = \phi * \phi'(z), \qquad g_t(z) = \int_0^t p_v(z)\, dv,$$

$\phi'(x) = \phi(-x)$ and $*$ represents convolution.

Denoting Fourier transforms in the usual way and writing $h_\alpha(p)$ for

$$\|p\|^2/(1+\|p\|^2)^{2-2\alpha},$$

we can write the integrals in the last line of (5.17) as

$$\int_{\mathbb{R}^2} h_\alpha(p)(\hat{f} * \hat{g}_t(p))\, dp. \tag{5.18}$$

Clearly, if we can establish that this integral is finite for some $\alpha < 1$, then the $\mathscr{L}^2$ convergence of $J_\varepsilon(\phi)$ follows from (5.15)–(5.17). A standard inequality on convolutions [e.g., Reed and Simon (1975), page 29] gives us that

$$\int_{\mathbb{R}^2} h_\alpha(p)(\hat{f} * \hat{g}_t(p))\, dp \le \|h_\alpha\|_s \|f\|_q \|g\|_1, \tag{5.19}$$

where $1/s + 1/q = 1$. We leave it to the reader to show that if we choose $\alpha < \frac{1}{2}$ and $s > (1-2\alpha)^{-1}$ then each of the three norms in (5.19) is finite. This completes the proof of Theorem 3.3. □

Only one task remains to complete our work:

Proof of Lemma 3.5. The proof centers on noting that the equivalence (3.9), which is what we must establish, is almost identical to that established in Lemma 5.3, with the function g in the former replaced by K^ε in the latter. If we can send $\varepsilon \to 0$, and show that all terms in Lemma 5.3 converge in $\mathscr{L}^2$

to the corresponding term in (3.9), then we shall be done. This, however, is not too difficult, since at this stage the density parameter λ is still finite.

We shall consider only one term of Lemma 5.3, and, following our established practice, shall choose the most difficult term. Set

$$F_\varepsilon^\lambda(\phi) = \int_0^t \int_0^u A_{uv}^\lambda(K^\varepsilon(x-y)\phi(y))\,du\,dv.$$

We need to show that as $\varepsilon \to 0$ this converges in $\mathscr{L}^2$ to first term of (3.9). Note that for $\varepsilon, \delta > 0$,

$$\begin{aligned}
&E\Big\{\big|F_\varepsilon^\lambda(\phi) - F_\delta^\lambda(\phi)\big|^2\Big\} \\
&\quad = E\left[\int_0^t du \int_0^u dv\,\lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j (K^\varepsilon - K^\delta)(X_u^i - X_v^j) \cdot \phi(X_v^j)\right]^2 \\
&\quad \le \int_0^t du \int_0^u dv\,\lambda^{-2} E\left[\sum_{i \neq j} \sigma^i \sigma^j (K^\varepsilon - K^\delta)(X_u^i - X_v^j) \cdot \phi(X_v^j)\right]^2 \\
&\quad \le \int_0^t du \int_0^u dv\,\lambda^{-2}\lambda^2 E\Big\{\big|(K^\varepsilon - K^\delta)(X_u^i - X_v^j)\big|^2 \cdot \big|\phi(X_v^j)\big|^2\Big\} \\
&\quad = \int_0^t du \int_0^u dv \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz (K^\varepsilon - K^\delta)^2(y-z) \cdot \phi^2(z) \\
&\quad \le t^2 \int_{\mathbb{R}^2} \phi^2(z)\,dz \int_{\mathbb{R}^2} (K^\varepsilon - K^\delta)^2(y)\,dy.
\end{aligned}$$

The important thing to note at this stage is that we have managed to reduce the computations to the point that they are of the same nature as those which prove the existence of intersection local time for two independent Brownian motions, as in Rosen (1986). In fact, they are very similar to those at the end of the previous proof. We thus leave it to the interested reader to satisfy himself that it is now not hard to show that

$$\lim_{\varepsilon, \delta \to 0} E\Big\{\big|F_\varepsilon^\lambda(\phi) - F_\delta^\lambda(\phi)\big|^2\Big\} = 0,$$

which completes the proof of the convergence of the first term of the representation of Lemma 5.3 to that of (3.9). The remaining terms can be handled similarly, and this completes the proof of Lemma 3.5. □

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