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LARGE DEVIATIONS THEOREMS FOR EMPIRICAL MEASURES IN FREIDLIN–WENTZELL EXIT PROBLEMS

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We consider the jump-type Markov processes which are small random perturbations of dynamical systems and their empirical processes. We prove large deviations theorems for empirical measures which are marginal measures of empirical processes at the exit time of Markov processes from a bounded domain in a d-dimensional Euclidean space $\mathscr{R}^d$.

1. Introduction. The exit problems in Freidlin–Wentzell theory have been considered by many authors [cf. Day (1983, 1987, 1988), Freidlin and Wentzell (1984), Galves, Olivieri and Vares (1987) and Martinelli and Scoppola (1988)]. In this paper we prove large deviations theorems for the empirical measures

$$\mu^\varepsilon(dy) = \int_0^{\tau_D^\varepsilon} 1_{\{dy\}}(X^\varepsilon(s))\,ds/\tau_D^\varepsilon, \quad \varepsilon \to 0, \tag{1.1}$$

where 1_A is the indicator function of the set A, D is a bounded domain of $\mathscr{R}^d$, $X^\varepsilon(t)$ are $\mathscr{R}^d$-valued Markov processes stated below and

$$\tau_D^\varepsilon \equiv \inf\{t > 0; X^\varepsilon(t) \notin D\} \tag{1.2}$$

is the exit time of X^ε from D.

We consider the following conservative Markov processes which are right continuous and have left-hand limits; for each $\varepsilon > 0$, $(X^\varepsilon(t), P_x)_{0 \le t,\, x \in \mathscr{R}^d}$ is a strong Markov process whose infinitesimal generator $\mathscr{A}^\varepsilon$ is defined, on the set of C^2 functions f of $\mathscr{R}^d$ to $\mathscr{R}$ with compact supports, by

$$\begin{aligned}\mathscr{A}^\varepsilon f(x) \equiv{}& \sum_{i=1}^d b^i(x)^\varepsilon\, \partial f(x)/\partial x_i + \sum_{i,j=1}^d a^{ij}(x)^\varepsilon\, \partial^2 f(x)/\partial x_i\,\partial x_j \\ &+ \int_{\beta \ne o}\left[f(\beta + x) - f(x) - \sum_{i=1}^d \beta^i\,\partial f(x)/\partial x_i\right]\nu_x^\varepsilon(d\beta),\end{aligned} \tag{1.3}$$

where o is the origin in $\mathscr{R}^d$, $\nu_x^\varepsilon(d\beta)$ is a nonnegative measure on $\mathscr{R}^d \setminus \{o\}$ and $a(x)^\varepsilon \equiv (a^{ij}(x)^\varepsilon)_{i,j=1}^d$ is a symmetric, nonnegative definite matrix [cf. Komatsu (1973) and Stroock (1975)].

The cumulant is defined by

$$\begin{aligned}H(x;z)^\varepsilon \equiv{}& \langle b(x)^\varepsilon, z\rangle + \langle a(x)^\varepsilon z, z\rangle \\ &+ \int_{\beta \ne o}[\exp\langle z,\beta\rangle - 1 - \langle z,\beta\rangle]\nu_x^\varepsilon(d\beta),\end{aligned} \tag{1.4}$$

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where we put $b(x)^\varepsilon \equiv (b^i(x)^\varepsilon)_{i=1}^d$ and $\langle \cdot , \cdot \rangle$ denotes the inner product in $\mathscr{R}^d$.

The following assumption determines Hamiltonian.

(C) There exist a bounded domain Ω which contains D and a function $H(x;z)$ such that for any $C > 0$,

$$\varepsilon H(x;z/\varepsilon)^\varepsilon \to H(x;z) \quad \text{as } \varepsilon \to 0, \tag{1.5}$$

uniformly with respect to $x \in \Omega$ and $|z| < C$.

REMARK 1.1. Assumption (C) implies that $X^\varepsilon(t)$ can be decomposed, for sufficiently small $\varepsilon > 0$, on $[0, \tau_D^\varepsilon]$ in the manner

$$X^\varepsilon(t) = X^\varepsilon(0) + \int_0^t b(X^\varepsilon(s))^\varepsilon \, ds + M^{\varepsilon,[c]}(t) + M^{\varepsilon,[d]}(t), \tag{1.6}$$

where $M^{\varepsilon,[c]}(t)$ and $M^{\varepsilon,[d]}(t)$ are square integrable continuous and purely discontinuous martingales, respectively [cf. Lemma 2.7 and Meyer (1976)].

Under some assumptions on $H(x;z)$, $X^\varepsilon(t)$ can be considered as small random perturbations of dynamical systems, i.e., there exist dynamical systems $\{X(t,x)\}_{0 \le t,\, x \in \mathscr{R}^d}$ such that for all $\delta > 0$, $x \in \mathscr{R}^d$, $T > 0$,

$$\lim_{\varepsilon \to 0} P_x\Big(\sup_{0 \le t \le T} |X^\varepsilon(t) - X(t,x)| > \delta \Big) = 0 \tag{1.7}$$

[cf. Freidlin and Wentzell (1984)].

Let us give some notation. Let $L(x; \cdot\,)$ be the Legendre transformation of $H(x; \cdot\,)$,

$$L(x;u) \equiv \sup_{z \in \mathscr{R}^d} \{\langle z, u\rangle - H(x;z)\}, \tag{1.8}$$

which is nonnegative since $H(x;o) = 0$ from (1.4) and (1.5), and let $S_{0T}(\varphi)$ denote the action functional

$$S_{oT}(\varphi) = \begin{cases} \int_0^T L(\varphi(t);\dot\varphi(t))\,dt, & \text{for an absolutely continuous function } \varphi(\cdot)\colon [0,T] \mapsto \mathscr{R}^d, \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.9}$$

where we put $\dot\varphi(t) \equiv d\varphi(t)/dt$. Put

$$\mu_{\varphi,T}(dy) \equiv \int_0^T 1_{\{dy\}}(\varphi(t))\,dt/T. \tag{1.10}$$

Let $\overline{D}$ denote the closure of D, D° denote the interior of D, Id denote a $d \times d$ identity matrix and ∇ denote the d-dimensional gradient. For a $d \times d$ matrix $A \equiv (A^{ij})_{i,j=1}^d$, put $\|A\| \equiv (\sum_{i,j=1}^d (A^{ij})^2)^{1/2}$.

In this paper we consider two cases:

CASE 1.

(A.0) D is a bounded domain in $\mathscr{R}^d$ which contains o with a C^2 boundary ∂D.

$$(1.11)\qquad \begin{aligned} &b(x) \equiv \nabla_z H(x;o) \quad \text{exists and is Lipschitz continuous in } \Omega,\\ &b(o) = o,\\ &\langle b(x), n_x\rangle < 0 \quad \text{for } x \in \partial D, \end{aligned}$$

where n_x is the normalized outward normal at $x \in \partial D$. The solutions of the following ordinary differential equations are attracted to the origin o. For $x \in \overline{D}$,

$$(1.12)\qquad \begin{aligned} dX(t,x)/dt &= b(X(t,x)),\\ X(0,x) &= x. \end{aligned}$$

(A.1) $\sup\{H(x;z);\ x \in \Omega\}$ is finite for all $z \in \mathscr{R}^d$. For each $x \in \Omega$, $D_z^2 H(x;z) \equiv (\partial^2 H(x;z)/\partial z_i\, \partial z_j)_{i,j=1}^d$ exists and is continuous, and

$$(1.13)\qquad m \equiv \inf\{\langle D_z^2 H(x;z)e, e\rangle; x \in \Omega, z \in \mathscr{R}^d, |e| = 1, e \in \mathscr{R}^d\} > 0.$$

(A.2) There exist $\delta_1 > 0$, $c_1 > 0$ such that for all $\delta < \delta_1$,

$$(1.14)\qquad \sup\{[L(y;u) - L(x;u)]/(|x-y|[1 + L(x;u)])\} < c_1,$$

where the supremum is taken over all $x, y \in \Omega$ for which $|x - y| < \delta$ and $u \in \mathscr{R}^d$.

(A.3) $L(\cdot\,;\cdot)$ and $H(\cdot\,;\cdot)$ are once and twice continuously differentiable in $\overline{\Omega} \times \mathscr{R}^d$, respectively.

(A.4)

$$(1.15)\qquad \begin{aligned} &\limsup_{n\to\infty} \big(\sup\{L(x; u + Au)/(1 + L(x;u));\\ &\qquad\qquad x \in \overline{D}, u \in \mathscr{R}^d, \|A\| \le 1/n\}\big) < +\infty. \end{aligned}$$

CASE 2.

(H.0) D is a bounded domain in $\mathscr{R}^d$ with a C^2 boundary ∂D. $b(x) \equiv \nabla_z H(x;o)$ exists and is Lipschitz continuous in Ω. $X(t,x)$ exits Ω within finite time uniformly with respect to $x \in \overline{D}$.

(H.1) = (A.1).

(H.2) Put

$$(1.16)\qquad L(\delta) \equiv \sup\{[L(y;u) - L(x;u)]/[1 + L(x;u)]\},$$

where the supremum is taken over all $x, y \in \Omega$ for which $|x - y| < \delta$ and $u \in \mathscr{R}^d$. Then $L(\delta) \to 0$ as $\delta \to 0$.

(H.3) = (A.4).

We define the action functional $L_x(\cdot)$ on the set $\mathscr{P}(\overline{D})$ of probability measures on $\overline{D}$ with Prohorov metric ρ [cf. Stroock and Varadhan (1979)] by

$$L_x(\mu) = \begin{cases} 0, & \text{if } \mu = \delta_o, \\ \inf\{S_{0T}(\varphi); \varphi(0) = x, \varphi(T) \in \partial D, \\ \qquad \mu_{\varphi,T} = \mu, T > 0\}, & \text{otherwise}, \end{cases} \tag{1.17}$$

where δ_o is the delta measure concentrated on o and the infimum is $+\infty$ if the set over which it is taken is empty.

Then we obtain the following results.

THEOREM 1.1. *Suppose that the conditions* (C) *and* (A.0)–(A.4) *hold. Then*:

(O) $\Phi_K(s) \equiv \{\mu \in \mathscr{P}(\overline{D});\ L_x(\mu) \le s,\ x \in K\}$ *is compact in* $\mathscr{P}(\overline{D})$ *for any compact subset* K *of* D *and* $s > 0$.

(I) *For any Borel subset* B *of* $\mathscr{P}(\overline{D})$ *and* $x \in D$,

$$-\inf\{L_x(\mu); \mu \in B^\circ\} \le \liminf_{\varepsilon \to 0} \varepsilon \log P_x(\mu^\varepsilon \in B^\circ), \tag{1.18}$$

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_x(\mu^\varepsilon \in \overline{B}) \le -\inf\{L_x(\mu); \mu \in \overline{B}\}. \tag{1.19}$$

THEOREM 1.2. *Suppose that the conditions* (C) *and* (H.0)–(H.3) *hold. Then* (O) *and* (I) *in Theorem* 1.1 *hold with the action functional* $\tilde{L}_x(\cdot)$ *defined by*

$$\tilde{L}_x(\mu) = \inf\{S_{0T}(\varphi); \varphi(0) = x, \varphi(T) \in \partial D, \mu_{\varphi,T} = \mu, T > 0\}, \tag{1.20}$$

where the infimum is $+\infty$ *if the set over which it is taken is empty.*

REMARK 1.2. The assumption (A.1) = (H.1) implies [cf. Rockafellar (1970)] that for each $x \in \Omega$, $L(x; \cdot\,)$ is twice continuously differentiable and

$$1/m = \sup\{\langle D_u^2 L(x;u)e, e\rangle; x \in \Omega, u \in \mathscr{R}^d, |e| = 1, e \in \mathscr{R}^d\}. \tag{1.21}$$

REMARK 1.3. The assumptions (A.0)–(A.2) imply that

$$\lim_{\varepsilon \to 0} \varepsilon \log E_x[\tau_D^\varepsilon] = \inf\{V(x); x \in \partial D\} \tag{1.22}$$

and (H.0)–(H.2) imply that

$$\lim_{\varepsilon \to 0} P_x(|\tau_D^\varepsilon - T(X(\cdot, x))| > \delta) = 0 \quad \text{for any } \delta > 0, \tag{1.23}$$

where

$$V(x) \equiv \inf\{S_{0T}(\varphi); \varphi(0) = o, \varphi(T) = x, T > 0\} \tag{1.24}$$

is, so called, Freidlin–Wentzell quasipotential and

$$T(\varphi) \equiv \inf\{t > 0; \varphi(t) \notin D\} \tag{1.25}$$

is the first exit of $\varphi(\cdot)$ from D [cf. Freidlin and Wentzell (1984)].

REMARK 1.4. (A.4) is equivalent to the condition: There exist n_o and $C > 0$ such that for all $n \geq n_o$, $x \in \overline{D}$, $z \in \mathscr{R}^d$ and $\|A\| \leq 1/n$,

$$CH(x;(z + Az)/C) \leq C + H(x;z). \tag{1.26}$$

For

$$\begin{aligned} H(x;z) \equiv {} & \langle b(x)^0, z\rangle + \langle a(x)^0 z, z\rangle \\ & + \int_{\beta \neq o} [\exp\langle z, \beta\rangle - 1 - \langle z, \beta\rangle] \nu_x^0(d\beta), \end{aligned} \tag{1.27}$$

(A.4) is satisfied, if

1. $b(x)^0$ is bounded in $\overline{D}$,
2. $a(x)^0$ is bounded and uniformly positive definite in $\overline{D}$,
3. $\nu_x^0(d\beta)/\nu_x^0(d(A\beta))$ is bounded uniformly with respect to $\beta \neq o$, rotation matrix A and $x \in \overline{D}$.

This is true, since for $C > \|Id + A\|$,

$$\begin{aligned} & C\int_{\beta \neq o} [\exp(\langle (Id + A)z/C, \beta\rangle) - 1 - \langle (Id + A)z/C, \beta\rangle] \nu_x^0(d\beta) \\ & \leq C\int_{\beta \neq o} [\exp(|\langle (Id + A)z/\|Id + A\|, \beta\rangle| \times \|Id + A\|/C) - 1 \\ & \qquad - |\langle (Id + A)z/\|Id + A\|, \beta\rangle| \times \|Id + A\|/C] \nu_x^0(d\beta) \\ & \leq \|Id + A\|^2/C \sup_{|\alpha| \leq |z|} \int_{\beta \neq o} [\exp[(|\langle \beta, \alpha\rangle|)] - 1 - |\langle \beta, \alpha\rangle|] \nu_x^0(d\beta) \\ & \leq 2\|Id + A\|^2/C \sup_{|\alpha| = |z|} \int_{\beta \neq o} [\exp(\langle \beta, \alpha\rangle) - 1 - \langle \beta, \alpha\rangle] \nu_x^0(d\beta) \\ & \leq 2\|Id + A\|^2/C \sup\{\nu_x^0(d\beta)/\nu_x^0(d(A\beta)); \\ & \qquad\qquad x \in \overline{D}, \beta \neq o, A \text{ is a rotation matrix}\} \\ & \quad \times \int_{\beta \neq o} [\exp(\langle \beta, z\rangle) - 1 - \langle \beta, z\rangle] \nu_x^0(d\beta). \end{aligned} \tag{1.28}$$

Here we used the facts

$$0 \leq \exp(x) - 1 - x \leq \exp(|x|) - 1 - |x| \quad \text{for } x \in \mathscr{R}, \tag{1.29}$$

$$\exp(r|x|) - 1 - r|x| \leq r^2(\exp|x| - 1 - |x|) \quad \text{for } x \in \mathscr{R}, 0 \leq r \leq 1. \tag{1.30}$$

REMARK 1.5. Since we consider the process $X^\varepsilon(\cdot)$ in $[0, \tau_D^\varepsilon)$, we can assume, from (C), (H.1) and (A.2) [or (H.2)], that the assumptions in Mikami (1988) hold and that $S_{0T}(\cdot)/\varepsilon$ is the action functional for $(X^\varepsilon(t), P_x)_{0 \leq t \leq T,\ x \in \mathscr{R}^d}$ uniformly with respect to the initial point [see also Wentzell (1979)].

For the sake of the proof of our results, we state large deviations theorems whose form is different from that in Wentzell (1979) but is equivalent to it [cf. Freidlin and Wentzell (1984), Chapter 3].

THEOREM 1.3 [cf. Mikami (1988)]. *Suppose that* (C), (A.1) *and* (A.2) [*or* (H.2)] *hold. Then*:

(O) *For any compact subset K of $\mathscr{R}^d$ and $s > 0$, the set $\{\varphi \in C([0,T];\mathscr{R}^d);\ S_{0T}(\varphi) \le s,\ \varphi(0) \in K\}$ is a compact subset of $C([0,T];\mathscr{R}^d)$ whose topology is that induced by the sup norm. In particular, $S_{0T}(\cdot)$ is lower semicontinuous on $C([0,T];\mathscr{R}^d)$.*

(I) *For any $x \in \mathscr{R}^d$ and open set $O \in C([0,T];\mathscr{R}^d)$,*

$$\liminf_{\varepsilon\to 0} \varepsilon \log P_x(X^\varepsilon \in O) \ge -\inf\{S_{0T}(\varphi);\ \varphi(0) = x,\ \varphi \in O\}.$$

(II) *For any $x \in \mathscr{R}^d$ and closed set $A \in C([0,T];\mathscr{R}^d)$,*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P_x(X^\varepsilon \in A) \le -\inf\{S_{0T}(\varphi);\ \varphi(0) = x,\ \varphi \in A\}.$$

In Section 2 we give lemmas necessary for the proof of our results. In Section 3 we prove the theorems.

2. Lemmas. In this section we give lemmas necessary for the proof of our results.

Before we proceed to lemmas, we mention that the condition (H.2) is weaker than (A.2).

Lemma 2.1 is given in Freidlin and Wentzell [(1984), page 110, Lemma 2.2].

LEMMA 2.1. *Suppose that* (A.0)–(A.2) *hold. Then for any $\alpha > 0$, there exist $a = a(\alpha)$ and $T_o = T_o(\alpha) > 0$ such that*

$$S_{0T}(\varphi) > a(T - T_o) \tag{2.1}$$

for all $\varphi(\cdot)$ for which $\varphi(t) \in \overline{D} \setminus U_\alpha(o)$ for all $0 \le t \le T$ [*see* (1.9)], *where $U_\alpha(o)$ denotes an α neighborhood of o.*

The following lemma is used to prove the upper estimate of (I) in Theorem 1.1.

LEMMA 2.2. *Suppose that* (A.0), (A.1) *and* (A.2) [*or* (H.2)] *hold. Then for any $\mu \in \mathscr{P}(\overline{D}) \setminus \{\delta_o\}$ and $x \in D$,*

$$\liminf_{\alpha\to 0} S_{\alpha,x}(\mu) \ge L_x(\mu) \tag{2.2}$$

[*see* (1.17)], *where we put*

$$\begin{aligned} S_{\alpha,x}(\mu) = \inf\{&S_{0T}(\varphi);\ \varphi(0) = x,\ \varphi(T) \in \partial D,\ \varphi(t) \in \overline{D} \\ &\text{for all } 0 \le t \le T,\ \rho(\mu_{\varphi,T},\mu) < \alpha,\ T > 0\} \end{aligned} \tag{2.3}$$

[*see* (1.9) *and* (1.10)]. *Here ρ denotes Prohorov metric.*

PROOF. Suppose that for all $n \ge 1$, there exist $T_n > 0$ and φ_n for which $\varphi_n(0) = x$, $\varphi_n(T_n) \in \partial D$, $\varphi_n(t) \in \overline{D}$ for all $0 \le t \le T_n$ such that

$$\lim_{n\to\infty} \rho(\mu_{\varphi_n,T_n},\mu) = 0, \tag{2.4}$$

$$\lim_{n\to\infty} S_{0,T_n}(\varphi_n) < +\infty. \tag{2.5}$$

Then there exists $R > 0$ such that $T_n < R$ for all $n \geq 1$, since $\mu \neq \delta_o$. In fact, for sufficiently small, arbitrary $\alpha > 0$,

$$\inf_{|x|=\alpha} V(x) > 0 \tag{2.6}$$

[cf. (1.24) and Freidlin and Wentzell (1984), Chapter 4].

If $\alpha' > 0$ is sufficiently small compared to $\alpha > 0$, then

$$\inf_{|x|=\alpha} V(x) > \sup_{|x|=\alpha'} V(x) > 0, \tag{2.7}$$

since $V(x)$ is Lipschitz continuous in $\overline{D}$ and

$$\begin{gathered}\lim_{x \to o} V(x) = 0,\\ V(x) > 0 \quad \text{for all } x \in D \setminus \{o\}\end{gathered} \tag{2.8}$$

[cf. Freidlin and Wentzell (1984), Chapter 4].

The $\varphi_n(\cdot)$ hit an α' neighborhood within bounded time, uniformly for sufficiently large n from Lemma 2.1 and (2.5). But they cannot cross the sets $\{x;\ |x| = \alpha\}$ and $\{x;\ |x| = \alpha'\}$ infinitely often, from (2.5) and (2.7). Therefore if

$$\lim_{n \to \infty} T_n = \infty, \tag{2.9}$$

then the supports of μ_{φ_n, T_n} tend to concentrate to an α neighborhood of o. Since $\alpha > 0$ can be arbitrarily small,

$$\mu = \delta_o, \tag{2.10}$$

which is a contradiction.

Put

$$\tilde{\varphi}_n(t) \equiv \begin{cases} \varphi_n(t), & \text{if } 0 \leq t \leq T_n, \\ X(t - T_n, \varphi_n(T_n)), & \text{if } T_n \leq t \leq R \end{cases} \tag{2.11}$$

[see (1.12)]. Then there exist a convergent subsequence $\{\tilde{\varphi}_{n_k}(t)\}_{k=1}^{\infty}$, $0 \leq t \leq R$, a function $\tilde{\varphi}(t)$, $0 \leq t \leq R$, and $0 < T' \leq R$ such that

$$\begin{gathered}\liminf_{k \to +\infty} S_{0R}(\tilde{\varphi}_{n_k}) \geq S_{0R}(\tilde{\varphi}) \geq L_x(\mu),\\ \mu_{\tilde{\varphi}, T'} = \mu,\end{gathered} \tag{2.12}$$

from lower semicontinuity of $S_{0R}(\cdot)$ [cf. Theorem 1.3(O)]. □

The following lemma is used to prove Lemmas 2.4 and 2.5.

Lemma 2.3. *Suppose that* (A.1) *holds. Then for all* $\varphi(\cdot)$ *for which* $S_{0T}(\varphi) < +\infty$ [*see* (1.9)] *and* $\varphi(t) \in \overline{D}$ *for all* $0 \leq t \leq T$,

$$\int_0^T |\nabla_u L(\varphi(s); \dot{\varphi}(s))|\, ds < +\infty. \tag{2.13}$$

Proof. Put

$$p(t) \equiv \nabla_u L(\varphi(t); \dot{\varphi}(t)). \tag{2.14}$$

Then we have

$$\begin{aligned}(2.15)\quad &\int_0^T |\dot\varphi(s)|\,ds\\ &= \sup\left\{\int_0^T \langle z(s), \dot\varphi(s)\rangle\,ds;\ \operatorname*{ess\,sup}_{0\le t\le T}|z(t)| = 1\right\}\\ &\le \int_0^T L(\varphi(s);\dot\varphi(s))\,ds + T\cdot\sup\{H(x;z);\ x\in\overline{D},\ |z|=1\}\\ &< +\infty\end{aligned}$$

from (1.8) and (A.1). On the other hand,

$$\begin{aligned}&\sup\left\{\int_0^T \langle z(s), \dot\varphi(s)\rangle\,ds;\ \operatorname*{ess\,sup}_{0\le t\le T}|z(t)| = 1\right\}\\ &\quad= \sup\left\{\int_0^T \langle z(s), \nabla_z H(\varphi(s);p(s))\rangle\,ds;\ \operatorname*{ess\,sup}_{0\le t\le T}|z(t)| = 1\right\}\\ (2.16)\quad &\quad\ge -\int_{0\le s\le T,\,p(s)=o} |\nabla_z H(\varphi(s);o)|\,ds\\ &\qquad+\int_{0\le s\le T,\,p(s)\ne o}\langle p(s)/|p(s)|, \nabla_z H(\varphi(s);o)\rangle\,ds + m\int_0^T |p(s)|\,ds\\ &\quad\ge -T\sup_{x\in D}|b(x)| + m\int_0^T |p(s)|\,ds\end{aligned}$$

from (A.1), since for $0\le s\le T$ for which $p(s)\ne o$,

$$\begin{aligned}&\langle p(s)/|p(s)|, \nabla_z H(\varphi(s);p(s))\rangle\\ (2.17)\quad &\quad= \langle p(s)/|p(s)|, \nabla_z H(\varphi(s);o)\rangle\\ &\qquad+ \langle p(s)/|p(s)|, D_z^2 H(\varphi(s);\theta(s)p(s))p(s)\rangle,\end{aligned}$$

for some $0\le\theta(s)\le 1$, by the mean value theorem. □

The following two lemmas are used to prove the lower estimate of (I) in Theorem 1.1.

LEMMA 2.4. *Suppose that* (A.0), (A.1), (A.2) [*or* (H.2)] *and* (A.4) *hold. Then for any function* $\varphi(\cdot)$ *for which* $\varphi(0)\in D$, $\varphi(T)\in\partial D$, $\varphi(t)\in\overline{D}$ *for all* $0\le t\le T$ *and* $S_{0T}(\varphi) < +\infty$ [*see* (1.9)], *there exist functions* $\varphi_n(\cdot)$ *such that*

$$\sup_{0\le t\le T}|\varphi_n(t)-\varphi(t)|\to 0 \quad as\ n\to+\infty, \tag{2.18}$$

$$S_{0T}(\varphi_n)\to S_{0T}(\varphi) \quad as\ n\to+\infty, \tag{2.19}$$

$$T(\varphi_n)\to T \quad as\ n\to+\infty. \tag{2.20}$$

Moreover, for $x \equiv \varphi(0)$,

$$\limsup_{\alpha \to 0} \tilde{S}_{\alpha, x}(\mu_{\varphi, T}) \le L_x(\mu_{\varphi, T}) \tag{2.21}$$

[*see* (1.10) *and* (1.17)], *where we put*

$$\begin{aligned} \tilde{S}_{\alpha, x}(\mu) = \inf\{S_{0T(\phi)}(\phi); \phi(0) = x, T(\phi) < +\infty, \\ \rho(\mu_{\phi, T(\phi)}, \mu) < \alpha\}. \end{aligned} \tag{2.22}$$

PROOF. Since ∂D is of class C^2, there exist $\delta > 0$ such that for all $x \in \overline{D}$ for which $\operatorname{dist}(x, \partial D) \equiv \inf\{|x - y|; y \in \partial D\} < 3\delta$, the mapping

$$x \mapsto -n_{y(x)} \equiv \nu_x \quad \text{is of class } C^1, \tag{2.23}$$

where $y(x) \in \partial D$ is determined by

$$|x - y(x)| = \operatorname{dist}(x, \partial D). \tag{2.24}$$

Put

$$\begin{aligned} &\sigma_0 = 0, \\ &T_n \equiv \inf\{t > \sigma_{n-1}; \varphi(t) \in \partial D\}, \\ &\tau_n \equiv \sup\{T_n > t; \operatorname{dist}(\varphi(t), \partial D) \ge \delta\}, \\ &\sigma_n \equiv \inf\{t > T_n; \operatorname{dist}(\varphi(t), \partial D) \ge 2\delta\}, \quad \text{for } n \ge 1. \end{aligned} \tag{2.25}$$

Then there exists $n_o > 0$ such that $T_{n_o} = T$, since

$$\begin{gathered} S_{0T}(\varphi) < +\infty, \\ \inf\{S_{0T}(\phi); \operatorname{dist}(\phi(0), \partial D) = 2\delta, \\ \phi(0) \in D, \phi(T) \in \partial D, T > 0\} > 0 \end{gathered} \tag{2.26}$$

from (A.0) [cf. Freidlin and Wentzell (1984), Chapter 4].

For $1 \le k \le n_o$, put

$$\varphi_n(t) \equiv \begin{cases} \varphi(t), & \text{if } \sigma_{k-1} \le t \le \tau_k, \\ \varphi(t) + (t - \tau_k)\nu_{\varphi(t)}/(nR), & \text{if } \tau_k \le t \le \sigma_k - 1/n, \\ \varphi_n(\sigma_k - 1/n) & \\ \quad + n(\varphi(\sigma_k) - \varphi_n(\sigma_k - 1/n)) & \\ \qquad \times (t - \sigma_k + 1/n), & \text{if } \sigma_k - 1/n \le t \le \sigma_k, \end{cases} \tag{2.27}$$

where we put $\sigma_{n_o} \equiv T$ for convenience. Here we take $R > 1$ sufficiently large so that

$$\sum_{i=1}^{n_o} \int_{\tau_i}^{\sigma_i} L(\varphi(t); \dot{\varphi}(t) + (t - \tau_i)\, \partial\nu_{\varphi(t)}\, \dot{\varphi}(t)/R)\, dt < +\infty \tag{2.28}$$

[cf. (A.4)], where we put $\partial\nu_x \equiv (\partial\nu_x^i/\partial x_j)_{i,j=1}^d$.

First we prove (2.18). Put

$$\delta^n \equiv \sup_{0\le t\le T} |\varphi_n(t) - \varphi(t)|. \tag{2.29}$$

Then

$$\begin{aligned}\delta^n &\le 2\sup\{|\varphi(t) - \varphi(s)|; 0 \le t, s \le T, |t-s| < 1/n\} \\ &\quad + 2\sup\{T \cdot |\nu_x|/(nR); X \in D, \operatorname{dist}(x, \partial D) \le 2\delta\} \to 0 \\ &\qquad \text{as } n \to \infty,\end{aligned} \tag{2.30}$$

since for $\tau_k \le t \le \sigma_k - 1/n$ $(k = 1, \dots, n_o)$,

$$\begin{aligned}|\varphi(t) - \varphi_n(t)| &\le T \cdot |\nu_{\varphi(t)}|/(nR) \\ &\le T \sup\{|\nu_x|/(nR); x \in D, \operatorname{dist}(x, \partial D) \le 2\delta\}\end{aligned} \tag{2.31}$$

and for $\sigma_k - 1/n \le t \le \sigma_k$ $(k = 1, \dots, n_o)$,

$$\begin{aligned}&|\varphi_n(t) - \varphi(t)| \\ &= |\varphi_n(\sigma_k - 1/n) + n(\varphi(\sigma_k) - \varphi_n(\sigma_k - 1/n)) \\ &\qquad \times (t - \sigma_k + 1/n) - \varphi(t)| \\ &\le |\varphi_n(\sigma_k - 1/n) - \varphi(t)| + |\varphi_n(\sigma_k - 1/n) - \varphi(\sigma_k)| \\ &\le |\varphi_n(\sigma_k - 1/n) - \varphi(\sigma_k - 1/n)| + |\varphi(\sigma_k - 1/n) - \varphi(t)| \\ &\quad + |\varphi_n(\sigma_k - 1/n) - \varphi(\sigma_k - 1/n)| + |\varphi(\sigma_k - 1/n) - \varphi(\sigma_k)| \\ &\le 2T \sup\{|\nu_x|/(nR); x \in D, \operatorname{dist}(x, \partial D) \le 2\delta\} \\ &\quad + 2\sup\{|\varphi(t) - \varphi(s)|; 0 \le t, s \le T, |t-s| < 1/n\}\end{aligned} \tag{2.32}$$

from (2.31).

By the lower semicontinuity of $S_{0T}(\cdot)$ [cf. Theorem 1.3(O)], to prove (2.19), we only have to show

$$\limsup_{n\to\infty} S_{0T}(\varphi_n) \le S_{0T}(\varphi). \tag{2.33}$$

In fact, for $k = 1, \dots, n_o$,

$$\begin{aligned}&\int_{\tau_k}^{\sigma_k - 1/n} L(\varphi_n(s); \dot\varphi_n(s))\, ds \\ &\le (1 - 1/n) \int_{\tau_k}^{\sigma_k - 1/n} L(\varphi_n(s); \dot\varphi(s))\, ds \\ &\quad + \int_{\tau_k}^{\sigma_k - 1/n} L\big(\varphi_n(s); \dot\varphi(s) + (s - \tau_k)\, \partial\nu_{\varphi(s)}\, \dot\varphi(s)/R + \nu_{\varphi(s)}/R\big)\, ds/n\end{aligned} \tag{2.34}$$

from the convexity of $L(x; \cdot\,)$ [cf. (1.8)];

$$\begin{aligned}\int_{\tau_k}^{\sigma_k - 1/n} L(\varphi_n(s); \dot\varphi(s))\, ds &\le (\sigma_k - 1/n - \tau_k) L(\delta^n) \\ &\quad + (1 + L(\delta^n)) \int_{\tau_k}^{\sigma_k - 1/n} L(\varphi(s); \dot\varphi(s))\, ds\end{aligned} \tag{2.35}$$

from (A.2) [or (H.2)] [see (1.16)];

$$\begin{aligned}
&\int_{\tau_k}^{\sigma_k - 1/n} L\big(\varphi_n(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R + \nu_{\varphi(s)}/R\big)\,ds \\
&\qquad \le (\sigma_k - 1/n - \tau_k)L(\delta^n) + (1 + L(\delta^n)) \\
&\qquad\qquad \times \int_{\tau_k}^{\sigma_k - 1/n} L\big(\varphi(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R + \nu_{\varphi(s)}/R\big)\,ds
\end{aligned} \tag{2.36}$$

from (A.2) [or (H.2)] [see (1.16)];

$$\begin{aligned}
&\int_{\tau_k}^{\sigma_k - 1/n} L\big(\varphi(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R + \nu_{\varphi(s)}/R\big)\,ds \\
&\quad \le \int_{\tau_k}^{\sigma_k - 1/n} L\big(\varphi(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R\big)\,ds \\
&\qquad + \int_{\tau_k}^{\sigma_k - 1/n} \langle \nabla_u L\big(\varphi(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R\big), \nu_{\varphi(s)}/R\rangle\,ds \\
&\qquad + \int_{\tau_k}^{\sigma_k - 1/n} |\nu_{\varphi(s)}/R|^2/(2m)\,ds \quad [\text{from } (1.21)] \\
&\quad \le \int_{\tau_k}^{\sigma_k} L\big(\varphi(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R\big)\,ds \\
&\qquad + \sup\{|\nu_x|/R;\, x\in D,\, \mathrm{dist}(x,\partial D) \le 2\delta\} \\
&\qquad\quad \times \int_{\tau_k}^{\sigma_k} |\nabla_u L\big(\varphi(s); \dot\varphi(s) + (s-\tau_k)\,\partial\nu_{\varphi(s)}\,\dot\varphi(s)/R\big)|\,ds \\
&\qquad + (\sigma_k - \tau_k)\sup\big\{(|\nu_x|/R)^2/(2m);\, x\in D,\, \mathrm{dist}(x,\partial D) \le 2\delta\big\} < +\infty
\end{aligned} \tag{2.37}$$

from the mean value theorem, (2.28), Lemma 2.3 and (A.1);

$$\begin{aligned}
&\int_{\sigma_k - 1/n}^{\sigma_k} L(\varphi_n(s); \dot\varphi_n(s))\,ds \\
&\quad \le L\Big(\sup_{|t-s|\le 1/n} |\varphi_n(s) - \varphi_n(t)|\Big)/n \\
&\qquad + \Big(1 + L\Big(\sup_{|t-s|\le 1/n} |\varphi_n(s) - \varphi_n(t)|\Big)\Big) \\
&\qquad\quad \times \int_{\sigma_k - 1/n}^{\sigma_k} L\Big(\varphi_n(\sigma_k);\, n\int_{\sigma_k - 1/n}^{\sigma_k} \big[\dot\varphi(t) - (\sigma_k - 1/n - \tau_k) \\
&\qquad\qquad\qquad \times \nu_{\varphi(\sigma_k - 1/n)}/R\big]\,dt\Big)\,ds
\end{aligned} \tag{2.38}$$

from (A.2) [or (H.2)] [see (1.16)];

$$\int_{\sigma_k - 1/n}^{\sigma_k} L\left(\varphi_n(\sigma_k);\, n\int_{\sigma_k - 1/n}^{\sigma_k} \left[\dot{\varphi}(t) - (\sigma_k - 1/n - \tau_k)\nu_{\varphi(\sigma_k - 1/n)}/R\right] dt \right) ds$$

$$\leq \int_{\sigma_k - 1/n}^{\sigma_k} L\big(\varphi(\sigma_k); \dot{\varphi}(t) - (\sigma_k - 1/n - \tau_k)\nu_{\varphi(\sigma_k - 1/n)}/R\big)\, dt$$

$$\begin{aligned}(2.39)\quad &\leq \left(1 + L\left(\sup_{|t-s|\leq 1/n} |\varphi(s) - \varphi(t)|\right)\right) \\ &\quad \times \int_{\sigma_k - 1/n}^{\sigma_k} L\big(\varphi(t); \dot{\varphi}(t) - (\sigma_k - 1/n - \tau_k)\nu_{\varphi(\sigma_k - 1/n)}/R\big)\, dt \\ &\quad + L\left(\sup_{|t-s|\leq 1/n} |\varphi(s) - \varphi(t)|\right)/n\end{aligned}$$

from Jensen's inequality [cf. Rockafellar (1970)] and (A.2) [or (H.2)] [see (1.16)];

$$\int_{\sigma_k - 1/n}^{\sigma_k} L\big(\varphi(t); \dot{\varphi}(t) - (\sigma_k - 1/n - \tau_k)\nu_{\varphi(\sigma_k - 1/n)}/R\big)\, dt$$

$$\begin{aligned}&\leq \int_{\sigma_k - 1/n}^{\sigma_k} L(\varphi(t); \dot{\varphi}(t))\, dt \\ &\quad + \int_{\sigma_k - 1/n}^{\sigma_k} \langle \nabla_u L(\varphi(t); \dot{\varphi}(t)), -(\sigma_k - 1/n - \tau_k)\nu_{\varphi(\sigma_k - 1/n)}/R\rangle\, dt \\ (2.40)\quad &\quad + |-(\sigma_k - 1/n - \tau_k)\nu_{\varphi(\sigma_k - 1/n)}/R|^2/(2mn) \quad [\text{from } (1.21)] \\ &\leq \int_{\sigma_k - 1/n}^{\sigma_k} L(\varphi(t); \dot{\varphi}(t))\, dt \\ &\quad + T\sup\{|\nu_x|/R;\, x \in D,\, \operatorname{dist}(x, \partial D) \leq 2\delta\} \int_{\sigma_k - 1/n}^{\sigma_k} |\nabla_u L(\varphi(t); \dot{\varphi}(t))|\, dt \\ &\quad + T^2 \sup\big\{(|\nu_x|/R)^2/(2mn);\, x \in D,\, \operatorname{dist}(x, \partial D) \leq 2\delta\big\} \to 0 \quad \text{as } n \to \infty\end{aligned}$$

from the mean value theorem, (A.1) and Lemma 2.3.

From the construction of φ_n, it is easy to see that (2.20) is satisfied.

(2.21) is true from the definition of $L_x(\cdot)$ [see (1.17)], since (2.18)–(2.20) mean that any function $\varphi(\cdot)$ for which $\varphi(0) \in D$, $\varphi(T) \in \partial D$, $\varphi(t) \in \overline{D}$ for all $0 \leq t \leq T$ and $S_{0T}(\varphi) < +\infty$ can be approximated by the functions $\varphi_n(\cdot)$ such that $\varphi_n(0) = \varphi(0) \in D$, $\varphi_n(T) \in \partial D$, $\varphi_n(t) \in D$ for all $0 \leq t < T - 1/n$ and that satisfy (2.19). □

Lemma 2.5 can be proved in the same way as in Freidlin and Wentzell [(1984), pages 88–89].

LEMMA 2.5. *Suppose that* (A.1) *and* (A.2) [*or* (H.2)] *hold. Then for any* $R > 0$ *and any* $\{\varphi(t)\}_{0 \le t \le R}$ *for which* $\varphi(0) \in D$, $T(\varphi) < R$ [*see* (1.25)] *and* $S_{0R}(\varphi) < \infty$, *there exist functions* $\{\varphi_n(t)\}_{0 \le t \le R,\, 1 \le n}$ *which exit* $\overline{D}$ *such that*

$$\varphi_n(0) = \varphi(0), \tag{2.41}$$

$$\sup_{0 \le t \le R} |\varphi_n(t) - \varphi(t)| \to 0 \quad \text{as } n \to +\infty, \tag{2.42}$$

$$S_{0R}(\varphi_n) \to S_{0R}(\varphi) \quad \text{as } n \to +\infty. \tag{2.43}$$

PROOF. Put

$$\varphi_k(t) \equiv \varphi(t) + n_{\varphi(T(\varphi))} t/k \quad \text{for all } 0 \le t \le R \tag{2.44}$$

[cf. (1.11) and (1.25)]. Then clearly (2.41) and (2.42) are satisfied. For sufficiently large $k \ge 1$, $\varphi_k(T(\varphi)) \notin \overline{D}$ since ∂D is of class C^2. From the lower semicontinuity of $S_{0R}(\cdot)$ [cf. Theorem 1.3(O)], we only have to show

$$\limsup_{n \to \infty} S_{0R}(\varphi_n) \le S_{0R}(\varphi). \tag{2.45}$$

This is true, since

$$\begin{aligned} &\int_0^R L(\varphi_k(s); \dot\varphi_k(s))\, ds \\ &\le RL\Big(\sup_{0 \le t \le R} |\varphi_k(t) - \varphi(t)|\Big) \\ &\quad + \Big(1 + L\Big(\sup_{0 \le t \le R} |\varphi_k(t) - \varphi(t)|\Big)\Big) \int_0^R L\big(\varphi(s); \dot\varphi(s) + n_{\varphi(T(\varphi))}/k\big)\, ds \end{aligned} \tag{2.46}$$

from (A.2) [or (H.2)] [see (1.16)] and

$$\begin{aligned} &\int_0^R L\big(\varphi(s); \dot\varphi(s) + n_{\varphi(T(\varphi))}/k\big)\, ds \\ &\le \int_0^R L(\varphi(s); \dot\varphi(s))\, ds + \int_0^R \langle \nabla_u L(\varphi(s); \dot\varphi(s)), n_{\varphi(T(\varphi))}/k \rangle\, ds \\ &\quad + R|n_{\varphi(T(\varphi))}/k|^2/(2m) \quad \text{[from (1.21)]} \end{aligned} \tag{2.47}$$

by the mean value theorem and (A.1). From Lemma 2.3, the proof is over. □

Lemmas 2.6–2.8 are used to prove Key Step of the proof of Theorem 1.1.

LEMMA 2.6. *Suppose that* (A.0)–(A.3) *hold. Then there exists a family* $\{V^\alpha(\cdot)\}_{\alpha > 0}$ *of* C^∞ *functions such that for any* $r_0 > 0$,

$$\limsup_{\alpha \to 0} \big(\sup\{\langle b(x), \nabla V^\alpha(x)\rangle;\ |x| \ge 2r_0,\ x \in \overline{D}\}\big) < 0. \tag{2.48}$$

PROOF. Take $\phi \in C_o^\infty(|x| < 1; [0,\infty))$ and put

$$\phi_\alpha(z) \equiv \phi(z/\alpha) \Big/ \int_{\mathscr{R}^d} \phi(y/\alpha)\, dy, \tag{2.49}$$

$$V^\alpha(x) \equiv \int_{\mathscr{R}^d} \phi_\alpha(x-y) V(y)\, dy \tag{2.50}$$

[see (1.24)]. Let D_α denote an α-neighborhood of D. Take a sufficiently small open neighborhood Ω_o, Ω_1 for which $\Omega \supset \Omega_o \supset \Omega_1 \supset D$ and change L outside Ω_1 so that

$$\begin{gathered} \sup_{x \in \partial D} V(x) < \inf_{x \in \partial \Omega_o} V(x), \\ \lim_{t \to +\infty} X(t,x) = o, \quad \text{for } x \in \overline{\Omega}_o \end{gathered} \tag{2.51}$$

[see (1.12)]. We only have to show

$$\lim_{\alpha \to 0} \operatorname{ess\,inf}\left\{|\nabla V(x)|^2;\, x \in \overline{D}_\alpha,\, |x| > r_0\right\} > 0, \tag{2.52}$$

since V is Lipschitz continuous [cf. Freidlin and Wentzell (1984), page 112, Lemma 2.3] and for $x \in \overline{D}$ for which $|x| \geq 2r_0$ and α $(< r_0)$ for which $\sup_{x \in \partial D_\alpha} V(x) < \inf_{x \in \partial \Omega_o} V(x)$,

$$\begin{aligned} \langle b(x), \nabla V^\alpha(x) \rangle &= \int_{\mathscr{R}^d} \langle b(x) - b(y), \phi_\alpha(x-y)\, \nabla V(y) \rangle\, dy \\ &\quad + \int_{\mathscr{R}^d} \langle b(y), \phi_\alpha(x-y)\, \nabla V(y) \rangle\, dy \\ &\leq \sup\left\{|b(x) - b(y)|;\, |x-y| < \alpha,\, x, y \in \overline{D}_\alpha\right\} \\ &\quad \times \operatorname{ess\,sup}\left\{|\nabla V(y)|;\, y \in \overline{D}_\alpha\right\} \\ &\quad + \operatorname{ess\,sup}\left\{\langle b(x), \nabla V(x) \rangle;\, x \in \overline{D}_\alpha,\, |x| > r_0\right\} \end{aligned} \tag{2.53}$$

[cf. Fleming and Vermes (1988), Lemma 3.2] and for almost every $x \in \overline{D}_\alpha$,

$$H(x; \nabla V(x)) = 0 \tag{2.54}$$

[cf. Fleming (1969), Theorem 1],

$$\begin{aligned} \langle b(x), \nabla V(x) \rangle &= -(H(x; \nabla V(x)) - \langle b(x), \nabla V(x) \rangle) \\ &\leq -m|\nabla V(x)|^2/2 \end{aligned} \tag{2.55}$$

from (A.1) and the mean value theorem.

Suppose that V is differentiable at $x_n \in D_\alpha$ $(|x_n| > r_0)$ and $\nabla V(x_n) \to o$ as $n \to +\infty$. Take the minimizing functions $\varphi_n(\cdot)$ such that

$$\begin{gathered} \lim_{t \to -\infty} \varphi_n(t) = o, \\ \varphi_n(0) = x_n, \\ S_{-\infty,0}(\varphi_n) = V(x_n) \end{gathered} \tag{2.56}$$

[cf. Wentzell and Freidlin (1970), Lemma 3.3].

Then $p_n(t) \equiv \nabla_u L(\varphi_n(t); \dot{\varphi}_n(t))$ is equicontinuous [cf. Day and Darden (1985), (3.2) and (3.5); (3.5) in their paper implies the boundedness of $p_n(\cdot)$ and (3.2) in their paper implies the boundedness of $\dot{p}_n(\cdot)$ under our assumptions (A.0)–(A.3), since $\varphi_n(\cdot)$ are bounded] and

$$p_n(0) = \nabla_u L(\varphi_n(0); \dot{\varphi}_n(0)) = \nabla V(x_n) \tag{2.57}$$

[cf. Fleming (1969), Theorem 1].

Since

$$S_{-1,0}(\varphi_n) \leq \sup_{x \in \overline{\Omega}} V(x) < +\infty \quad \text{for all } n \geq 1, \tag{2.58}$$

there exist, from Theorem 1.3(O), convergent subsequences $p_{n_k}(\cdot), \varphi_{n_k}(\cdot)$ $(k \geq 1)$, functions $\varphi(t), p(t), -1 \leq t \leq 0$, and $x, y \in \overline{D}$ such that

$$\begin{aligned}
&\sup_{-1 \leq t \leq 0} |\varphi_{n_k}(t) - \varphi(t)| \to 0, \quad \text{as } k \to +\infty, \\
&\sup_{-1 \leq t \leq 0} |p_{n_k}(t) - p(t)| \to 0, \quad \text{as } k \to +\infty,
\end{aligned}$$

$$\begin{gathered}
\varphi(0) = x, \qquad \varphi(-1) = y, \\
S_{-1,0}(\varphi) + V(y) = V(x), \\
L(\varphi(0); \dot{\varphi}(0)) = \langle \dot{\varphi}(0), p(0) \rangle = 0,
\end{gathered} \tag{2.59}$$

which is a contradiction [cf. Day and Darden (1985), page 267, Corollary 3]. □

Put

$$F(\varepsilon) \equiv \sup\left\{ \|a(x)^{\varepsilon}\| + \int_{\beta \neq o} |\beta|^2 \nu_x^{\varepsilon}(d\beta)/2;\ x \in \overline{D} \right\} \tag{2.60}$$

[see below (1.10) for notation]. Then we get the following fact that implies Remark 1.1.

Lemma 2.7. *Suppose that* (C) *holds. Then*

$$\lim_{\varepsilon \to 0} F(\varepsilon) = 0, \tag{2.61}$$

$$\limsup_{\varepsilon \to 0} \sup_{x \in D} |b(x)^{\varepsilon}| < +\infty. \tag{2.62}$$

Moreover, $X^{\varepsilon}(t)$ can be decomposed, for sufficiently small $\varepsilon > 0$, on $[0, \tau_D^{\varepsilon}]$ in the following way:

$$X^{\varepsilon}(t) = X^{\varepsilon}(0) + \int_0^t b(X^{\varepsilon}(s))^{\varepsilon}\, ds + M^{\varepsilon,[c]}(t) + M^{\varepsilon,[d]}(t), \tag{2.63}$$

where $M^{\varepsilon,[c]}(t)$ and $M^{\varepsilon,[d]}(t)$ are square integrable continuous and purely discontinuous martingales, respectively [*cf. Meyer* (1976)].

PROOF. For basis vector $e_i = (\delta_{ik})_{k=1}^d$ ($\delta_{ik} = 0$ if $i \neq k$, $=1$ if $i = k$) and $i, j = 1, \dots, d$,

$$\begin{aligned} |a^{ij}(x)^\varepsilon| &= |\langle a(x)^\varepsilon(e_i + e_j), (e_i + e_j)\rangle/2 \\ &\quad -\big(\langle a(x)^\varepsilon e_i, e_i\rangle + \langle a(x)^\varepsilon e_j, e_j\rangle\big)/2| \\ &\leq 2 \sup_{|z|=1} \langle a(x)^\varepsilon z, z\rangle. \end{aligned} \tag{2.64}$$

For $x \in \overline{D}$ and $|z| = 1$,

$$\begin{aligned} &\langle a(x)^\varepsilon z, z\rangle + \int_{\beta \neq o} |\beta|^2 \nu_x^\varepsilon(d\beta)/2 \\ &\leq \varepsilon^2 \bigg(\sup\bigg\{ \langle a(x)^\varepsilon z/\varepsilon, z/\varepsilon\rangle \\ &\qquad + d\int_{\beta \neq o} [\exp(\langle z/\varepsilon, \beta\rangle) - 1 - \langle z/\varepsilon, \beta\rangle] \nu_x^\varepsilon(d\beta) \\ &\qquad + d\int_{\beta \neq o} [\exp(\langle -z/\varepsilon, \beta\rangle) - 1 - \langle -z/\varepsilon, \beta\rangle] \nu_x^\varepsilon(d\beta);\ |z| = 1 \bigg\}\bigg) \\ &\leq 2d\big(\sup\{\varepsilon H(x, z/\varepsilon)^\varepsilon - \langle b(x)^\varepsilon, z\rangle;\ |z| = 1,\ x \in \overline{D}\}\big)\varepsilon \\ &\leq 4d\big(\sup\{\varepsilon H(x, z/\varepsilon)^\varepsilon;\ |z| = 1,\ x \in \overline{D}\}\big)\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \end{aligned} \tag{2.65}$$

from (C), since

$$\varepsilon H(x; -z/\varepsilon)^\varepsilon \geq \langle b(x)^\varepsilon, -z\rangle. \tag{2.66}$$

If $b(x)^\varepsilon \neq o$, then for $x \in \overline{D}$,

$$\begin{aligned} |b(x)^\varepsilon| &= \langle b(x)^\varepsilon, b(x)^\varepsilon/|b(x)^\varepsilon|\rangle \\ &\leq \varepsilon H\big(x; b(x)^\varepsilon/(|b(x)^\varepsilon|\varepsilon)\big)^\varepsilon \\ &\leq \sup\{\varepsilon H(y; z/\varepsilon)^\varepsilon;\ y \in \overline{D},\ |z| = 1\} \end{aligned} \tag{2.67}$$

from (2.66), which proves (2.62) by (C). (2.63) can be proved from (2.61) and (2.62) [cf. Meyer (1976)]. □

LEMMA 2.8. *Suppose that* (C) *holds. Change* $V(\cdot)$ *outside* Ω *so that* V *is uniformly Lipschitz continuous in* $\mathscr{R}^d$. *Put*

$$V_\infty \equiv \operatorname{ess\,sup}\{|\partial V(x)/\partial x_i|;\ x \in \mathscr{R}^d,\ i = 1, \dots, d\}, \tag{2.68}$$

$$\begin{aligned} G(\varepsilon) \equiv \sup\bigg\{ \varepsilon \int_{\beta \neq o} &[\exp(r[V^\alpha(x+\beta) - V^\alpha(x)]/\varepsilon) - 1 \\ &- r[V^\alpha(x+\beta) - V^\alpha(x)]/\varepsilon] \nu_x^\varepsilon(d\beta)/r^2 \bigg\}, \end{aligned} \tag{2.69}$$

where the supremum is taken over all $x \in D$, $0 < r < 1$ and $\alpha > 0$. Then

$$\limsup_{\varepsilon \to 0} G(\varepsilon) \le 2^{d+1} \sup\{H(x;z);\, x \in D,\, |z| \le d^{1/2} V_\infty\}. \tag{2.70}$$

PROOF. For $x \in D$, $0 < r < 1$ and $\alpha > 0$,

$$\begin{aligned}
&\varepsilon \int_{\beta \ne o} [\exp(r[V^\alpha(x+\beta) - V^\alpha(x)]/\varepsilon) - 1 \\
&\qquad\qquad - r[V^\alpha(x+\beta) - V^\alpha(x)]/\varepsilon]\nu_x^\varepsilon(d\beta) \\
&\le \varepsilon \int_{\beta \ne o} \left[\exp\left(rV_\infty \sum_{i=1}^d |\beta_i|/\varepsilon\right) - 1 - rV_\infty \sum_{i=1}^d |\beta_i|/\varepsilon\right] \nu_x^\varepsilon(d\beta) \\
&\le r^2 \varepsilon \int_{\beta \ne o} \left[\exp\left(V_\infty \sum_{i=1}^d |\beta_i|/\varepsilon\right) - 1 - V_\infty \sum_{i=1}^d |\beta_i|/\varepsilon\right] \nu_x^\varepsilon(d\beta) \\
&\le r^2 \sum_{z \in \{-V_\infty, V_\infty\}^d} \varepsilon \int_{\beta \ne o} [\exp\langle z/\varepsilon, \beta\rangle - 1 - \langle z/\varepsilon, \beta\rangle] \nu_x^\varepsilon(d\beta)
\end{aligned} \tag{2.71}$$

from (1.29) and (1.30).

For $z \in \{-V_\infty, V_\infty\}^d$ and $x \in D$,

$$\begin{aligned}
&\varepsilon \int_{\beta \ne o} [\exp\langle z/\varepsilon, \beta\rangle - 1 - \langle z/\varepsilon, \beta\rangle] \nu_x^\varepsilon(d\beta) \\
&\qquad \le \varepsilon H(x; z/\varepsilon)^\varepsilon - \langle b(x)^\varepsilon, z\rangle \\
&\qquad \le \varepsilon H(x; z/\varepsilon)^\varepsilon + \varepsilon H(x; -z/\varepsilon)^\varepsilon,
\end{aligned} \tag{2.72}$$

which completes the proof from (C). □

Lemma 2.9 can be proved in the same way as in Freidlin and Wentzell [(1984), page 110, Lemma 2.2].

LEMMA 2.9. *Suppose that* (H.0)–(H.2) *hold. Then there exist a_1 and $T_1 > 0$ such that*

$$S_{0T}(\varphi) > a_1(T - T_1), \tag{2.73}$$

for all $\varphi(\cdot)$ for which $\varphi(t) \in \overline{D}$ for all $0 \le t \le T$.

3. Proofs of theorems. In this section we prove our results.

PROOF OF THEOREM 1.1. (O) Since $\mathscr{P}(\overline{D})$ [see below (H.3) for notation] is compact [cf. Stroock and Varadhan (1979)], we only have to prove that $\Phi_K(s)$ is closed for any $s > 0$ and compact subset K of D. Suppose that for some

$\mu \in \mathscr{P}(\overline{D})$

$$\lim_{n \to +\infty} \mu_{\varphi_n, T_n} = \mu, \quad S_{0T_n}(\varphi_n) \le s \tag{3.1}$$

[see (1.9) and (1.10)]. If $\limsup_{n \to +\infty} T_n = +\infty$, then $\mu = \delta_o$ from the proof of Lemma 2.2.

If $\limsup_{n \to +\infty} T_n < +\infty$, then $\mu = \mu_{\varphi, T}$ for some for φ which $\varphi(0) \in K$, $\varphi(T) \in \partial D$, $\varphi(t) \in \overline{D}$ for all $0 \le t \le T$ and $S_{0T}(\varphi) \le s$ (cf. proof of Lemma 2.2).

(I) We prove the following inequalities: For any $\mu \in \mathscr{P}(\overline{D})$,

$$-L_x(\mu) \le \lim_{\eta \to 0} \liminf_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta), \tag{3.2}$$

$$\lim_{\eta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta) \le -L_x(\mu) \tag{3.3}$$

[see (1.1) and (1.17)].

We divide the proof into four steps.

Key Step. Super large deviations. For any $\mu \in \mathscr{P}(\overline{D}) \setminus \{\delta_o\}$, there exists $\eta = \eta(\mu) > 0$ such that for any $x \in D$,

$$\limsup_{R \to +\infty} \limsup_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta,\ R \le \tau_D^\varepsilon) = -\infty \tag{3.4}$$

[see (1.1) and (1.2)].

Step 1. For any $x \in D$, $\mu \in \mathscr{P}(\overline{D})$,

$$\lim_{\eta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta) \le -L_x(\mu). \tag{3.5}$$

Step 2. For any $x \in D$, $\mu \in \mathscr{P}(\overline{D}) \setminus \{\delta_o\}$,

$$\lim_{\eta \to 0} \liminf_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta) \ge -L_x(\mu). \tag{3.6}$$

Step 3. For any $x \in D$, $\eta > 0$,

$$\lim_{\varepsilon \to 0} P_x(\rho(\mu^\varepsilon, \delta_o) < \eta) = 1. \tag{3.7}$$

First we prove Steps 1–3.

Proof of Step 1. For $\eta > 0$, put

$$A_\eta = A_\eta(\mu, x) \equiv \{\varphi;\ \varphi(0) = x,\ \rho(\mu_{\varphi, T(\varphi)}, \mu) < \eta\} \tag{3.8}$$

[see (1.10) and (1.25)]. Then

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta) \\ &\quad \le \max\Big(-\inf\{S_{0R}(\varphi);\ \varphi \in \overline{A}_\eta,\ T(\varphi) \le R\}, \\ &\qquad \limsup_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta,\ R \le \tau_D^\varepsilon)\Big) \end{aligned} \tag{3.9}$$

[see below (1.10) for notation] from Theorem 1.3(II) since

$$\begin{aligned} &P_x(\rho(\mu^\varepsilon, \mu) < \eta) \\ &\quad \le P_x(\rho(\mu^\varepsilon, \mu) < \eta,\, \tau_D^\varepsilon \le R) + P_x(\rho(\mu^\varepsilon, \mu) < \eta,\, R \le \tau_D^\varepsilon). \end{aligned} \tag{3.10}$$

Let $R \to +\infty$. Then we get, from Key Step,

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta) \\ &\quad \le -\liminf_{R \to +\infty} \{S_{0R}(\varphi);\, \varphi \in \bar{A}_\eta,\, T(\varphi) \le R\} \\ &\quad \le -S_{\eta, x}(\mu). \end{aligned} \tag{3.11}$$

Let $\eta \to 0$ in (3.11). Then from Lemma 2.2, the proof is complete. □

PROOF OF STEP 2. Take $\mu \in \mathscr{P}(\bar{D}) \setminus \{\delta_o\}$ for which $L_x(\mu) < +\infty$. Then from (1.17), $\mu \equiv \mu_{\varphi_o, T_o}$ for some $T_o > 0$ and $\{\varphi_o(t)\}_{0 \le t \le T_o}$ for which $\varphi_o(0) = x$ and $\varphi_o(T_o) \in \partial D$.

For $\eta > 0$, from Theorem 1.3(I)

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta) \\ &\quad \ge \liminf_{\varepsilon \to 0} \varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta,\, \tau_D^\varepsilon \le R) \\ &\quad \ge -\inf\{S_{0R}(\varphi);\, \varphi \in A_\eta^\circ \cap \{\phi;\, T(\phi) \le R\}^\circ\} \\ &\quad \ge -\inf\{S_{0R}(\varphi);\, \varphi(0) = x,\, T(\varphi) < R,\, \varphi(t) \text{ exit } \bar{D},\, \rho(\mu_{\varphi, T(\varphi)}, \mu) < \eta\} \\ &\quad = -\inf\{S_{0T(\varphi)}(\varphi);\, \varphi(0) = x,\, T(\varphi) < R,\, \rho(\mu_{\varphi, T(\varphi)}, \mu) < \eta\} \\ &\qquad\qquad \text{(from Lemma 2.5)} \\ &\quad \to -\tilde{S}_{\eta, x}(\mu) \quad \text{as } R \to +\infty \end{aligned} \tag{3.12}$$

[see below (1.10) for notation]. From Lemma 2.4, the proof is complete. □

PROOF OF STEP 3. For $\eta > 0$, there exist $n_o > 0$, and $\mu_i \in \mathscr{P}(\bar{D})$, $\ne \delta_o (i = 1, \dots, n_o)$ such that

$$P_x(\rho(\mu^\varepsilon, \delta_o) > \eta) \le \sum_{i=1}^{n_o} P_x(\rho(\mu^\varepsilon, \mu_i) < \eta(\mu_i)), \tag{3.13}$$

which tends to 0 as $\varepsilon \to 0$ from Key Step and (3.11), since $S_{\eta, x}(\mu) > 0$ for $\mu \ne \delta_o$ [cf. Freidlin and Wentzell (1984)] and $\mathscr{P}(\bar{D})$ is compact [cf. Stroock and Varadhan (1979)]. □

PROOF OF KEY STEP.

$$\begin{aligned} &P_x(\rho(\mu^\varepsilon, \mu) < \eta,\, R \le \tau_D^\varepsilon) \\ &\quad \le P_x(\tau_D^\varepsilon \ge E_x[\tau_D^\varepsilon] \exp(R/\varepsilon)) \\ &\qquad + P_x(\rho(\mu^\varepsilon, \mu) < \eta,\, R \le \tau_D^\varepsilon \le E_x[\tau_D^\varepsilon] \exp(R/\varepsilon)) \\ &\quad \le \exp(-R/\varepsilon) + P_x(\rho(\mu^\varepsilon, \mu) < \eta,\, R \le \tau_D^\varepsilon \le E_x[\tau_D^\varepsilon] \exp(R/\varepsilon)). \end{aligned} \tag{3.14}$$

Change $V(\cdot)$ outside Ω so that $V(\cdot)$ is bounded and uniformly Lipschitz continuous on $\mathscr{R}^d$. Put for $t \in [0, \tau_D^\varepsilon]$ and sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
Y^\varepsilon(t) = Y^{\varepsilon,r}(t) \equiv \int_0^t & \langle r\,\nabla^\alpha(X^\varepsilon(s)), b(X^\varepsilon(s))^\varepsilon\rangle\, ds \\
&+ \int_0^t \langle r\,\nabla V^\alpha(X^\varepsilon(s)), dM^{\varepsilon,[c]}(s)\rangle \\
&+ r\int_0^{t+}\int_{\beta\neq o} [V^\alpha(X^\varepsilon(s-)+\beta) \\
&\qquad - V^\alpha(X^\varepsilon(s-))]\,\tilde{N}^\varepsilon(ds\,d\beta),
\end{aligned} \tag{3.15}
$$

where $M^{\varepsilon,[c]}(t)$ is a continuous martingale part and $\int_0^{t+}\int_{\beta\neq o}\beta\tilde{N}^\varepsilon(ds\,d\beta) \equiv M^{\varepsilon,[d]}(t)$ is a purely discontinuous martingale part of $X^\varepsilon(t)$ [cf. Lemma 2.7 and Meyer (1976)]. Put for $t \in [0, \tau_D^\varepsilon]$ and sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
\Pi^\varepsilon(t) \equiv Y^\varepsilon(t) &- \int_0^t \langle r\,\nabla V^\alpha(X^\varepsilon(s)), b(X^\varepsilon(s))^\varepsilon\rangle\, ds \\
&- \int_0^t r^2\langle a(X^\varepsilon(s))^\varepsilon \nabla V^\alpha(X^\varepsilon(s)), \nabla V^\alpha(X^\varepsilon(s))\rangle\, ds/\varepsilon \\
&- \varepsilon\int_0^t\int_{\beta\neq o} [\exp(r[V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s))]/\varepsilon) - 1 \\
&\qquad - r[V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s))]/\varepsilon]\,\nu^\varepsilon_{X^\varepsilon(s)}(d\beta)\, ds,
\end{aligned} \tag{3.16}
$$

$$
C(\varepsilon) \equiv \min(\tau_D^\varepsilon, E_x[\tau_D^\varepsilon]\exp(R/\varepsilon)). \tag{3.17}
$$

Then $\exp(\Pi^\varepsilon(t)/\varepsilon)$ is an exponential martingale, for sufficiently small $\varepsilon > 0$, on $[0, \tau_D^\varepsilon]$ from Lemma 2.7 and (3.15) [cf. Meyer (1976)].

By the Itô formula,

$$
\begin{aligned}
-Y^\varepsilon(\tau_D^\varepsilon) &= -rV^\alpha(X^\varepsilon(\tau_D^\varepsilon)) + rV^\alpha(X^\varepsilon(0)) \\
&\quad + r\int_0^{\tau_D^\varepsilon} \sum_{i,j=1}^d a^{ij}(X^\varepsilon(s))^\varepsilon\, \partial^2 V^\alpha(X^\varepsilon(s))/\partial x_i\,\partial x_j\, ds \\
&\quad + r\int_0^{\tau_D^\varepsilon}\int_{\beta\neq o} [V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s)) \\
&\qquad - \langle\beta, \nabla V^\alpha(X^\varepsilon(s))\rangle]\,\nu^\varepsilon_{X^\varepsilon(s)}(d\beta)\, ds \\
&\le 2r\sup_{x\in\mathscr{R}^d} V^\alpha(x) + rF(\varepsilon)\tau_D^\varepsilon \sup_{x\in\mathscr{R}^d}\|D^2V^\alpha(x)\|
\end{aligned} \tag{3.18}
$$

from Lemma 2.7 [cf. below (1.10) for notation].

Take $\alpha > 0$ so that

$$
\int_{\overline{D}} \langle\nabla V^\alpha(x), b(x)\rangle\mu(dx) < 0. \tag{3.19}
$$

This is possible. In fact, take $r_1 > 0$ sufficiently small so that

$$\mu\big(\overline{D} \setminus \{x; |x| < r_1\}\big) > 0. \tag{3.20}$$

Take $r_2 > 0$ sufficiently small so that $r_1 > r_2$ and that

$$\begin{aligned} &\int_{|x|<r_2} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx) \\ &\quad \le \sup_{x\in\mathscr{R}^d} |\nabla V(x)| \cdot \sup_{|x|<r_2} |b(x)| \\ &\quad < -\limsup_{\alpha\to 0} \sup\{\langle \nabla V^\alpha(x), b(x)\rangle; |x| \ge r_1, x \in \overline{D}\} \\ &\qquad \times \mu\big(\overline{D} \setminus \{x; |x| < r_1\}\big)/2, \end{aligned} \tag{3.21}$$

which is possible from Lemma 2.6 and (1.11). Then from Lemma 2.6,

$$\begin{aligned} \int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx) &= \int_{\overline{D}\cap\{x;\, |x|<r_2\}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx) \\ &\quad + \int_{\overline{D}\cap\{x;\, r_2<|x|<r_1\}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx) \\ &\quad + \int_{\overline{D}\cap\{x;\, r_1<|x|\}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx) \\ &\le \limsup_{\alpha\to 0} \sup\{\langle \nabla V^\alpha(x), b(x)\rangle; |x| \ge r_1, x \in \overline{D}\} \\ &\quad \times \mu\big(\overline{D} \setminus \{x; |x| < r_1\}\big)/2 < 0, \end{aligned} \tag{3.22}$$

for sufficiently small $\alpha > 0$.

Take $\eta = \eta(\mu) > 0$ sufficiently small so that if $\rho(\mu^\varepsilon, \mu) < \eta$, then

$$\begin{aligned} &\int_0^{\tau_D^\varepsilon} \langle \nabla V^\alpha(X^\varepsilon(s)), b(X^\varepsilon(s))\rangle\, ds/\tau_D^\varepsilon \\ &\quad \le \int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/2. \end{aligned} \tag{3.23}$$

Take $r > 0$ sufficiently small so that

$$\begin{aligned} &rM \sup\{|\nabla V^\alpha(x)|^2; x \in \mathscr{R}^d\}/2 \\ &\qquad + r2^{d+2} \sup\{H(x;z); x \in D, |z| \le d^{1/2}V_\infty\} \\ &\quad \le -\int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/8 \end{aligned} \tag{3.24}$$

[cf. (2.68)], where we put

$$M = M(\alpha, r) \equiv \sup\{\|D_z^2 H(x;z)\|; |z| < r|\nabla V^\alpha(x)|, x \in \Omega\} \tag{3.25}$$

[see below (1.10) for notation].

Put

$$\tilde{H}^\varepsilon(C) = \sup\{|\varepsilon H(x;z/\varepsilon)^\varepsilon - H(x;z)|;\, x \in \overline{D},\, |z| < C\}. \tag{3.26}$$

Then for sufficiently small $\varepsilon > 0$, on the set $\{\rho(\mu^\varepsilon, \mu) < \eta\}$,

$$Y^\varepsilon(\tau_D^\varepsilon) - \Pi^\varepsilon(\tau_D^\varepsilon) \le r\tau_D^\varepsilon \int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/4, \tag{3.27}$$

since from (3.16),

$$\begin{aligned}
&Y^\varepsilon(\tau_D^\varepsilon) - \Pi^\varepsilon(\tau_D^\varepsilon)\\
&\quad= \int_0^{\tau_D^\varepsilon} \varepsilon H(X^\varepsilon(s); r\nabla V^\alpha(X^\varepsilon(s))/\varepsilon)^\varepsilon\, ds\\
&\qquad - \varepsilon \int_0^{\tau_D^\varepsilon}\int_{\beta\neq o} [\exp(r\langle \nabla V^\alpha(X^\varepsilon(s))/\varepsilon, \beta\rangle) - 1\\
&\qquad\qquad - r\langle \nabla V^\alpha(X^\varepsilon(s))/\varepsilon, \beta\rangle] \nu_{X^\varepsilon(s)}^\varepsilon(d\beta)\, ds\\
&\qquad + \varepsilon \int_0^{\tau_D^\varepsilon}\int_{\beta\neq o} [\exp(r[V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s))]/\varepsilon) - 1\\
&\qquad\qquad - r[V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s))]/\varepsilon] \nu_{X^\varepsilon(s)}^\varepsilon(d\beta)\, ds\\
&\quad= \int_0^{\tau_D^\varepsilon} \big[\varepsilon H(X^\varepsilon(s); r\nabla V^\alpha(X^\varepsilon(s))/\varepsilon)^\varepsilon - H(X^\varepsilon(s); r\nabla V^\alpha(X^\varepsilon(s)))\big]\, ds\\
&\qquad + \int_0^{\tau_D^\varepsilon} [H(X^\varepsilon(s); r\nabla V^\alpha(X^\varepsilon(s))) - \langle r\nabla V^\alpha(X^\varepsilon(s)), b(X^\varepsilon(s))\rangle]\, ds\\
&\qquad + r\int_0^{\tau_D^\varepsilon} \langle \nabla V^\alpha(X^\varepsilon(s)), b(X^\varepsilon(s))\rangle\, ds\\
&\qquad - \varepsilon \int_0^{\tau_D^\varepsilon}\int_{\beta\neq o} [\exp(r\langle \nabla V^\alpha(X^\varepsilon(s))/\varepsilon, \beta\rangle) - 1\\
&\qquad\qquad - r\langle \nabla V^\alpha(X^\varepsilon(s))/\varepsilon, \beta\rangle] \nu_{X^\varepsilon(s)}^\varepsilon(d\beta)\, ds\\
&\qquad + \varepsilon \int_0^{\tau_D^\varepsilon}\int_{\beta\neq o} [\exp(r[V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s))]/\varepsilon) - 1\\
&\qquad\qquad - r[V^\alpha(X^\varepsilon(s)+\beta) - V^\alpha(X^\varepsilon(s))]/\varepsilon] \nu_{X^\varepsilon(s)}^\varepsilon(d\beta)\, ds\\
&\quad\le r\tau_D^\varepsilon \Big\{ \tilde{H}^\varepsilon\Big(r \sup_{x\in\mathscr{R}^d} |\nabla V^\alpha(x)|\Big)/r + rM \sup_{x\in\mathscr{R}^d} |\nabla V^\alpha(x)|^2/2\\
&\qquad + \int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/2 + rG(\varepsilon)\Big\},
\end{aligned} \tag{3.28}$$

from (3.26),

$$
\begin{aligned}
&H(x;z) - \langle z, b(x)\rangle \\
(3.29)\qquad &= H(x;z) - H(x;o) - \langle z, \nabla_z H(x;o)\rangle \\
&= \langle z, D_z^2 H(x;\theta z)z\rangle/2 \quad \text{for some } 0 \le \theta \le 1
\end{aligned}
$$

[from (1.4), (1.5), (1.11) and (A.1)] and (3.25), (3.23), (1.29) and (2.69).

Therefore on the set $\{\rho(\mu^\varepsilon, \mu) < \eta,\ R \le \tau_D^\varepsilon\}$, for sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
-\Pi^\varepsilon(\tau_D^\varepsilon) &= -Y^\varepsilon(\tau_D^\varepsilon) + \{Y^\varepsilon(\tau_D^\varepsilon) - \Pi^\varepsilon(\tau_D^\varepsilon)\} \\
&\le 2r \sup_{x\in\mathscr{R}^d} V^\alpha(x) + r\tau_D^\varepsilon \Big\{ F(\varepsilon) \sup_{x\in\mathscr{R}^d} \|D^2 V^\alpha(x)\| \\
(3.30)\qquad &\qquad + \int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/4 \Big\} \\
&\le 2r \sup_{x\in\mathscr{R}^d} V^\alpha(x) + rR\int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/8
\end{aligned}
$$

from (3.18), (3.27), Lemma 2.7 and (3.19).

Hence we have

$$
\begin{aligned}
&\varepsilon \log P_x(\rho(\mu^\varepsilon, \mu) < \eta,\ R \le \tau_D^\varepsilon \le E_x[\tau_D^\varepsilon]\exp(R/\varepsilon)) \\
&\quad = \varepsilon \log E_x[\exp(\Pi^\varepsilon(\tau_D^\varepsilon)/\varepsilon)\exp(-\Pi^\varepsilon(\tau_D^\varepsilon)/\varepsilon); \\
&\qquad\qquad \rho(\mu^\varepsilon, \mu) < \eta,\ R \le \tau_D^\varepsilon \le E_x[\tau_D^\varepsilon]\exp(R/\varepsilon)] \\
(3.31)\qquad &\quad \le r\left(2 \sup_{x\in\mathscr{R}^d} V^\alpha(x) + R\int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/8\right) \\
&\qquad + \varepsilon \log E_x[\exp(\Pi^\varepsilon(C(\varepsilon))/\varepsilon)] \\
&\quad = r\left(2 \sup_{x\in\mathscr{R}^d} V^\alpha(x) + R\int_{\overline{D}} \langle \nabla V^\alpha(x), b(x)\rangle \mu(dx)/8\right) \to -\infty \\
&\qquad\qquad\qquad\qquad \text{as } R \to +\infty,
\end{aligned}
$$

uniformly with respect to sufficiently small $\varepsilon > 0$, since $\exp(\Pi^\varepsilon(t)/\varepsilon)$ is an exponential martingale, for sufficiently small $\varepsilon > 0$, on $[0, \tau_D^\varepsilon]$ [see below (3.17)]. □

REMARK 3.1. Step 3 can be proved in the same way as in the proof of the upper bound of Donsker and Varadhan (1975) [see also Galves, Olivieri and Vares (1987), Lemma 6].

PROOF OF THEOREM 1.2. From Lemma 2.9, we can prove, as in the proof of Theorem 1.1, that (2.2) holds for all $\mu \in \mathscr{P}(\overline{D})$ under (H.0)–(H.2), that (2.21) holds for all functions φ for which $\varphi(0) \in D$, $\varphi(T) \in \partial D$, $\varphi(t) \in \overline{D}$ for all $0 \le t \le T$ and $S_{0T}(\varphi) < +\infty$ under (H.0)–(H.3) and that (O) holds under (H.0)–(H.2). Moreover

$$\limsup_{R \to +\infty} \limsup_{\varepsilon \to 0} \varepsilon \log P_x(\tau_D^\varepsilon > R) = -\infty \tag{3.32}$$

[cf. Freidlin and Wentzell (1984), page 168, Lemma 1.9].

From this, (I) can be proved in the same way as in Steps 1 and 2 in the proof of Theorem 1.1. □

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