The Annals of Probability
1991, Vol. 19, No. 1, 160–191

SELF-INTERSECTIONS AND LOCAL NONDETERMINISM OF GAUSSIAN PROCESSES[1]

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Let $\mathbf{X}(t)$, $t \geq 0$, be a vector Gaussian process in R^m whose components are i.i.d. copies of a real Gaussian process $X(t)$ with stationary increments. Under specified conditions on the spectral distribution function used in the representation of the incremental variance function, it is shown that the self-intersection local time of multiplicity r of the vector process is jointly continuous. The dimension of the self-intersection set is estimated from above and below. The main tool is the concept of local nondeterminism.

1. Introduction and summary. Let $\mathbf{X}(t)$, $t \geq 0$, be a vector random process in R^m. For r closed disjoint intervals $I_1, \dots, I_r$, define the process

$$(1.1)\quad (\mathbf{X}(t_2) - \mathbf{X}(t_1), \dots, \mathbf{X}(t_r) - \mathbf{X}(t_{r-1})), \qquad t_i \in I_i,\ i = 1, \dots, r,$$

from $I_1 \times \cdots \times I_r$ into $R^{m(r-1)}$. The local time of this process, if it exists, is called the r-multiple intersection local time of the original process $\mathbf{X}(t)$. The zeros of the process (1.1) are points $t_1 < \cdots < t_r$ on the time axis, where $\mathbf{X}(t_1) = \cdots = \mathbf{X}(t_r)$, and the set of such r-tuples of points is the r-multiple intersection set. The focus of this work is the Hausdorff dimension of the multiple intersection set for a particular class of vector Gaussian processes. In line with current work in this area, the main tool in the determination of this dimension is the intersection local time.

Let $X(t)$, $t \geq 0$, be a real Gaussian process with mean 0, stationary increments and with $X(0) = 0$ a.s. Then the vector process $\mathbf{X}(t)$ is defined as having m independent component processes which are distributed as $X(t)$. The major step in obtaining the results on the dimension of the self-intersection set is showing that the intersection local time is jointly continuous and satisfies a certain uniform Holder condition. This is achieved by showing that a particular multiple integral whose integrand contains the modulus of the joint characteristic function of the random vectors (1.1) is finite. The tool used in the evaluation of this integral is the local nondeterminism of the underlying real process, introduced by Berman (1973). The latter property implies the following relation: For each $k \geq 2$ there exists a constant $K > 0$ such that if

Received December 1988; revised August 1989.

[1]This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of National Science Foundation Grant DMS-88-01188 and the U.S. Army Research Office, Contract DAAL 03-86-K-0127.

AMS 1980 *subject classifications*. 60G15, 60G17, 60J55.

Key words and phrases. Intersections of sample paths, local nondeterminism, local time, Gaussian processes, spectral distribution.

$t_1 < \cdots < t_k$, and $t_k - t_1$ is sufficiently small, then

$$\operatorname{Var}\left[u_1 X(t_1) + \sum_{j=1}^{k} u_j\big(X(t_j) - X(t_{j-1})\big)\right] \geq K\left[u_1^2 \operatorname{Var} X(t_1) + \sum_{j=2}^{k} u_j^2 \operatorname{Var}\big(X(t_j) - X(t_{j-1})\big)\right] \tag{1.2}$$

for all $(u_1, \ldots, u_k)$. This inequality has a key role in the evaluation of the multiple integral.

The present work depends heavily on the sufficient conditions for local nondeterminism given by Berman (1987). These conditions are closely related to the upper and lower bounds for the dimension of the self-intersection set obtained here. Suppose that the process is real, and let $\sigma^2(t) = EX^2(t)$ be the incremental variance function, and $F(x)$ the corresponding spectral distribution function, so that

$$\sigma^2(t) = \int_0^\infty (1 - \cos xt) \frac{1 + x^2}{x^2}\, dF(x).$$

If F is absolutely continuous, let f be its density function, and assume that

$$\beta_1 = \liminf_{x \to \infty} \frac{xf(x)}{F(\infty) - F(x)} > 0 \tag{1.3}$$

and

$$\beta_2 = \limsup_{x \to \infty} \frac{xf(x)}{F(\infty) - F(x)} < 2. \tag{1.4}$$

Our main results are stated in Theorem 9.2 and Corollaries 9.1 and 9.2.

With probability 1 *the intersection local time is jointly continuous, and the dimension of the r-multiple intersection set is at most equal to*

$$r - \tfrac{1}{2}\beta_1 m(r-1). \tag{1.5}$$

With positive probability, the dimension is at least equal to

$$r - \tfrac{1}{2}\beta_2 m(r-1). \tag{1.6}$$

There is an analogous result when F is a discrete distribution on the nonnegative integers. Here the sample function is periodic, so that the time domain is restricted to $[0, 2\pi)$. This case is discussed in Section 10.

Rosen (1984) considered the dimension of the self-intersection set for particular kinds of Gaussian processes, including one in which the underlying component is a fractional Brownian motion process of index β, which, in our notation, corresponds to 2β. This is a real Gaussian process $X(t)$, $t \in R^N$, such that $X(0) = 0$, a.s., $EX(t) \equiv 0$ and $E(X(t) - X(s))^2 = \|s - t\|^{2\beta}$, where $\|\cdot\|$ is the Euclidean norm in R^N. Rosen's result is that, with probability 1, the dimension is at most

$$Nr - \beta m(r - 1) \tag{1.7}$$

and, with positive probability, at least equal to the same. Thus our result is a generalization of Rosen's for $N = 1$. The extension of our results to $N > 1$ would be based on the corresponding extension of our previous results on local nondeterminism [Berman (1987)], but this would represent a project for another time and place.

The significance of the current work is that the general kind of local nondeterminism implied by conditions (1.3) and (1.4) is used in the place of the special property of the fractional Brownian process used by Rosen. Pitt (1978) had shown that (1.2) holds for a fractional Brownian motion with vector time t for all $t_1, \dots, t_k$ such that

$$\|t_{j+1} - t_j\| \le \|t_{j+1} - t_i\|, \qquad 1 \le i \le j \le k,$$

$$\varepsilon < \|t_j\| < \varepsilon^{-1}, \qquad j = 1, \dots, k,$$

for any $\varepsilon > 0$. His argument employs the self-similarity of this process. By contrast our hypothesis, stated as (1.3) and (1.4), involves only the tail of the spectrum, which governs the local behavior of the process. This hypothesis implies only the local version (1.2) for $t_k - t_1 \to 0$. Much of the effort in this paper goes to extending (1.2) to a version which holds globally, without the special feature of self-similarity. The key to the verification of (1.2) is to show that the joint distribution of increments $X(t_j) - X(t_{j-1})$, $j = 1, \dots, k$, each divided by its respective standard deviation, for $t_0 < t_1 < \cdots < t_k$, is "bounded away" from a singular distribution. While our earlier work (1987) showed that the latter is true for the same points (t_j) converging to a common limit, the current arguments now demonstrate that under the same assumptions, the conclusion about nonsingularity continues to hold without the convergence to a common point. This implies the desired global version of (1.2). The latter inequality is then employed to estimate the key integral which arises in Gaussian local time theory, and this enables us to prove the joint continuity of the intersection local time.

The results for Gaussian processes can be extended to certain non-Gaussian processes by the method of "subordination" introduced by Berman (1982) and developed further by Berman (1987). It is shown, in particular, in Section 10, that the results can be extended to random Fourier series with coefficients which are symmetric stable random variables.

Section 3 contains some improvements of the theory of local nondeterminism which are also of independent interest. Theorem 3.1 is a Tauberian theorem for characteristic functions which answers a question raised by Berman (1987), and which is crucial for the main results which are obtained here.

A related subject, that of the existence of multiple points of a given order, was also studied by Cuzick (1982).

2. Local nondeterminism and preliminary results about second-order moment inequalities. We recall the concept of local nondeterminism of a Gaussian process $X(t)$, $t \in J$, where J is a real open interval, introduced by Berman (1973). Assume that $EX(t) = 0$ and $EX^2(t) > 0$ for $t \in J$, and that there exists $d > 0$ such that $E(X(t) - X(s))^2 > 0$ whenever $0 < |t - s| < d$, for $s, t \in J$. For $k \geq 2$, let $t_1 < \cdots < t_k$ be points in J, and put

$$V_k = \frac{\operatorname{Var}\{X(t_k) - X(t_{k-1}) | X(t_1), \dots, X(t_{k-1})\}}{\operatorname{Var}\{X(t_k) - X(t_{k-1})\}}.$$

The process is called locally nondeterministic on J if, for each $k \geq 2$,

$$\lim_{c \to 0} \inf_{t_k - t_1 < c} V_k > 0.$$

It can be shown by means of the arguments in Berman (1969), Lemma 5.1, and Berman (1973), Lemma 8.1, that local nondeterminism implies the following result: For every $k \geq 2$, there exists $C_k > 0$ such that whenever $t_k - t_1$ is sufficiently small, we have

$$\begin{aligned} &\operatorname{Var}\left[u_1 X(t_1) + \sum_{j=2}^{k} u_j\big(X(t_j) - X(t_{j-1})\big)\right] \\ (2.1) \qquad &\geq C_k\left[u_1^2 \operatorname{Var} X(t_1) + \sum_{j=2}^{k} u_j^2 \operatorname{Var}\big(X(t_j) - X(t_{j-1})\big)\right] \end{aligned}$$

for all $u_1, \dots, u_k$.

This condition can be put in a convenient matrix form. Let $A = (a_{ij})$ be the $k \times k$ matrix defined as

$$(2.2) \qquad a_{ij} = \delta_{ij} - \delta_{i,j+1},$$

$i, j = 1, \dots, k$. (A has 1's on the diagonal, and -1's as entries just below the diagonal entries.) Put $X_j = X(t_j)$, and define the random k-component vector $\mathbf{X} = (X_j)$. (The dependence on the t's is implicitly understood.) Then $A\mathbf{X}$ has the components $(A\mathbf{X})_1 = X_1$, $(A\mathbf{X})_j = X_j - X_{j-1}$, $2 \leq j \leq k$, and (2.1) may be written in the form

$$(2.3) \qquad \operatorname{Var}(\mathbf{U}'A\mathbf{X}) \geq C_k \sum_{j=1}^{k} u_j^2 \operatorname{Var}(A\mathbf{X})_j,$$

where $\mathbf{U} = (u_j)$, and $\mathbf{U}'$ is its transpose. Since (2.3) holds for all $\mathbf{U}$ as a consequence of local nondeterminism, it holds in particular if $\mathbf{U}$ is replaced by $(A')^{-1}U$,

$$(2.4) \qquad \operatorname{Var}(\mathbf{U}'\mathbf{X}) \geq C_k \sum_{j=1}^{k} (A'^{-1}\mathbf{U})_j^2 \operatorname{Var}(A\mathbf{X})_j$$

for all $\mathbf{U}$.

The following preliminary results, dealing with general second-order moments of random variables are intended for applications to the extensions of (2.3) and (2.4).

LEMMA 2.1. *Let* $Y_1, \dots, Y_r$ *be random variables with variances* $\sigma_1^2, \dots, \sigma_r^2$, *and put* $\rho = \max_{i \neq j} |\text{Correlation}(Y_i, Y_j)|$; *then*

$$\operatorname{Var}(Y_1 + \cdots + Y_r) \geq (1 - \rho(r-1))(\sigma_1^2 + \cdots + \sigma_r^2). \tag{2.5}$$

PROOF. This follows from the elementary results that

$$\operatorname{Var}\left(\sum_{i=1}^r Y_i\right) \geq \sum_{i=1}^r \sigma_i^2 - \rho \sum_{i \neq j} \sigma_i \sigma_j,$$

$$\sum_{i \neq j} \sigma_i \sigma_j = \left(\sum_{i=1}^r \sigma_i\right)^2 - \sum_{i=1}^r \sigma_i^2$$

and

$$\left(\sum_{i=1}^r \sigma_i\right)^2 \leq r \sum_{i=1}^r \sigma_i^2. \qquad \square$$

LEMMA 2.2. *Let* $\mathbf{Y} = (Y_j)$ *and* $\mathbf{Z} = (Z_j)$ *be k-component random vectors such that for some* $C > 0$,

$$\begin{aligned} \operatorname{Var}(\mathbf{U}'\mathbf{Y}) &\geq C \sum_{j=1}^k U_j^2 \operatorname{Var} Y_j, \\ \operatorname{Var}(\mathbf{U}'\mathbf{Z}) &\geq C \sum_{j=1}^k U_j^2 \operatorname{Var} Z_j \end{aligned} \tag{2.6}$$

for all $\mathbf{U}$; *and for every pair of components* Y_i *and* Z_j,

$$|\text{Correlation}(Y_i, Z_j)| \leq \delta \tag{2.7}$$

for some $0 \leq \delta < 1$. *Then for every* $\mathbf{U}$ *and* $\mathbf{V}$,

$$|\text{Correlation}(U'Y, V'Z)| \leq \delta r / C. \tag{2.8}$$

PROOF. By an elementary formula and the Cauchy–Schwarz inequality we have

$$\begin{aligned} &|\text{Covariance}(\mathbf{U}'\mathbf{Y}, \mathbf{V}'\mathbf{Z})| \\ &\quad = \left| \sum_{i,j} U_i V_j (\operatorname{Var} Y_i \cdot \operatorname{Var} Z_j)^{1/2} \text{Correlation}(Y_i, Z_j) \right| \\ &\quad \leq \delta \left(\sum_i |U_i| \sqrt{\operatorname{Var} Y_i} \right) \left(\sum_j |V_j| \sqrt{\operatorname{Var} Z_j} \right) \\ &\quad \leq \delta r \left(\sum_i U_i^2 \operatorname{Var} Y_i \right)^{1/2} \left(\sum_j V_j^2 \operatorname{Var} Z_j \right)^{1/2}. \end{aligned}$$

By the assumption (2.6), the latter product is at most equal to

$$(\delta r/C)[\operatorname{Var} \mathbf{U}'\mathbf{Y} \cdot \operatorname{Var} \mathbf{V}'\mathbf{Z}]^{1/2},$$

which confirms the claim (2.8). □

Our main result here is Lemma 2.3.

LEMMA 2.3. *Let* $\mathbf{Y}_1, \dots, \mathbf{Y}_r$ *be k-component random vectors, and put* $Y_{ij} =$ *component j of* $\mathbf{Y}_i$. *Suppose that there exists* $C > 0$ *such that for any set of k-vectors* $\mathbf{U}_1, \dots, \mathbf{U}_r$,

$$\operatorname{Var}(\mathbf{U}'_i\mathbf{Y}_i) \geq C \sum_{j=1}^{k} U_{ij}^2 \operatorname{Var} Y_{ij}, \qquad i = 1, \dots, r. \tag{2.9}$$

Put

$$\rho = \max_{\substack{i \neq i',\, i, j = 1, \dots, k \\ j, j' = 1, \dots, r}} |\operatorname{Correlation}(Y_{ij} Y_{i'j'})|; \tag{2.10}$$

then, for any $\mathbf{U}_1, \dots, \mathbf{U}_r$,

$$\operatorname{Var}\left(\sum_{i=1}^{r} \mathbf{U}'_i\mathbf{Y}_i\right) \geq (C - r(r-1)\rho) \sum_{i=1}^{r} \sum_{j=1}^{k} U_{ij}^2 \operatorname{Var} Y_{ij}. \tag{2.11}$$

PROOF. By Lemma 2.2, with ρ in the place of δ, we have $|\operatorname{Correlation}(\mathbf{U}'_i\mathbf{Y}_i, \mathbf{U}'_j\mathbf{Y}_j) \leq \rho r/C$ for $i \neq j$. Hence, by Lemma 2.1, with $\mathbf{U}'_i\mathbf{Y}_i$ in the place of the random variable Y_i, $i = 1, \dots, r$, and $\rho r/C$ in the place of ρ, we have

$$\operatorname{Var}\left(\sum_{i=1}^{r} \mathbf{U}'_i\mathbf{Y}_i\right) \geq (1 - r(r-1)\rho/C) \sum_{i=1}^{r} \operatorname{Var}(\mathbf{U}'_i\mathbf{Y}_i),$$

and the latter, by (2.9), is at least equal to the right-hand member of (2.11). □

3. Local nondeterminism for Gaussian processes with stationary increments. Let $X(t)$, $t \geq 0$, be a separable Gaussian process with mean 0 and stationary increments, and with incremental variance function $\sigma^2(t) = E(X(s+t) - X(s))^2$, which is continuous for $t \geq 0$. For simplicity we assume that $X(0) = 0$ a.s. so that $EX^2(t) = \sigma^2(t)$, $t \geq 0$. Then $\sigma^2(t)$ has the well-known representation

$$\sigma^2(t) = \tfrac{1}{4}\int_{-\infty}^{\infty} |e^{ixt} - 1|^2 (1 + x^2) x^{-2}\, dF(x), \tag{3.1}$$

where F is the spectral distribution function. Since we are assuming that X is real, F is symmetric, and (3.1) becomes

$$\sigma^2(t) = \int_0^{\infty} (1 - \cos xt)(1 + x^2) x^{-2}\, dF(x). \tag{3.2}$$

Suppose that F is absolutely continuous with density f:

$$F(x) = \int_0^x f(y)\,dy. \tag{3.3}$$

Define

$$\phi(t) = F(\infty) - F(1/t), \qquad t > 0. \tag{3.4}$$

Following the definition of Cuzick (1978), we say that X is ϕ-locally nondeterministic if for $t_1 < \cdots < t_k$, the determinant of the covariance matrix of the standardized increments

$$\frac{X(t_1)}{\phi(t_1)}, \frac{X(t_2) - X(t_1)}{\phi(t_2 - t_1)}, \dots, \frac{X(t_k) - X(t_{k-1})}{\phi(t_k - t_{k-1})} \tag{3.5}$$

is bounded away from 0 for $t_k - t_1 \to 0$. The main result in Berman (1987) is that X is ϕ-locally nondeterministic if the spectrum satisfies the conditions

$$\begin{aligned} \beta_1 &= \liminf_{x\to\infty} \frac{xf(x)}{F(\infty) - F(x)} > 0, \\ \beta_2 &= \limsup_{x\to\infty} \frac{xf(x)}{F(\infty) - F(x)} < 2. \end{aligned} \tag{3.6}$$

The definition of ordinary local nondeterminism in Berman (1973) is that this condition holds in the case where ϕ is replaced in (3.5) by the function σ^2. [Cuzick's definition is not restricted to the particular function ϕ in (3.4) but may be employed for any nonnegative nondecreasing function ϕ.] I noted in Berman (1987) that I did not know the exact relation between ordinary local nondeterminism and ϕ-local nondeterminism for ϕ in (3.4). It will now be shown that the two are actually equivalent. The following theorem is a "characteristic function version" of a Tauberian theorem of de Haan and Stadtmüller (1985) for Laplace transforms.

THEOREM 3.1. *Under* (3.6)

$$\limsup_{t\to 0} \frac{\sigma^2(t)}{\phi(t)} < \infty \tag{3.7}$$

and

$$\liminf_{t\to 0} \frac{\sigma^2(t)}{\phi(t)} > 0. \tag{3.8}$$

PROOF. The inequality (3.8) was proved in Berman (1987), page 81. We now prove (3.7). The representation (3.2) implies that $\sigma^2(t)/\phi(t)$ is, for fixed

$T > 0$ and all t such that $Tt < 1$, equal to the sum of three integrals,

$$\int_0^T (1 - \cos xt) \frac{1 + x^2}{x^2} \frac{f(x)}{\phi(t)}\, dx, \tag{3.9}$$

$$\int_T^{1/t} (1 - \cos xt) \frac{1 + x^2}{x^2} \frac{f(x)}{\phi(t)}\, dx, \tag{3.10}$$

$$\int_{1/t}^{\infty} (1 - \cos xt) \frac{1 + x^2}{x^2} \frac{f(x)}{\phi(t)}\, dx. \tag{3.11}$$

The integral (3.9) is at most equal to a constant times $t^2/\phi(t)$, which, by Berman (1987), Lemma 2.1, converges to 0 for $t \to 0$.

Since T is arbitrary, we fix it to be so large that, under (3.6), $f(x) < 3\phi(1/x)(1/x)$, for all $x > T$, so that (3.10) is at most equal to $3T^{-2}(1 + T^2)$ times

$$\int_T^{1/t} (1 - \cos xt) \frac{\phi(1/x)}{\phi(t)} \frac{dx}{x}.$$

Since $1/x > t$ on the domain of integration above, the first inequality in Berman (1987), formula (2.6), implies

$$\frac{\phi(1/x)}{\phi(t)} \le (xt)^{-2+\varepsilon}$$

for some $\varepsilon > 0$ arbitrarily small. Hence the integral above is at most equal to

$$\int_T^{1/t} \frac{1 - \cos xt}{(xt)^{2-\varepsilon}} \frac{dx}{x} \le \int_0^1 \frac{1 - \cos x}{x^{3-\varepsilon}}\, dx < \infty.$$

Finally, we estimate (3.11). It is obviously less than

$$3(1 + t^2) \int_{1/t}^{\infty} (1 - \cos xt) \frac{\phi(1/x)}{\phi(t)} \frac{dx}{x}.$$

Since $1/x < t$ on the domain of integration it follows from the second inequality in Berman (1987), formula (2.6), that

$$\frac{\phi(1/x)}{\phi(t)} \le (xt)^{-\varepsilon},$$

so that the integral displayed above is at most equal to a constant times

$$\int_{1/t}^{\infty} \frac{dx}{(xt)^{\varepsilon} x} = \int_1^{\infty} \frac{dx}{x^{1+\varepsilon}} < \infty.$$

We have shown that (3.9) tends to 0 for $t \to 0$, and that (3.10) and (3.11) are bounded. □

As noted before the statement of Theorem 3.1, the significance of the result is that the function $\sigma^2(t)$ may be interchanged with $\phi(t)$ in the criterion for

ϕ-local nondeterminism based on the family of random variables (3.5). Thus (3.6) is also sufficient for ordinary local nondeterminism.

We also deduce the following improvements in the content of Berman (1987), due to Theorem 3.1.

COROLLARY 3.1. *Under* (3.6), *for any* $\beta_1' < \beta_1$ *and* $\beta_2' > \beta_2$,

$$t^{\beta_2'} \le \sigma^2(t) \le t^{\beta_1'} \tag{3.12}$$

for all sufficiently small $t > 0$.

PROOF. This is a consequence of Berman (1987), formula (2.6), and Theorem 3.1. □

COROLLARY 3.2. *Under* (3.6) *the local time of the sample function exists a.s.*

PROOF. In view of Corollary 3.1, the condition in the hypothesis of Berman (1987), Theorem 3.1, is fulfilled, and the conclusion is immediate. □

4. Preservation of the hypothesis under certain changes in the spectrum. The condition (3.6) implies that, for all sufficiently large x, the density $f(x)$ is exactly of the same asymptotic order as $(1/x)\phi(1/x)$. In the following lemmas we give a more precise version of this result.

LEMMA 4.1. *Let* f *satisfy* (3.6), *and define*

$$\begin{aligned} g(x) &= x^{-1}(F(\infty) - F(x)),\\ G(x) &= \int_x^\infty g(u)\,du \end{aligned} \tag{4.1}$$

for all sufficiently large x; *then*

$$\beta_1 \le \liminf_{x\to\infty} \frac{xg(x)}{G(x)} \tag{4.2}$$

and

$$\limsup_{x\to\infty} \frac{xg(x)}{G(x)} \le \beta_2. \tag{4.3}$$

PROOF. First of all, by Theorem 3.1 and Corollary 3.1, $F(\infty) - F(x) \le x^{-\beta_1'}$, for every $\beta_1' < \beta_1$, for all large x. Therefore $g(x) \le x^{-1-\beta_1'}$, and so the integral $G(x)$ in (4.1) is finite. Writing the ratio $xg(x)/G(x)$ in terms of F,

$$\frac{xg(x)}{G(x)} = \frac{F(\infty) - F(x)}{\int_x^\infty u^{-1}(F(\infty) - F(u))\,du}, \tag{4.4}$$

and applying (3.6), we see that for each $\beta_1' < \beta_1$ and all sufficiently large x, the denominator in (4.4) is at most equal to $\int_x^\infty (\beta_1')^{-1} f(u)\,du = (\beta_1')^{-1}(F(\infty) - F(x))$, so that the ratio is at least equal to β_1'. Therefore

$$\liminf_{x\to\infty} \frac{xg(x)}{G(x)} \geq \beta_1'$$

for all $\beta_1' \leq \beta_1$, and (4.2) follows.

By the assumption (3.6), for every $\beta_2' > \beta_2$, the denominator in (4.4) is at least equal to $\int_x^\infty (\beta_2')^{-1} f(u)\,du = (\beta_2')^{-1}(F(\infty) - F(x))$, for all sufficiently large x, so that the ratio is at most equal to β_2'. Therefore

$$\limsup_{x\to\infty} \frac{xg(x)}{G(x)} \leq \beta_2'$$

for all $\beta_2' \geq \beta_2$, and (4.3) follows. □

LEMMA 4.2. *Let the spectral distribution F in the representation* (3.2) *of σ^2 have a density f satisfying* (3.6). *For arbitrary $x_0 > 0$, let $f(x)$ be replaced by the function $g(x)$ in* (4.1) *for all $x \geq x_0$, and let $\bar{\sigma}^2(t)$ be the transformed integral* (3.2) *corresponding to the altered density. Then*

$$0 < \liminf_{t\to 0} \frac{\sigma^2(t)}{\bar{\sigma}^2(t)}, \qquad \limsup_{t\to 0} \frac{\sigma^2(t)}{\bar{\sigma}^2(t)} < \infty.$$

PROOF. By Lemma 4.1, the condition (3.6) is preserved under the given change in the spectral distribution, with a possibly larger lim inf and a possibly smaller lim sup. By Theorem 3.1, applied to the new spectral distribution, $\bar{\sigma}^2(t)/G(1/t)$ is bounded away from 0 and ∞ for $t \to 0$. By Lemma 4.1, it follows that $G(1/t)$ is exactly of the same asymptotic order as $t^{-1}g(1/t) = F(\infty) - F(1/t) = \phi(1/t)$, which, by Theorem 3.1, is exactly of the same asymptotic order as $\sigma^2(t)$. □

5. Nonsingularity of finite-dimensional distributions. In this section we prove a general inequality for the variance of a linear combination of random variables of a Gaussian process with stationary increments. This inequality does not depend on (3.6). Furthermore, it holds also for a process whose spectrum is supported by the positive integers. This case is included here because it is to be applied in Section 10 to the case of the discrete spectrum and its extension to random Fourier series.

If the spectral distribution in the representation (3.2) of $\sigma^2(t)$ has a density, then, for every $0 < t_1 < \cdots < t_k$, the joint distribution of $X(t_1), \dots, X(t_k)$ is nonsingular. This fact is well known. For the benefit of the reader we state and prove an analogous result in the discrete case which may be less well known.

LEMMA 5.1. *Suppose that F has support on $\{0, 1, 2 \dots\}$ with f_n defined as the mass at n, for $n \geq 0$. If $f_n > 0$ for all large n, then, for $0 < t_1 < \cdots < t_k < 2\pi$, the joint distribution of $X(t_1), \dots, X(t_k)$ is nonsingular.*

PROOF. It suffices to show that

$$\operatorname{Var}\left(\sum_{j=1}^{k} b_j X(t_j)\right) > 0 \tag{5.1}$$

for any nonzero vector $(b_1, \dots, b_k)$. The spectral representation (3.1) implies [under the assumption $X(0) = 0$ a.s.] that the variance in (5.1) is equal to $\frac{1}{4}$ times

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \sum_{j=1}^{k} b_j (e^{int_j} - 1) \right|^2 \frac{1+n^2}{n^2} f_{|n|}.$$

If this were to be equal to 0, then, under the assumption of the lemma, we would have

$$\sum_{j=1}^{k} b_j (e^{int_j} - 1) = 0$$

for all sufficiently large $|n|$. This implies

$$\sum_{j=1}^{k} b_j \int_0^{t_j} e^{int}\, dt = 0$$

for all sufficiently large $|n|$. The left-hand member above represents 2π times the nth Fourier coefficient of the function

$$\sum_{j=1}^{k} b_j \mathbf{1}_{[0, t_j]}(x).$$

On the one hand, we have just shown that this function has at most finitely many nonzero Fourier coefficients. On the other hand, by its definition, it is a step function assuming the values $b_1 + \cdots + b_k, b_1 + \cdots + b_{k-1}, \dots, b_1$ on the intervals $[0, t_1), [t_1, t_2), \dots, [t_{k-1}, t_k)$, respectively, and 0 on $[t_k, 2\pi)$. These two conclusions are reconcilable only if $b_1 = \cdots = b_k = 0$, so that (5.1) must hold for every nonzero vector $(b_1, \dots, b_k)$. □

THEOREM 5.1. *Let F be absolutely continuous or be discrete and satisfy the assumption of Lemma 5.1. For $0 < t_1 < \cdots < t_k$ (with $t_k < 2\pi$ in the discrete case) there exists a constant $K > 0$, which may depend on $t_1, \dots, t_k$, such that*

$$\operatorname{Var}\left(\sum_{j=1}^{k} u_j X(t_j)\right) \geq K \sum_{j=1}^{k} u_j^2 \sigma^2(t_j) \tag{5.2}$$

for all $u_1, \dots, u_k$.

PROOF. For $1 \le i \le k$,

$$\operatorname{Var}\left(\sum_{j=1}^{k} u_j X(t_j)\right) \ge \operatorname{Var}\left(\sum_{j=1}^{k} u_j X(t_j) | X(t_h),\ h \ne i\right)$$
$$= u_i^2 \operatorname{Var}(X(t_i) | X(t_h),\ h \ne i).$$

Averaging over $i = 1, \dots, k$, we obtain

$$\operatorname{Var}\left(\sum_{j=1}^{k} u_j X(t_j)\right) \ge k^{-1} \min_{1 \le i \le k} \operatorname{Var}(X(t_i) | X(t_h),\ h \ne i) \sum_{j=1}^{k} u_j^2$$
$$\ge \frac{\min_i \operatorname{Var}(X(t_i) | X(t_h),\ h \ne i)}{k \max_i \sigma^2(t_i)} \sum_{j=1}^{k} u_j^2 \sigma^2(t_j).$$

The ratio on the right-hand side above is positive by virtue of Lemma 5.1 in the discrete case, and follows from the established nonsingularity of the distribution in the absolutely continuous case. □

6. Asymptotic properties of the standardized increments. As is well known, the representation (3.1) of $\sigma^2(t)$ is equivalent to the orthogonal decomposition representation of the process itself,

$$X(t) = \tfrac{1}{2} \int_{-\infty}^{\infty} (e^{ixt} - 1)|x|^{-1}(1 + x^2)\xi(dx), \tag{6.1}$$

where $\xi(x)$ has orthogonal Gaussian increments, with $\operatorname{Var}(\xi(dx)) = dF(x)$. The point of the following lemma is to identify the asymptotically dominant x-frequency band in the representation of a small increment obtained from (6.1).

LEMMA 6.1. *Under* (3.6), *for arbitrary* $\delta > 0$, $h > 0$, *the random variable* $(X(t + h) - X(t))/\sigma(h)$ *is equal to*

$$\frac{1}{2\sigma(h)} \int_{\delta/h \le |x| \le 1/\delta h} e^{ixt}(e^{ixh} - 1)|x|^{-1}(1 + x^2)^{1/2} \xi(dx) \tag{6.2}$$

plus an independent random variable $\eta = \eta(\delta, h, t)$ *having a normal distribution with mean* 0 *and such that for each* t,

$$\lim_{\delta \to 0} \limsup_{h \to 0} E(\eta(\delta, h, t))^2 = 0. \tag{6.3}$$

PROOF. The proof relies only on the properties of ϕ stated in Berman (1987), Lemma 2.1, and the current Theorem 3.1. The random variable η is taken as the stochastic integral (6.2) with the complementary domain $\{|x| < \delta/h\} \cup \{|x| > 1/\delta h\}$, so that the variance of η is

$$\frac{1}{4\sigma^2(h)} \int_{\{|x| < \delta/h\} \cup \{|x| > 1/\delta h\}} |e^{ixh} - 1|^2 \frac{1 + x^2}{x^2} f(x)\, dx. \tag{6.4}$$

For arbitrary fixed $M > 0$,

$$\frac{1}{\sigma^2(h)}\int_{|x|\le M}|e^{ixh}-1|^2\frac{1+x^2}{x^2}f(x)\,dx$$

$$\le \frac{1+M^2}{\sigma^2(h)}\int_{|x|\le M}h^2 f(x)\,dx = \text{constant}\frac{h^2}{\sigma^2(h)},$$

which, by Theorem 3.1 above, and Berman (1987), Lemma 2.1, converges to 0 for $h \to 0$. Thus, in estimating (6.4), we may replace the domain of integration by its subset where $|x| > M$, for arbitrary fixed $M > 0$. In particular, for each $\beta_2' > \beta_2$, we may take M so large that $f(x) \le \beta_2'\phi(1/x)(1/x)$, for $x > M$ [by (3.6)], so that the expression (6.4) is at most equal to

$$\frac{\beta_2'(1+M^2)}{4\sigma^2(h)M^2}\int_{\{|x|<\delta/h\}\cup\{|x|>1/\delta h\}}|e^{ixh}-1|^2\phi\left(\frac{1}{|x|}\right)\frac{dx}{|x|}.$$

By Theorem 3.1 and a change of variable of integration, this expression is exactly of the same asymptotic order as

$$\int_{\{|x|<\delta\}\cup\{|x|>1/\delta\}}|e^{ix}-1|^2\frac{\phi(h/|x|)}{\phi(h)}\frac{dx}{|x|}. \tag{6.5}$$

If $\delta < 1$, then there exists ε, $0 < \varepsilon < 1$, such that

$$\frac{\phi(h/|x|)}{\phi(h)} \le |x|^{-2+\varepsilon}$$

for all $|x| < \delta$ and all small $h > 0$ [Berman (1987), formula (2.6)]. Therefore

$$\int_{\{|x|<\delta\}}|e^{ix}-1|^2\frac{\phi(h/|x|)}{\phi(h)}\frac{dx}{|x|} \le \int_{|x|<\delta}|x|^{-1+\varepsilon}\,dx = (2/\varepsilon)\delta^{\varepsilon}.$$

Furthermore, there exists ε, $0 < \varepsilon < 1$, such that

$$\frac{\phi(h/|x|)}{\phi(h)} \le |x|^{-\varepsilon}$$

for all $|x| > 1$ and small $h > 0$ [Berman (1987), formula (2.6)]. Therefore

$$\int_{|x|>1/\delta}|e^{ix}-1|^2\frac{\phi(h/|x|)}{\phi(h)}\frac{dx}{|x|}$$

$$\le 2\int_{|x|>1/\delta}|x|^{-1-\varepsilon}\,dx = (4/\varepsilon)\delta^{\varepsilon}.$$

It follows that the integral (6.5) is at most equal to $(6/\varepsilon)\delta^{\varepsilon}$, and so, by the reasoning leading from (6.4) to (6.5), it follows that (6.3) holds. □

Our first application of Lemma 6.1 is Lemma 6.2.

LEMMA 6.2. *Under* (3.6) *let* (s, s') *and* (t, t') *be a family of real pairs such that*

$$|s - s'| \to 0, \qquad |t - t'| \to 0, \qquad |s - s'|/|t - t'| \to 0. \tag{6.6}$$

Under this limiting operation, the random variables

$$\frac{X(s) - X(s')}{\sigma(|s - s'|)}, \frac{X(t) - X(t')}{\sigma(|t - t'|)} \tag{6.7}$$

have a joint limiting distribution which is a product of standard normal distributions.

PROOF. It suffices to show that the limit of the covariance of the pair is equal to 0. According to Lemma 6.1, we may estimate the covariance by putting $h_1 = |s - s'|$ and $h_2 = |t - t'|$, and computing the limit of the covariance of the pair

$$\frac{1}{2\sigma(h_i)} \int_{\{\delta/h_i \le |x| \le 1/\delta h_i\}} e^{ixt}(e^{ixh} - 1)|x|^{-1}(1 + x^2)^{1/2} \xi(dx)$$

for $i = 1, 2$, for $h_i \to 0$ and then $\delta \to 0$. The covariance of the pair above is equal to

$$\frac{1}{4\sigma(h_1)\sigma(h_2)} \int_{\{[\delta/\min(h_1, h_2)] \le |x| \le 1/[\delta \max(h_1, h_2)]\}} (e^{ixs} - e^{ixs'})(e^{-ixt} - e^{-ixt'}) \times \frac{1 + x^2}{x^2}\, dF(x).$$

By the assumption (6.6), h_1 is eventually smaller than h_2, and, for fixed $\delta > 0$, $1/\delta h_2$ is eventually smaller than δ/h_1, so that the domain of integration is eventually empty. It follows that the limit of the covariance of (6.7) is equal to 0. □

LEMMA 6.3. *Assume* (3.6). *Let I be a fixed real interval. Under the limiting operation*

$$|t - t'| \to 0, \qquad t, t' \in I, \qquad \inf_{s \in I} \sigma(s) > 0, \tag{6.8}$$

the random variables

$$\frac{X(s)}{\sigma(s)}, \frac{X(t) - X(t')}{\sigma(|t - t'|)} \tag{6.9}$$

have a joint limiting distribution which is a product of standard normal distributions.

PROOF. For arbitrary $s \in I$ and $T > 0$, it is easily seen that $X(s)$ is mean square approximable by

$$X_T(s) = \tfrac{1}{2} \int_{|x| \le T} e^{ixs}|x|^{-1}(1 + x^2)^{1/2} \xi(dx)$$

in the sense that

$$\lim_{T\to\infty} \sup_{s\in I} E(X_T(s) - X(s))^2 = 0.$$

By Lemma 6.1, $(X(t) - X(t'))/\sigma(|t - t'|)$ is mean square approximable by (6.2) with $h = |t - t'| \to 0$. By the orthogonality of the increments of ξ, the covariance of the approximating random variables is

$$\frac{1}{4\sigma(h)} \int_{\{x:\ |x| \le T,\ \delta/h \le |x| \le 1/\delta h\}} e^{ixs}(e^{-ixt} - e^{-ixt'})|x|^{-2}(1 + x^2)\, dF(x).$$

If, in the approximation of the pair (6.9) we choose h and T to be related as $h = o(1/T)$, then, for fixed $\delta > 0$, h is ultimately smaller than δ/T. Hence the domain of integration above is empty, and therefore the limit of the covariance (and the correlation) is equal to 0. □

7. Asymptotic independence of local increments for processes with transformed spectral distributions. In this section we establish some limiting distributional properties of processes whose spectra are of the special type considered in Section 4.

THEOREM 7.1. *Suppose that F is absolutely continuous with a density f satisfying* (3.6). *Let $X(t)$ be a process having a spectral distribution with a density of the form*

$$g(x) = x^{-1}(F(\infty) - F(x))$$

for all sufficiently large x. Let (s, s') and (t, t') be a family of pairs of real numbers such that

(7.1) $|s - s'| \to 0,\quad |t - t'| \to 0$ *and* $|s - t|$ *is bounded away from* 0.

Then the random variables

$$\frac{X(s) - X(s')}{\sigma(|s - s'|)} \quad \text{and} \quad \frac{X(t) - X(t')}{\sigma(|t - t'|)} \tag{7.2}$$

have a joint limiting distribution equal to a product of standard normal distributions.

PROOF. The cases $|s - s'|/|t - t'| \to 0$ or ∞ are covered by Lemma 6.2, so that it remains to consider only the case $|s - s'|/|t - t'| \to c$, $0 < c < \infty$. For simplicity, we will consider just the case $h = |s - s'| = |t - t'| \to 0$. By Lemma 6.1, the covariance of the pair (7.2) is approximately equal to

$$\frac{1}{4\sigma^2(h)} \int_{\delta/h \le |x| \le 1/h\delta} e^{ix(t-s)}|e^{ixh} - 1|^2 \frac{1 + x^2}{x^2} \phi\left(\frac{1}{|x|}\right) \frac{dx}{|x|}$$

for small $\delta > 0$ and $h \to 0$. The latter is asymptotically equal to

$$\frac{1}{4\sigma^2(h)} \int_{\delta \le |x| \le 1/\delta} e^{ix(t-s)/h} |e^{ix} - 1|^2 \phi\left(\frac{h}{|x|}\right) \frac{dx}{|x|},$$

which, by Theorem 3.1, is of exactly the same order as

$$\int_{\delta \le |x| \le 1/\delta} e^{ix(t-s)/h} |e^{ix} - 1|^2 \frac{\phi(h/|x|)}{\phi(h)} \frac{dx}{|x|}. \tag{7.3}$$

We will show: For every sequence (h) converging to 0, there is a subsequence over which (7.3) has the limit 0; this will complete the proof of the lemma. Observe that $\phi(h/x)/\phi(h)$ is monotonic in x, and, by Berman (1987), Lemma 2.1, is bounded on $\delta \le x \le 1/\delta$, uniformly in h. Hence there are a subsequence (h') of (h) and a monotonic bounded function $q(x)$ such that $\phi(h'/x)/\phi(h) \to q(x)$ at all points of continuity in $(\delta, 1/\delta)$. From this and the bounded convergence theorem it follows that

$$\int_{\delta \le |x| \le 1/\delta} \left| q(|x|) - \frac{\phi(h'/|x|)}{\phi(h')} \right| dx \to 0.$$

Therefore, if h in (7.3) is replaced by h', the latter integral has the same limit as

$$\int_{\delta \le |x| \le 1/\delta} e^{ix(t-s)/h'} |e^{ix} - 1|^2 q(|x|)\, dx,$$

which, by the Riemann–Lebesgue lemma, has the limit 0. □

8. Extension of local nondeterminism to r distinct points. In the case of a process with stationary increments, the limiting operation $t_k - t_1 \to 0$ in the definition of local nondeterminism may be replaced by the simpler operation $t_k \to \tau$, $t_1 \to \tau$, for each fixed τ. Accordingly, the inequality (2.1) holds in a neighborhood of each fixed point τ. We will now formulate an inequality which extends (2.1) to neighborhoods of r distinct points. This will provide the inequality needed in the estimation of the local time integral for multiple self-intersections.

Let $0 < \tau_1 < \cdots < \tau_r$ be ordered real points. For each i, let t_{ij}, $j = 1, \ldots, k$, be points such that $t_{i1} < \cdots < t_{ik}$, for each i. We assume that t_{ij} has an implicit parameter h such that

$$t_{ij} = t_{ij}(h) \to \tau_i, \qquad i = 1, \ldots, r,\ j = 1, \ldots, k, \tag{8.1}$$

for $h \to 0$.

Consider the random $r \times k$ matrix (X_{ij}) defined by $X_{ij} = X(t_{ij})$, and define $\mathbf{X}_i$ as the transpose of the ith row vector, that is, a column vector with components X_{ij}, $j = 1, \ldots, k$.

THEOREM 8.1. *Let $X(t)$, $t \ge 0$, be a Gaussian process with stationary increments whose spectral distribution satisfies* (3.6). *Let (τ_i) and $(t_{ij}(h))$ be* *defined as above. Then there exists a number $C > 0$ which may depend on $\tau_1, \dots, \tau_r$ and on k such that if h is sufficiently small in the operation* (8.1) *then*

$$\text{Var}\left(\sum_{i=1}^{r} \mathbf{U}_i'(A\mathbf{X}_i) \right) \geq C \sum_{i=1}^{r} \sum_{j=1}^{k} (\mathbf{U}_i)_j^2 \text{Var}(A\mathbf{X}_i)_j, \tag{8.2}$$

where A is the matrix (2.2), *for all k-vectors $\mathbf{U}_1, \dots, \mathbf{U}_r$, and $\mathbf{U}'$ denotes the transpose of $\mathbf{U}$.*

PROOF. The left-hand member of (8.2) may be written in the form

$$\text{Var}\left\{ \sum_{i=1}^{r} u_{i1} X(t_{i1}) + \sum_{i=1}^{r} \sum_{j=2}^{k} u_{ij} \left(X(t_{ij}) - X(t_{i,j-1}) \right) \right\}. \tag{8.3}$$

By the spectral representation (3.1), the variance (8.3) is representable as

$$\frac{1}{4} \int_{-\infty}^{\infty} \left| \sum_{h=1}^{r} u_{h1} (e^{ixt_{h1}} - 1) + \sum_{h=1}^{r} \sum_{j=2}^{k} u_{hj} \left(e^{ixt_{hj}} - e^{ixt_{h,j-1}} \right) \right|^2 \frac{1 + x^2}{x^2} f(x)\, dx,$$

which, for any $T > 0$, is at least equal to

$$\frac{1}{4} \int_{|x| > T} \left| \sum_{h=1}^{r} u_{h1} (e^{ixt_{h1}} - 1) + \sum_{h=1}^{r} \sum_{j=2}^{k} u_{hj} \left(e^{ixt_{hj}} - e^{ixt_{h,j-1}} \right) \right|^2 f(x)\, dx. \tag{8.4}$$

For every $\beta_1' < \beta_1$, there exists T sufficiently large so that a lower estimate of (8.4) is obtained by replacing $f(x)$ by $\beta_1' |x|^{-1} \phi(1/|x|)$. Lemma 4.1 implies that the condition (3.6) holds for the modified spectrum with a possibly larger β_1 and possibly smaller β_2. Therefore the process with the modified spectrum inherits the local nondeterminism of the original process, including the properties expressed by the inequality (2.1) and the lemmas of Section 6. Lemma 4.3 states that the same variance function σ may be used for the altered process. Furthermore, the process with the modified spectrum has the additional asymptotic independence property expressed in the statement of Theorem 7.1.

We proceed to find the appropriate lower bound for (8.3). By Theorem 5.1 there exists $K > 0$ such that if h is sufficiently small in (8.1), that is, t_{i1} is sufficiently close to τ_i, $i = 1, \dots, r$, then

$$\text{Var}\left(\sum_{i=1}^{r} u_{i1} X(t_{i1}) \right) \geq K \sum_{i=1}^{r} u_{i1}^2 \sigma^2(t_{i1}) \tag{8.5}$$

for all (u_{i1}).

Let $\mathbf{Y}_1, \dots, \mathbf{Y}_r$ be the $(k-1)$-component random vectors obtained from $A\mathbf{X}_1, \dots, A\mathbf{X}_r$ by eliminating the first component of the latter: $\mathbf{Y}_i = (X(t_{ij}) - X(t_{i,j-1}),\ j = 2, \dots, k)$. By the ordinary local nondeterminism of the process [Theorem 3.1 above and Berman (1987)] and by the inequality (2.1) above with $j = 2, \dots, k$, it follows that there exists $C > 0$ such that (2.9) holds. Furthermore, by Theorem 7.1, the maximum modulus ρ of the correlation coefficient in (2.10) converges to 0. Therefore, by (2.11), there exists $K > 0$, that is, $K = C - r(r-1)\rho$ for $\rho < C/(r(r-1))$, such that

$$\operatorname{Var}\left\{\sum_{i=1}^{r}\sum_{j=2}^{k} u_{ij}\big(X(t_{ij}) - X(t_{i,j-1})\big)\right\} \ge K\sum_{i=1}^{r}\sum_{j=2}^{k} u_{ij}^2\sigma^2(t_{ij} - t_{i,j-1}) \tag{8.6}$$

for all (u_{ij}). Here K is not necessarily the same as in (8.5).

By Lemma 6.3, every random variable $X(t_{i1})$ and every increment $X(t_{l,j}) - X(t_{l,j-1})$, $i, l = 1, \dots, r$ and $j = 2, \dots, k$ have limiting correlations equal to 0. Hence, from (8.5) and (8.6) by another application of Lemma 2.3, the variance (8.3) is at least equal to a constant times the sum of the right-hand members of (8.5) and (8.6). □

9. Joint continuity of the intersection local time of multiplicity r. Let $X(t)$ be a Gaussian process with stationary increments satisfying (3.6). Let $X_h(t)$, $h = 1, \dots, m$, be independent copies of $X(t)$. For disjoint, compact intervals of positive length in the time domain, $I_1, \dots, I_r$, consider the $m(r-1)$-component vector process

$$\big(X_h(t_{i+1}) - X_h(t_i)\colon i = 1, \dots, r-1;\ h = 1, \dots, m\big) \tag{9.1}$$

defined for $t_i \in I_i$, $i = 1, \dots, r$. The process obviously has an r-component time parameter.

To simplify the notation here, we define $t_{i0} = 0$ for all i. Since $X_h(0) = 0$ a.s. by assumption, it follows that $X_h(t_{i0}) = 0$, a.s. Following the outline of Rosen (1984), we prove Theorem 9.1.

THEOREM 9.1. *The integral*

$$\begin{aligned}\int_{I_1^k}\cdots\int_{I_r^k}\int\cdots\int_{R^{mk(r-1)}} &\prod_{j=1}^{k}\prod_{i=1}^{r-1}\left(\sum_{h=1}^{m} u_{ijh}^2\right)^{\gamma/2}\\ &\times\exp\left[-\tfrac{1}{2}\operatorname{Var}\left\{\sum_{h=1}^{m}\sum_{i=1}^{r-1}\sum_{j=1}^{k} u_{ijh}\big[X_h(t_{i+1,j}) - X_h(t_{i,j})\big]\right\}\right]\\ &\times\prod_{i=1}^{r-1}\prod_{j=1}^{k}\prod_{h=1}^{m} du_{ijh}\prod_{i=1}^{r}\prod_{j=1}^{k} dt_{ij}\end{aligned} \tag{9.2}$$

is finite for all $\gamma > 0$, $m \ge 1$, $r > 1$ and $k \ge 2$ such that

$$\frac{\beta_2}{2}\frac{r-1}{r}\big(m + 2\gamma(r-1)\big) < 1. \tag{9.3}$$

PROOF. As we noted in Section 1, the distinction between this work and that of Rosen (1984) is that we are using a more general form of local nondeterminism than that of the fractional Brownian motion process. Thus, while there is a similarity between the outline of the proof of this theorem and corresponding proofs in Rosen (1984) and Geman, Horowitz and Rosen (1984), I have presented a complete proof of this theorem because it differs from the earlier ones in the details related to the features representing the generality of this work.

In the course of this proof, the index $i = 1, \dots, r$ is a label for the intervals (I_i), the index $j = 1, \dots, k$ is a label for the random variables $X(t_{ij})$, $t_{ij} \in I_i$, and $h = 1, \dots, m$ is a label for the components of the vector process.

Since $(X_h(t))$ are independent copies of $X(t)$, the summation operator over h may be placed before the variance in the exponent in (9.2), and the subscript h on X_h may be omitted:

$$\sum_{h=1}^{m} \operatorname{Var}\left\{ \sum_{i=1}^{r-1} \sum_{j=1}^{k} u_{ijh} \left[X(t_{i+1,j}) - X(t_{ij}) \right] \right\}. \tag{9.4}$$

For each i, consider the subset of I_i^k for which

$$t_{i,1} < t_{i,2} < \cdots < t_{ik} \quad \text{for } t_{ij} \in I_i,\ i = 1, \dots, r. \tag{9.5}$$

Then, with the exception of a set of measure 0, I_i^k is the union of the images of the subset (9.5) under the family of all permutations of $(1, \dots, k)$. Let $\mathbf{X}_i$, $i = 1, \dots, r$, be the k-component random vectors defined in Section 8, and let P_i be a $k \times k$ row permutation matrix, that is, one obtained from the identity by a permutation of rows. Then the integral (9.2) may be expressed as the sum of $(k!)^r$ integrals of the form

$$\begin{aligned} &\int \cdots \int_{\times_{i=1}^{r}\{t_{i1} < \cdots < t_{ik}\}} \int \cdots \int_{R^{mk(r-1)}} \prod_{j=1}^{k} \prod_{i=1}^{r-1} \left(\sum_{h=1}^{m} u_{ijh}^2 \right)^{\gamma/2} \\ &\qquad \times \exp\left\{ -\tfrac{1}{2} \sum_{h=1}^{m} \operatorname{Var}\left[\sum_{i=1}^{r-1} \sum_{j=1}^{k} u_{ijh}\big((P_{i+1}X_{i+1})_j - (P_i X_i)_j \big) \right] \right\} \\ &\qquad \times \prod du_{ijh} \prod dt_{ij}, \end{aligned} \tag{9.6}$$

where $(P_i\mathbf{X}_i)_j$ is the jth component of $\mathbf{X}_i$ after the permutation of coordinates defined by P_i.

For $h = 1, \dots, m$, let $\mathbf{U}_{ih}$ be a k-component column vector with components u_{ijh}, $j = 1, \dots, k$. Then the variance in the exponent in (9.6) may be written in the vector form

$$\operatorname{Var}\left[\sum_{i=1}^{r-1} \mathbf{U}_{ih}'(P_{i+1}\mathbf{X}_{i+1} - P_i\mathbf{X}_i) \right],$$

which, by summation by parts, may be written as

$$\operatorname{Var}\left\{-U_{1h}'P_1\mathbf{X}_1+\sum_{i=2}^{r-1}(\mathbf{U}_{i-1,h}-\mathbf{U}_{ih})'P_i\mathbf{X}_i+U_{r-1}'P_r\mathbf{X}_r\right\}. \tag{9.7}$$

Define the k-vectors

$$\begin{aligned}\mathbf{V}_{ih}&=\mathbf{U}_{1h}, & i&=1,\\ &=\mathbf{U}_{ih}-\mathbf{U}_{i-1,h}, & i&=2,\dots,r-1;\end{aligned} \tag{9.8}$$

then (9.7) is equal to

$$\operatorname{Var}\left\{-\sum_{i=1}^{r-1}\mathbf{V}_{ih}'P_i\mathbf{X}_i+\left(\sum_{i=1}^{r-1}\mathbf{V}_{ih}'\right)P_r\mathbf{X}_r\right\}. \tag{9.9}$$

We aim to find a lower bound for (9.9) of the type described in Lemma 2.3. For any nonsingular $k\times k$ matrix A, the variance (9.9) may be written as

$$\operatorname{Var}\left\{-\sum_{i=1}^{r-1}(A'^{-1}P_i'\mathbf{V}_{ih})'A\mathbf{X}_i+\left(A'^{-1}P_r'\left(\sum_{i=1}^{r-1}\mathbf{V}_{ih}\right)\right)'A\mathbf{X}_r\right\}. \tag{9.10}$$

Take A as the matrix (2.2). For the purpose of proving that (9.2) is finite, it suffices to consider intervals $I_1,\dots,I_r$ of arbitrarily small length. Hence we may consider I_i to be a small neighborhood of a point τ_i, $i=1,\dots,r$, and then apply Theorem 8.1: There exists a constant $C>0$ such that the variance (9.10) is at least equal to C times

$$\sum_{i=1}^{r-1}\sum_{j=1}^{k}(A'^{-1}P_i'\mathbf{V}_{ih})_j^2\operatorname{Var}(A\mathbf{X}_i)_j$$
$$+\sum_{j=1}^{k}\left(A'^{-1}P_r'\left(\sum_{i=1}^{r-1}\mathbf{V}_{ih}\right)\right)_j^2\operatorname{Var}(AX_r)_j.$$

The latter is equal to

$$\begin{aligned}&\sum_{i=1}^{r-1}\sum_{j=1}^{k}(A'^{-1}P_i'\mathbf{V}_{ih})_j^2\sigma^2(t_{ij}-t_{i,j-1})\\ &\quad+\sum_{j=1}^{k}\left(A'^{-1}P_r'\left(\sum_{i=1}^{r-1}\mathbf{V}_{ih}\right)\right)_j^2\sigma^2(t_{rj}-t_{r,j-1}).\end{aligned} \tag{9.11}$$

Now we use the transformation (9.8) as a change of variables of integration in (9.6):

$$\begin{aligned}v_{ijh}&=u_{ijh}, & i&=1,\\ &=u_{ijh}-u_{i-1,j,h}, & i&=2,\dots,r-1,\ j=1,\dots,k,\ h=1,\dots,m.\end{aligned}$$

The Jacobian is equal to 1. It also follows from the definition that

$$u_{ijh} = \sum_{l=1}^{i} v_{ljh},$$

and so, by the Cauchy–Schwarz inequality,

$$u_{ijh}^2 \leq (r-1) \sum_{i=1}^{r-1} v_{ijh}^2, \qquad i = 1, \dots, r-1.$$

It follows that

$$\prod_{i=1}^{r-1} \left(\sum_{h=1}^{m} u_{ijh}^2 \right)^{\gamma/2} \leq (r-1)^{\gamma(r-1)/2} \left(\sum_{i=1}^{r-1} \sum_{h=1}^{m} v_{ijh}^2 \right)^{\gamma(r-1)/2}$$

for $1 \leq j \leq k$. From this and the lower bound (9.11) for the variance, we obtain the following upper bound for the integrand in (9.6) with integration variables v_{ijh}:

$$\begin{aligned}
&(r-1)^{\gamma k(r-1)/2} \prod_{j=1}^{k} \left(\sum_{i=1}^{r-1} \sum_{h=1}^{m} v_{ijh}^2 \right)^{\gamma(r-1)/2} \\
&\times \exp\left\{ -\tfrac{1}{2} C \left[\sum_{h=1}^{m} \sum_{i=1}^{r-1} \sum_{j=1}^{k} (A'^{-1} P_i' \mathbf{V}_{ih})_j^2 \sigma^2(t_{ij} - t_{i,j-1}) \right. \right. \\
&\left. \left. + \sum_{h=1}^{m} \sum_{j=1}^{k} \left(A'^{-1} P_r' \left(\sum_{i=1}^{r-1} \mathbf{V}_{ih} \right) \right)_j^2 \sigma^2(t_{rj} - t_{r,j-1}) \right] \right\}.
\end{aligned} \tag{9.12}$$

Now we are going to find r different upper bounds for (9.12), and their corresponding integrals over the variables (v_{ijh}). Our first bound is obtained by omitting the second sum from the exponent. It is a constant times

$$\begin{aligned}
&\prod_{j=1}^{k} \left(\sum_{i=1}^{r-1} \sum_{h=1}^{m} v_{ijh}^2 \right)^{\gamma(r-1)/2} \\
&\times \exp\left\{ -\frac{C}{2} \sum_{h=1}^{m} \sum_{i=1}^{r-1} \sum_{j=1}^{k} (A'^{-1} P_i' \mathbf{V}_{ih})_j^2 \sigma^2(t_{ij} - t_{i,j-1}) \right\}.
\end{aligned} \tag{9.13}$$

The integral of this function is not changed if the permutation matrix P_i' is removed from the exponent. Indeed, any permutation matrix has Jacobian 1, and the product $\prod_j (\sum_i \sum_h v_{ijh}^2)$ is invariant under permutations of the indices $j = 1, \dots, k$. Thus the integral of (9.13) is equal to the integral of

$$\begin{aligned}
&\prod_{j=1}^{k} \left(\sum_{i=1}^{r-1} \sum_{h=1}^{m} v_{ijh}^2 \right)^{\gamma(r-1)/2} \\
&\times \exp\left\{ -\frac{C}{2} \sum_{h=1}^{m} \sum_{i=1}^{r-1} \sum_{j=1}^{k} (A'^{-1} \mathbf{V}_{ih})_j^2 \sigma^2(t_{ij} - t_{i,j-1}) \right\}.
\end{aligned} \tag{9.14}$$

Define the vectors $\mathbf{W}_{ih} = A'^{-1}\mathbf{V}_{ih}$, where A is the matrix (2.2). Then, with the convention $(\mathbf{W}_{ih})_j = 0$ for $j = 0, k+1$, we have $\mathbf{V}_{ih} = A'\mathbf{W}_{ih}$, and so

$$|v_{ijh}| \le |w_{ijh}| + |w_{i,j+1,h}|,$$

and, as a consequence,

$$v_{ijh}^2 \le 2w_{ijh}^2 + 2w_{i,j+1,h}^2,$$

and so

$$\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1}\sum_{h=1}^{m} v_{ijh}^2\right)^{\gamma(r-1)/2}$$
$$\le 2^{k\gamma(r-1)/2}\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1}\sum_{h=1}^{m} w_{ijh}^2 + \sum_{i=1}^{r-1}\sum_{h=1}^{m} w_{i,j+1,h}^2\right)^{\gamma(r-1)/2}.$$

The latter is at most equal to the sum of 2^k terms of the form

$$2^{k\gamma(r-1)/2}\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1}\sum_{h=1}^{m} w_{ijh}^2\right)^{\theta_j\gamma(r-1)/2}, \tag{9.15}$$

where $\theta_j = 0$, 1 or 2, for $j = 1, \dots, k$.

It follows from (9.15) that, after the change of variables from (v) to (w), the integrand (9.14) is bounded above by a sum of 2^k terms of the form

$$\begin{aligned} &\text{constant}\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1}\sum_{h=1}^{m} w_{ijh}^2\right)^{\theta_j\gamma(r-1)/2} \\ &\qquad\times\exp\left\{-\frac{C}{2}\sum_{j=1}^{k}\sum_{i=1}^{r-1}\left(\sum_{h=1}^{m} w_{ijh}^2\right)\sigma^2(t_{ij} - t_{i,j-1})\right\}. \end{aligned} \tag{9.16}$$

Let us now integrate over the variables (w_{ijh}). For fixed (i, j) we first integrate over spheres in R^m, $w_{ij}^2 = \sum_{h=1}^m w_{ijh}^2$, and then over w_{ij}. The integral of (9.16) over $R^{mk(r-1)}$ is equal to a constant times

$$\begin{aligned} &\int\cdots\int_{R^{k(r-1)}}\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1} w_{ij}^2\right)^{\theta_j\gamma(r-1)/2} \\ &\qquad\times\exp\left\{-\frac{C}{2}\sum_{j=1}^{k}\sum_{i=1}^{r-1} w_{ij}^2\sigma^2(t_{ij} - t_{i,j-1})\right\} \\ &\qquad\times\prod_{i=1}^{r-1}\prod_{j=1}^{k}|w_{ij}|^{m-1}\prod_{i=1}^{r-1}\prod_{j=1}^{k} dw_{ij}. \end{aligned} \tag{9.17}$$

By the elementary inequality,

$$\left(\sum_{i=1}^{r-1} w_i^2\right)^p \le \prod_{i=1}^{r-1}(1 + w_i^2)^p$$

for $p \geq 1$, and from the fact that $0 \leq \theta_j \leq 2$, the integral (9.17) is at most equal to the product of $k(r-1)$ single integrals,

$$\prod_{i=1}^{r-1} \prod_{j=1}^{k} \int_{-\infty}^{\infty} (1+w^2)^{\gamma(r-1)} |w|^{m-1} \exp\left(-\frac{C}{2} w^2 \sigma^2(t_{ij} - t_{i,j-1})\right) dw.$$

By a change of variable and a simple computation, the latter product is seen to be bounded above by a constant times

$$\prod_{i=1}^{r-1} \prod_{j=1}^{k} \left(\sigma(t_{ij} - t_{i,j-1})\right)^{-m-2\gamma(r-1)}. \tag{9.18}$$

This is our first bound for the integral of (9.12) over the variables (v_{ijh}).

Now we construct $r-1$ other upper bounds. For an integer q, $1 \leq q \leq r-1$, we obtain a lower bound for the sum in the exponent in (9.12) by dropping all terms of index $i = q$, for any j and h, from the first sum. Thus the vector $\mathbf{V}_{qh}$ appears only in $\sum_{i=1}^{r-1} \mathbf{V}_{ih}$, in the second sum in the exponent. We define new variables

$$\begin{aligned} \mathbf{W}_{ih} &= \mathbf{V}_{ih} \qquad \text{for } i \neq q,\ i = 1, \ldots, r-1, \\ &= \sum_{l=1}^{r-1} \mathbf{V}_{lh} \quad \text{for } i = q. \end{aligned} \tag{9.19}$$

The Jacobian of the transformation is equal to 1, and the sum in the exponent in (9.12) becomes

$$\begin{aligned} &\sum_{h=1}^{m} \sum_{i=1,\, i \neq q}^{r-1} \sum_{j=1}^{k} \left(A'^{-1} P_i' \mathbf{W}_{ih}\right)_j^2 \sigma^2(t_{ij} - t_{i,j-1}) \\ &\qquad + \sum_{h=1}^{m} \sum_{j=1}^{k} \left(A'^{-1} P_r' \mathbf{W}_{qh}\right)_j^2 \sigma^2(t_{rj} - t_{r,j-1}). \end{aligned} \tag{9.20}$$

For fixed (j, h) consider the linear transformation $v_{ijh} \to w_{ijh}$ induced by (9.19). It has an $(r-1) \times (r-1)$ matrix of the form: 1's on the diagonal and on row q, and 0's elsewhere. The inverse has 1's on the diagonal, and -1's throughout row q with the exception of the diagonal entry. It follows that

$$\begin{aligned} \sum_{i=1}^{r-1} v_{ijh}^2 &\leq \sum_{i=1}^{r-1} \left(\sum_{i'=1}^{r-1} |w_{i'jh}| \right)^2 \\ &= (r-1) \left(\sum_{i=1}^{r-1} |w_{ijh}| \right)^2, \end{aligned}$$

which, by the Cauchy–Schwarz inequality, is at most equal to

$(r-1)^2\sum_{i=1}^{r-1} w_{ijh}^2$. Therefore the product appearing in (9.12) satisfies

$$\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1}\sum_{h=1}^{m} v_{ijh}^2\right)^{\gamma(r-1)/2} \le (r-1)^{\gamma k(r-1)}\prod_{j=1}^{k}\left(\sum_{i=1}^{r-1}\sum_{h=1}^{m} w_{ijh}^2\right)^{\gamma(r-1)/2}. \tag{9.21}$$

Applying (9.21), and using the lower bound (9.20) for the sum in the exponent in (9.12), we see that the upper bound we obtain for (9.12) is the same as our first bound (9.13) except that the index $i = q$ of P_i and $t_{i,j}$ has been replaced by r. Thus, by the same calculations leading to (9.18), we obtain, for each q, $1 \le q \le r-1$, the bound

$$\text{constant} \prod_{i=1, i\ne q}^{r}\prod_{j=1}^{k}\sigma(t_{ij}-t_{i,j-1})^{-m-2\gamma(r-1)} \tag{9.22}$$

for the integral of (9.12) over the variables (v_{ijh}).

Since the integral is bounded by a constant times the function (9.18) of the t's and is also bounded by a constant times each function (9.22), for $1 \le q \le r-1$, it follows that the geometric mean of the r bounds is also a bound. Therefore

$$\text{constant} \prod_{i=1}^{r}\prod_{j=1}^{k}\left[\sigma(t_{ij}-t_{i,j-1})\right]^{-((r-1)/r)(m+2\gamma(r-1))} \tag{9.23}$$

is also a bound for the integral of (9.12) over the variables (v_{ijh}).

Now we integrate (9.23) over the t-domain indicated in (9.6). The integral is finite if

$$\int_0^1 [\sigma(s)]^{-((r-1)/r)(m+2\gamma(r-1))}\,ds < \infty. \tag{9.24}$$

This holds if (9.3) is assumed. Indeed, if the function $\sigma^2(t)$ is based on the original spectral density f, or the original mass function (f_n), then, for any $\beta_2' > \beta_2$, it follows from the first inequality in (3.14) that the integral (9.24) is finite if

$$\frac{\beta_2'}{2}\frac{r-1}{r}(m+2\gamma(r-1)) < 1.$$

Since β_2' is an arbitrary number greater than β_2, it follows that (9.3) is sufficient. If the function $\sigma^2(t)$ is based on the modified spectral distribution described in Lemma 4.1, then, by Lemma 4.2, $\sigma^2(t)$ is, for $t \to 0$, of exactly the same asymptotic order as the one based on the original spectral distribution, and so (9.3) is still sufficient for (9.24). □

Now we apply Theorem 9.1 to the local time of the process (9.1), denoted $\alpha(\mathbf{x}, B)$, where $\mathbf{x} \in R^{m(r-1)}$ and B is a rectangle in $I_1 \times \cdots \times I_r$.

THEOREM 9.2. *Let the real underlying process* $X_h(t)$ *in* (9.1) *satisfy the spectral conditions of Section* 3, *and suppose that*

$$\frac{\beta_2(r-1)m}{2r} < 1. \tag{9.25}$$

Then the local time $\alpha(\mathbf{x}, B)$ *exists and is jointly continuous, a.s.*

PROOF. If (9.25) holds, then there exists $\gamma > 0$ sufficiently small, so that (9.3) also holds. Then the integral (9.2) is finite for the given γ, and the conclusion of joint continuity is a consequence of Geman, Horowitz and Rosen (1984), Theorem 2.8. □

COROLLARY 9.1. *Under the condition* (9.25) *the dimension of the set of zeros of the process* (9.1) *is at most equal to*

$$r - \tfrac{1}{2}m\beta_1(r-1) \tag{9.26}$$

almost surely.

PROOF. It is well known that the right-hand inequality in (3.12) implies that the component sample function $X_h(t)$ satisfies a Hölder condition of order less than $\beta_1'/2$. It then follows that the same is true for the process (9.1). Noting the joint continuity of $\alpha(\mathbf{x}, B)$, and applying Lemma 8.2.2 of Adler (1981), we conclude that the dimension is at most equal to $r - \frac{1}{2}\beta_1' m(r-1)$. Since $\beta_1' < \beta_1$ is arbitrary, we obtain the bound (9.26). □

COROLLARY 9.2. *Under the condition* (9.25) *the dimension of the set of zeros of the process* (9.1) *is at least equal to*

$$r - \tfrac{1}{2}m\beta_2(r-1) \tag{9.27}$$

with positive probability.

PROOF. Following the notation of Geman, Horowitz and Rosen (1984), we call the integral (9.2) $V_{k\gamma}(I_1 \times \cdots \times I_r)$. Let J_i be the subset of I_i for which $t_{i1} < \cdots < t_{ik}$. In proving Theorem 9.1 above we showed that the integral (9.2) is dominated by

$$\int_{J_1^k} \cdots \int_{J_r^k} [\text{function (9.23)}] \prod_{i,j} dt_{ij}.$$

A standard integration shows that this is at most equal to a constant times

$$\prod_{i=1}^{r}\left(\int_{I_i}(\sigma(t))^{-((r-1)/r)(m+2\gamma(r-1))}\,dt\right)^{k-1}.$$

By the first inequality in (3.14), the latter integral is at most equal to a constant times

$$\left[\prod_{i=1}^{r}(\text{mes } I_i)\right]^{(k-1)[1-(\beta_2'/2)((r-1)/r)(m+2\gamma(r-1))]}$$

for any $\beta_2' > \beta_2$. Putting

$$\xi = (k-1)\left[1-\frac{\beta_2'}{2}\left(\frac{r-1}{r}\right)(m+2\gamma(r-1))\right],$$

we observe that we have just shown that

$$V_{k\gamma}(B) \le \text{constant}(\text{mes } B)^{\xi}$$

for every rectangle B in the r-dimensional time domain. It follows from Geman and Horowitz (1980), Theorem (27.1), and Geman, Horowitz and Rosen (1984), Section 2, that the local time $\alpha(\mathbf{x}, B)$ satisfies a uniform (in $\mathbf{x}$) Holder condition in B of every order $\zeta < (\xi - 1)/k$. By Adler (1981), Theorem 8.7.4, it follows that the Hausdorff dimension of the set of zeros of the process (9.1) is at least equal to $r(\xi - 1)/k$ whenever the local time at 0 is positive. By the definition of ξ we have

$$r(\xi-1)/k = \frac{k-1}{k}\{r - \tfrac{1}{2}\beta_2'(r-1)(m+2\gamma(r-1))\}.$$

In the expression above, k may be taken arbitrarily large, $\gamma > 0$ arbitrarily small and $\beta_2' > \beta_2$ arbitrary. Thus $r(\xi - 1)/k$ may be replaced by the expression (9.27). The proof is now completed by showing that $\alpha(0, B) > 0$ with positive probability for any rectangle B. Indeed, $E\alpha(0, B)$ is equal to the integral over the time parameter of the joint density of the random variables (9.1) at 0. Thus $E\alpha(0, B) > 0$, and so $\alpha(0, B) > 0$ with positive probability. □

10. Application to random Fourier series. While the results above have been formulated in the case where the spectral distribution is absolutely continuous, there is also a version in the discrete case where the spectrum has support on the nonnegative integers. The significance of this case is illustrated by the application in this section to random Fourier series. Suppose that $F(x)$ is of the form

$$F(x) = \sum_{0\le n\le x} f_n,$$

where $f_n \geq 0$ and $\Sigma_n f_n < \infty$. In the place of (3.6), we assume

$$\beta_1 = \liminf_{n \to \infty} \frac{nf_n}{\Sigma_{j>n} f_j} > 0, \tag{10.1}$$
$$\beta_2 = \limsup_{n \to \infty} \frac{nf_n}{\Sigma_{j>n} f_j} < 2.$$

This condition is sufficient for ϕ-local nondeterminism in the discrete case [Berman (1987)]. Furthermore, it is sufficient for all of the results proved above in the absolutely continuous case under (3.6). The required modifications of the proofs above are uncomplicated. The functions g and G in (4.1) are taken as sequences (g_n) and (G_n) defined as $g_n = n^{-1}\Sigma_{j>n} f_j$ and $G_n = \Sigma_{j>n} g_j$, respectively. The content of Section 5 includes the discrete case. The only point in the discrete case requiring a careful distinction from the density case is the proof of Theorem 7.1. In the place of the Riemann–Lebesgue lemma, one should use the technique of Berman (1987), page 80, for the evaluation of the trigonometric sum which arises as the analog of the Fourier integral (7.3).

Let $X(t)$, $0 \leq t < 2\pi$, be a real process of the form

$$X(t) = \sum_{n=0}^{\infty} a_n (X_n \cos nt + Y_n \sin t), \tag{10.2}$$

where (X_n) and (Y_n) are all standard normal independent random variables, and

$$\sum_{n=0}^{\infty} a_n^2 < \infty. \tag{10.3}$$

Then $X(t)$ is a stationary Gaussian process with mean 0 and a spectral distribution consisting of masses $f_n = a_n^2$ at $n = 0, 1, \ldots$. If we assume that (f_n) satisfies (10.1) for some $\beta_1 < \beta_2 < 2$, then the theorems and corollaries of Section 9 apply to the vector process (9.1) defined in terms of $X(t)$.

These results can be extended to series of the form (10.2), where the coefficients (X_n) and (Y_n) are not necessarily Gaussian. In Berman (1982, 1987) it was shown that the local time properties of a series (10.2) with Gaussian coefficients are inherited by series with non-Gaussian coefficients under "subordination." In Berman (1987) we considered X_n and Y_n with symmetric stable distributions of index α, $0 < \alpha < 2$. In the place of (10.3), we assume the stronger condition

$$\sum_{n=0}^{\infty} |a_n|^{\alpha} < \infty. \tag{10.4}$$

[The factor 2 should be removed from the exponent in Berman (1987), formula

(4.15).] The latter condition assures the a.s. convergence of the series (10.2) when the coefficients are α-stable. Let us refer to the series on the right-hand side of (10.2) as $Y(t)$ when the coefficients are standard normal. By the form of the stable characteristic function, we have for $u_1, \ldots, u_k$ and $t_1, \ldots, t_k$,

$$
\begin{aligned}
-\log E \exp\left[i \sum_{j=1}^{k} u_j X(t_j) \right]
&= \sum_{n=0}^{\infty} |a_n|^{\alpha} \left[\left| \sum_{j=1}^{k} u_j \cos nt_j \right|^{\alpha} + \left| \sum_{j=1}^{k} u_j \sin nt_j \right|^{\alpha} \right] \\
&= \sum_{n=0}^{\infty} (a_n^2)^{\alpha/2} \left[\left(\sum_{j=1}^{k} u_j \cos nt_j \right)^{2(\alpha/2)} + \left(\sum_{j=1}^{k} u_j \sin nt_j \right)^{2(\alpha/2)} \right] \\
&\geq \left\{ \sum_{n=0}^{\infty} a_n^2 \left[\left(\sum_{j=1}^{k} u_j \cos nt_j \right)^{2} + \left(\sum_{j=1}^{k} u_j \sin nt_j \right)^{2} \right] \right\}^{\alpha/2} \\
&= \left\{ \operatorname{Var}\left(\sum_{j=1}^{k} u_j Y(t_j) \right) \right\}^{\alpha/2}.
\end{aligned}
$$

Hence

$$
E\left[\exp\left(i \sum_{j=1}^{k} u_j X(t_j) \right) \right] \leq \exp\left\{ - \left[\operatorname{Var} \sum_{j=1}^{k} u_j Y(t_j) \right]^{\alpha/2} \right\}. \tag{10.5}
$$

Therefore, by the calculations in the proof of Theorem 9.1 leading to the finiteness of the integral (9.2), we conclude that the analogous integral for the characteristic function of the stable process (10.2) is finite whenever it is true for the Gaussian version $Y(t)$ of (10.2). [See Berman (1982) for this "subordination" principle.] Thus the conclusions of Theorem 9.2 and of Corollary 9.2 hold for the stable version of (10.2) as long as the coefficients (a_n) and the numbers m and r satisfy the conditions stated for the Gaussian version of (10.2).

The conclusion of Corollary 9.1 requires a major modification because we employ the Hölder continuity of the sample function. Here the stable case does not easily follow from the Gaussian case. Indeed, the stable sample functions have a much smaller Hölder index.

For $0 < \alpha < 2$, define the function

$$
R(t) = \left(\sum_{n=1}^{\infty} |a_n|^{\alpha} (1 - \cos nt) \right). \tag{10.6}
$$

For $h = |t - s|$ it follows that

$$
\begin{aligned}
-&\log E(e^{iu(X(t)-X(s))}) \\
&= |u|^\alpha \sum_{n=0}^{\infty} |a_n|^\alpha [|\cos nt - \cos ns|^\alpha + |\sin nt - \sin nt|^\alpha] \\
&\le 2^{1/2-1/\alpha}|u|^\alpha \sum_{n=0}^{\infty} |a_n|^\alpha \left((\cos nt - \cos ns)^2 + (\sin nt - \sin ns)^2\right)^{\alpha/2} \qquad (10.7)\\
&= 2^{1/2-1/\alpha+\alpha/2}|u|^\alpha \sum_{n=0}^{\infty} |a_n|^\alpha (1 - \cos n(t-s))^{\alpha/2}.
\end{aligned}
$$

By the moment inequality we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} |a_n|^\alpha (1 - \cos n(t-s))^{\alpha/2} \\
&\le \left(\sum_{n=0}^{\infty} |a_n|^\alpha\right)^{1-\alpha/2} \left(\sum_{n=0}^{\infty} |a_n|^\alpha (1 - \cos n(t-s))\right)^{\alpha/2}.
\end{aligned}
$$

Hence, from (10.6) and (10.7),

$$
-\log E(e^{iu(X(t)-X(s))}) \le C|u|^\alpha (R(t-s))^{\alpha/2} \qquad (10.8)
$$

for some $C > 0$.

The ratio

$$
\frac{X(t) - X(s)}{\{-\log E[e^{i(X(t)-X(s))}]\}^{1/\alpha}}
$$

has a (stable) characteristic function independent of (s,t) for $s \ne t$. Hence, by the known form of the stable distribution tail, we obtain

$$
P\left(\frac{|X(t) - X(s)|}{\{-\log E[e^{i(X(t)-X(s))}]\}^{1/\alpha}} > x\right) \sim Kx^{-\alpha}
$$

for $x \to \infty$ and some fixed $K > 0$, uniformly in (s,t) for $s \ne t$. From this and from (10.8) we infer the relation

$$
P\left(\frac{|X(t) - X(s)|}{(R(t-s))^{1/2}} > x\right) \le Kx^{-\alpha} \qquad (10.9)
$$

for all $x > 1$ and some $K > 0$ not depending on (s,t) for $s \ne t$.

Let us now assume that

$$
R(t) \sim b|t|^\gamma, \qquad t \to 0, \qquad (10.10)
$$

for some $b > 0$ and $\gamma > 0$. It then follows from (10.9) that

$$P\big(|X(t) - X(s)| > b^{\delta/2}|t - s|^{\gamma\delta/2}\big)$$

$$\sim P\left(\frac{|X(t) - X(s)|}{(R(t-s))^{1/2}} > b^{(\delta-1)/2}|t - s|^{\gamma(\delta-1)/2}\right) \tag{10.11}$$

$$\leq Kb^{\alpha(1-\delta)/2}|t - s|^{\gamma\alpha(1-\delta)/2}$$

for arbitrary δ, $0 < \delta < 1$, for $|t - s|$ positive and sufficiently small.

Let us now apply Loève (1978), page 183, with $g(h) = \text{constant}|h|^{\gamma\delta/2}$ and $q(h) = \text{constant}|h|^{\gamma\alpha(1-\delta)/2}$. If

$$\gamma\alpha(1 - \delta)/2 > 1, \tag{10.12}$$

then $X(t)$ is almost surely Hölder continuous of order $\gamma\delta/2$. This implies the following proposition.

If

$$\gamma\alpha > 2, \tag{10.13}$$

then $X(t)$ is almost surely Hölder continuous of order less than

$$\frac{\gamma}{2} - \frac{1}{\alpha}. \tag{10.14}$$

Indeed, if (10.13) holds, then (10.12) holds for any $0 < \delta < 1 - 2/\gamma\alpha$, and $X(t)$ is Hölder continuous of order $\gamma\delta/2$, which is less than (10.14).

We mention that there has been much recent research on the continuity of stable series of the form (10.2); for example, see Marcus and Pisier (1984). However, for simplicity, we have chosen to apply the elementary general theorem stated in Loève (1978) to obtain the specific result on Hölder continuity.

We now apply the results above to the stable series (10.2), where we take the sequence (a_n) to satisfy

$$a_n \sim n^{-\theta}, \qquad n \to \infty. \tag{10.15}$$

THEOREM 10.1. *Let $X(t)$ be the process represented by* (10.2), *where* (X_n) *and* (Y_n) *are i.i.d. symmetric and α-stable, and a_n satisfies* (10.15), *with*

$$\frac{1}{\alpha} < \theta < \frac{3}{2}. \tag{10.16}$$

If

$$(2\theta - 1)(r - 1)m < 2r, \tag{10.17}$$

then the conclusion of Theorem 9.2 *holds, as well as that of Corollary* 9.2 *with the dimension*

$$r - \tfrac{1}{2}m(r - 1)(2\theta - 1) \tag{10.18}$$

in the place of (9.27). *If, in addition,*

$$\alpha(\alpha\theta - 1) > 2 \tag{10.19}$$

and

$$m(r-1)\left[\frac{\alpha\theta-1}{2}-\frac{1}{\alpha}\right]<r, \tag{10.20}$$

then the conclusion of Corollary 9.1 *holds with the dimension*

$$r-m(r-1)\left[\frac{\alpha\theta-1}{2}-\frac{1}{\alpha}\right] \tag{10.21}$$

in the place of (9.26).

PROOF. Let $Y(t)$ be the process represented by the right-hand member of (10.2) when (X_n) and (Y_n) are independent standard normal. The process is stationary with spectral sequence $f_n = a_n^2 > n^{-2\theta}$. By a standard calculation, it is seen that condition (10.1) holds with

$$\beta_1 = \beta_2 = 2\theta - 1. \tag{10.22}$$

The first inequality in (10.16) implies (10.4), which is sufficient for the almost sure convergence of the series (10.2) in both the Gaussian and stable cases. Both inequalities (10.16) also imply the requirement $0 < \beta_1 = \beta_2 < 2$. The statement of Theorem 9.1 holds for the Gaussian process $Y(t)$. Therefore, by the remarks following (10.5), Theorem 9.1 also holds for the stable version of (10.2), and so the conclusions of Theorem 9.2 and Corollary 9.2 hold in the modified forms stated above.

In order to obtain the conclusion of Corollary 9.1, we derive the Hölder index of the sample function. For (a_n) of the form (10.15), it follows that the sequence $f_n = |a_n|^\alpha$ satisfies condition (10.1) with $\beta_1 = \beta_2 = \alpha\theta - 1$, so that $0 < \beta_1 = \beta_2 < 2$ under the hypothesis (10.16). Thus the Tauberian theorem implies

$$R(t) = \sum_{n=0}^{\infty} |a_n|^\alpha (1-\cos nt) \sim \text{constant}|t|^{\alpha\theta-1}$$

for $t \to 0$. Put $\gamma = \alpha\theta - 1$; then the hypothesis (10.19) implies that (10.13) holds, and so (10.14) is the index of Hölder continuity. Substituting

$$\frac{\gamma}{2}-\frac{1}{\alpha} \quad \text{or} \quad \frac{\alpha\theta-1}{2}-\frac{1}{\alpha}$$

for the Hölder index $\beta_1/2$ in (9.26), we obtain (10.21). □

The parameter θ in (10.15) is identified as $\beta/2$ in Berman (1987), page 83. The member $\max(1, \alpha^{-1})$ on the bottom line of that page is incorrect and should, in accordance with (10.16) above, be replaced by $2/\alpha$.

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