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EMBEDDING THE FINITE SAMPLING PROCESS AT A RATE[1]

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A huge body of if and only if theorems can be obtained based on certain strong embedding theorems for the partial sum process $\mathbb{S}_n$ and the uniform empirical and quantile processes $\mathbb{U}_n$ and $\mathbb{V}_n$. This embedding was accomplished in 1986 by M. Csörgő, S. Csörgő, L. Horváth and D. Mason. Their embedding is beautifully formulated so that many necessary and sufficient type results can be established using it. It is worthwhile to have an accessible proof. Indeed, these authors have since produced two papers that obtain the essential form of their result without appealing to earlier work of J. Komlós, P. Major and G. Tusnády. Indeed, this present paper does this for the finite sampling process $\mathbb{R}_n$ and the weighted empirical process $\mathbb{W}_n$. These latter results are entirely new and are the main objectives of the present paper. The applications of these latter results will appear elsewhere.

1. Statement of results. Let $(c_{n1}, \dots, c_{nn})$ denote a vector of known constants that satisfies the *uan condition*, that is,

$$(1.1)\quad \max_{1\le i\le n} \frac{|c_{ni}-\bar{c}_n|}{\sqrt{n}\,\sigma_{c,n}} = \max_{1\le i\le n} \frac{|c_{ni}-\bar{c}_n|}{\left[\sum_{i=1}^n (c_{ni}-\bar{c}_n)^2\right]^{1/2}} \to 0 \quad \text{as } n\to\infty,$$

where $\bar{c}_n \equiv \sum_{i=1}^n c_{ni}/n$ and $\sigma_{c,n}^2 \equiv \sum_{i=1}^n (c_{ni}-\bar{c}_n)^2/n$. We assume throughout that the c_{ni}'s have been normalized so that

$$(1.2)\quad \bar{c}_n = 0 \quad\text{and}\quad \sigma_{c,n}^2 = 1.$$

Let $(D_{n1}, \dots, D_{nn})$ denote a random permutation of $(1, \dots, n)$ in which each of the $n!$ permutations is equally likely. We will call

$$(1.3)\quad \mathbb{R}_n(t) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{[(n+1)t]} \frac{c_{nD_{ni}} - \bar{c}_n}{\sigma_{c,n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[(n+1)t]} c_{nD_{ni}} \quad \text{for } 0\le t\le 1$$

the *finite sampling process*. Let I denote the identity function, and let $\|f\|_a^b \equiv \sup\{|f(t)|: a\le t\le b\}$, with $\|f\| = \|f\|_0^1$.

THEOREM 1.1. *Let $\mathbb{W}$ be a Brownian bridge. Suppose $c_{n1}, \dots, c_{nn}$, $n \ge 1$, satisfy $\bar{c}_n = 0$, $\sigma_{c,n}^2 = 1$ and*

$$(1.4)\quad \sum_{i=1}^n \frac{c_{ni}^4}{n} \le (\textit{some } M) < \infty \quad \textit{for all } n.$$

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Then the random permutation $D_{n1}, \dots, D_{nn}$ *can be defined in such a way that*

$$(1.5)\qquad n^{\nu}\left\| \frac{\mathbb{R}_n - \mathbb{W}}{[I(1-I)]^{1/2-\nu}} \right\|_{1/(n+1)}^{n/(n+1)} = O_p(1) \quad \textit{for each } 0 \le \nu < \frac{1}{4}.$$

REMARK 1.1. We may replace (1.5) by

$$(1.6)\qquad \frac{n^{1/4}}{\log n}\|\mathbb{R}_n - \mathbb{W}\| = O_p(1).$$

Of course, it has long been known that

$$(1.7)\qquad \|\mathbb{R}_n - \mathbb{W}\| = o_p(1)$$

is possible.

Let $\xi_{n1}, \dots, \xi_{nn}$, $n \ge 1$, be a triangular array of row independent Uniform(0, 1) rv's having order statistics $0 \le \xi_{n:1} \le \cdots \le \xi_{n:n} \le 1$, ranks $R_{n1}, \dots, R_{nn}$ and antiranks $D_{n1}, \dots, D_{nn}$. Thus, $\xi_{nD_{ni}} = \xi_{n:i}$ and $\xi_{ni} = \xi_{n:R_{ni}}$. Let $\mathbb{G}_n$ denote the empirical df and let $\mathbb{G}_n^{-1}$ denote the left continuous version of its inverse. The *weighted uniform empirical process* is then

$$(1.8)\qquad \begin{aligned} \mathbb{W}_n(t) &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{c_{ni} - \bar{c}_n}{\sigma_{c,n}} \left[1_{[\xi_{ni} \le t]} - t\right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_{ni}\left[1_{[\xi_{ni} \le t]} - t\right] \quad \text{for } 0 \le t \le 1. \end{aligned}$$

The key relationship (since $\bar{c}_n = 0$) is

$$(1.9)\qquad \mathbb{W}_n(t) = \mathbb{R}_n(\mathbb{G}_n(t)) \quad \text{for } 0 \le t \le 1.$$

THEOREM 1.2. *Suppose* $\bar{c}_n = 0$, $\sigma_{c,n}^2 = 1$ *and* (1.4) *holds. Then the rv's* $\xi_{n1}, \dots, \xi_{nn}$ *can be defined in such a way that* (1.5) *still holds and*

$$(1.10)\qquad n^{\nu}\left\| \frac{\mathbb{W}_n - \mathbb{W}}{[I(1-I)]^{1/2-\nu}} \right\|_{1/n}^{(n-1)/n} = O_p(1) \quad \textit{for each } 0 \le \nu < \frac{1}{4}.$$

We may replace $1/n$ *and* $(n-1)/n$ *by* $\xi_{n:1}$ *and* $\xi_{n:n}$ *in* (1.10).

REMARK 1.2. We may replace (1.10) by

$$(1.11)\qquad \frac{n^{1/4}}{\log n}\|\mathbb{W}_n - \mathbb{W}\| = O_p(1).$$

Skorokhod (1965) showed that the partial sum process $\mathbb{S}_n$ of iid rv's with finite variance can be embedded in Brownian motion $\mathbb{S}$ on $[0, \infty)$, and from this it was possible to establish [as in Breiman (1968)] the uniform convergence of $\mathbb{S}_n$ to $\mathbb{S}$. Strassen (1967) established that the rate of convergence is almost $n^{-1/4}$ when the rv's have a finite fourth moment, and he generalized such results to martingales. Breiman [(1968), Chapter 13] used identities relating the empirical process $\mathbb{U}_n$ and the quantile process $\mathbb{V}_n$ of Uniform(0, 1) rv's to the partial sum process of Exponential(1) rv's in a fashion that implies uniform convergence of $\mathbb{U}_n$ and $-\mathbb{V}_n$ to the same Brownian bridge $\mathbb{U}$, while Rosenkrantz and O'Reilly (1972) established a rate of convergence of almost $n^{-1/4}$ (note our Remark 1.1). Komlós, Major and Tusnády (1975, 1976) (KMT) developed embeddings in which convergence takes place at strong, or even best possible, rates. In this KMT construction, the "limiting" Brownian bridge $\mathbb{B}_n$ is indexed by n, which is merely a matter of phrasing.

Csörgő, Csörgő, Horváth and Mason (1986) (CsCsHM) established a rate of convergence of $\mathbb{U}_n$ and $-\mathbb{V}_n$ to $\mathbb{B}_n$ when measured in certain weighted supremum metrics, and they used these to establish anew many previously well-known *best possible* limit theorems for empirical processes. Other best possible results based on this CsCsHM formulation soon followed. Csörgő, Haeussler and Mason (1988a, b) used the CsCsHM construction to establish again all the classical results on necessary and sufficient conditions for a df to belong to the domain of attraction of any stable rv; moreover, they were able to reframe the classical condition into a very natural and useful condition on the quantile function F^{-1}. Mason and Shorack (1988) were able to extend these results to give the first theorem in the literature on *necessary* and sufficient conditions for asymptotic normality, and even for stochastic compactness, of L-statistics. These examples show that the CsCsHM embedding is powerful, and perfectly *formulated* for certain kinds of results! The CsCsHM construction, like Breiman's work cited previously, used identities relating $\mathbb{U}_n$ and $\mathbb{V}_n$ to the partial sum process $\mathbb{S}_n$ of iid Exponential(1) rv's. Since an Exponential(1) rv has finite moments beyond the second, it was soon realized that simplified proofs of the CsCsHM construction could be given. Mason (1986) gives one, while Csörgő and Horváth (1986) give another; both rely on the result of Breiman (1967) or on the KMT construction. (Of course, there are theorems that require the added power of the KMT construction. The present paper can surely be strengthened in that direction.)

We now turn to the contents of the present paper. Our *main goal* is to establish Theorem 1.1 for the partial sum process $\mathbb{R}_n$ that arises from a finite sampling situation. A suitably modified version of $\mathbb{R}_n$ is a martingale that can be embedded into Brownian motion, and by this means Theorem 1.1 can be established for $\mathbb{R}_n$. This is an important new theorem. Our proof is technically simple in the sense that it uses nothing more sophisticated than the Hájek–Rényi inequality, the monotone inequality, the one-sided reflection principle for Brownian motion and Mill's ratio. We chose to present Theorem 1.1 in terms of a fixed Brownian bridge $\mathbb{U}$, rather than a sequence of Brownian bridges $\mathbb{B}_n$ as was typically done in earlier papers, as it is more convenient for our purposes. The work by Rosenkrantz and O'Reilly (1972) was very helpful in getting started toward the right form for the inequalities used. One would certainly suspect from KMT results that (1.6) could be improved to an almost sure result in which $n^{1/2}$ replaces $n^{1/4}$ provided the c_{ni}'s are restricted sufficiently beyond (1.4). However, the $\log n$ term in such a result would preclude the applications we have in mind. This is why we say the CsCsHM result is beautifully formulated. One also suspects that a proof not based on Skorokhod embedding would allow a strengthened version of (1.5) under a relaxation of (1.4).

2. Proof of Theorem 1.1. We begin by noting that

$$Z_{ni} \equiv \frac{R_{ni}}{1 - i/n} \equiv \frac{\mathbb{R}_n(i/(n+1))}{1 - i/n}, \qquad 0 \le i \le n-1, \tag{2.1}$$

is a martingale with respect to the σ-field $\mathscr{F}_{ni} \equiv \sigma[D_{n1}, \dots, D_{ni}]$. This is true since

$$Y_{ni} \equiv \Delta Z_{ni} \equiv Z_{ni} - Z_{n,i-1} = \left[\frac{n}{n-i} R_{ni} - \frac{n}{n-i}\frac{n-i}{n-i+1} R_{n,i-1}\right] \tag{2.2}$$

$$= \frac{n}{n-i+1}\left[\frac{n-i+1}{n-i}\frac{1}{\sqrt{n}} c_{nD_{ni}} + \frac{1}{n-i} R_{n,i-1}\right] \tag{2.3}$$

$$= \frac{1}{\sqrt{n}}\frac{n}{n-i}\left[c_{nD_{ni}} + \frac{1}{n-i+1}\sum_{j=1}^{i-1} c_{nD_{nj}}\right]$$

$$= \frac{1}{\sqrt{n}}\frac{n}{n-i}\left[c_{nD_{ni}} - \frac{1}{n-i+1}\sum_{j=i}^{n} c_{nD_{nj}}\right] \tag{2.4}$$

(since $\bar{c}_n = 0$) has [using (2.4)]

$$E(Y_{ni}|\mathscr{F}_{n,i-1}) = \frac{1}{\sqrt{n}}\frac{n}{n-i}\left[\frac{1}{n-i+1}\sum_{j=i}^{n} c_{nD_{nj}} - \frac{1}{n-i+1}\sum_{j=i}^{n} c_{nD_{nj}}\right] = 0. \tag{2.5}$$

It follows from standard finite sampling results [see Shorack and Wellner (1986), page 135, for example] applied to (2.3) that

$$\operatorname{Var}[Y_{ni}] = \frac{\sigma_{c,n}^2}{n}\left(\frac{n}{n-i+1}\right)^2\left\{\left(\frac{n-i+1}{n-i}\right)^2 - 2\frac{i-1}{n-1}\frac{n-i+1}{(n-i)^2} + \frac{1}{(n-i)^2}(i-1)\left[1 - \frac{(i-1)-1}{n-1}\right]\right\}$$

$$= \frac{\sigma_{c,n}^2}{n-1}\frac{n^2}{(n-i)(n-i+1)} = \frac{1}{n-1}\frac{n^2}{(n-i)(n-i+1)}, \tag{2.6}$$

since $\sigma^2_{c,n} = 1$. A trivial calculation based on (2.4) shows that

$$E\left(Y_{ni}^2 \mid \mathscr{F}_{n,i-1}\right) = \frac{1}{n}\left(\frac{n}{n-i}\right)^2 \frac{1}{n-i+1}\sum_{j=i}^{n}\left[c_{nD_{nj}} - \frac{1}{n-i+1}\sum_{j=i}^{n} c_{nD_{nj}}\right]^2 \tag{2.7}$$

$$\equiv \frac{1}{n-i+1}\sum_{j=i}^{n} g_{ni}(j)^2 \quad \text{with} \tag{2.8}$$

$$g_{ni}(j) \equiv \frac{1}{\sqrt{n}}\frac{n}{n-i}\left[c_{nD_{nj}} - \frac{1}{n-i+1}\sum_{j=i}^{n} c_{nD_{nj}}\right]$$

$$= \begin{array}{l}\text{(the variance of the normed numbers}\\ g_{ni}(j) \text{ that are still in the urn).}\end{array} \tag{2.9}$$

Now let $\mathbb{Z}$ denote a Brownian motion on $[0, \infty)$. We will use Skorokhod's embedding technique to embed the martingale Z_{ni} in $\mathbb{Z}$. [This is a corollary to Theorem 4.3 in Strasen (1967), but we wouldn't gain by an appeal to it.] Thus, $0 \le \tau_{n0} \le \tau_{n1} \le \cdots \le \tau_{n,n-1} \equiv \tau_{nn} \equiv \tau_{n,n+1}$ denotes a sequence of stopping times for which

(2.10) Z_{ni} and $Z^*_{ni} \equiv \mathbb{Z}(\tau_{ni}), 0 \le i \le n-1$, have the same joint distribution.

Let $Y^*_{ni} \equiv \Delta Z^*_{ni}$, $R^*_{ni} \equiv (1 - i/n)Z^*_{ni}$, $T_{ni} \equiv \Delta\tau_{ni}$ and $\mathscr{F}^0_{ni} \equiv \sigma[\mathbb{Z}(t)\colon 0 \le t \le \tau_{ni}]$. It follows that

$$E\left(T_{ni} \mid \mathscr{F}^0_{n,i-1}\right) \cong E\left(Y^2_{ni} \mid \mathscr{F}_{n,i-1}\right) = \frac{1}{n-i+1}\sum_{j=i}^{n} g^2_{ni}(j), \tag{2.11}$$

and Kendall, Stuart and Ord [1987, (12.107)] give

$$t_{ni} \equiv ET_{ni} = EY^2_{ni} = \frac{1}{n-1}\frac{n^2}{(n-i)(n-i+1)}. \tag{2.12}$$

Thus,

$$\begin{aligned} E\tau_{ni} &= \sum_{j=1}^{i} t_{nj} \\ &= \frac{1}{n-1}\sum_{j=1}^{i}\frac{n^2}{(n-j)(n-j+1)} \\ &= \frac{n}{n-1}\frac{i}{n}\Big/\left(1 - \frac{i}{n}\right) \doteq \frac{i}{n}\Big/\left(1 - \frac{i}{n}\right). \end{aligned} \tag{2.13}$$

Let us briefly review Skorokhod embedding, and how the preceding came about. Suppose $\tau_{n,i-1}$ has been defined. Consider the Brownian motion $\mathbb{Z}_{ni}(t) \equiv \mathbb{Z}(\tau_{n,i-1} + t) - \mathbb{Z}(\tau_{n,i-1})$ for $t \ge 0$. Let (A_{ni}, B_{ni}) be rv's independent of $\mathbb{Z}_{ni}$ having joint df H_{ni} specified by

$$dH_{ni}(a, b) = (b + a)\, dF_{ni}(-a)\, dF_{ni}(b)/\mu^+_{ni} \quad \text{for } a \ge 0,\ b > 0, \tag{2.14}$$

where F_{ni} is the df of the equally likely distribution on the $n - i + 1$ numbers $g_{ni}(j)$ with $i \le j \le n$ and where $\mu_{ni}^{+} \equiv \int_{(0,\infty)} x\, dF_{ni}(x)$. One chooses $A_{ni} \ge 0$ and $B_{ni} > 0$ at random according to the df H_{ni} and then runs the Brownian motion $\mathbb{Z}_{ni}$ until it hits one of the two levels $-A_{ni}$ or B_{ni}. The time required is denoted by T_{ni}, and we then let $Y_{ni}^{*} \equiv \mathbb{Z}_{ni}(T_{ni})$. This distribution is exactly the same as that of a rv G_{ni} that has the equally likely distribution on the random numbers $g_{ni}(j)$, $i \le j \le n$. Moreover,

$$E_i T_{ni} = E_i(G_{ni}^2) \quad \text{and} \quad \operatorname{Var}_i[T_{ni}] \le E_i T_{ni}^2 \le 64 E_i G_{ni}^4, \tag{2.15}$$

where $E_i(\ ) \equiv E(\ |\mathscr{F}_{n,i-1}^{0})$, as in (2.5) of Rosenkrantz and O'Reilly (1972).

We now note that

$$\sum_{j=1}^{i} \left[T_{nj} - E\left(T_{nj} | \mathscr{F}_{n,j-1}^{0}\right)\right] \equiv \sum_{j=1}^{i} \left[T_{nj} - E_j(T_{nj})\right], \qquad 1 \le i \le n-1, \tag{2.16}$$

is a martingale. Moreover,

$$W_{ni}^2 \equiv \operatorname{Var}\left[T_{ni} | \mathscr{F}_{n,i-1}^{0}\right] = E_i\left[T_{ni} - E_i(T_{ni})\right]^2 \le 64 E_i G_{ni}^4, \tag{2.17}$$

by (2.15). We thus claim that

$$\operatorname{Var}[\tau_{ni}] = E\left\{\sum_{j=1}^{i} \left[T_{nj} - E_j(T_{nj}) + E_j(T_{nj}) - t_{nj}\right]\right\}^2$$

$$\le 2E\left\{\sum_{j=1}^{i} \left[T_{nj} - E_j(T_{nj})\right]\right\}^2 + 2E\left\{\sum_{j=1}^{i} \left[E_j(T_{nj}) - t_{nj}\right]\right\}^2 \tag{2.18}$$

$$\begin{aligned} &= 2\sum_{j=1}^{i} E\left[T_{nj} - E_j(T_{nj})\right]^2 \\ &\quad + 2\sum_{j=1}^{i} \sum_{k=1}^{i} E\left\{\left[E_j(T_{nj}) - t_{nj}\right]\left[E_k(T_{nk}) - t_{nk}\right]\right\} \end{aligned} \tag{2.19}$$

$$\le 2\sum_{j=1}^{i} EW_{nj}^2 + 2\sum_{j=1}^{i} \sum_{k=1}^{i} \left(E\left[E_j(T_{nj}) - t_{nj}\right]^2 E\left[E_k(T_{nk}) - t_{nk}\right]^2\right)^{1/2} \tag{2.20}$$

$$\le 128 \sum_{j=1}^{i} EG_{nj}^4 + 2\left\{\sum_{j=1}^{i} \left(E\left[E_j(T_{nj}) - t_{nj}\right]^2\right)^{1/2}\right\}^2 \tag{2.21}$$

$$= 128 \sum_{j=1}^{i} EG_{nj}^4 + 2\left\{\sum_{j=1}^{i} \left(E\left[E_j(G_{nj}^2) - t_{nj}\right]^2\right)^{1/2}\right\}^2, \tag{2.22}$$

using the c_r-inequality for (2.18), using the martingale structure of (2.16) for (2.19), averaging conditionally as in (2.17) for the first term of (2.20) and using the Cauchy–Schwarz inequality for the second term of (2.20), using (2.17) for

(2.21) and using (2.15) for (2.22). We consider the two terms in (2.22) separately. Consider $E[E_i(G_{ni}^2) - t_{ni}]^2$. From (2.9) we see that $E_i(G_{ni}^2)$ is just the sample variance in a finite sampling situation in which $n - i + 1$ are sampled from a population consisting of the n numbers $n^{-1/2}(n/(n-i))c_{nk}$, $1 \le k \le n$, given in (2.8). Thus Kendall, Stuart and Ord [1987, (12.129)] show that

$$E\left[E_i(G_{ni}^2) - t_{ni}\right]^2 \le \frac{1}{n^2}\left(\frac{n}{n-i}\right)^4 \frac{12}{n-i+1}\left[\frac{1}{n}\sum_{j=1}^{n} c_{nj}^4\right]$$

$$\le \frac{12}{n^3}\left(\frac{n}{n-i}\right)^5 \overline{c_n^4}. \tag{2.23}$$

In like fashion, Kendall, Stuart and Ord [1987, (12.120)] give

$$EG_{ni}^4 = E\left(E_i\left(G_{ni}^4\right)\right) = E(\text{the sample fourth central moment})$$

$$\le \frac{32}{n^2}\left(\frac{n}{n-i}\right)^4 \overline{c_n^4}. \tag{2.24}$$

Thus, (2.17), and then (2.23) and (2.24), show that

$$\operatorname{Var}[T_{ni}] = E(\operatorname{Var}_i[T_{ni}]) + \operatorname{Var}[E_i(T_{ni})]$$

$$\le 64EG_{ni}^4 + E\left[E_i(G_{ni}^2) - t_{ni}\right]^2$$

$$\le \frac{2054}{n^2}\left(\frac{n}{n-i}\right)^4 \overline{c_n^4}. \tag{2.25}$$

Likewise, using (2.23) and (2.24) in (2.22) gives

$$\operatorname{Var}[\tau_{ni}] \le 4096\sum_{j=1}^{i}\frac{1}{n^2}\left(\frac{n}{n-j}\right)^4 \overline{c_n^4} + 2\left\{\sum_{j=1}^{i}\left[\frac{12}{n^3}\left(\frac{n}{n-j}\right)^5 \overline{c_n^4}\right]^{1/2}\right\}^2$$

$$\le \frac{4096}{n}\overline{c_n^4}\int_0^{i/n}\frac{2^4}{(1-x)^4}\,dx + \frac{24}{n}\overline{c_n^4}\left\{\int_0^{i/n}\frac{2^{5/2}}{(1-x)^{5/2}}\,dx\right\}^2$$

$$\le \frac{21846}{n}\overline{c_n^4}\frac{1}{(1-i/n)^3} + \frac{342}{n}\overline{c_n^4}\left\{\frac{1}{(1-i/n)^{3/2}}\right\}^2$$

$$\le \frac{22188}{n}\overline{c_n^4}\frac{1}{(1-i/n)^3} \tag{2.26}$$

$$\le 22188\left[\frac{\max c_{ni}^2}{n}\right]\left[\frac{1}{n}\sum_{i=1}^{n} c_{ni}^2\right]\frac{1}{(1-i/n)^3}$$

$$\to 0 \quad \text{provided } \frac{i}{n} \le 1 - \varepsilon \text{ for some } \varepsilon > 0, \tag{2.27}$$

using (1.1) in the final step.

DIGRESSION TO PROVE (1.7). Combining (2.13) and (2.27) to give a weak law of large numbers (WLLN), we have

$$\tau_{n,[ns]} \to_p s/(1-s) \quad \text{as } n \to \infty \text{ for each } 0 \le s < 1. \tag{2.28}$$

Using the diagonalization technique we can extract from any subsequence a further subsequence n' on which

$$\tau_{n',[n's]} \to s/(1-s) \quad \text{a.s. as } n' \to \infty \text{ for each } 0 \le s < 1. \tag{2.29}$$

Since both sides of (2.29) are monotone, and since the limit on the right is continuous, we have from (2.29) that

$$\tau_{n',[n's]} \to s/(1-s) \quad \text{uniformly on each } [0, 1-\delta] \text{ as } n' \to \infty. \tag{2.30}$$

Since the sample paths of $\mathbb{Z}$ are uniformly continuous on any $[0, M]$, we thus have

$$\max_{1 \le i \le n(1-\delta)} \left| \mathbb{Z}(\tau_{ni}) - \mathbb{Z}\left(\frac{i/n}{1 - i/n} \right) \right| \to_p 0 \tag{2.31}$$

$$\text{as } n \to \infty \text{ for each } 0 < \delta < 1.$$

We next note that

$$\mathbb{W}(s) \equiv (1-s)\mathbb{Z}(s/(1-s)), \qquad 0 \le s \le 1, \tag{2.32}$$

is a Brownian bridge. Moreover, (2.31) is just a statement that

$$\max_{1 \le i \le n(1-\delta)} |R^*_{ni} - \mathbb{W}(i/n)| \to_p 0 \quad \text{as } n \to \infty \text{ for each } 0 < \delta < 1. \tag{2.33}$$

Now,

$$\begin{aligned} &P\Big(\max_{n(1-\delta) \le i \le n} |R^*_{ni}| \ge 2\varepsilon \Big) \\ &\quad = P\Big(\max_{1 \le i \le n\delta} |R^*_{ni}| \ge 2\varepsilon \Big) \\ &\quad \le P(\|\mathbb{W}\|_0^\delta \ge \varepsilon) + P\Big(\max_{1 \le i \le n\delta} |R^*_{ni} - \mathbb{W}(i/n)| \ge \varepsilon \Big) \\ &\quad \to 0 \quad \text{as } \delta \to 0. \end{aligned} \tag{2.34}$$

Combining (2.33) and (2.34) gives

$$\max_{1 \le i \le n} |R^*_{ni} - \mathbb{W}(i/n)| \to_p 0 \quad \text{as } n \to \infty. \tag{2.35}$$

It is an elementary application of Mills' ratio to "fill in the gaps" in (2.35) by showing that both

$$\max_{1 \le i \le n} |\mathbb{W}(i/(n+1)) - \mathbb{W}(i/n)| \to_p 0 \quad \text{and} \quad \|\mathbb{W} - \overline{\mathbb{W}}_n\| \to_p 0, \tag{2.36}$$

where $\overline{\mathbb{W}}_n(t) \equiv \mathbb{W}([(n+1)t]/(n+1))$. These combine with (2.35) to give

$$\|\mathbb{R}^*_n - \mathbb{W}\| \to_p 0 \quad \text{as } n \to \infty, \tag{2.37}$$

where $\mathbb{R}^*_n \equiv R^*_{n,[(n+1)t]}$ for $0 \le t \le 1$. This is (1.7). □

We now return to the proof of Theorem 1.1.

Now, for $m \equiv [(1-\theta)n]$ the Shorack–Smythe monotone inequality [see Shorack and Wellner (1986), page 844] gives (with $\operatorname{Log} k \equiv 1 \vee \log k$)

$$A_n^c \equiv \left[\max_{1 \le k \le m} \frac{|\Sigma_{i=1}^k (T_{ni} - t_{ni})|}{(\sqrt{k}\,\operatorname{Log} k)/n} \ge 4M_\varepsilon \right] \subset (A_{1n}^c \cup A_{2n}^c), \tag{2.38}$$

where

$$A_{1n}^c \equiv \left[\max_{1 \le k \le m} \left| \sum_{i=1}^k \frac{n[T_{ni} - E_i(T_{ni})]}{\sqrt{i}\,\operatorname{Log} i} \right| \ge M_\varepsilon \right]$$

and

$$A_{2n}^c \equiv \left[\max_{1 \le k \le m} \left| \sum_{i=1}^k \frac{n[E_i(T_{ni}) - t_{ni}]}{\sqrt{i}\,\operatorname{Log} i} \right| \ge M_\varepsilon \right].$$

Applying the Hájek–Rényi inequality [see Shorack and Wellner (1986), page 873] to the martingale of (2.16) and then, using (2.15) and (2.24), we have

$$\begin{aligned} P(A_{1n}^c) &\le \frac{1}{M_\varepsilon^2} \sum_{i=1}^m \frac{n^2 E[T_{ni} - E_i(T_{ni})]^2}{i(\operatorname{Log} i)^2} \\ &\le \frac{64}{M_\varepsilon^2} \sum_{i=1}^m \frac{n^2}{i(\operatorname{Log} i)^2} E G_{ni}^4 \\ &= \frac{2048}{M_\varepsilon^2} \sum_{i=1}^m \frac{n^2}{i(\operatorname{Log} i)^2} \frac{1}{n^2} \left(\frac{n}{n-i} \right)^4 \overline{c_n^4} \\ &\le \frac{2048}{\theta^4 M_\varepsilon^2} \left(\sum_{i=1}^\infty \frac{1}{i(\operatorname{Log} i)^2} \right) \overline{c_n^4} \\ &< \varepsilon, \end{aligned} \tag{2.39}$$

for M_ε large enough, provided (1.4) holds. [*Note:* It is only in (2.39) that (1.4) is invoked anywhere in this proof.] We now turn to A_{2n}^c. Now, for $k \le m \equiv [(1-\theta)n]$, we have

$$\begin{aligned} &\sum_{i=1}^k \frac{n[E_i(T_{ni}) - t_{ni}]}{\sqrt{i}\,\operatorname{Log} i} \\ &\quad = \sum_{i=1}^k \frac{n[E_i(G_{ni}^2) - t_{ni}]}{\sqrt{i}\,\operatorname{Log} i} \\ &\quad = \frac{n}{n} \sum_{i=1}^k \frac{n\left[\frac{1}{n-i+1} \sum_{j=i}^n g_{ni}^2(j) - t_{ni} \right]}{\sqrt{i}\,\operatorname{Log} i} \end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{k} n^2 \Biggl\{ \Bigl(\frac{n}{n-i}\Bigr)^2 \frac{n}{n-i+1} \times \frac{1}{n}\sum_{j=i}^{n}\Biggl[\frac{c_{nD_{nj}}}{\sqrt{n}} - \frac{1}{n-i+1}\sum_{j=i}^{n}\frac{c_{nD_{nj}}}{\sqrt{n}}\Biggr]^2 - t_{ni}\Biggr\}\Bigg/ (\sqrt{i}\operatorname{Log} i)$$

$$= \frac{1}{n}\sum_{i=1}^{k} n^2 \Biggl\{ \Bigl(\frac{n}{n-i}\Bigr)^2 \frac{n}{n-i+1} \times \frac{1}{n}\Biggl[\sum_{j=i}^{n}\frac{c^2_{nD_{nj}}}{n} - \frac{1}{n-i+1}\Biggl(\sum_{j=i}^{n}\frac{c_{nD_{nj}}}{\sqrt{n}}\Biggr)^2\Biggr] - t_{ni}\Biggr\}\Bigg/ (\sqrt{i}\operatorname{Log} i)$$

$$= \frac{1}{n}\sum_{i=1}^{k} n^2 \Biggl\{ \Bigl(\frac{n}{n-i}\Bigr)^2 \frac{n}{n-i+1} \times \frac{1}{n}\Biggl[1 - \sum_{j=1}^{i-1}\frac{c^2_{nD_{nj}}}{n} - \frac{1}{n-i+1}\mathbb{R}_n^2\Bigl(\frac{i-1}{n+1}\Bigr)\Biggr] - t_{ni}\Biggr\}\Bigg/ (\sqrt{i}\operatorname{Log} i)$$

$$= -\frac{1}{n}\sum_{i=1}^{k}\Bigl(\frac{n}{n-i}\Bigr)^2\Bigl(\frac{n}{n-i+1}\Bigr)^2\mathbb{R}_n^2\Bigl(\frac{i-1}{n+1}\Bigr)\Bigg/ (\sqrt{i}\operatorname{Log} i)$$

$$+ \frac{1}{n}\sum_{i=1}^{k} n^2 \Biggl\{ \Bigl(\frac{n}{n-i}\Bigr)^2\Bigl(\frac{n}{n-i+1}\Bigr)\frac{1}{n} - \Bigl(\frac{n}{n-i}\Bigr)\Bigl(\frac{n}{n-i+1}\Bigr)\frac{1}{n-1} - \sum_{j=1}^{i-1}\frac{c^2_{nD_{nj}}}{n}\Bigl(\frac{n}{n-i}\Bigr)^2\Bigl(\frac{n}{n-i+1}\Bigr)\frac{1}{n}\Biggr\}\Bigg/ (\sqrt{i}\operatorname{Log} i)$$

$$= O_p(1) + \frac{1}{n}\sum_{i=1}^{k}\frac{n}{\sqrt{i}\operatorname{Log} i}\Bigl(\frac{n}{n-i}\Bigr)^2\Bigl(\frac{n}{n-i+1}\Bigr) \times \Biggl\{\Bigl[1 - \frac{n-i}{n-1}\Bigr] - \sum_{j=1}^{i-1}\frac{c^2_{nD_{nj}}}{n}\Biggr\} \tag{2.40}$$

$$= O_p(1) - \frac{1}{n}\sum_{i=1}^{k}\frac{n}{\sqrt{i}\operatorname{Log} i}\Bigl(\frac{n}{n-i}\Bigr)^2\Bigl(\frac{n}{n-i+1}\Bigr) \times \Biggl[\sum_{j=1}^{i-1}\frac{c^2_{nD_{nj}} - 1}{n} - \frac{(i-1)}{n(n-1)}\Biggr]$$

$$= O_p(1) - \frac{1}{n}\sum_{i=1}^{k}\frac{n}{\sqrt{i}\operatorname{Log} i}\Bigl(\frac{n}{n-i}\Bigr)^2\Bigl(\frac{n}{n-i+1}\Bigr) \times \sum_{j=1}^{i-1}\frac{c^2_{nD_{nj}} - 1}{n} + O(1)$$

$$= -\sum_{i=1}^{k}\left[\frac{1}{\sqrt{ni}\,\operatorname{Log} i}\left(\frac{n}{n-i}\right)^2\right]\left[\frac{\dfrac{1}{\sqrt{n}}\sum_{j=1}^{i-1} d_{nD_{nj}}}{1-\dfrac{i-1}{n}}\right] + O_p(1)$$

$$(2.41)\qquad \equiv -\sum_{i=1}^{k} b_{ni} Z_{n,i-1}^{(d)} + O_p(1),$$

where $d_{nj} \equiv c_{nj}^2 - 1$ and where we note (2.1) to understand the choice of the notation $Z_{ni}^{(d)}$ in (2.41). We can apply the Kolmogorov and Hájek–Rényi inequalities to conclude, since the $O_p(1)$ term in (2.41) is $O_p(1)$ uniformly in n, that for M_ε large enough we have, with $e_{ni} \equiv \sum_{j=i+1}^{m} b_{nj} \searrow$ and $Y_{ni}^{(d)} \equiv Z_{ni}^{(d)} - Z_{n,i-1}^{(d)}$, that

$$P(A_{2n}^c) \le P\left(\max_{1\le k\le m}\left|\sum_{i=1}^{k} b_{ni} Z_{n,i-1}^{(d)} + O_p(1)\right| \ge M_\varepsilon\right)$$

$$\le P\left(\max_{1\le k\le m}\left|\sum_{i=1}^{k} b_{ni} Z_{n,i-1}^{(d)}\right| \ge \frac{M_\varepsilon}{2}\right) + \varepsilon$$

$$= P\left(\max_{2\le k\le m}\left|\sum_{i=1}^{k-1} e_{ni} Y_{ni}^{(d)} - e_{nk}\sum_{i=1}^{k-1} Y_{ni}^{(d)}\right| \ge \frac{M_\varepsilon}{2}\right) + \varepsilon$$

$$\le P\left(\max_{1\le k\le m-1}\left|\sum_{i=1}^{k} e_{ni} Y_{ni}^{(d)}\right| \ge \frac{M_\varepsilon}{4}\right)$$

$$+ P\left(\max_{1\le k\le m-1} e_{n,k+1}\left|\sum_{i=1}^{k} Y_{ni}^{(d)}\right| \ge \frac{M_\varepsilon}{4}\right) + \varepsilon$$

$$\le \frac{16}{M_\varepsilon^2}\sum_{i=1}^{m-1} e_{ni}^2 \operatorname{Var}\left[Y_{ni}^{(d)}\right] + \frac{16}{M_\varepsilon^2}\sum_{i=1}^{m-1} e_{n,i+1}^2 \operatorname{Var}\left[Y_{ni}^{(d)}\right] + \varepsilon$$

$$\le \frac{32}{M_\epsilon^2} e_{n1}^2 \sum_{i=1}^{m-1} \operatorname{Var}\left[Y_{ni}^{(d)}\right] + \epsilon$$

$$\le \frac{32\sigma_{d,n}^2}{M_\varepsilon^2}\left[\sum_{j=2}^{m}\frac{1}{\sqrt{nj}\,\operatorname{Log} j}\left(\frac{n}{n-j}\right)^2\right]^2 \sum_{i=1}^{m-1}\frac{1}{n-1}\,\frac{n}{n-i}\,\frac{n}{n-i+1} + \varepsilon$$

$$(2.42)\qquad \le \frac{320\sigma_{d,n}^2}{\theta^6 M_\varepsilon^2}\left[\frac{1}{n}\sum_{j=2}^{m}\frac{1}{\sqrt{j/n}}\right]^2 + \varepsilon$$

$$\le \frac{1280\sigma_{d,n}^2}{\theta^6 M_\varepsilon^2} + \varepsilon$$

$$(2.43)\qquad < 2\varepsilon$$

for M_ε sufficiently large, by (1.1). [The technicalities of the steps leading to (2.42) were redone after D. Mason pointed out the need.] Thus

$$(2.44)\qquad P(A_n^c) < 3\varepsilon \quad \text{for all } n, \text{ provided } M_\varepsilon \text{ is large enough}$$

and (1.4) holds.

We thus have

$$P(B_n) \equiv P\left(\max_{1 \le k \le m} \frac{n^\nu |\mathbb{Z}(\Sigma_{i=1}^k T_{ni}) - Z(\Sigma_{i=1}^k t_{ni})|}{(k/n)^{1/2-\nu}} \ge M_\varepsilon \right)$$
$$\le P(B_n \cap A_n) + P(A_n^c)$$
$$\le \sum_{k=1}^m P\left(\left[\frac{|\mathbb{Z}(\Sigma_1^k T_{ni}) - \mathbb{Z}(\Sigma_1^k t_{ni})|}{[\sqrt{k}(\operatorname{Log} k) 4M_\varepsilon/n]^{1/2}} \ge M_\varepsilon \frac{k^{1/2-\nu}}{\sqrt{n}} \frac{1}{[\sqrt{k}(\operatorname{Log} k)4M_\varepsilon/n]^{1/2}} \right] \cap A_n \right) + 3\varepsilon$$
$$\le 12 \sum_{k=1}^m P\left(N(0,1) \ge \frac{1}{3} \frac{\sqrt{M_\varepsilon}}{2} \frac{k^{1/4-\nu}}{\sqrt{\operatorname{Log} k}} \right) + 3\varepsilon$$
$$\le 12 \sum_{k=1}^m \exp\left(-\frac{M_\varepsilon}{72} \frac{k^{1/2-2\nu}}{\operatorname{Log} k} \right) + 3\varepsilon$$
$$(2.45)\qquad < 4\varepsilon,$$

if M_ε is large enough and if $0 \le \nu < \frac{1}{4}$. Thus, (2.45) and (2.1) show that

$$(2.46)\qquad \max_{1 \le k \le m} \frac{n^\nu |\mathbb{R}_n^*(k/(n+1)) - (1-k/n)\mathbb{Z}(\Sigma_1^k t_{ni})|}{(k/n)^{1/2-\nu}} = O_p(1).$$

"Filling in the gaps" in (2.46) and replacing $\Sigma_{i=1}^k t_{ni}$ by $(k/n)/(1-k/n)$ is simple; it is essentially the same as showing (2.36). We thus have shown that

$$(2.47)\qquad n^\nu \left\| \frac{\mathbb{R}_n^* - (1-I)\mathbb{Z}(I/(1-I))}{I^{1/2-\nu}} \right\|_{1/(n+1)}^{1-\theta} = O_p(1),$$

for any $0 < \theta < 1$, where

$$(2.48)\qquad \mathbb{W}_0(t) \equiv (1-t)\mathbb{Z}(t/(1-t)) \quad \text{for } 0 \le t \le 1 \text{ is a Brownian bridge.}$$

We will now digress and verify that $\mathbb{R}_n^*$ satisfies (1.6). Note that establishing (1.6) carries with it the implication that $\theta = 0$ is permitted in (2.47).

DIGRESSION TO PROVE THAT $\mathbb{R}_N^*$ SATISFIES (1.6). Now

$$P(B_n) \equiv P\left(\max_{1\le k\le n-1} \frac{n^{1/4}}{\log n}\left|\mathbb{R}_n^*\left(\frac{k}{n+1}\right) - \left(1-\frac{k}{n}\right)\mathbb{Z}\left(\sum_{i=1}^k t_{ni}\right)\right| \ge 2M\right)$$

$$= P\left(\max_{1\le k\le n-1} \frac{|\mathbb{R}_n^*(k/(n+1))/(1-k/n) - \mathbb{Z}(\sum_{i=1}^k t_{ni})|}{1/(1-k/n)} \ge 2M\frac{\log n}{n^{1/4}}\right)$$

$$= P\left(\max_{1\le k\le n-1} \frac{|\mathbb{Z}(\sum_{i=1}^k T_{ni}) - \mathbb{Z}(\sum_{i=1}^k t_{ni})|}{1/(1-k/n)} \ge 2M\frac{\log n}{n^{1/4}}\right)$$

$$\le \sum_{k=1}^{n-1} P\left(\left[\left|\mathbb{Z}\left(\sum_{i=1}^k T_{ni}\right) - \mathbb{Z}\left(\sum_{i=1}^k t_{ni}\right)\right| \ge \frac{2M}{(1-k/n)}\frac{\log n}{n^{1/4}}\right] \cap A_n\right) \tag{2.49}$$

$$+ P(A_n^c),$$

where the Shorack–Smythe monotone inequality [see Shorack and Wellner (1986), page 844] gives

$$A_n^c \equiv \left[\max_{1\le k\le n-1}\left|\sum_{i=1}^k (T_{ni} - t_{ni})\right| \Big/ \left[\sqrt{\frac{1}{n}\frac{1}{(1-k/n)^3}}\ \mathrm{Log}(n-k)\right] \ge 4M_\varepsilon\right] \tag{2.50}$$

$$\subset (A_{1n}^c \cup A_{2n}^c),$$

where

$$A_{1n}^c \equiv \left[\max_{1\le k\le n-1}\left|\sum_{i=1}^k \{T_{ni} - E_i(T_{ni})\} \Big/ \left[\sqrt{\frac{1}{n}\frac{1}{(1-i/n)^3}}\ \mathrm{Log}(n-i)\right]\right| \ge M_\varepsilon\right]$$

and

$$A_{2n}^c \equiv \left[\max_{1\le k\le n-1}\left|\sum_{i=1}^k \{E_i(T_{ni}) - t_{ni}\} \Big/ \left[\sqrt{\frac{1}{n}\frac{1}{(1-i/n)^3}}\ \mathrm{Log}(n-i)\right]\right| \ge M_\varepsilon\right].$$

Repeating the argument of (2.38) to (2.44) with $(\sqrt{i}\ \mathrm{Log}\, i)/n$ replaced by $\{n^{-1}(1-i/n)^{-3}\}^{1/2}\,\mathrm{Log}(n-i)$ gives

$$P(A_n^c) < 3\varepsilon \quad \text{for all } n \text{ provided } M_\varepsilon \text{ is large enough.} \tag{2.51}$$

As in (2.45), we find from (2.49) that

$$P(B_n) \le \sum_{k=1}^{n-1} P\left(\left[\left\| \mathbb{Z}\left(\sum_{i=1}^{k} T_{ni} \right) - \mathbb{Z}\left(\sum_{i=1}^{k} t_{ni} \right) \right\| \Big/ \left[4M \sqrt{\frac{\operatorname{Log}^2(n-k)}{n(1-k/n)^3}} \right]^{1/2} \right.\right.$$

$$\left.\left. \ge \frac{2M}{(1-k/n)} \frac{\log n}{n^{1/4}} \Big/ \left[4M \sqrt{\frac{\operatorname{Log}^2(n-k)}{n(1-k/n)^3}} \right]^{1/2} \right] \cap A_n \right) + P(A_n^c)$$

$$\le 12 \sum_{k=1}^{n-1} P\left(N(0,1) \ge \frac{\sqrt{M}}{3} \frac{1}{(1-k/n)^{1/4}} \frac{\log n}{(\operatorname{Log}(n-k))^{1/2}} \right) + 3\varepsilon$$

$$\le 12 \sum_{k=1}^{n-1} \exp\left(-\frac{M}{18} \frac{1}{\sqrt{1-k/n}} \frac{(\log n)^2}{\operatorname{Log}(n-k)} \right) + 3\varepsilon$$

$$< 4\varepsilon \tag{2.52}$$

if $M = M_\varepsilon$ is large enough. We leave "filling in the gaps" à la (2.36) to the reader. Thus $\mathbb{R}_n^*$ satisfies (1.6). □

As indicated earlier, this implies that (2.47) can be improved to

$$n^\nu \|(\mathbb{R}_n^* - \mathbb{W}_0)/I^{1/2-\nu}\|_{1/(n+1)}^1 = O_p(1) \quad \text{for all } 0 \le \nu < \tfrac{1}{4} \tag{2.53}$$

for the Brownian bridge $\mathbb{W}_0$. But we must be more precise than this. Note from (2.39), (2.40) and Shorack and Wellner [(1986), (11) on page 134 with all $q_i = 1$], (2.42), (2.45), (2.51) and (2.52) that we in fact have

$$P\left(n^\nu \|(\mathbb{R}_n^* - \mathbb{W}_0)/I^{1/2-\nu}\|_{1/(n+1)}^1 \ge M_\varepsilon\right)$$

$$\le \varepsilon + (\text{Absolute constant})\overline{c_n^4}/M_\varepsilon^2 < 2\varepsilon \tag{2.54}$$

for M_ε large enough; and note that the first ε in (2.54) [see the series prior to (2.45) and (2.52)] is completely independent of the c_{ni}'s.

We now return to the proof of Theorem 1.1. We need only symmetrize (2.53).

Consider an urn containing the numbers $c_{n1}, \dots, c_{nn}$. Let $m_{0n} = [(n+1)/2]$ and $m_{1n} = n - m_{0n}$. Let $a_{n1}, \dots, a_{nm_{0n}}$ denote a random sample of size m_{0n} from $c_{n1}, \dots, c_{nn}$; and let $b_{n1}, \dots, b_{nm_{1n}}$ denote the c_{ni}'s not sampled. Let

$$\tilde{a}_{ni} \equiv (a_{ni} - \bar{a}_n)/\sigma_{a,n} \quad \text{and} \quad \tilde{b}_{ni} \equiv \left(b_{ni} - \bar{b}_n\right)/\sigma_{b,n} \tag{2.55}$$

[recall the notation used in (1.1)]. Now put the $\tilde{a}_{ni}$'s in Urn_0 and the $\tilde{b}_{ni}$'s in Urn_1. We first embed the $\tilde{a}_{ni}$'s in a Brownian bridge $\mathbb{B}_0$, as in (2.54); but note that this embedding is conditional on the values of the a_{ni}'s sampled from the c_{ni}'s. That is, if

$$\mathbb{R}_{0n}(t) \equiv \sum_{i=1}^{[(m_{0n}+1)t]} \frac{\tilde{a}_{ni}}{\sqrt{m_{0n}}} \quad \text{for } 0 \le t \le 1, \tag{2.56}$$

then by (2.54) we may suppose that $\mathbb{R}_{0n}$ is embedded in a Brownian bridge $\mathbb{B}_0$ in such a way that

$$P\Big(m_{0n}^{\nu}\|(\mathbb{R}_{0n}-\mathbb{B}_0)/I^{1/2-\nu}\|_{1/(m_{0n}+1)}^{1}\geq M_\varepsilon|a_{ni}\text{'s}\Big)$$

$$\leq \varepsilon + K\overline{\tilde{a}_n^4}/M_\varepsilon^2 < 2\varepsilon \tag{2.57}$$

for M_ε large. However, for M'_ε large enough we have

$$P\Big(\overline{\tilde{a}_n^4}\geq M'_\varepsilon\overline{c_n^4}\Big)\leq\varepsilon. \tag{2.58}$$

Thus, the conditional probability statement in (2.57) can be improved to

$$P\Big(n^{\nu}\|(\mathbb{R}_{0n}-\mathbb{B}_0)/I^{1/2-\nu}\|_{1/(n+1)}^{1}\geq M_\varepsilon\Big)\leq 2\varepsilon + KM'_\varepsilon\overline{c_n^4}/M_\varepsilon^2\leq 3\varepsilon \tag{2.59}$$

for M_ε large enough; it is understood in this that $\mathbb{R}_{0n}$ is the embedded finite sampling process of randomly chosen $a_{n1},\dots,a_{nm_{0n}}$ that have been centered and scaled to produce $\tilde{a}_{ni}$'s.

We similarly embed

$$\mathbb{R}_{1n}\equiv\sum_{i=1}^{[(m_{1n}+1)t]}\frac{\tilde{b}_{ni}}{\sqrt{m_{1n}}}\quad\text{for } 0\leq t\leq 1 \tag{2.60}$$

in an independent Brownian bridge $\mathbb{B}_1$; that is, subscript 1 may replace subscript 0 in (2.59). We may also suppose that

$$P\left(n^{\nu}\left|\frac{2\sum_{i=1}^{m_{0n}}a_{ni}}{\sqrt{n}}-Z\right|\geq M_\varepsilon\right)\leq\varepsilon+(\text{absolute constant})\overline{c_n^4}/M_\varepsilon^2 \tag{2.61}$$

for a $N(0,1)$ rv Z independent of $\mathbb{B}_0$ and $\mathbb{B}_1$. Now, for $0\leq t\leq\frac{1}{2}$, we have

$$\mathbb{R}_n(t)\equiv\sum_{i=1}^{[(n+1)t]}\frac{c_{ni}}{\sqrt{n}}=\sum_{i=1}^{[(n+1)t]}\frac{a_{ni}}{\sqrt{n}} \tag{2.62}$$

$$=\sqrt{\frac{m_{0n}}{n}}\left\{\sum_{i=1}^{[(n+1)t]}\frac{a_{ni}-\bar{a}_n}{\sqrt{m_{0n}}\,\sigma_{a,n}}\right\}\sigma_{a,n}+\frac{n}{m_{0n}}\frac{\sum_{i=1}^{m_{0n}}a_{ni}}{\sqrt{n}}\frac{[(n+1)t]}{n}$$

$$=\sqrt{\frac{m_{0n}}{n}}\left\{\mathbb{R}_{0n}\left(\frac{n+1}{m_{0n}}t\right)\right\}\sigma_{a,n}+\frac{n}{2m_{0n}}\frac{2\sum_{i=1}^{m_{0n}}a_{ni}}{\sqrt{n}}\frac{[(n+1)t]}{n} \tag{2.63}$$

$$=\frac{1}{\sqrt{2}}\{\mathbb{B}_0(2t)\}\cdot 1+1\cdot Zt+\mathbb{E}_{0n}(t)$$

$$=\frac{1}{\sqrt{2}}\mathbb{B}_0(2t)+Zt+\mathbb{E}_{0n}(t), \tag{2.64}$$

where (2.59) and (2.61) show that

$$P\Big(n^{\nu}\|\mathbb{E}_{0n}/I^{1/2-\nu}\|_{1/(n+1)}^{1/2} \geq M_{\varepsilon}\Big) \leq 3\varepsilon + (\text{absolute constant})\overline{c_n^4}/M_{\varepsilon}^2 \leq 4\varepsilon \tag{2.65}$$

for M_{ε} sufficiently large. Also, for $\frac{1}{2} < t \leq 1$, we have

$$\mathbb{R}_n(t) \equiv \sum_{i=1}^{[(n+1)t]} \frac{c_{ni}}{\sqrt{n}} = -\sum_{i=[(n+1)t]+1}^{n+1} \frac{c_{ni}}{\sqrt{n}} = -\sum_{i=1}^{[(n+1)(1-t)]} \frac{b_{ni}}{\sqrt{n}} \tag{2.66}$$

$$= \sqrt{\frac{m_{1n}}{n}}\left\{-\sum_{i=1}^{[(n+1)(1-t)]} \frac{b_{ni}-\bar{b}_n}{\sqrt{m_{1n}}\,\sigma_{b,n}}\right\}\sigma_{b,n} - \frac{n}{m_{1n}}\frac{\sum_{i=1}^{m_{1n}} b_{ni}}{\sqrt{n}}\frac{[(n+1)(1-t)]}{n}$$

$$= \sqrt{\frac{m_{1n}}{n}}\left\{-\mathbb{R}_{1n}\left(\frac{n+1}{m_{1n}}(1-t)\right)\right\}\sigma_{b,n} + \frac{n}{2m_{1n}}\frac{2\sum_{i=1}^{m_{0n}} a_{ni}}{\sqrt{n}}\frac{[(n+1)(1-t)]}{n} \tag{2.67}$$

$$= -\frac{1}{\sqrt{2}}\mathbb{B}_1(2(1-t)) + Z(1-t) + \mathbb{E}_{1n}(t), \tag{2.68}$$

where we likewise have E_{1n} satisfying (2.65). Now,

$$\mathbb{W}(t) \equiv \begin{cases} \dfrac{1}{\sqrt{2}}\mathbb{B}_0(2t) + tZ & \text{for } 0 \leq t \leq \dfrac{1}{2}, \\ -\dfrac{1}{\sqrt{2}}\mathbb{B}_1(2(1-t)) + (1-t)Z & \text{for } \dfrac{1}{2} \leq t \leq 1 \end{cases} \tag{2.69}$$

is easily shown to be a Brownian bridge; recall that the Brownian bridges $\mathbb{B}_0$ and $\mathbb{B}_1$ and the $N(0,1)$ rv Z are independent. If we also let

$$\mathbb{E}_n(t) = \begin{cases} \mathbb{E}_{0n}(t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \mathbb{E}_{1n}(t) & \text{for } \frac{1}{2} < t \leq 1 \end{cases} \tag{2.70}$$

incorporate the error terms of (2.64) and (2.68), then (2.65) shows that

$$P\Big(n^{\nu}\|\mathbb{E}_n/(I(1-I))^{1/2-\nu}\|_{1/(n+1)}^{n/(n+1)} \geq M_{\varepsilon}\Big) \leq 4\varepsilon + 2KM_{\varepsilon}'\overline{c_n^4}/M_{\varepsilon}^2 \leq 6\varepsilon \tag{2.71}$$

for M_{ε} sufficiently large. Since

$$\mathbb{E}_n = \mathbb{R}_n - \mathbb{W}, \tag{2.72}$$

this establishes (1.5).

Thus, all results for $\mathbb{R}_n$ have been proved. To prove (1.10) and (1.11) for $\mathbb{W}_n$, use the identity (1.9) and follow the short Csörgő and Horváth (1986) proof that the version of (1.10) with $\mathbb{W}_n - \mathbb{W}$ replaced by $\mathbb{W}(\mathbb{G}_n) - \mathbb{W}$ holds. You will also need to use the "in-probability linear bound on $\mathbb{G}_n$" of Shorack and Wellner [(1986), page 419] to replace $I(1 - I)$ by $\mathbb{G}_n(1 - \mathbb{G}_n)$. □

The referee has indicated that Theorem 1B of Komlós, Major and Tusnády (1976) gives "an almost complete proof of Theorem 1.1 in a special case."

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