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A UNIFORM CENTRAL LIMIT THEOREM FOR NONUNIFORM ϕ-MIXING RANDOM FIELDS[1]

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A sufficient condition is given for a sequence of partial-sum set-indexed processes with nonuniform ϕ-mixing condition to converge to Brownian motion. The main result (Theorem 1.1) is an extension of the similar results of Goldie and Greenwood by weakening the ϕ-mixing condition. An application (Corollary 4.2) to certain Gibbs fields is given.

1. Introduction. At first we introduce some notation. Let Z be the set of all integers, $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ be a random field on a d-dimensional integer lattice and Z_+ be the set of all positive integers. Let $\mathbf{t} + \mathbf{s} := (t_1 + s_1, \dots, t_d + s_d)$, for $\mathbf{t} = (t_1, \dots, t_d)$ and $\mathbf{s} = (s_1, \dots, s_d) \in Z^d$. For $m \in Z_+$, let

$$J_m := \{(l_1/m, \dots, l_d/m): l_j \in Z_+, l_j \le m, j = 1, \dots, d\},$$

$$G := \{(\mathbf{a}, \mathbf{b}]: \mathbf{a}, \mathbf{b} \in [0,1]^d\},$$

$$C_{m,\mathbf{j}} := \prod_{i=1}^{d} (j_i - m^{-1}, j_i], \qquad \mathbf{j} = (j_1, \dots, j_d) \in J_m,$$

where $\mathbf{a} = (a_1, \dots, a_d)$, $\mathbf{b} = (b_1, \dots, b_d)$, $a_i, b_i \in [0,1]$, $i = 1, \dots, d$, and $(\mathbf{a}, \mathbf{b}] = \prod_{i=1}^d (a_i, b_i]$. For A, $B \in G$, we define $d_L(A, B) := |A \vartriangle B|$, with $A \vartriangle B$ the symmetric difference and $|A|$ the Lebesgue measure of A. It can be proved that $d_L(\cdot, \cdot)$ is a metric on G and that G is a complete metric space. Now, from a random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ we form the partial-sum process of nth level as

$$Z_n(A) := n^{-(d/2)} \sum_{\mathbf{j} \in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}), \qquad A \in G, n \in Z_+,$$

where $n\mathbf{j} = (nj_1, \dots, nj_d)$. Let $C(G)$ be the space of all continuous functions on G with supremum norm $\|f\|_G = \sup_{A \in G} |f(A)|$, $f \in C(G)$. It is known that $C(G)$ is a complete countably compact space. Let $CA(G)$ be the set of everywhere additive elements in $C(G)$, namely, elements f such that $f(A \cup B) = f(A) + f(B) - f(A \cap B)$ whenever A, B, $A \cap B$, $A \cup B \in G$. It can be shown that for fixed ω, $Z_n(\cdot) \in CA(G)$. A Brownian motion on G with parameter σ is a random element $Z(\cdot)$ of $CA(G)$ whose finite-dimensional laws are Gaussian

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with $EZ(A) = 0$ and $EZ(A)Z(B) = \sigma^2|A \cap B|$. While $\sigma = 1$, we call $Z(\cdot)$ a standard Brownian motion on G.

Dudley [6] showed that if G satisfies a certain metric entropy condition, the Brownian motion on G is well defined. For every $\varepsilon > 0$ there is a finite set $N(G, \varepsilon) \subset G$, which we take to have minimal cardinality $e^{H(\varepsilon)}$, such that for every $A \in G$, there exist A^+, $A^- \in N(G, \varepsilon)$, with $A^- \subseteq A \subseteq A^+$ and $|A^+ \setminus A^-| \le \varepsilon$. The function H is called the metric entropy (with inclusion). Its exponent is

$$r = \limsup_{\varepsilon \searrow 0} \frac{\log H(\varepsilon)}{\log(1/\varepsilon)}.$$

In what follows, for our G with metric $d_L(\cdot, \cdot)$, its metric entropy exponent $r = 0$, and the Brownian motion on G is well-defined. A survey of metric entropy appears in the introduction of [1].

Now let us introduce the nonuniform ϕ-mixing condition. The condition was first introduced by Dobrushin and Nahapetian [5] and it is stronger than the Bolthausen mixing condition [3]. The nonuniform mixing condition is considered because of Dobrushin's assertion [4] that the uniform condition (cf. [8, 9]) is not satisfied even for some simple examples of Gibbs random fields.

DEFINITION 1.1. We say that the random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ satisfies the nonuniform ϕ-mixing condition if, for $\Lambda_i \subset Z^d$, $|\Lambda_i| < \infty$, $i = 1, 2$, there exists a nonnegative real function $\phi_{|\Lambda_1|}(\cdot)$ depending only on $|\Lambda_1|$, such that

$$\sup_{E \in \sigma(\Lambda_1),\, F \in \sigma(\Lambda_2),\, P(F) > 0} |P(E|F) - P(E)| \le \phi_{|\Lambda_1|}(d(\Lambda_1, \Lambda_2))$$

and $\phi_{|\Lambda_1|}(a) \to 0$ as $a \to \infty$, where $|\Lambda|$ is the cardinality of Λ, $\sigma(\Lambda)$ is the σ-algebra generated by $\xi_{\mathbf{t}}$, $\mathbf{t} \in \Lambda$, and $d(\Lambda_1, \Lambda_2) = \min\{\|\mathbf{t}_1 - \mathbf{t}_2\|: \mathbf{t}_i \in \Lambda_i, i = 1, 2\}$ with Euclidean norm $\|\cdot\|$.

Now we can state our main theorem.

THEOREM 1.1. *Suppose the strictly stationary random field* $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ *satisfies the nonuniform* ϕ*-mixing condition and*

(1.1.1) *there exists a nonnegative real function* $\phi(\cdot)$ *on* R^1*, such that for any* $\Lambda \subset Z^d$, $|\Lambda| < \infty$, $\phi_{|\Lambda|}(\cdot) \le |\Lambda|\phi(\cdot)$, *and for some* $\delta > 0$,

$$\limsup_{r \to \infty,\, r \in R^+} (\phi(r))^{1/2} r^{(3\delta+4)d/\delta} < \infty,$$

(1.1.2) $$E|\xi_{\mathbf{0}}|^{2+\delta} < \infty,$$

(1.1.3) $$0 < \sigma^2 := \sum_{\mathbf{t} \in Z^d} \mathrm{Cov}(\xi_{\mathbf{0}}, \xi_{\mathbf{t}}) < \infty,$$

where $\mathbf{0}$ *is the zero vector in* Z^d.

Then the partial-sum processes of the random field converge weakly to the Brownian motion with parameter σ, as $n \to \infty$. The weak convergence is in $CA(G)$.

Theorem 1.1 is an extension of the similar results of [8] and [9] by weakening the ϕ-mixing condition and omitting the β-mixing condition. A similar result for uniform ρ-mixing random fields with finite $2 + \delta$ moment was obtained by Goldie and Greenwood [8, 9] as well. To the author's knowledge it is an open problem whether the similar result is held under the Bolthausen's mixing condition in [3]. There is a survey paper [10] of the central limit problem for set-indexed partial-sum processes, which was written by Goldie and Morrow.

In Section 2 we shall prove the convergence of finite dimensional distributions by a result of [8] and an idea of Nahapetian [11]. In Section 3 we shall show tightness by modifying the method of Goldie and Greenwood [8, 9]. The method was initiated by Bass [2]. In Section 4 we shall give an application to certain Gibbs fields.

2. Convergence of finite dimensional distributions.

The main result of this section is the following theorem.

THEOREM 2.1. *Suppose $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ satisfies the conditions of Theorem 1.1. Then the finite dimensional distributions of the partial-sum processes of the random field converge to the corresponding finite dimensional distributions of Brownian motion with parameter σ.*

Without loss of generality we assume that $\sigma = 1$ in what follows.

PROPOSITION 2.2. *Under the conditions of Theorem 1.1 we have*

$$\sum_{\mathbf{k} \in Z^d \setminus \mathbf{0}} (\phi(\|\mathbf{k}\|))^{1/2} < \infty.$$

PROOF. Without loss of generality we assume that $\phi(\cdot)$ is nonincreasing. Then

$$\sum_{\mathbf{k} \in Z^d \setminus \mathbf{0}} (\phi(\|\mathbf{k}\|))^{1/2} = \sum_{r=1}^{\infty} \sum_{r \le \|\mathbf{k}\| < r+1} (\phi(\|\mathbf{k}\|))^{1/2}$$
$$\le \sum_{r=1}^{\infty} C r^{d-1} (\phi(r))^{1/2},$$

where C is a constant depending only on d. By (1.1.1), if r is large enough, there is a D such that

$$r^{d-1}(\phi(r))^{1/2} \le D r^{-1-q}, \qquad q = 2(d-1) + (4/\delta) > 0.$$

Then the proposition is obtained by $\sum_{r=1}^{\infty} r^{-(1+q)} < \infty$. $\square$

Now we give some more notation. For $\mathbf{n} = (n_1, \dots, n_d) \in Z_+^d$, denote by $\Lambda_\mathbf{n}$ the rectangle with center at the origin and with length of sides $2n_i$, $|n_i/n_j| < C$, $i, j = 1, \dots, d$, where C is a constant; $\mathbf{n} \to \infty$ means $\min\{n_i\colon i = 1, \dots, d\} \to \infty$.

PROPOSITION 2.3. *Suppose $\{\xi_\mathbf{t}, \mathbf{t} \in Z_+^d\}$ satisfies the conditions of Theorem 1.1. Then for any series $\Lambda_\mathbf{n}$, $\mathbf{n} \to \infty$, we have*

$$\lim_{\mathbf{n}\to\infty} (|\Lambda_\mathbf{n}|)^{-1} \operatorname{Var}\Big(\sum_{\mathbf{t}\in\Lambda_\mathbf{n}} \xi_\mathbf{t}\Big) = 1,$$

where $\operatorname{Var}(\cdot)$ *is the variance of the random variable.*

PROOF. Let $\Lambda'_\mathbf{n} := \{\mathbf{s} \in \Lambda_\mathbf{n}\colon |s_i| \le (1 - n_i^{-1/2})n_i,\ i = 1, \dots, d\}$. Obviously,

$$\lim_{\mathbf{n}\to\infty} |\Lambda'_\mathbf{n}|/|\Lambda_\mathbf{n}| = 1.$$

By the stationary condition,

$$\begin{aligned}
|\Lambda_\mathbf{n}|^{-1} \operatorname{Var}\Big(\sum_{\mathbf{t}\in\Lambda_\mathbf{n}} \xi_\mathbf{t}\Big) &= |\Lambda_\mathbf{n}|^{-1} \sum_{\mathbf{t}\in\Lambda_\mathbf{n}} \sum_{\mathbf{s}\in\Lambda_\mathbf{n}} \operatorname{Cov}(\xi_\mathbf{0}, \xi_{\mathbf{s}-\mathbf{t}}) \\
&= |\Lambda_\mathbf{n}|^{-1} \sum_{\mathbf{t}\in\Lambda'_\mathbf{n}} \sum_{\mathbf{s}\colon |s_i - t_i| \le n_i^{1/2},\, i=1,\dots,d,\, \mathbf{s}\in\Lambda_\mathbf{n}} \operatorname{Cov}(\xi_\mathbf{0}, \xi_{\mathbf{s}-\mathbf{t}}) \\
&\quad + |\Lambda_\mathbf{n}|^{-1} \sum_{\mathbf{t}\in\Lambda_\mathbf{n}\setminus\Lambda'_\mathbf{n}} \sum_{\mathbf{s}\in\Lambda_\mathbf{n}} \operatorname{Cov}(\xi_\mathbf{0}, \xi_{\mathbf{s}-\mathbf{t}}) \\
&\quad + |\Lambda_\mathbf{n}|^{-1} \sum_{\mathbf{t}\in\Lambda'_\mathbf{n}} \sum_{\mathbf{s}\colon |s_i - t_i| > n_i^{1/2},\, i=1,\dots,d,\, \mathbf{s}\in\Lambda_\mathbf{n}} \operatorname{Cov}(\xi_\mathbf{0}, \xi_{\mathbf{s}-\mathbf{t}}) \\
&:= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

Condition (1.1.3) implies

$$\mathrm{I}_1 = (|\Lambda'_\mathbf{n}|/|\Lambda_\mathbf{n}|) \sum_{\mathbf{k}\colon |k_i| \le n_i^{1/2},\, i=1,\dots,d} \operatorname{Cov}(\xi_\mathbf{0}, \xi_\mathbf{k}) \to 1,$$

$$\sum_{\mathbf{k}\colon |k_i| > n_i^{1/2},\, i=1,\dots,d} \operatorname{Cov}(\xi_\mathbf{0}, \xi_\mathbf{k}) \to 0,$$

$$\mathrm{I}_3 \le (|\Lambda'_\mathbf{n}|/|\Lambda_\mathbf{n}|) \sum_{\mathbf{s}\colon |s_i - t_i| > n_i^{1/2},\, i=1,\dots,d} \operatorname{Cov}(\xi_\mathbf{0}, \xi_{\mathbf{s}-\mathbf{t}}) \to 0,$$

as $\mathbf{n} \to \infty$. Using Lemma 1 in [11], Proposition 2.1 and condition (1.1.1),

$$|\operatorname{Cov}(\xi_\mathbf{0}, \xi_{\mathbf{s}-\mathbf{t}})| \le 2(\phi(\|\mathbf{s} - \mathbf{t}\|))^{1/2} \operatorname{Var}(\xi_\mathbf{0}), \qquad \mathbf{s}, \mathbf{t} \in Z^d,$$

$$\sum_{\mathbf{s}\in\Lambda_\mathbf{n}} (\phi(\|\mathbf{s} - \mathbf{t}\|))^{1/2} < \infty, \qquad \mathbf{s}, \mathbf{t} \in Z^d.$$

Then, as $\mathbf{n} \to \infty$,

$$\mathrm{I}_2 \le (|\Lambda_{\mathbf{n}} \setminus \Lambda'_{\mathbf{n}}|/|\Lambda_{\mathbf{n}}|) \sum_{\mathbf{s} \in Z^d \setminus \mathbf{0}} (\phi(\|s\|))^{1/2} \to 0.$$

Combining all of these, we complete the proof of the proposition. □

LEMMA 2.4. *Suppose $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ satisfies the conditions of Theorem* 1.1. *Then there exists a constant $C^{(1)}$ such that*

$$EZ_n^2(A)/|A| \le C^{(1)} \operatorname{Var}(\xi_0), \qquad A \in G.$$

PROOF. For $A \in G$,

$$\begin{aligned} Z_n(A) &= n^{-(d/2)} \sum_{\mathbf{j} \in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}) \\ &= n^{-(d/2)} \sum_{\mathbf{j} \in J_n} |nA \cap nC_{n,\mathbf{j}}| (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}), \end{aligned}$$

where $nA = \{n\mathbf{t}\colon \mathbf{t} \in A\}$. Noting that $\#\{\mathbf{j} \in J_n\colon A \cap C_{n,\mathbf{j}} \ne \Phi\}/|nA| \to 1$ as $n \to \infty$, by a method analogous to that used in the proof of Proposition 2.3, we get that there exists a constant $C^{(1)}$ such that, for large n,

$$EZ_n^2(A) \le n^{-d} C^{(1)} |nA| \operatorname{Var}(\xi_0) = C^{(1)} |A| \operatorname{Var}(\xi_{\mathbf{0}}). \qquad \square$$

LEMMA 2.5. *Assume $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ satisfies the conditions of Theorem* 1.1. *Then there exists a constant $C^{(2)}$ such that, for large n,*

$$E|Z_n(A)|^{2+\delta} \le C^{(2)} |A|^{(2+\delta)/2}, \qquad A \in G.$$

PROOF. Let

$$\begin{aligned} \Lambda'_n &:= \left\{\mathbf{j} \in J_n\colon 0 < |A \cap C_{n,\mathbf{j}}| < n^{-d}\right\}, \\ \Lambda_n &:= \left\{\mathbf{j} \in J_n\colon |A \cap C_{n,\mathbf{j}}| = n^{-d}\right\}. \end{aligned}$$

For any $A \in G$, Λ_n is a rectangle and

$$\begin{aligned} Z_n(A) &= n^{-(d/2)} \sum_{\mathbf{j} \in J_n} |nA \cap nC_{n,\mathbf{j}}| (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}) \\ &= n^{-(d/2)} \left[\sum_{\mathbf{j} \in \Lambda_n} (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}) + \sum_{\mathbf{j} \in \Lambda'_n} |nA \cap nC_{n,\mathbf{j}}| (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}) \right] \\ &:= \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

By Lemma 5 in [11], there exists a constant C_d (depending on d) such that

$$E\left| \sum_{\mathbf{j} \in \Lambda_n} (\xi_{n\mathbf{j}} - E\xi_{n\mathbf{j}}) \right|^{2+\delta} \le C_d |\Lambda_n|^{(2+\delta)/2}.$$

Noting that Λ'_n is a union of $(d-1)$-dimensional rectangles, we denote these rectangles by $\Lambda^{(i)}$, $i = 1, \dots, 2d$. Thus, $\Lambda'_n = \sum_{i=1}^{2d} \Lambda^{(i)}$, and for each $\Lambda^{(i)}$ we can apply Lemma 5 of [11], that is, there exists a constant C'_d such that

$$E\left|\sum_{\mathbf{j}\in\Lambda^{(i)}}(\xi_{n\mathbf{j}}-E\xi_{n\mathbf{j}})|nA\cap nC_{n,\mathbf{j}}|\right|^{2+\delta}\le C'_d|\Lambda^{(i)}|^{(2+\delta)/2},\qquad i=1,\dots,2d.$$

Thus,

$$Z_n(A) := \mathrm{I}_1+\sum_{i=1}^{2d}\mathrm{II}_i,$$

$$\begin{aligned}E|Z_n(A)|^{2+\delta}&\le E\left(\mathrm{I}_1+\sum_{i=1}^{2d}\mathrm{II}_i\right)^2\left|\mathrm{I}_1+\sum_{i=1}^{2d}\mathrm{II}_i\right|^{\delta}\\&\le C_{\delta,d}E\left(\mathrm{I}_1^2+\sum_{i=1}^{2d}\mathrm{II}_i^2\right)\left(|\mathrm{I}_1|^{\delta}+\sum_{i=1}^{2d}|\mathrm{II}_i|^{\delta}\right),\end{aligned}$$

where $C_{\delta,d}$ is a constant depending on δ and d. Therefore,

$$\begin{aligned}E|Z_n(A)|^{2+\delta}\le C_{\delta,d}\Bigg(&E|\mathrm{I}_1|^{2+\delta}+\sum_{i=1}^{2d}E|\mathrm{II}_i|^{2+\delta}+\sum_{i=1}^{2d}E\big(\mathrm{I}_1^2|\mathrm{II}_i|^{\delta}\big)\\&+\sum_{i\ne j,\,i,j=1}^{2d}E\big(\mathrm{II}_i^2|\mathrm{II}_j|^{\delta}\big)+\sum_{i=1}^{2d}E\big(|\mathrm{I}_1|^{\delta}\mathrm{II}_i^2\big)\Bigg).\end{aligned}$$

But

$$\begin{aligned}E|\mathrm{I}_1|^{2+\delta}&\le C_d\big(n^{-(d/2)}\big)^{2+\delta}|\Lambda_n|^{(2+\delta)/2},\\E|\mathrm{II}_i|^{2+\delta}&\le C'_d\big(n^{-(d/2)}\big)^{2+\delta}|\Lambda^{(i)}|^{(2+\delta)/2},\\E\big(\mathrm{I}_1^2|\mathrm{II}_i|^{\delta}\big)&\le\big(E|\mathrm{I}_1|^{2+\delta}\big)^{2/(2+\delta)}\big(E|\mathrm{II}_i|^{2+\delta}\big)^{\delta/(2+\delta)}\\&\le(C_d)^{2/(2+\delta)}|\Lambda_n|(C'_d)^{\delta/(2+\delta)}|\Lambda^{(i)}|^{\delta/2}\big(n^{-(d/2)}\big)^{2+\delta},\\E\big(\mathrm{II}_i^2|\mathrm{II}_j|^{\delta}\big)&\le C'_d|\Lambda^{(i)}|\,|\Lambda^{(j)}|^{\delta/2}\big(n^{-(d/2)}\big)^{2+\delta},\\E\big(\mathrm{II}_i^2|\mathrm{I}_1|^{\delta}\big)&\le(C_d)^{\delta/(2+\delta)}(C'_d)^{2/(2+\delta)}|\Lambda^{(i)}|\,|\Lambda_n|^{\delta/2}\big(n^{-(d/2)}\big)^{2+\delta},\\&\qquad\qquad i,j=1,\dots,2d.\end{aligned}$$

These imply that

$$\begin{aligned}E|\mathrm{I}_1|^{2+\delta}|A|^{-(2+\delta)/2}&\le C_\delta\big(|\Lambda_n|/|nA|\big)^{(2+\delta)/2}\to C_\delta,\\E|\mathrm{II}_i|^{2+\delta}|A|^{-(2+\delta)/2}&\le C_\delta\big(|\Lambda^{(i)}|/|nA|\big)^{(2+\delta)/2}\to 0,\\E\big(\mathrm{I}_1^2|\mathrm{II}_j|^{\delta}\big)|A|^{-(2+\delta)/2}&\to 0,\\E\big(\mathrm{II}_i^2|\mathrm{II}_j|^{\delta}\big)|A|^{-(2+\delta)/2}&\to 0,\\E\big(|\mathrm{I}_1|^{\delta}\mathrm{II}_i^2\big)|A|^{-(2+\delta)/2}&\to 0,\qquad i,j=1,\dots,2d,\end{aligned}$$

as $n \to \infty$. Then, if n is large enough, there exists a constant $C^{(2)}$ depending on d and δ, such that

$$E|Z_n(A)|^{2+\delta} \le C^{(2)} A^{(2+\delta)/2}. \qquad \square$$

LEMMA 2.6. *Suppose* $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ *satisfies the condition of Theorem* 1.1. *Then* $\{Z_n^2(A)/|A|\}_{A \in G,\, n \ge 1}$ *is uniformly integrable.*

PROOF. The lemma follows immediately from $E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$ and Lemma 2.5. $\square$

THEOREM 2.7. *Suppose* $\{Z_n(\cdot)\}_{n \ge 1}$ *is a series of additive processes in* $CA(G)$ *and satisfies the following conditions*:

$$\{Z_n^2(A)/|A|\}_{n \ge 1,\, A \in G} \text{ is uniformly integrable.} \tag{2.7.1}$$

$$EZ_n(C) \to 0 \quad \text{as } n \to \infty, C \in G. \tag{2.7.2}$$

$$EZ_n^2(C) \to |C| \quad \text{as } n \to \infty, C \in G. \tag{2.7.3}$$

(2.7.4) *For any* $C_1, \dots, C_k \in G$, $d(C_i, C_j) > 0$, $i \ne j$, $i, j = 1, \dots, k$, *for any* $z_1, \dots, z_k \in R^1$, *as* $n \to \infty$,

$$P(Z_n(C_1) \le z_1, \dots, Z_n(C_k) \le z_k) - \prod_{i=1}^{k} P(Z_n(C_i) \le z_i) \to 0,$$

where $d(C_i, C_j) = \inf\{\|\mathbf{t}_i - \mathbf{t}_j\|: \mathbf{t}_i \in C_i, \mathbf{t}_j \in C_j\}$.

Then the finite dimensional distributions of $Z_n(\cdot)$ *converge to the corresponding finite dimensional distribution of standard Brownian motion.*

PROOF. See Corollary 3.3 of [8]. $\square$

PROOF OF THEOREM 2.1. We assume that $\sigma = 1$, $E\xi_{\mathbf{0}} = 0$. We need only to check the conditions of Theorem 2.7 for the partial-sum processes of the random field. Lemma 2.6 implies that condition (2.7.1) is satisfied. Condition (2.7.2) obviously is satisfied. The proof of condition (2.7.3) is an analogue of the proof of Corollary 1.4 of [9]. Now we check condition (2.7.4). It is enough to check for the case $k = 2$. For the case $k > 2$, it is analogous. Let $r := d(C_1, C_2) > 0$, $A_n := \{Z_n(C_1) \le z_1\}$ and $B_n := \{Z_n(C_2) \le z_2\}$. By Lemma 1 of [11],

$$|E(I_{A_n} I_{B_n}) - EI_{A_n} EI_{B_n}| \le 2\left(\phi_{|nC_1|}(nr)\right)^{1/2} \le 2n|C_1|\left(\phi(nr)\right)^{1/2},$$

where I_A is the characteristic function of the set A. By condition (1.1.1),

$$n^d\left(\phi(nr)\right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

Then, as $n \to \infty$,

$$|P(Z_n(C_1) \le z_1, Z_n(C_2) \le z_2) - P(Z_n(C_1) \le z_1) P(Z_n(C_2) \le z_2)| \to 0. \qquad \square$$

3. The proof of tightness.

LEMMA 3.1. *Suppose* $\{\xi_{\mathbf{t}},\ \mathbf{t} \in Z^d\}$ *satisfies the conditions of Theorem* 1.1. *Then for any* $\lambda > 0$,

$$\lim_{\nu \searrow 0} \limsup_{n \to \infty} P\Big(\sup_{A, B \in G, |A \triangle B| \le \nu} |Z_n(A) - Z_n(B)| > \lambda \Big) = 0. \tag{3.1.1}$$

This is the main result of this section. The proof is lengthy. We assume that $\sigma = 1$, $E\xi_{\mathbf{t}} = 0$. For $0 \le u \le v < \infty$, define

$$\eta_{n,\mathbf{j}}(u,v) := n^{-(d/2)} \xi_{n\mathbf{j}} I\big\{u \le n^{d\delta/(2(1+\delta))} n^{-(d/2)} |\xi_{n\mathbf{j}}| < v\big\}, \qquad \mathbf{j} \in J_n,$$

$$Z_n(A,u,v) := \sum_{\mathbf{j} \in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} \big(\eta_{n,\mathbf{j}}(u,v) - E\eta_{n,\mathbf{j}}(u,v)\big).$$

PROPOSITION 3.2. *Assume the conditions of Lemma* 3.1 *are satisfied. Then as* $n \to \infty$, $U_n(I^d, a, \infty) \to 0$, *a.s.*, $EU_n(I^d, a, \infty) \to 0$, *where* $I^d = [0,1]^d$, $a > 0$ *and*

$$U_n(A,u,v) := \sum_{\mathbf{j} \in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} |\eta_{n,\mathbf{j}}(u,v)|.$$

PROOF. We use Bass's technique [2]. For $\mathbf{k} = (k_1, \dots, k_d) \in Z_+^d$, let $\mu(\mathbf{k}) := \max\{k_1, \dots, k_d\}$ and $\Psi(i,a) := \sup\{k \in Z_+ \colon a k^{d/(2(1+\delta))} < i+1\}$. It is well known that $\#\{\mathbf{k} \in Z_+^d \colon \mu(\mathbf{k}) = r\} \le C r^{d-1}$, where C is a positive constant depending on d.

$$\begin{aligned}
&\sum_{\mathbf{k} \in Z_+^d} I\big\{i + 1 > a(\mu(\mathbf{k}))^{d/(2(1+\delta))}\big\} (\mu(\mathbf{k}))^{-(d/2)} \\
&\quad = \sum_{r=1}^{\infty} \sum_{\{\mathbf{j} \colon \mu(\mathbf{j}) = r\}} I\{i + 1 > a r^{d/(2(1+\delta))}\} r^{-(d/2)} \\
&\quad = C \sum_{r=1}^{\infty} I\{i + 1 > a r^{d/(2(1+\delta))}\} r^{(d-2)/d} \\
&\quad \le C(\Psi(i,a))^{d/2} \le C a^{-1} (i+1)^{1+\delta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{\mathbf{k} \in Z_+^d} E|\xi_{\mathbf{k}}| I\big\{a(\mu(\mathbf{k}))^{d/(2(1+\delta))} \le |\xi_{\mathbf{k}}| < \infty\big\} (\mu(\mathbf{k}))^{-(d/2)} \\
&\quad \le \sum_{\mathbf{k}} \sum_{i+1 > a(\mu(\mathbf{k}))^{d/(2(1+\delta))}} (i+1) P(i < |\xi_{\mathbf{k}}| \le i+1) (\mu(\mathbf{k}))^{-(d/2)} \\
&\quad \le \sum_{i=0} \Big[\sum_{\mathbf{k}} I\big\{i+1 > a(\mu(\mathbf{k}))^{d/(2(1+\delta))}\big\} (\mu(\mathbf{k}))^{-(d/2)} \Big] \\
&\qquad \times (i+1) P(i < \xi_{\mathbf{0}} \le i+1) \\
&\quad \le C a^{-1} \sum_{i=0} (i+1)^{2+\delta} P(i < |\xi_{\mathbf{0}}| \le i+1) \\
&\quad \le C a^{-1} E(|\xi_{\mathbf{0}}| + 1)^{2+\delta} < \infty.
\end{aligned} \tag{3.2.1}$$

Then for $\varepsilon > 0$, there exists $n_1(\omega)$, such that

$$\sum_{\mathbf{k}:\,\mu(\mathbf{k})>n_1} |\xi_{\mathbf{k}} I\{a(\mu(\mathbf{k}))^{d/(2(1+\delta))} \le |\xi_{\mathbf{k}}| < \infty\}|/(\mu(\mathbf{k}))^{d/2} < \infty.$$

From this,

$$U_n(I^d, a, \infty) \le \sum_{\mu(\mathbf{k}) \le n_1} |\xi_{\mathbf{k}} I\{an^{d/(2(1+\delta))} \le |\xi_{\mathbf{k}}| < \infty\}| n^{-(d/2)}.$$

Thus, as $n \to \infty$, $U_n(I^d, a, \infty) \to 0$, a.s. Analogously, we can obtain

$$EU_n(I^d, a, \infty) \to 0. \qquad \square$$

For convenience let $G_\nu := \{A \setminus B\colon A, B \in G, |A \setminus B| \le \nu\}$. Then, in order to prove (3.1.1), we need only prove

$$\lim_{\nu \searrow 0} \limsup_{n\to\infty} P(\|Z_n\|_{G_\nu} > \lambda) = 0. \tag{3.2.2}$$

Since $Z_n(A) = Z_n(A, 0, a) + Z_n(A, a, \infty)$, moreover,

$$|Z_n(A, a, \infty)| \le U_n(I^d, a, \infty) + EU_n(I^d, a, \infty).$$

Therefore by Proposition 3.2, in order to prove (3.2.2), we need only prove

$$\lim_{\nu \searrow 0} \limsup_{n\to\infty} P(\|Z_n(A, 0, a)\|_{G_\nu} > \lambda) = 0. \tag{3.2.3}$$

Let $p_n := [n^{(2+\delta)/(2(1+\delta))}]$, $[r]$ is the integer part of r and $m_n := n/(2p_n)$. We divide I^d in the following two ways: $C_{p_n,\mathbf{l}}$, $\mathbf{l} \in J_{p_n}$ and $C_{2p_n,\mathbf{l}}$, $\mathbf{l} \in J_{2p_n}$. There are $2^d C_{2p_n,\mathbf{l}}$ in each $C_{p_n,\mathbf{l}}$. Denote by $I_{n,\mathbf{l},i}$ the ith $C_{2p_n,\mathbf{j}}$ in $C_{p_n,\mathbf{l}}$, $\mathbf{l} \in J_{p_n}$, $\mathbf{j} \in J_{2p_n}$. Let

$$I_{n,i} := \bigcup_{\mathbf{l}\in J_{p_n}} I_{n,\mathbf{l},i}, \qquad i = 1, \dots, 2^d.$$

Then

$$Z_n(\cdot, 0, a) = \sum_{i=1}^{2^d} Z_n(\cdot \cap I_{n,i}, 0, a).$$

Here, we follow the notation and method of [9]. To prove (3.2.3), we need only prove for certain i, $i = 1, \dots, 2^d$,

$$\lim_{a,\nu \searrow 0} \limsup_{n\to\infty} P(\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{G_\nu} > \lambda) = 0. \tag{3.2.4}$$

Now,

$$\begin{aligned} Z_n(\cdot \cap I_{n,i}, 0, a) &= \sum_{\mathbf{l}\in J_{p_n}} Z_n(\cdot \cap I_{n,\mathbf{l},i}, 0, a) \\ &= \sum_{\mathbf{l}\in J_{p_n}} \sum_{\mathbf{j}\in S(n,\mathbf{l},i)} \frac{|\cdot \cap I_{n,\mathbf{l},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} \\ &\qquad \times (\eta_{n,\mathbf{j}}(0,a) - E\eta_{n,\mathbf{j}}(0,a)), \end{aligned}$$

where $S(n,\mathbf{l},i) := \{\mathbf{j} \in J_n\colon C_{n,\mathbf{j}} \cap I_{n,\mathbf{l},i} \neq \Phi\}$. It is obvious that for $\mathbf{j}_s \in S(n,\mathbf{l},i)$, $\mathbf{l}_s \in J_{p_n}$, $s = 1,2$, $\mathbf{l}_1 \neq \mathbf{l}_2$, we have $\|\mathbf{j}_1 - \mathbf{j}_2\| \geq (m_n/n)$.

We construct a new series of processes $\{\bar{\xi}_{n\mathbf{j}}\}_{\mathbf{j} \in S(n,\mathbf{l},i)}$, $\mathbf{l} \in J_{p_n}$, such that the distribution of each process $\{\bar{\xi}_{n\mathbf{j}}\}_{\mathbf{j} \in S(n,\mathbf{l},i)}$ coincides with that of $\{\xi_{n\mathbf{j}}\}_{\mathbf{j} \in S(n,\mathbf{l},i)}$, and for different $\mathbf{l} \in J_{p_n}$, the new processes are independent of each other. Let

$$\bar{\eta}_{n,\mathbf{j}}(u,v) := n^{-(d/2)}\bar{\xi}_{n\mathbf{j}} I\left\{u \leq n^{d\delta/(2(1+\delta))} n^{-(d/2)} |\bar{\xi}_{n\mathbf{j}}| \leq v\right\}, \qquad \mathbf{j} \in J_n,$$

$$\bar{V}_{n,\mathbf{l}}(A,u,v) := \sum_{\mathbf{j} \in S(n,\mathbf{l},i)} \frac{|A \cap I_{n,\mathbf{l},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} \left(\bar{\eta}_{n,\mathbf{j}}(u,v) - E\bar{\eta}_{n,\mathbf{j}}(u,v)\right), \qquad \mathbf{l} \in J_{p_n},$$

$$\bar{Z}_n(A,u,v) := \sum_{\mathbf{l} \in J_{p_n}} \bar{V}_{n,\mathbf{l}}(A,u,v),$$

$$W_{n,\mathbf{l}}(A,u,v) := \sum_{\mathbf{j} \in S(n,\mathbf{l},i)} \frac{|A \cap I_{n,\mathbf{l},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} |\bar{\eta}_{n,\mathbf{l}}(u,v)|, \qquad \mathbf{l} \in J_{p_n},$$

$$\bar{U}_n(A,u,v) := \sum_{\mathbf{l} \in J_{p_n}} W_{n,\mathbf{l}}(A,u,v).$$

PROPOSITION 3.3. *Assume the condition of Lemma* 3.1 *is satisfied. Then*

$$\|L(Z_n(\cdot \cap I_{n,i}, 0, a)) - L(\bar{Z}_n(\cdot, 0, a))\|_{\mathrm{Var}} \to 0,$$

as $n \to \infty$, where $L(\xi)$ is the distribution function of ξ and $\|\cdot\|_{\mathrm{Var}}$ is the total variance norm of the distribution function.

REMARK. Proposition 3.3 replaces the same result of [9], page 834, but uses condition (1.1.1) (nonuniform ϕ-mixing).

PROOF OF PROPOSITION 3.3. By definition,

$$\#S(n,\mathbf{l},i) \leq \left[\frac{1/(2p_n)}{1/n}\right]^d \sim 2^{-d} n^{d\delta/(2(1+\delta))},$$

where $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. By Lemma 1 of [11],

$$\begin{aligned}
&\|L\left(V_{n,\mathbf{l}}(\cdot,u,v), \mathbf{l} \in J_{p_n}\right) \\
&\qquad - L\left(V_{n,\mathbf{l}_1}(\cdot,u,v)\right) L\left(V_{n,\mathbf{l}}(\cdot,u,v), \mathbf{l} \neq \mathbf{l}_1, \mathbf{l} \in J_{p_n}\right)\|_{\mathrm{Var}} \\
&\leq 2 \sup_{\substack{A \in \sigma(V_{n,\mathbf{l}_1}(\cdot,u,v)),\\ B \in \sigma(V_{n,\mathbf{l}}(\cdot,u,v)),\, \mathbf{l} \neq \mathbf{l}_1,\, \mathbf{l} \in J_{p_n}}} |E(I_A I_B) - E(I_A)E(I_B)| \\
&\leq 2C\left(\phi_{\#S(n,\mathbf{l},i)}(m_n - 1)\right)^{1/2} \leq 2C\#S(n,\mathbf{l},i)\left(\phi(m_n - 1)\right)^{1/2} \\
&\leq 2C\left(2^{-d} n^{d\delta/(2(1+\delta))}\right)\left(\phi\left(2^{-d} n^{\delta/(2(1+\delta))}\right)\right)^{1/2},
\end{aligned}$$

where

$$V_{n,\mathbf{l}}(\cdot,u,v) := \sum_{\mathbf{j}\in S(n,\mathbf{l},i)} \frac{|\cdot\cap I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\big(\eta_{n,\mathbf{j}}(u,v) - E\eta_{n,\mathbf{j}}(u,v)\big).$$

Analogously,

$$\begin{aligned}
&\|L\big(V_{n,\mathbf{l}}(\cdot,u,v),\mathbf{l}\in J_{p_n}\big) - \prod_{\mathbf{l}\in J_{p_n}} L\big(V_{n,\mathbf{l}}(\cdot,u,v)\big)\|_{\mathrm{Var}}\\
&\quad\le 2C2^{-d}n^{d\delta/(2(1+\delta))}\big(\phi(2^{-d}n^{\delta/(2(1+\delta))})\big)^{1/2}2\sum_{k=1}^{[(p_n^d-1)/2]}k\\
&\quad\le C'2^{-d}n^{(4+3\delta)d/(2(1+\delta))}\big(\phi(2^{-d}n^{\delta/(2(1+\delta))})\big)^{1/2}\\
&\quad\le C'K^{(4+3\delta)d/\delta}\big(\phi(K)\big)^{1/2}, \qquad K := 2^{-d}n^{\delta/(2(1+\delta))}.
\end{aligned}$$

From condition (1.1.1), we obtain that as $n\to\infty$, the last term converges to zero. Thus, as $n\to\infty$,

$$\|L\big(V_{n,\mathbf{l}}(\cdot,u,v),\mathbf{l}\in J_{p_n}\big) - \prod_{\mathbf{l}\in J_{p_n}} L\big(\overline{V}_{n,\mathbf{l}}(\cdot,u,v)\big)\|_{\mathrm{Var}} \to 0.$$

This implies

$$\|L\big(Z_n(\cdot\cap I_{n,i},0,a)\big) - L\big(\overline{Z}_n(\cdot,0,a)\big)\|_{\mathrm{Var}} \to 0. \qquad \square$$

By Proposition 3.3, in order to prove (3.2.4), we need only prove

$$\lim_{a,\nu\searrow 0}\ \limsup_{n\to\infty} P\big(\|\overline{Z}_n(\cdot,0,a)\|_{G_\nu} > \lambda\big) = 0. \tag{3.2.5}$$

For the purpose of using the Bernstein inequality (cf. [9]), we need an almost sure bound. By definition,

$$\begin{aligned}
|\overline{V}_{n,\mathbf{l}}(A,u,v)| &\le 2vn^{-d\delta/(2(1+\delta))}\sum_{\mathbf{j}\in S(n,\mathbf{l},i)} |I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}|/|C_{n,\mathbf{j}}|\\
&= 2vn^{-d\delta/(2(1+\delta))}n^d|I_{n,\mathbf{l},i}|\\
&= 2vn^{-d\delta/(2(1+\delta))}n^d(2p_n)^{-d} \le 2v.
\end{aligned}$$

By Lemma 2.4

$$\begin{aligned}
\mathrm{Var}\big(\overline{Z}_n(A,u,v)\big) &= \sum_{\mathbf{l}\in J_{p_n}} \mathrm{Var}\big(\overline{V}_{n,\mathbf{l}}(A,u,v)\big)\\
&= \sum_{\mathbf{l}\in J_{p_n}} \mathrm{Var}\big(Z_n(A\cap I_{n,\mathbf{l},i},u,v)\big)\\
&\le \sum_{\mathbf{l}\in J_{p_n}} C|A\cap I_{n,\mathbf{l},i}| \le C|A|.
\end{aligned}$$

Analogously, we have

$$|W_{n,\mathbf{l}}(A,u,v)| \le v \quad \text{a.s.}$$

and

$$\operatorname{Var}\big(\overline{U}_n(A,u,v)\big) \le C|A|.$$

Finally,

$$\begin{aligned} E\overline{U}_n(A,u,v) &= EU_n(A\cap I_{n,i},u,v)\\ &\le \sum_{\mathbf{j}\in J_n}\big(|A\cap C_{n,\mathbf{j}}|/|C_{n,\mathbf{j}}|\big)E|\eta_{n,\mathbf{j}}(u,v)|^{2+\delta}\\ &\le |A|u^{-(1+\delta)}C'E|\eta_{n,\mathbf{j}}(u,v)|^{2+\delta}\\ &\le |A|u^{-(1+\delta)}C'E|\xi_{\mathbf{0}}|^{2+\delta}. \end{aligned}$$

Now we return to the proof of (3.2.5). We shall choose λ_i, a_i, δ_i, $i = 0, 1, 2, \dots,$ later so that

$$\begin{aligned} &\lambda_i \searrow 0, \qquad 4\sum_{i=0}^{\infty}\lambda_i < \lambda,\\ (3.2.6)\qquad &\delta_0 = \nu, \qquad \delta_i \searrow 0,\\ &a_0 = a, \qquad a_i \searrow 0. \end{aligned}$$

For $A \in G$, there exist A_i, $A_i^+ \in N(G, \delta_i)$ (cf. the definition of metric entropy) so that $A_i \subset A \subset A_i^+$, $|A_i^+ \setminus A_i| \le \delta_i$. Then

$$\begin{aligned} \overline{Z}_n(A,0,a) = \overline{Z}_n(A_0,0,a) &+ \sum_{i=1}^{\infty}\big(\overline{Z}_n(A_{i+1},0,a_i) - \overline{Z}_n(A_i,0,a_i)\big)\\ &+ \sum_{i=1}^{\infty}\big(\overline{Z}_n(A,a_i,a_{i-1}) - \overline{Z}_n(A_i,a_i,a_{i-1})\big). \end{aligned}$$

The remainder of the proof is the analogue of the nesting part of the tightness proof in Section 5.6 of [9]. We need only note that for our G the metric entropy exponent $r = 0$. We omit the details. □

PROOF OF THEOREM 1.1. Analogously to the proof of Theorem 1.1 of [9] we need only prove the following:

(a) for each element A of some countably dense set in G, the family $\{Z_n(A)\}_{n\ge 1}$ is tight;

(b) for every $\lambda > 0$,

$$\lim_{\nu\searrow 0}\limsup_{n\to\infty} P\big(\|Z_n\|_{G_\nu} > \lambda\big) = 0.$$

(a) follows from Theorem 2.1 and (b) follows from Lemma 3.1. □

4. An application. In this section we use Theorem 1.1 to prove a uniform central limit theorem for certain Gibbs fields. We consider the Gibbs fields of the Ising model. The Ising model is a famous model in statistical physics and has great physical significance.

Let $S = \{-1, +1\}$, μ be the counting measure on S and f be a positive number. Let $D = \{\Lambda \subset Z^d\colon |\Lambda| < \infty\}$. We need to introduce the measurable space (S^{Z^d}, F), where F is the product σ-algebra. For any $\Lambda \in D$ and $\xi \in S^{Z^d}$, define

$$\Phi_\Lambda(\xi) := \begin{cases} f\xi_{\mathbf{s}}\xi_{\mathbf{t}} & \text{if } \Lambda = \{\mathbf{s}, \mathbf{t}\}, \|\mathbf{s} - \mathbf{t}\| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi_{\mathbf{s}}$ is the projection of ξ at site $\mathbf{s}$. We call $\{\Phi_\Lambda, \Lambda \in D\}$ the potential of the Ising model. Let $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$ be the potential, such that

$$(\beta\Phi)_\Lambda(\xi) = \beta\Phi_\Lambda(\xi), \qquad \Lambda \in D, \xi \in S^{Z^d}, \beta > 0.$$

Definition 4.1. A probability measure P on (S^{Z^d}, F) is called a Gibbs field of the potential $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$, if for any $\mathbf{t} \in Z^d$, $\xi \in S^{Z^d}$, $\beta > 0$,

$$P\big(\xi_{\mathbf{t}} | \xi_{\mathbf{s}}, \mathbf{s} \neq \mathbf{t}, \mathbf{s} \in Z^d\big) = \left(1 + \exp\left(2 \sum_{\mathbf{t} \in \Lambda} \beta\Phi_\Lambda(\xi)\right)\right)^{-1}.$$

Theorem 4.1 (cf. [7], Chapter V). *For $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$, $\beta > 0$, there exists a $\beta_{cr} > 0$ $(d > 1)$ such that for each β, $0 < \beta < \beta_{cr}$, the Gibbs field of $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$ exists uniquely and is strictly stationary.*

Corollary 4.2. *Suppose the distribution of the random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$ is the Gibbs field of the potential $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$, $0 < \beta < \beta_{cr}$. Then, the partial-sum processes of $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}Z_n(A)$, $A \in G$, converge weakly to the Brownian motion on G with parameter σ. Here,*

$$0 < \sigma^2 = \sum_{\mathbf{t} \in Z^d} \operatorname{Cov}(\xi_{\mathbf{0}}, \xi_{\mathbf{t}}) < \infty.$$

Proof. We need only check that the conditions of Theorem 1.1 are satisfied for $\{\xi_{\mathbf{t}}, \mathbf{t} \in Z^d\}$. By Lemma V.7.1 of [7], we know $0 < \sigma < \infty$. The stationary property follows from Theorem 4.1. Condition (1.1.2) follows from the result of Chapter V of [7]. In what follows we check condition (1.1.1).

We check that the potential $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$, $0 < \beta < \beta_{cr}$, is in the class N_μ (cf. [11], page 534). Note that

$$\|\beta\Phi\| = \sum_{\mathbf{k}\colon \|\mathbf{k}\| = 1, \mathbf{k} \in Z^d} 2\beta f = 4\, d\beta\, f < \infty,$$

$$\sum_{\Lambda\colon \mathbf{0} \in \Lambda \in D, |\Lambda| > 2} |\Lambda| \sup|\beta\Phi_\Lambda(\cdot)| = 0.$$

Then we know that conditions 1 and 2 (cf. [11], page 534) of N_μ are satisfied. Thus $\{(\beta\Phi)_\Lambda, \Lambda \in D\} \in N_\mu$. Furthermore, $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$, $0 < \beta < \beta_{cr}$, are finite range potentials. Thus, by the discussion on page 534 of [11], we know that the Gibbs fields of $\{(\beta\Phi)_\Lambda, \Lambda \in D\}$, $0 < \beta < \beta_{cr}$, satisfy the nonuniform ϕ-mixing condition (i.e., the u.s.m. condition of [11]) and $\phi_{|\Lambda|}(r) \le |\Lambda|\phi(r)$, where $\phi(r)$ is decreasing exponentially to zero as $r \to \infty$, which means that condition (1.1.1) is satisfied. This ends the proof of the corollary. □

REMARK. To the author's knowledge there is no similar result of Corollary 4.2 for other Gibbs fields in the class N_μ. The difficulty is to prove $0 < \sigma < \infty$.

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