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LOCAL TIME AND STOCHASTIC AREA INTEGRALS

By L. C. G. Rogers and J. B. Walsh

University of Cambridge and University of British Columbia

If $(B_t)_{t\geq 0}$ is Brownian motion on $\mathbb{R}$, if $A(t,x) \equiv \int_0^t I_{\{B_s \leq x\}}\, ds$ and if $\tau(\cdot, x)$ is the right-continuous inverse to $A(\cdot, x)$, then the process $\tilde{B}(t,x) \equiv B(\tau(t,x))$ is a reflecting Brownian motion in $(-\infty, x]$. If $\mathscr{E}_x$ denotes the σ-field generated by $\tilde{B}(\cdot, x)$, then $(\mathscr{E}_x)_{x\in\mathbb{R}}$ forms a filtration. It has been proved recently that all $(\mathscr{E}_x)$-martingales are continuous, in common with the martingales on the Brownian filtration. Here we shall prove that, as with the Brownian filtration, all $(\mathscr{E}_x)$-martingales can be written as stochastic area integrals with respect to local time. This requires a theory of such integrals to be developed; the first version of this was given by Walsh some years ago, but we consider the account presented here to be definitive. We apply this theory to an investigation of stochastic line integrals of local time along curves which need not be adapted processes and illustrate these constructs by identifying the compensator of the supermartingale $(L(\tau(t,x),x))_{x\geq a}$ previously studied by McGill.

1. Introduction. Let B_t denote a standard motion, $B_0 = 0$, defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and let $L(t,x)$ denote its local time and $(\mathscr{F}_t)$ its filtration, completed as usual, so that $\mathscr{F}_0$ contains all null sets of $\mathscr{F}$. We assume that $\mathscr{F} = \vee_t \mathscr{F}_t$. Let

$$A(t,x) = \int_0^t I_{\{B_s \leq x\}}\, ds = \int_{-\infty}^x L(t,y)\, dy. \tag{1.1}$$

For each x, let $\tau(\cdot, x)$ be the inverse of $A(\cdot, x)$:

$$\tau(t,x) = \inf\{u : A(u,x) > t\}. \tag{1.2}$$

Let $\tilde{B}(t,x) = B_{\tau(t,x)}$ be the time change of B by τ and let $\tilde{L}$ be the time change of local time given by

$$\tilde{L}(t,x) = L(\tau(t,x), x). \tag{1.3}$$

$\tilde{L}$ is called the *intrinsic local time* and is jointly continuous; see Corollary 1.9 of McGill (1986). Finally, define a σ-field $\mathscr{E}_x$ by

$$\mathscr{E}_x^0 = \sigma\{\tilde{B}(t,x), t \geq 0\} \tag{1.4}$$

and let $\mathscr{E}_x$ be the usual augmentation of $\mathscr{E}_x^0$ obtained by adjoining all null sets of $\mathscr{F}$ to $\mathscr{E}_{x+}^0$.

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Changing time by $\tau(\cdot, x)$ wipes out the time B spends in (x, ∞) so that to get $\tilde{B}$ from B we delete all excursions of B above x and close up the gaps in the time axis. The resulting process is a reflecting Brownian motion on $(-\infty, x]$. The family $(\mathscr{E}_x)$ is a filtration, called the filtration of *excursion fields*. It was introduced in Walsh (1978) to study Ray–Knight theorems. The parameter of the excursion fields is x, not t. Nevertheless the two filtrations enjoy many of the same characteristics. Some examples of this were given in Walsh (1983) and it was conjectured that they shared two of the most important properties of $(\mathscr{F}_t)$:

1. All $\mathscr{E}_x$ martingales are continuous.
2. All square-integrable $\mathscr{E}_x$ martingales can be written as stochastic integrals.

These conjectures have been nearly settled. Williams (1979) produced a dense set of continuous $(\mathscr{E}_x)$-martingales to show that (1) was true. Walsh (1983) then gave a theory of stochastic area integrals with respect to local time and showed how to write Williams' martingales as stochastic integrals. This implied the conjectured representation theorem, which in turn implied (1). In a later paper, McGill (1986) approached stochastic integration from a different viewpoint, using $\tilde{L}$ instead of L and gave another representation of Williams' martingales, arriving at yet another proof of (1) and (2).

As it happens, none of these proofs was totally satisfactory. Williams' proof relied on what he called CMO formulas. He gave an elegant proof of the first order ones, but the higher order formulas were too complicated to prove carefully. Walsh's proof ran into a similar problem in a different spot. It used Green's formulas and the derivations of the higher order equations were more heuristic than rigorous. McGill's proof relied on an explicit decomposition of the supermartingale $\{\tilde{L}(t, x) + 2x^-, \mathscr{E}_x, x \geq a\}$ into its martingale and bounded variation parts; we could not see how this followed from the arguments in the paper. (We hasten to add that the decomposition is correct and can doubtless be proved by his methods. However, the attempt to understand it motivated much of the research of this paper, so we will come back to this point and give a different proof of the decomposition.)

More recently, Rogers (1987) gave a complete and elementary proof of the martingale continuity, laying to rest any remaining doubts about (1). We intend to do the same for (2) in Section 2 of this paper. In order to make the paper self-contained, we will begin with an account of the stochastic integral with respect to local time and then proceed to a direct and relatively simple proof of Theorem 7.1 in Walsh (1983). This is based on Rogers' continuity proof, which constructed a dense family of continuous martingales. We simply show here that they can all be written as stochastic integrals.

We will turn to questions of stochastic line integrals and their relation with Brownian local time on certain curves in Section 3. We show that it is possible to define a local time on a class of curves in space-time; the class of curves is quite different from the classes usually treated and the results connect closely with both the line integrals and the stochastic area integrals of Section 2.

We apply this to the supermartingale $\{\tilde{L}(x,t) + 2x^-, \mathscr{E}_x, x \geq a\}$ in the final section, giving a new proof of the Meyer decomposition and showing several probabilistic interpretations of the martingale part and particularly of the increasing process.

2. Stochastic area integrals.

The main object of this section is to explain and prove Theorem 2.1.

THEOREM 2.1. *Each $X \in L^2(\Omega, \mathscr{F}, P)$ can be represented as*

$$X = E\{X\} + \int\int \phi(t,x) L(dt,dx), \tag{2.1}$$

where ϕ is an identifiable process satisfying

$$\|\phi\|^2 \equiv 4E\left\{\int_0^\infty \phi(t,B_t)^2\,dt\right\} < \infty. \tag{2.2}$$

Moreover, ϕ is unique up to equivalence.

Identifiable processes and the double integral in (2.1) were introduced in Walsh (1983), but rather than refer to that, we will carry out the construction of the stochastic area integral with respect to L, taking the opportunity to simplify the approach a little. The simplification comes about largely because we use the characterization of identifiability given in the appendix of Walsh (1983), which is easier to work with than the original definition.

Let us first note some facts about the function A and its inverse τ. The following are evident from (1.1). For a.e. ω:

1. $A(t,x)$ is jointly continuous in (t,x).
2. For fixed x, A is an increasing Lipschitz continuous function of t.
3. For fixed t, A is an increasing $C^{(1)}$ function of x with

$$\frac{\partial A}{\partial x} = L(t,x).$$

Moreover, as a function of x,

$$A(t,x) = \begin{cases} t & \text{if } x \geq M_t, \\ \text{strictly increasing} & \text{if } m_t \leq x \leq M_t, \\ 0 & \text{if } x \leq m_t, \end{cases} \tag{2.3}$$

where

$$m_t = \inf\{B_s, s \leq t\}, \qquad M_t = \sup\{B_s, s \leq t\}.$$

The properties of τ, the inverse of A, are easily derived from this. We will leave the proof of the following as an exercise.

PROPOSITION 2.2.

(i) τ *is finite on* $\mathbf{R}_+ \times \mathbf{R}$ *and* $\tau(t, x) = t$ *if* $x > M_t$;
(ii) $t \mapsto \tau(t, x)$ *is right-continuous and increasing*;
(iii) $x \mapsto \tau(t, x)$ *is left-continuous and decreasing*;
(iv) $A(\tau(t, x), x) = t$ *and* $t \leq \tau(A(t, x), x) = \inf\{s > t: A(s, x) > A(t, x)\}$.

Let $\Omega' = \Omega \times (0, \infty)$ and let $(\mathscr{E}_x')_{x \in \mathbf{R}}$ be the filtration $\mathscr{E}_x' = \mathscr{E}_x \times \mathscr{B}((0, \infty))$ on Ω'. A process $\{X(\omega, t, x): \omega \in \Omega,\ t > 0, x \in \mathbf{R}\}$ is *$(\mathscr{E}_x')$-adapted* if for each x, $(\omega, t) \mapsto X(\omega, t, x)$ is $(\mathscr{E}_x')$-measurable and it is *left-continuous* if for almost every ω, $x \mapsto X(\omega, t, x)$ is left-continuous for all t. By analogy with the usual one-parameter theory we coin the following terms.

DEFINITION. The *(excursion) predictable* sigma field $\mathscr{E}\mathscr{P}$ on $\Omega' \times \mathbf{R}$ is the sigma field generated by all left-continuous $\mathscr{E}_x'$-adapted processes. A process $\{X(\omega, t, x): \omega \in \Omega,\ t > 0, x \in \mathbb{R}\}$ is *predictable* if it is $\mathscr{E}\mathscr{P}$-measurable.

Note that a predictable process is only defined for $t > 0$ and that the predictable sets are subsets of the open half plane $\mathbf{R}_{++} \times \mathbf{R}$, where $\mathbf{R}_{++} = (0, \infty)$. In our theory as in the one parameter theory, the point $t = 0$ often has to be discussed separately as a special case. We will minimize these discussions by simply eliminating the origin from the definition. We shall usually suppress the ω from our notation and write $X(t, x)$ instead of $X(\omega, t, x)$. The following result may be useful.

PROPOSITION 2.3. *$\mathscr{E}\mathscr{P}$ is generated by sets of either of the two forms*

(i) $(0, T) \times (a, b]$, *where* $a < b$ *and where* $T \geq 0$ *is* $\mathscr{E}_a$-*measurable*;
(ii) $(0, T] \times (a, b]$, *where* $a < b$ *and where* $T \geq 0$ *is* $\mathscr{E}_a$-*measurable*.

Of course $(0, T] \times (a, b]$ is short for the set $\{(\omega, t, x): 0 < t \leq T(\omega), a < x \leq b\}$. The proof of Proposition 2.3 is elementary and we will leave it to the reader.

Let $\mathfrak{R}$ be the set

$$\begin{aligned}\mathfrak{R} &= \{(\omega, t, x): t > \tau(\omega, 0, x), \omega \in \Omega, x \in \mathbf{R}\}\\ &= \{(\omega, t, x): A(\omega, t, x) > 0, \omega \in \Omega, x \in \mathbf{R}\}.\end{aligned}$$

Define the map Γ: $\mathfrak{R} \mapsto \Omega \times \mathbf{R}_{++} \times \mathbf{R}$ by

$$\Gamma(\omega, t, x) = (\omega, A(t, x, \omega), x).$$

We can now say what identifiability means in terms of the map Γ.

DEFINITION. A process ϕ: $\mathfrak{R} \mapsto \mathbf{R}$ is *identifiable* if there exists an (excursion) predictable process $\tilde{\phi}$ such that

$$\phi = \tilde{\phi} \circ \Gamma.$$

The σ-field $\mathscr{I}$ of subsets of $\mathfrak{R}$ defined by

$$\mathscr{I} = \Gamma^{-1}(\mathscr{E}\mathscr{P})$$

is called the σ-field of *identifiable sets*.

It is easy to see that an identifiable process ϕ: $\mathfrak{R} \mapsto \mathbf{R}$ is measurable with respect to $\mathscr{I}$ and that $\mathscr{I}$ is generated by the identifiable processes.

Notice that an identifiable process $\phi(t, x)$ is only defined for $t > \tau(0, x)$. If $x > 0$, then $\tau(0, x) = 0$, but if $x < 0$, then $\tau(0, x)$ is almost surely the first time the Brownian path reaches x, so that $\phi(\cdot, x)$ is not defined during the first excursion from the origin to x. Just as the theory of predictable processes is complicated by the inclusion of $t = 0$ in the parameter set, the theory of identifiable processes can be complicated by the inclusion of the first excursion from 0 to a point $x < 0$. The definitions we have made avoid this.

By definition then, a process ϕ is identifiable iff $\phi(t, x) = \tilde{\phi}(A(t, x), x)$ for some predictable $\tilde{\phi}$, in which case $\tilde{\phi}$ is given by $\tilde{\phi}(t, x) = \phi(\tau(t, x), x)$. In practice, to test whether a given process ϕ is identifiable, we form the process $\tilde{\phi}(t, x) \equiv \phi(\tau(t, x), x)$ and try to decide whether or not it is predictable. If so, we check that $\phi = \tilde{\phi} \circ \Gamma$ and the job is done. In view of Proposition 2.2(iv), this last equation holds if $t \mapsto \phi(t, x)$ is constant on each (closed) interval of constancy of A. We can express the latter condition more neatly by introducing an equivalence relation. We say two points (s, x) and (t, y) are *equivalent* and we write $(s, x) \sim (t, x)$, if $x = y$ and $A(s, x) = A(t, y)$. Then the above remarks show the following.

PROPOSITION 2.4. *ϕ is identifiable iff both*

(a) $\tilde{\phi}(t, x) =_{\text{def}} \phi(\tau(t, x), x)$ *is predictable*;
(b) $z \sim z' \Rightarrow \phi(z) = \phi(z')$.

The next result contains some elementary but fundamental facts.

PROPOSITION 2.5. (i) *Fix $x \geq a$ and let T be a nonnegative $\mathscr{E}_a$-measurable random variable. For any $t \geq 0$, the three events*

$$(2.4)\quad \{A(\tau(T, a), x) < t\}, \qquad \{\tau(t, x) > \tau(T, a)\}, \qquad \{A(\tau(t, x), a) > T\}$$

are a.s. equal and are $\mathscr{E}_x$-measurable.

(ii) *Fix $a \leq b$ and let $T > 0$ and Z be $\mathscr{E}_a$-measurable random variables. Then the process*

$$(2.5\text{a})\qquad \phi(t, x) = ZI_{[\tau(T, a), \infty)}(t)\, I_{(a, b]}(x)$$

is identifiable.

PROOF. (i) If $x = a$, the three events are clearly equal, so we suppose that $x > a$. It is evident that the second two sets are equal, since $\tau(t, x) > \tau(T, a)$ if and only if $A(\tau(t, x), a) > T$. It follows from the continuity of $A(\cdot, x)$ that

$A(u,x) < t$ implies that $\tau(t,x) > u$. Moreover, if $\tau(t,x) > \tau(T,a)$, then $A(\tau(T,a),x) \le t$; but in this case, $A(\cdot,x)$ must be strictly increasing at $\tau(T,a)$ since $B(\tau(T,a)) \le a < x$, hence there is strict inequality and the first two sets must be equal.

To prove that the events are $\mathscr{E}_x$-measurable, notice that

$$\begin{aligned} A(\tau(t,x),a) &= \int_0^{\tau(t,x)} I_{\{B_s \le a\}}\, ds \\ &= \int_0^{\tau(t,x)} I_{\{B_s \le a\}} I_{\{B_s \le x\}}\, ds \\ &= \int_0^t I_{\{\tilde{B}(u,x) \le a\}}\, du \end{aligned}$$

which is clearly an $\mathscr{E}_x$-measurable random variable.

(ii) By insisting that $T > 0$ we ensure that ϕ vanishes off $\mathfrak{R}$. To see that it is identifiable, notice that

$$\begin{aligned} \tilde{\phi}(t,x) &\equiv \phi(\tau(t,x),x) \\ &= Z I_{\{\tau(T,a) \le \tau(t,x)\}} I_{(a,b]}(x) \\ &= Z I_{\{A(\tau(T,a),x) \le t\}} I_{(a,b]}(x). \end{aligned}$$

Now $A(\tau(T,a),x)$ is $\mathscr{E}_x$-measurable by (2.4) if $x > a$ and it is continuous in x, so $\tilde{\phi}$ is predictable. To complete the proof that ϕ is identifiable, we check condition (b) of Proposition 2.4.

Suppose that $s < t$, $(s,x) \sim (t,x)$ and, without loss of generality, suppose that $a < x \le b$. Since $A(s,x) = A(t,x)$, it must be that $B_u \ge x$ for all $u \in [s,t]$, and hence it is impossible that $\tau(T,a) \in [s,t]$. Thus $\phi(s,x) = \phi(t,x)$ and the proof is complete. □

It is useful to give a name to times of the form $\tau(T,a)$.

Definition. Let $a \in \mathbf{R}$. A random variable S with values in $[0,\infty]$ is *$\mathscr{E}_a$-identifiable* if there is an $\mathscr{E}_a$-measurable random variable T with values in $[0,\infty]$ such that $S = \tau(T,a)$.

Remark 1. $\tau(\infty,x) = \infty$ by definition.

Remark 2. This definition agrees with that of Walsh (1983) except that an $\mathscr{E}_a$-identifiable time was allowed to take on the value zero there, while here it must be greater than or equal to $\tau(0,a)$. This is innocent if $a \ge 0$, but when $a < 0$, $\tau(0,a)$ is strictly positive.

Remark 3. If T is $\mathscr{E}_a$-identifiable and $a < y$, then T is also $\mathscr{E}_y$-identifiable. Indeed, if $T = \tau(T',a)$ for some $\mathscr{E}_a$-measurable random variable T', then $T = \tau(\sigma,y)$, where $\sigma = A(\tau(T',a),y)$, which is $\mathscr{E}_y$-measurable by (2.4).

REMARK 4. Let Z and T be $\mathscr{E}_a$-measurable random variables with $T \geq 0$. By considering $T + 1/n$, it follows from (2.5a) and the right continuity of $\tau(\cdot, a)$ that processes of the form

$$\phi(t, x) = ZI_{(\tau(T, a), \infty)}(t) I_{(a, b]}(x) \tag{2.5b}$$

and even of the form

$$\phi(t, x) = ZI_{(\tau(0, a), \tau(T, a)]}(t) I_{(a, b]}(x) \tag{2.5c}$$

are identifiable. More is true, however. $\mathscr{I}$ is generated by either of the following two collections of sets:

(a) $[\tau(T, a), \infty) \times (a, b]$, with T strictly positive and $\mathscr{E}_a$-measurable;
(b) $(\tau(T, a), \infty) \times (a, b]$ with T nonnegative and $\mathscr{E}_a$-measurable.

To see this, notice that from Proposition 2.3(ii), sets of the form

$$\Gamma^{-1}((T, \infty) \times (a, b]) = \{(t, x): \tau(T, x) < t, a < x \leq b\}$$

generate the identifiable σ-field $\mathscr{I}$, where $T \geq 0$ is $\mathscr{E}_a$-measurable. This set is the union of Λ_n, where

$$\Lambda_n = \bigcup_{j=0}^{2^n - 1} \{(t, x): \tau(T, a + j\delta_n) < t, a + j\delta_n < x \leq a + (j + 1)\delta_n\}$$

and $\delta_n = (b - a)2^{-n}$. But each Λ_n is a union of sets of the form (b) above and thus the collection of such set generates $\mathscr{I}$. Finally, each set of the form (a) is identifiable by (2.5a) and each set of the form (b) is a limit of sets of the form (a), which proves the assertion.

Let $\mathscr{C}$ be the class of processes

$$\mathscr{C} = \{ZI_{(\tau(S, a), \tau(T, a)]}: a < b, Z \in b\mathscr{E}_a, 0 \leq S \leq T \in b\mathscr{E}_a^+\} \tag{2.6}$$

and let $\mathscr{V}$ be the linear span of $\mathscr{C}$. If we use Remark 3, it is not hard to see that $\mathscr{C}$ is closed under multiplication, so that $\mathscr{V}$ is an algebra. For $\phi = \phi(t, x)$, let

$$\|\phi\|^2 \equiv 4E\left\{\int_0^\infty \phi(t, B_t)^2 \, dt\right\}.$$

Define

$$L^2(\mathscr{E}) = \{\phi: \phi \text{ is identifiable and } \|\phi\| < \infty\}. \tag{2.7}$$

We will show that $\mathscr{V}$ is dense in $L^2(\mathscr{E})$. We can now say what we mean by $\int\int \phi \, dL$, defined firstly for $\phi \in \mathscr{C}$.

DEFINITION. For $\phi \in \mathscr{C}$ of the form $\phi(t, x) = ZI_{(S, T] \times (a, b]}(t, x)$, we define

$$\begin{aligned} \int\int \phi \, dL &\equiv \int\int \phi(t, x) L(dt, dx) \\ &\equiv Z[L(T, b) - L(T, a) - L(S, b) + L(S, a)]. \end{aligned} \tag{2.8}$$

The following theorem establishes the extension and basic properties of $\int\int \phi \, dL$.

THEOREM 2.6. *We extend the stochastic area integral $\int\int \phi \, dL$ to $\phi \in \mathscr{V}$ by linearity. The vector space $\mathscr{V}$ is contained in $L^2(\mathscr{E})$ and the map $\phi \mapsto \int\int \phi \, dL$ is an isometry between $\mathscr{V}$ and a subspace of $L^2(\mathscr{F})$:*

$$E\left\{\left(\int\int \phi \, dL\right)^2\right\} = \|\phi\|^2, \qquad \phi \in \mathscr{V}.$$

This extends to an isometry between $L^2(\mathscr{E})$ and a closed subspace of $L^2(\mathscr{F})$. For $\phi \in L^2(\mathscr{E})$, the process

$$\phi \cdot L_x =_{\text{def}} \int\int \phi(t, y) I_{\{y \le x\}} L(dt, dy) \tag{2.9}$$

is a continuous L^2-bounded $(\mathscr{E}_x)$-martingale whose increasing process is

$$\langle \phi \cdot L \rangle_x = 4 \int_0^\infty \phi(s, B_s)^2 I_{\{B_s \le x\}} \, ds. \tag{2.10}$$

NOTES. (i) By polarization, if ϕ and ψ are in $L^2(\mathscr{E})$,

$$\langle \phi \cdot L, \psi \cdot L \rangle_x = 4 \int_0^\infty \phi(s, B_s) \psi(s, B_s) I_{\{B_s \le x\}} \, ds. \tag{2.11}$$

(ii) The process $\phi(t, x)$ is only defined if $(t, x) \in \mathfrak{R}$, where $t > \tau(0, x)$. Nevertheless, the right-hand sides of (2.10) and (2.11) are well-defined, for $(s, B_s) \in \mathfrak{R}$ for a.e. (Lebesgue) s. Indeed, $\int I_{\mathfrak{R}^c}(s, B_s) \, ds = \int_{-\infty}^{\infty} (L(\tau(0, x), x) - L(0, x)) \, dx$ and this vanishes since $L(\tau(0, x), x) = 0$.

Let us recall two further results before proving Theorem 2.6. The first is a variation on the Ray–Knight theorem proved in Walsh (1978) which is the basis for this integral. We will state it here for ease of reference.

THEOREM 2.7. (i) *Suppose $S \le T \le U \le V$ are $\mathscr{E}_a$-identifiable times and let $M_x = L(T, x) - L(S, x)$ and $N_x = L(V, x) - L(U, x)$. Then $\{M_x, \mathscr{E}_x, x \ge a\}$ is a positive continuous local martingale with increasing process*

$$\langle M \rangle_x = 4 \int_0^x M_y \, dy. \tag{2.12}$$

Moreover, M and N are orthogonal: $\langle M, N \rangle \equiv 0$.

(ii) *For each $p \ge 1$, if $M_a \in L^p$, then for each $x \ge a$, $\sup_{a \le y \le x} M_y \in L^p$, and if $\exp(\varepsilon M_a) \in L^p$, then for each $x \ge a$, there is an $\varepsilon' > 0$ such that $\sup_{a \le y \le x} \exp(\varepsilon' M_y) \in L^p$. In particular, $\{M_x, x \ge a\}$ is a continuous L^p-martingale if $M_a \in L^p$.*

PROOF. (i) was proved in Walsh (1978). To see (ii), note that by (2.12) there is a Brownian motion W_y independent of M_a such that

$$M_x = M_a + 2 \int_a^x (M_y^+)^{1/2} \, dW_y.$$

Pathwise uniqueness holds for this stochastic differential equation, so by the strong comparison theorem [see, for example, Ikeda & Watanabe (1981)], M is dominated by the solution $\tilde{M}$ of

$$\tilde{M}_x = M_a + 2\int_a^x \left(\tilde{M}_y^+\right)^{1/2} dW_y + 2(x-a).$$

But this process has the same distribution as $\{|Z_x|^2, x \geq a\}$, where Z is a complex Brownian motion with $|Z_a|^2 = M_a$. It is immediate that $\sup_{a \leq y \leq x}|Z_y|^2 \leq 2|Z_y|^2 + 2\sup_{a \leq y \leq x}|Z_y - Z_a|^2$. The result now follows since the square of the maximum of a Brownian motion on a finite time interval has exponential moments of order ε for small enough ε. □

Now we can prove Theorem 2.6.

PROOF OF THEOREM 2.6. Let $\phi(t,x) = ZI_{(S,T]}(t)I_{(a,b]}(x) \in \mathscr{C}$. Then Z is bounded, say $|Z| \leq K$, and $\tau(0,a) \leq S \leq T \leq \tau(t_0, a)$ are $\mathscr{E}_a$-identifiable times. To see that $\phi \in L^2(\mathscr{E})$, note that

$$\begin{aligned} E\left\{\int_0^\infty \phi(t,B_t)^2\,dt\right\} &\leq E\left\{K^2\int_S^T I_{(a,b]}(B_t)\,dt\right\} \\ &= E\left\{K^2\int_a^b (L(T,x) - L(S,x))\,dx\right\} \\ &\leq K^2\int_a^b E\{L(\tau(t_0,a),x) - L(\tau(0,a),x)\}\,dx \\ &\leq K^2(b-a)E\left\{\sup_{a\leq x\leq b} N_x\right\}, \end{aligned}$$

where $N_x = L(\tau(t_0,a),x) - L(\tau(0,a),x)$. Now N_x is a continuous local martingale by Theorem 2.7. Its initial value N_a has the same distribution as $\max_{s\leq t_0} B_s$ by Lévy's representation of reflecting Brownian motion, which in turn has the same distribution as $|B_{t_0}|$. Thus $M_a \in L^p$ for all p and hence the above is finite by Theorem 2.7(ii). Since the elements of $\mathscr{C}$ are identifiable, this shows that $\mathscr{C}$ and $\mathscr{V}$ are contained in $L^2(\mathscr{E})$.

Now let $M_x = L(T,x) - L(S,x)$, $x \geq a$. Then M is a square-integrable $\mathscr{E}_x$-martingale and for $x \geq a$, by the definition (2.8)–(2.9) of $\phi \cdot L$, we have

$$\begin{aligned} \phi \cdot L_x &= \int\int \phi(t,y) I_{\{y\leq x\}} L(dt,dy) \\ &= \int\int ZI_{(S,T]}(t) I_{(a,b\wedge x]} L(dt,dy) \\ &= Z[L(T,b\wedge x) - L(S,b\wedge x) - L(T,a) + L(S,a)] \\ &= Z[M_{b\wedge x} - M_a] \\ &= \int_a^x ZI_{(a,b]}(y)\,dM_y. \end{aligned}$$

Thus $\{\phi \cdot L_x, x \geq a\}$ is a martingale whose quadratic variation process is

$$\begin{aligned}\langle \phi \cdot L \rangle_x &= \int_a^x Z^2 I_{(a,b]}(y)\, d\langle M \rangle_y \\ &= 4\int_a^x Z^2 I_{(a,b]}(y)[L(T,y) - L(S,y)]\, dy \\ &= 4\int_S^T Z^2 I_{(a,b]}(B_s) I_{(-\infty,x]}(B_s)\, ds \\ &= 4\int_0^\infty \phi(s, B_s)^2 I_{(-\infty,x]}(B_s)\, ds,\end{aligned}$$

which proves (2.10) for ϕ. It follows that

$$\begin{aligned}E\left\{\left(\int\int \phi\, dL\right)^2\right\} &= E\{\langle \phi \cdot L \rangle_\infty\} \\ &= \|\phi\|^2,\end{aligned}$$

and all the statements of the theorem are established for $\phi \in \mathscr{E}$.

Consider a second function $\psi \in \mathscr{E}$ of the form $\psi(t,x) = WI_{(U,V]}(t)I_{(c,d]}(x)$ and let $N_x = L(V,x) - L(U,x)$. Suppose without loss of generality that $a \leq c$. Then

$$\langle \phi \cdot L, \psi \cdot L \rangle_x = \int_a^x ZWI_{(a,b]}(y) I_{(c,d]}(y)\, d\langle M, N \rangle_y,$$

so that if *either* $(a,b] \cap (c,d] = \varnothing$ or $(U,V] \cap (S,T] = \varnothing$, then $\langle \phi \cdot L, \psi \cdot L \rangle = 0$, using Theorem 2.7(i) in the second case. That is, if $\phi\psi = 0$, then

(2.13) $$E\left\{\left(\int\int \phi\, dL\right)\left(\int\int \psi\, dL\right)\right\} = 0 \quad \text{and} \quad \langle \phi \cdot L, \psi \cdot L \rangle_x \equiv 0.$$

Note that ϕ can be split at any $y \in (a,b]$ into

$$\begin{aligned}\phi(t,x) &= ZI_{(S,T]}(t) I_{(a,y]}(x) + ZI_{(S,T]}(t) I_{(y,b]}(x) \\ &= \phi_1 + \phi_2.\end{aligned}$$

Since S and T are $\mathscr{E}_y$-identifiable by Remark 2 and since $Z \in b\mathscr{E}_y$, both ϕ_1 and ϕ_2 are in $\mathscr{E}$ and $\int\int \phi\, dL = \int\int \phi_1\, dL + \int\int \phi_2\, dL$. We can also split ϕ at points of the time axis: If U is $\mathscr{E}_a$-identifiable and if $S \leq U \leq T$,

$$\begin{aligned}\phi(t,x) &= ZI_{(S,U]}(t) I_{(a,b]}(x) + ZI_{(U,T]}(t) I_{(a,b]}(x) \\ &= \phi_1' + \phi_2'\end{aligned}$$

splits ϕ into functions in $\mathscr{E}$ with disjoint support. It follows that any $\phi, \psi \in \mathscr{V}$ can be written as

(2.14) $$\phi(t,x) = \sum_i Z_i I_{\Lambda_i}, \qquad \psi(t,x) = \sum_i W_i I_{\Lambda_i},$$

where the $\Lambda_i = (S_i, T_i] \times (a_i, b_i]$ are pairwise disjoint and $I_{\Lambda_i} \in \mathscr{E}$. Thus

$$\langle \phi \cdot L, \psi \cdot L \rangle_x = \sum_{i,j} \langle (Z_i I_{\Lambda_i}) \cdot L, (W_j I_{\Lambda_j}) \cdot L \rangle_x.$$

All but the diagonal terms vanish by (2.13), leaving

$$= \sum_i \langle (Z_i I_{\Lambda_i}) \cdot L, (W_i I_{\Lambda_i}) \cdot L \rangle_x$$

$$= \sum_i 4 \int_{-\infty}^{x} Z_i W_i I_{(a_i, b_i]}(y) [L(T_i, y) - L(S_i, y)] \, dy$$

$$= 4 \int_0^\infty \phi(s, B_s) \psi(s, B_s) I_{(-\infty, x]}(B_s) \, ds.$$

Note that it follows that for every $\phi, \psi \in \mathscr{V}$,

$$E\left\{\left(\int\int \phi \, dL\right)\left(\int\int \psi \, dL\right)\right\} = 4E\left\{\int_0^\infty \phi(s, B_s)\psi(s, B_s) \, ds\right\}$$
$$= (\phi, \psi)_{L^2(\mathscr{E})}$$

so that the map $\phi \mapsto \int\int \phi \, dL$ is an isometry from $\mathscr{V}$ into $L^2(\mathscr{F})$. Familiar continuity arguments now extend Theorem 2.6 from $\phi \in \mathscr{V}$ to $\phi \in \overline{\mathscr{V}}$, and all that remains is to check that the closure of $\mathscr{V}$ is $L^2(\mathscr{E})$.

In order to see this, define

$$\Lambda_N \equiv \bigcup_{-N2^N \le j \le N2^N} \{(\tau(0, j2^{-N}), \tau(N, j2^{-N})] \times (j2^{-N}, (j+1)2^{-N}]\}$$

and let $\xi_N = I_{\Lambda_N}$. Then $\Lambda_N \uparrow \mathfrak{R}$ and $\xi_N \in \mathscr{V}$ for each N. For each fixed N, if we let $H \equiv \{f \in L^\infty(\Omega \times \mathbf{R}_{++} \times \mathbf{R})\colon f\xi_N \in \overline{\mathscr{V}}\}$, then H is a vector space containing 1 which is closed under bounded monotone and uniform limits. It also contains the algebra $\mathscr{V}$, so by the monotone class theorem and Remark 4, it contains all bounded identifiable processes. The result follows.

This verifies (2.8)–(2.10) for $\phi \in \mathscr{V}$ and shows us that the integral gives an isometry of $\mathscr{V}$ into $L^2(\mathscr{F})$. We can extend this by continuity to the closure $\overline{\mathscr{V}}$ of $\mathscr{V}$ in $L^2(\mathscr{E})$. The usual arguments show that (2.8)–(2.10) remain true. But according to Remark 3, $\mathscr{V}$ generates the identifiable processes, so that a familiar functional completion argument shows that $\mathscr{V}$ is dense in $L^2(\mathscr{E})$. This completes the proof. □

LEMMA 2.8. *Fix $a \in \mathbb{R}$, let $T' \ge 0$ be an integrable $\mathscr{E}_a$-measurable random variable and put $T = \tau(T', a)$. Then the set*

$$H \equiv \{(t, x)\colon x > a, \ A(t, x) > 0, t \le T\}$$

is identifiable. If $b \ge a$, then $I_{H \cap \{x \le b\}} \in L^2(\mathscr{E})$ and

$$\int\int I_{H \cap \{x \le b\}} \, dL = L(T, b) - L(T, a) + 2b^- - 2a^-. \tag{2.15}$$

PROOF. Since $H = \{(t, x)\colon x > a, \ 0 < A(t, x) \le A(T, x)\}$ and since $A(T, x)$ is $\mathscr{E}_x$-measurable for $x \ge a$ by Proposition 2.5, we have that $I_H = I_{\tilde{H}} \circ \Gamma$, where

$\tilde{H} = \{(t, x): x > a,\ 0 < t \le A(T, x)\}$ and it is easy to see that $\tilde{H}$ is in $\mathscr{E}\mathscr{P}$. Thus $H \in \mathscr{I}$. To check the integrability of $I_{H \cap \{x \le b\}}$, note that

$$E\left\{\int_0^\infty I_H(t, B_t) I_{\{B_t \le b\}}\, dt\right\} = E\left\{\int_0^\infty I_{(a,b]}(B_t) I_{\{t \le \tau(T', a)\}}\, dt\right\}$$
$$= E\left\{\int_a^b L(\tau(T', a), y)\, dy\right\}.$$

But $L(\tau(T', a), a) \le 2 \sup_{s \le T'} \beta_s \equiv Y$, where β is the Brownian motion $\tilde{B}(t, a) + \frac{1}{2}\tilde{L}(t, a) - (a \wedge 0)$, and by the Burkholder–Davis–Gundy inequalities, Y is square-integrable, since T' is assumed integrable. Thus the previous expectation is finite by part (ii) of Theorem 2.7.

To prove (2.15), note that $H \cap \{x \le b\}$ is the increasing union of the sets H_n, where

$$H_n = \bigcup_{k=1}^{2^n} (\tau(0, a_{k-1}^n), T] \times (a_{k-1}^n, a_k^n]$$

and where $a_k^n \equiv a + k2^{-n}(b - a)$. Each H_n is the union of finitely many disjoint identifiable rectangles, so by (2.8),

$$\begin{aligned}\iint I_{H_n}\, dL &= L(T, b) - L(T, a)\\ &\quad - \sum_{k=1}^{2^n} \left[L(\tau(0, a_{k-1}^n), a_k^n) - L(\tau(0, a_{k-1}^n), a_{k-1}^n)\right].\end{aligned} \tag{2.16}$$

The right-hand side of (2.16) can be simplified by noting that $L(\tau(0, x), x) = 0$ a.s. for any x. For those k such that $a_k^n \le 0$, $\tau(0, a_k^n)$ is a.s. equal to the first time B_t hits a_k^n, which is a stopping time, so that the strong Markov property implies that the random variables $L(\tau(0, a_{k-1}^n), a_k^n)$ are independent and identically distributed, each having the same distribution as $L(\tau(0, 2^{-n}(b - a)), 0)$, which is exponential with mean $2(b - a)2^{-n}$ and variance $4(b - a)^2 2^{-2n}$. For those k for which $a_{k-1}^n \ge 0$, the corresponding terms in the sum vanish. Thus the sum of the right-hand side of (2.16) converges a.s. to $-2(b^- - a^-)$. But I_{H_n} converges to $I_{H \cap \{x \le b\}}$, so the left-hand side of (2.16) also converges to the left-hand side of (2.15). □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. The basic idea of the proof is to show that the class of martingales used in Rogers (1987) can all be represented as stochastic area integrals.

First, if M is an L^2-bounded $(\mathscr{E}_x)$-martingale which can be written in the form

$$\begin{aligned}M_x &= \phi \cdot L_x\\ &= \iint I_{\{y \le x\}} \phi(t, y) L(dt, dy)\end{aligned} \tag{2.17}$$

for some $\phi \in L^2(\mathscr{E})$ and if θ is a bounded left-continuous $(\mathscr{E}_x)$-adapted process, then $\psi(t, x) \equiv \theta_x \phi(t, x)$ is in $L^2(\mathscr{E})$ and

$$\psi \cdot L_x = \int_{-\infty}^{x} \theta_y \, dM_y. \tag{2.18}$$

[One verifies this as usual: it is true by (2.8) if ψ and θ are indicator functions, etc.] Thus if M has an integral representation, so does $\int \theta \, dM$.

Next, if $M^i = \phi_i \cdot L$, $i = 1, 2$, and if $\phi_1 \phi_2 = 0$, then $\langle M^1, M^2 \rangle = 0$ by Theorem 2.6, so integration by parts gives

$$M^1 M^2 = \left(M^1 \phi_2 + M^2 \phi_1\right) \cdot L. \tag{2.19}$$

The essential step in the proof is contained in the following result. Let γ be the *inverse local time*:

$$\gamma(t, x) = \inf\{s : L(s, x) > t\}. \tag{2.20}$$

PROPOSITION 2.9. *Let $a < b$, $0 \le s_0 \le t_0$ and put $T = \gamma(t_0, a)$. Let S be either identically zero or let $S = \gamma(s_0, a)$. Define $Z = L(T, b) - L(S, b)$. Then for every $k \in \mathbf{N}$, $Z^k \in L^2(\mathscr{F})$ and there exists $\phi_k \in L^2(\mathscr{E})$ such that*

$$Z^k = E\{Z^k\} + \int\int \phi_k \, dL. \tag{2.21}$$

Moreover, ϕ_k is supported in $(S, T] \times (a, b]$.

PROOF. First suppose that $S = \gamma(s_0, a)$ and put

$$Z_x = \begin{cases} L(T, x) - L(S, x) & \text{if } x \ge a, \\ t_0 - s_0 & \text{if } x < a. \end{cases} \tag{2.22}$$

We claim that for $k \ge 1$ and $x > a$ there are a jointly continuous $(\mathscr{E}_y)$-adapted process $\theta_k(y, x)$, $a \le y \le x$, and a continuous deterministic function ψ_k such that

$$Z_x^k = \psi_k(x) + \int_a^x \theta_k(y, x) \, dZ_y, \tag{2.23a}$$

$$\sup_{a \le y \le x \le b} |\theta_k(y, x)| \in \bigcap_p L^p \quad \text{for each } b > a. \tag{2.23b}$$

This is clear if $k = 1$, for then $\theta_1 \equiv 1$. Since Z_a is deterministic, $Z \in L^p$ for all p by Theorem 2.7. Suppose (2.23) holds for $k - 1$. Apply Itô's formula, noting that $d\langle Z \rangle_y = 4 Z_y \, dy$:

$$Z_x^k = (t_0 - s_0)^k + k \int_a^x Z_y^{k-1} \, dZ_y + 2k(k-1) \int_a^x Z_y^{k-1} \, dy. \tag{2.24}$$

Use (2.23) and change the order of integration to see that

$$\begin{aligned} \int_a^x Z_y^{k-1} \, dy &= \int_a^x E\{Z_y^{k-1}\} \, dy + \int_a^x \int_a^y \theta_{k-1}(v, y) \, dZ_v \, dy \\ &= \int_a^x E\{Z_y^{k-1}\} \, dy + \int_a^x \int_v^x \theta_{k-1}(v, y) \, dy \, dZ_v. \end{aligned}$$

Substitute this in (2.24) and compare. We see that (2.23) holds with

$$\psi_k(y) \equiv (t_0 - s_0)^k + 2k(k-1)\int_a^x E\{Z_y^{k-1}\}\,dy,$$

$$\theta_1(y,x) \equiv 1, \tag{2.25}$$

$$\theta_k(y,x) = kZ_y^{k-1} + 2k(k-1)\int_y^x \theta_{k-1}(y,z)\,dz.$$

Note that θ_k is jointly continuous, $\mathscr{E}_y$-adapted and L^p-bounded (thanks to the inductive hypothesis and Burkholder's inequality), so that the above manipulations were all justified. But now according to (2.18), we can write (2.23a) in the form (2.21) with $\phi_k(t,y) = I_{(S,T]}(t)I_{(a,x]}(y)\theta_k(y,x)$.

This proves the proposition if $S = \gamma(s_0, a)$. Let us now consider the case where $S \equiv 0$. This presents nothing new if $a \geq 0$, since then $L(0,b) = 0 = L(\gamma(0,a),b)$ for all $b \geq a \geq 0$. We can therefore assume that $a < 0$. We can write $L(T,x) = L(\gamma(0,a),x) + (L(T,x) - L(\gamma(0,a),x))$; as we have already handled the second term, we may assume without loss of generality that $T = \gamma(0,a) = \tau(0,a)$. Then we must prove (2.21) for $Z = L(T,b)$.

First assume that $b \leq 0$ and define $\zeta_x = L(T,x) + 2(x^- - a^-)$. By Lemma 2.8, $\zeta_x = \int\int \phi\, dL$, where $\phi(t,y) = I_{H\cap\{a<y\leq x\}}(t,y)$.

We can now prove that for each $k \geq 1$ and $x \in (a,0]$, there exists a jointly continuous $\mathscr{E}_y$-adapted process $\theta_k(y,x)$, $a \leq y \leq x \leq 0$ and a continuous deterministic function ψ_k such that

$$Z_x^k = \psi_k(x) + \int_a^x \theta_k(y,x)\,d\zeta_y, \tag{2.26a}$$

$$\sup_{a\leq y\leq x\leq b} |\theta_k(y,x)| \in \bigcap_p L^p \quad \text{for all } b > a. \tag{2.26b}$$

The case $k = 1$ is immediate; $\theta_1 \equiv 1$, $\psi_1(x) = 2(x-a)$ and the induction step proceeds as before.

Finally, if $b > 0$, write $L(T,b) = L(T,0) + (L(T,b) - L(T,0))$. We have just dealt with the first term, so only the second remains. If T were of the form $\gamma(t_0, 0)$ for some deterministic t_0, (2.21) would follow immediately. However, it need not be, so we approximate it by a step function of the form

$$T_n = \sum_{j=0}^{n2^n} \gamma(j2^{-n}, 0) I_{A_{jn}},$$

where A_{jn} is the event $\{j2^{-n} < L(T,0) \leq (j+1)2^{-n}\}$. The representation (2.21) for $Z_n' \equiv L(T_n,b) - L(T_n,a)$ is now immediate and since $L(T_n,x)$ increases to $L(T,x)$ for each $x \geq 0$, the Z_n' converge to $Z' \equiv L(T,b) - L(T,0)$ in every L^p. The representation (2.21) for Z' now follows easily. This finishes the proof of Proposition 2.9. □

Let us return to the proof of Theorem 2.1. Let $a_j = j2^{-n}$, $j = 0, \pm 1, \pm 2, \dots$, and let $\mathscr{D}_n$ be the class of random variables of the forms

$$L(\gamma(t,a_j), a_{j+1}) \quad \text{or} \quad L(\gamma(t,a_j), a_{j+1}) - L(\gamma(s,a_j), a_{j+1}),$$

where $0 \le s < t$ and $j = 0, \pm 1, \pm 2, \dots$. Define

$$H = \left\{ Y \in L^2(\mathscr{F}) : Y = E\{Y\} + \int\!\!\int \phi \, dL, \text{ some } \phi \in L^2(\mathscr{E}) \right\}.$$

Evidently H is a closed subspace of $L^2(\mathscr{F})$. We will be done if we can show that the two are equal. First, we claim that H contains the algebra generated by $\mathscr{D}_n$. Note that any product of elements of $\mathscr{D}_n$ can be written as a finite sum of elements of the form

$$Z = \prod_{i=1}^{p} Z_i^{k_i},$$

where $Z_i \in \mathscr{D}_n$, $i = 1, \dots, p$, and if $Z_i = E\{Z_i\} + \int\!\int \phi_i \, dL$, then the ϕ_i have disjoint supports. By Proposition 2.8, the $Z_i^{k_i}$ are in H and the supports of their representations are again disjoint, so by (2.19) their product Z is also in H.

Let $\mathscr{A}_n = \sigma(\mathscr{D}_n)$ and let $\tilde{\mathscr{D}}_n$ be the closure in L^2 of the algebra generated by $\mathscr{D}_n$. We cannot use the usual monotone class argument to show that $\tilde{\mathscr{D}}_n = L^2(\mathscr{A}_n)$, since the random variables in question are not bounded. However, they each have exponential moments of some order by Theorem 2.7, so that a modification of the monotone class theorem [see Rogers (1987), Lemma 3] applies and we can conclude that $L^2(\mathscr{A}_n) \subset H$.

We claim that $\mathscr{A}_n \subset \mathscr{A}_{n+1}$ and $\mathscr{A} = \bigvee_n \mathscr{A}_n = \mathscr{F}$. Let $x < y$ and put $v = L(\gamma(t, x), y)$. If $z > y$, then $L(\gamma(t, x), z) = L(\gamma(v, y), z)$.

Apply this to $x = a_j$, $y = a_{j+1}$, $z = a_{j+2}$. Then $v = L(\gamma(t, a_j), a_{j+1})$ is in $\mathscr{D}_n$, so it is $\mathscr{A}_n$-measurable; by the above,

$$L\big(\gamma(t, a_j), a_{j+2}\big) = L\big(\gamma(v, a_{j+1}), a_{j+2}\big),$$

so that it is also $\mathscr{A}_n$-measurable. But $\mathscr{A}_{n-1}$ is generated by these random variables, so $\mathscr{A}_{n-1} \subset \mathscr{A}_n$. Moreover, an induction shows that for $k \ge 1$, $L(\gamma(t, a_j), a_{j+k})$ is $\mathscr{A}_n$-measurable. Letting $n \to \infty$ we can see that if $x < y$, then $L(\gamma(t, x), y) \in \mathscr{A}$. Now let $X_t = L(\gamma(t, x), y)$. X is increasing and $\mathscr{A}$-measurable and $\inf\{t\colon X_t > s\} = L(\gamma(s, y), x)$. Thus $L(\gamma(t, y), x)$ as well as $L(\gamma(t, x), y)$ is $\mathscr{A}$-measurable. Consequently, $L(\gamma(t, x), y)$ is $\mathscr{A}$-measurable for all y, not just for $y \ge x$. Thus $\gamma(t, x) = \int_{-\infty}^{\infty} L(\gamma(t, x), y) \, dy$ is also $\mathscr{A}$-measurable. Now just invert this in t. We see that $L(t, x) \in \mathscr{A}$ for all x, which means that $B_t \in \mathscr{A}$ for all t, hence $\mathscr{F} \subset \mathscr{A}$. This does it: H contains $\bigvee_n L^2(\mathscr{A}_n) = L^2(\mathscr{A}) = L^2(\mathscr{F})$, hence $H = L^2(\mathscr{F})$. This completes the proof. □

This gives another proof of the continuity of $(\mathscr{E}_x)$-martingales, which we give here for completeness.

COROLLARY 2.10. *Let $M = \{M_x, \mathscr{E}_x, x \ge a\}$ be a local martingale. Then M has a continuous version.*

PROOF. If M is uniformly integrable, it is closed on the right by an L^1-random variable M_∞ which is approximated in L^1 by the bounded random variables $M_\infty^n \equiv (M_\infty \wedge n) \vee (-n)$. By Doob's submartingale inequality, some subsequence of the martingales $M_x^n \equiv E(M_\infty^n | \mathscr{E}_x)$ converges uniformly a.s. to M. Thus it is sufficient to take the case where M is a bounded martingale. In this case by Theorem 2.1 there exists $\phi \in L^2(\mathscr{E})$ such that

$$\begin{aligned} M_\infty &= \int\int \phi \, dL + E\{M_\infty\} \\ &= \phi \cdot L_\infty + E\{M_\infty\}. \end{aligned}$$

Thus $M_x = E\{M_\infty | \mathscr{E}_x\} = \phi \cdot L_x + E\{M_\infty\}$ and $\phi \cdot L$ is continuous by Theorem 2.6. □

3. Line integrals and local time on curves. We will look at the relations between three things in this section: Brownian local time on curves, stochastic area integrals over identifiable sets and stochastic line integrals.

There is a standard way of defining Brownian local time on a random curve $y = \phi(t)$ when the process ϕ_t is adapted and of finite variation. In that case, $B_t - \phi_t$ is a semimartingale and the local time can be defined as the limit

$$l_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I_{(0, \varepsilon]}(B_s - \phi_s) \, ds,$$

which exists almost surely for each t [see Yor (1978)]. We shall see in this section that it is possible to define the local time of B on a curve $\{\phi(t): t \in \Lambda\}$ in many other cases, cases in which the semimartingale construction does not work because ϕ is either not adapted or is of infinite variation. We shall show that provided ϕ satisfies an easily verified condition, there is a local time of B on ϕ in the sense that the limit

$$\lambda(\phi) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Lambda I_{(0, \varepsilon]}(B_s - \phi_s) \, ds$$

exists a.s. This defines a local time on the whole curve; it does not in general give the local time as a function of t.

We think of a curve as a measurable map $\phi: \Omega \times (0, \infty) \mapsto \mathbf{R} \cup \{\infty\}$. If we are given a curve $\{\phi(\omega, t), t \in \Lambda(\omega)\}$ which is not defined for all $t > 0$ we can extend ϕ to a map on all of $\Omega \times (0, \infty)$ by setting $\phi(\omega, t) = \infty$ on $(0, \infty) - \Lambda(\omega)$. The graph $\Xi(\phi)$ is the graph of the finite part of ϕ, defined by

$$\Xi(\phi) = \{(\omega, t, x) \in \Omega \times (0, \infty) \times \mathbf{R}: \phi(\omega, t) = x\}.$$

We shall always assume that there is some integer N such that

$$\Xi(\phi) \subset \mathfrak{R} \cap \{(0, \tau(N, -N)] \times (-N, N)\}. \tag{3.1}$$

This is no great restriction and it ensures that the various integrals we meet will be defined.

Let us begin with stochastic line integrals. We will connect them with local time on curves shortly. In the simplest special case, when the curve ϕ is a horizontal line segment, we can define the integral of the local time over ϕ to be the difference of the local time at the two endpoints. Slightly more generally, suppose that

$$\phi(t) = \sum_{j=1}^{n} a_j I_{(S_j, T_j]}(t) + \infty I_{\{t \notin \cup_j (S_j, T_j]\}},$$

where $0 \le S_1 \le T_1 \le S_2 \le \dots \le T_n$ and S_j and T_j are $\mathscr{E}_{a_j}$-identifiable for each j. Then we define the integral of L along ϕ by

$$\int_\phi \partial_t L = \sum_{j=1}^{n} \left(L(T_j, a_j) - L(S_j, a_j) \right).$$

By assumption there is an N such that $|a_j| < N$ for all j and $T_n \le \tau(N, -N)$. Thus the process

$$I_{(\phi, N]}(t, x) \equiv I_{\{\phi(t) < x \le N\}}$$

is in $L^2(\mathscr{E})$ and

$$\int_\phi \partial_t L = \int I_{\{\phi(t) < N\}} L(dt, N) - \int\int I_{(\phi, N]}\, dL. \tag{3.2}$$

This trivial observation is the basis for the extension of the integral to more complicated curves since each of the terms on the right-hand side makes sense for more general ϕ.

Definition. Let ϕ: $\Omega \times (0, \infty) \mapsto \mathbf{R} \cup \{\infty\}$ be a curve satisfying (3.1) relative to an integer N such that the set

$$(\phi, \infty) \equiv \{(\omega, t, x): \phi(\omega, t) < x\}$$

is identifiable. Then the *line integral of L on* ϕ is defined by

$$\int_\phi \partial_t L \equiv \int I_{\{\phi(t) < N\}} L(dt, N) - \int\int I_{(\phi, N]}\, dL. \tag{3.3}$$

Remarks. The line integral of L on ϕ is well-defined by (3.3). Both integrals on the right-hand side converge thanks to assumption (3.1) and the only thing to check is that the right-hand side is independent of N for large enough N. This comes down to showing that if N satisfies (3.1) and if $H = \{(\omega, t)$: $\phi(\omega, t) < N\}$, then for any $M > N$,

$$\int_{H(\omega)} \{L(dt, M) - L(dt, N)\} = \int\int I_{H \times (N, M]}\, dL, \tag{3.4}$$

where $H(\omega)$ is the ω-section of H.

We will prove this shortly, using Proposition 3.1. It is worth pointing out that the situation is more delicate than one might expect. For example, suppose that $G \subset \Omega \times (0, \infty)$ is such that $I_{G \times (a, b]} \in L^2(\mathscr{E})$. Then one would expect that

$$\int_{G(\omega)} \{L(dt, b) - L(dt, a)\} = \int\int I_{G \times (a, b]}\, dL.$$

(The right-hand side is certainly well-defined.) However, this is *not* true in general; identifiability of $G \times (a, b]$ is not sufficient.

Here is an example to show what can go wrong. Let

$$\phi(t) = \begin{cases} 0 & \text{if } B_t = 0 \text{ and } 0 < t \le \tau(1,0), \\ \infty & \text{otherwise,} \end{cases}$$

and let $G = \{(\omega, t): \phi(\omega, t) < \infty\}$. If we take $N = 0$ and $M = 1$ one can see that $G \times (0, 1]$ is identifiable and the norm of $I_{G\times(0,1]}$ in $L^2(\mathscr{E})$ is

$$\|I_{G\times(0,1]}\|^2 = 4E\left\{\int_0^\infty I_{G\times(0,1]}(\omega, s, B_s(\omega))^2\, ds\right\} \\ = 0,$$

since the integrand vanishes identically. Thus $\int\int I_{G\times(0,1]}\, dL = 0$ a.s., while the left-hand side of (3.4) equals $L(\tau(1,0), 0) \ne 0$.

We mention in passing that, even though $G \times (0, 1]$ is identifiable, $G \times [0, 1]$ is not (we leave the proof of this as an exercise), so that $\int\int I_{G\times[0,1]}\, dL$ is not defined.

The same example warns us not to jump to conclusions about the definition of $\int_\phi \partial_t L$. According to (3.3) (choose $N = 1$ for this), $\int_\phi \partial_t L = 0$, while $\int_G L(dt, 0) = L(\tau(1, 0), 0)$.

We can see from this that $\int_\phi \partial_t L$ does not always coincide with our intuitive notion of what a line integral over ϕ should be. In fact, what we will see is that $\int_\phi \partial_t L$ is really the local time, not on the curve itself, but on the topside of the curve ϕ in the following sense:

$$\int_\phi \partial_t L = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\infty I_{\{\phi_s < B_s \le \phi_s + \varepsilon\}}\, ds. \tag{3.5}$$

The principal aim of the rest of this section is to prove this.

PROPOSITION 3.1. *Fix $a < b$ and let $n > 0$. Let F be an $\mathscr{E}_a'$-measurable subset of $\Omega \times (0, n]$, let $G = \{(\omega, t): (\omega, A(\omega, t, a)) \in F\}$ and let $G(\omega)$ be the ω-section of G. Then*

$$I_{G\times(a,b]} \in L^2(\mathscr{E}); \tag{3.6a}$$

$$\int\int I_{G\times(a,b]}\, dL = \int_{G(\omega)} (L(dt, b) - L(dt, a)). \tag{3.6b}$$

PROOF. Let $\mathscr{G} \equiv \{F \in \mathscr{E}_a': F \subseteq \Omega \times (0, n]$ and (3.6) holds for $F\}$. From Theorem 2.6, $\mathscr{G}$ contains $(S, T]$, where $0 \le S \le T \le n$ are $\mathscr{E}_a$-measurable random variables. $\mathscr{G}$ contains finite unions and intersections of such sets and $\mathscr{G}$ is a monotone class. Therefore $\mathscr{G}$ contains every $\mathscr{E}_a'$-measurable $F \subseteq \Omega \times (0, n]$. $\square$

Let us now prove (3.4), using Proposition 3.1 with $a = N$ and $b = M$. Since (ϕ, ∞) is identifiable, there is some $V \in \mathscr{E}\mathscr{P}$ such that $I_{(\phi,\infty)} = I_V \circ \Gamma$, hence

$$I_{\{\phi(t) < N\}} = I_{(\phi,\infty)}(t, N) = I_V(A(t, N), N).$$

Since V is $\mathscr{E}_x'$-adapted, the set $F = \{(\omega, t): (\omega, t, N) \in V\}$ is in $\mathscr{E}_N'$. Truncate F if necessary to $F \cap (\Omega \times (0, n])$ and apply Proposition 3.1.

There is an interesting filtration lurking here. Let $\mathscr{A}_x$ be the σ-field of subsets of $\Omega \times (0, \infty)$ of the form $\{(\omega, t): (\omega, A(\omega, t, x)) \in F\}$, where $F \in \mathscr{E}_x'$. This is a filtration. We will content ourselves by just giving the following facts about it here.

PROPOSITION 3.2. (i) $\{\mathscr{A}_x, x \in \mathbf{R}\}$ *is a filtration.*

(ii) *The curve* ϕ *is an* $(\mathscr{A}_{x+})$*-stopping time* (*i.e.*, $\{(\omega, t): x > \phi(\omega, t)\} \in \mathscr{A}_x$ *for all* x) *if and only if* $(\phi, \infty) \equiv \{(\omega, t, x): \phi(\omega, t) < x\}$ *is identifiable.*

PROOF. (i) Fix $a < b$, let $T \geq 0$ be $\mathscr{E}_a$-measurable and let $F = (T, \infty)$. Then

$$\{(\omega, t): (\omega, A(\omega, t, a)) \in F\} = (\tau(T, a), \infty).$$

It is enough to prove that this is in $\mathscr{A}_b$. Let $\tilde{T} = A(\tau(T, a), b)$, which is $\mathscr{E}_b$-measurable by Proposition 2.5(i). Then

$$t > \tau(\tilde{T}, b) \Leftrightarrow A(t, b) > \tilde{T} \Leftrightarrow t > \tau(T, a),$$

where the last equivalence follows because $\tau(T, a)$ is a time of two-sided increase of $A(\cdot, b)$. Therefore $\tau(T, a) = \tau(\tilde{T}, b)$ so $(\tau(T, a), \infty) = (\tau(\tilde{T}, b), \infty) = \{(\omega, t): (\omega, A(\omega, t, b)) > \tilde{T}\} \in \mathscr{A}_b$ and $F = (\tau(\tilde{T}, b), \infty) \in \mathscr{A}_b$.

(ii) $\Rightarrow$: Define approximating curves ϕ_n by

$$\phi_n(\omega, t) = \begin{cases} j2^{-n} & \text{if } (j-1)2^{-n} \leq \phi(\omega, t) < j2^{-n}; \\ \infty & \text{if } \phi(\omega, t) = \infty. \end{cases}$$

Then $\phi_n \downarrow \phi$ and it is enough to prove that (ϕ_n, ∞) is identifiable. But

$$(\phi_n, \infty) = \bigcup_j \left(\{t: (j-1)2^{-n} \leq \phi(t) < j2^{-n}\} \times (j2^{-n}, \infty)\right),$$

and each of the sets on the right-hand side is identifiable by the hypothesis on ϕ.

$\Leftarrow$: If $\psi(t, x) = I_{\{\phi(t) < x\}}$, then ψ is identifiable and $\psi = I_G \circ \Gamma$ for some $G \in \mathscr{E}\mathscr{P}$. Since a predictable process is $\mathscr{E}_a'$-adapted,

$$\{(\omega, t): \phi(\omega, t) < x\} = \{(\omega, t): (\omega, A(\omega, t, x)) \in G_x\},$$

where $G_x = \{(\omega, t): (\omega, t, x) \in G\}$; this is in $\mathscr{E}_x'$. □

REMARKS. Let us mention without proof some facts which we will not need in our subsequent development, but which may help illuminate the hypotheses on our curve ϕ.

(i) The σ-field $\mathscr{A}_x$ can be characterized in terms of identifiable sets: A set $G \subset \Omega \times (0, \infty)$ is in $\mathscr{A}_x$ if and only if the set $G \times \{x\}$ is identifiable. This is a consequence of the continuity of the filtration $(\mathscr{E}_x)$; all $\mathscr{E}_x$-martingales are a.s. continuous by Corollary 2.10, or more specifically, of its quasi-left continuity, since what is involved here is really a kind of predictability.

(ii) We could have placed stronger hypotheses on the curve ϕ by requiring that $[\phi, \infty)$, rather than (ϕ, ∞), be identifiable. This would imply that (ϕ, ∞) is also identifiable and hence that the graph $\Xi(\phi)$ of ϕ is identifiable.

(iii) We could distinguish curves by calling them *optional* if (ϕ, ∞) is identifiable and *predictable* if $[\phi, \infty)$ is identifiable. The previous counterexample was a curve which was optional but not predictable in this sense and the local time of B on the topside of ϕ was not equal to the local time on ϕ. In this terminology, Proposition 3.1 [and by extension, equation (3.4)] concerns predictable curves ($G \times \{a\}$ is identifiable by the previous work), and it shows that if ϕ is predictable and takes on only a finite number of values, the two local times are indeed equal.

For $\varepsilon > 0$, let $\Xi_\varepsilon(\phi)$ be the upper ε-neighborhood of $\Xi(\phi)$:

$$\Xi_\varepsilon(\phi) = \{(t, x) \colon \phi(t) < x \le \phi(t) + \varepsilon\}.$$

Note that $\Xi_\infty(\phi) = (\phi, \infty)$ is the set of all points above the graph of ϕ.

DEFINITION. The *local time* of B on the (topside of the) curve ϕ is

$$\begin{aligned} \lambda(\phi) &= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_0^\infty I_{\Xi_\varepsilon(\phi)}(t, B_t)\, dt \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_0^\infty I_{\{\phi(t) < B_t \le \phi(t) + \varepsilon\}}\, dt, \end{aligned} \tag{3.7}$$

if the limit exists.

This is a one-sided local time in the sense that it involves the occupation time of one-sided neighborhoods of ϕ. It evidently equals the local time of $B_t - \phi(t)$ at 0^+. If ϕ is an adapted process of bounded variation, then $B_t - \phi(t)$ is a semimartingale and its local time exists by the general theory. But we are dealing with local time on a quite different class of ϕ.

Our aim is to prove (3.5), i.e., to show that $\lambda(\phi)$ exists and equals $\int_\phi \partial_t L$ whenever $(\phi, \infty) = \Xi_\infty(\phi)$ is identifiable and satisfies (3.1). We will approximate ϕ from above as follows. For $\varepsilon \in (0,1)$ and $x \in \mathbf{R}$, define the curve ϕ_ε^x by

$$\phi_\varepsilon^x(t) = \begin{cases} x + k\varepsilon & \text{if } x + (k-1)\varepsilon \le \phi(t) < x + k\varepsilon, \\ \infty & \text{if } \phi(t) = \infty. \end{cases} \tag{3.8}$$

Then evidently $\phi_\varepsilon^x(t)$ satisfies (3.1) with N replaced by $M \equiv N + 1$ and, moreover, $(\phi_\varepsilon^x, \infty)$ is identifiable. To see this, fix x and ε and let

$$H_k = \{(\omega, t) \colon x + (k-1)\varepsilon \le \phi(\omega, t) < x + k\varepsilon\}.$$

Since (ϕ, ∞) is identifiable, $H_k \in \mathscr{A}_{x + k\varepsilon}$ by Proposition 3.2. By Proposition 3.1 we see that $H_k \times (x + k\varepsilon, M]$ is identifiable and is in $L^2(\mathscr{E})$. In particular, $H_k \times (x + k\varepsilon, \infty)$ is identifiable and therefore $(\phi_\varepsilon^x, \infty)$, which is the union of these sets, is identifiable. Moreover, again by Proposition 3.1, we have

$$\iint I_{H_k \times (x+k\varepsilon, M]}\, dL = \int_{H_k} [L(dt, M) - L(dt, x + k\varepsilon)]. \tag{3.9}$$

THEOREM 3.3. *Let ϕ be a curve satisfying* (3.1) *such that* (ϕ, ∞) *is identifiable. Then*

(i) *for any x, $\lim_{\varepsilon \downarrow 0} \int_{\phi_\varepsilon^x} \partial_t L = \int_\phi \partial_t L$ in L^2;*
(ii) *the local time on the topside of ϕ exists and*

$$\lambda(\phi) = \lim_{\varepsilon \downarrow 0} \int_0^\infty I_{\{\phi(t) < B_t \le \phi(t) + \varepsilon\}}\, dt = \int_\phi \partial_t L.$$

PROOF. (i) From the definition,

$$\begin{aligned} \int_{\phi_\varepsilon^x} \partial_t L &= \int I_{\{\phi_\varepsilon^x(t) < M\}} L(dt, M) - \iint I_{(\phi_\varepsilon^x, M]}\, dL \\ &= \sum_k \left[\int_{H_k} L(dt, M) - \iint I_{H_k \times (x+k\varepsilon, M]}\, dL \right] \\ &= \sum_k \int_{H_k} L(dt, x + k\varepsilon), \end{aligned} \tag{3.10}$$

from (3.9). But the right-hand side of the first equation is

$$\int_{\{\phi_\varepsilon^x < M\}} L(dt, M) - \iint I_{(\phi_\varepsilon^x, Md]}\, dL,$$

which converges in L^2 as $\varepsilon \to 0$ to

$$\int_{\{\phi(t) < M\}} L(dt, M) - \iint I_{(\phi, M]}\, dL = \int_\phi \partial_t L.$$

(ii) Notice that the L^2-convergence of $\int_{\phi_\varepsilon^x} \partial_t L$ to $\int_\phi \partial_t L$ is uniform in x, since

$$\int_{\phi_\varepsilon^x} \partial_t L - \int_\phi \partial_t L = \iint I_{(\phi, \phi_\varepsilon^x]}\, dL,.$$

which is bounded in L^2 by

$$\left[4E\left\{ \int_0^\infty I_{(\phi, \phi_\varepsilon^x]}(t, B_t)\, dt \right\} \right]^{1/2} \le \left[4E\left\{ \int_0^\infty I_{\Xi_\varepsilon(\phi)}(t, B_t)\, dt \right\} \right]^{1/2}, \tag{3.11}$$

which tends to zero uniformly in x. Now

$$\begin{aligned}\int_0^\infty I_{\{\phi(t)<B_t\le\phi(t)+\varepsilon\}}\,dt &= \int dx \int_0^\infty L(dt,x)\,I_{\{\phi(t)<x\le\phi(t)+\varepsilon\}}\,dt\\ &= \sum_k \int_0^\varepsilon dx \int_0^\infty L(dt, x+k\varepsilon)\,I_{\{x+(k-1)\varepsilon\le\phi(t)<x+k\varepsilon\}}\,dt\\ &= \sum_k \int_0^\varepsilon dx \int_{H_k} L(dt, x+k\varepsilon)\\ &= \int_0^\varepsilon dx \int_{\phi_\varepsilon^x} \partial_t L,\end{aligned}$$

from (3.10). Thus

$$\begin{aligned}E\left\{\int_\phi \partial_t L - \varepsilon^{-1}\int_0^\infty I_{\{\phi(t)<B_t\le\phi(t)+\varepsilon\}}\,dt\right\}^2 &= E\left\{\varepsilon^{-1}\int_0^\varepsilon dx\left[\int_\phi \partial_t L - \int_{\phi_\varepsilon^x}\partial_t L\right]\right\}^2\\ &\le \varepsilon^{-1}\int_0^\varepsilon E\left\{\int_\phi \partial_t L - \int_{\phi_\varepsilon^x}\partial_t L\right\}^2 dx\\ &\le \varepsilon^{-1}\int_0^\varepsilon 4E\left\{\int_0^\infty I_{\Xi_\varepsilon(\phi)}(t, B_t)\,dt\right\}dx,\end{aligned}$$

by (3.11), and this tends to zero as $\varepsilon \to 0$. This finishes the proof. □

One special case which is interesting to calculate is that in which the curve ϕ is the graph of the minimum m of the Brownian motion: $m_t = \min_{0\le s\le t} B_s$. We have already calculated $\int_m \partial_t L$ in the proof of Lemma 2.8, when we computed $\int\int I_{H\cap\{x\le b\}}\,dL$. Indeed, the sum on the right-hand side of (2.16) is the approximation $\int_{\phi_\varepsilon^x}\partial_t L$, where $\phi(t) = m_t$ if $a < m_t \le b$; $= +\infty$ otherwise, when we take $x = 0$ and $\varepsilon = 2^{-n}(b-a)$. We found that the local time on the topside of m was deterministic. If $a < 0$ and $b = 0$, the local time on the topside of m is $-2a$. This is a fact we already know: it is Lévy's famous result that the local time of $B_t - m_t$ is $2m_t$.

4. The decomposition of $\tilde{L}$. The intrinsic local time is $\tilde{L}(t,x) = L(\tau(t,x),x)$. McGill (1986) noted that $\tilde{L}(t,x) + 2x^-$ is an $\mathscr{E}_x$-supermartingale in x for fixed t and gave a formula for its compensator. This compensator turns out to be an interesting object, for it has several distinct interpretations. McGill expressed it as the derivative of a certain integral involving the square of the intrinsic local time. However, it can also be expressed as a stochastic line integral and as a Brownian local time on certain curves. Moreover, the martingale part of the decomposition of $\tilde{L}$ is given by a stochastic area integral. We will derive the decomposition in this section.

There is in fact a further aspect of the decomposition, which relates it to the local time of the process $A(t, B_t)$, but the proof of this, or at least the proof we have now, involves different methods and we will leave this to a future article.

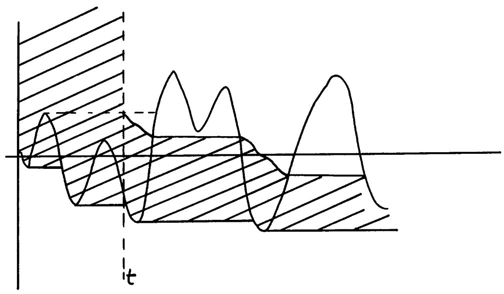

FIG. 1. $\mathscr{R}_t$ and its boundary.

Recall that $\mathfrak{R} = \{(s, x): A(s, x) > 0\}$. Let us define

$$\mathscr{R}_t = \{(s,x): 0 < A(s,x) \le t\},$$
$$\psi_s(t) = \inf\{x: A(s,x) > t\}.$$

The function ψ_s is just the inverse of $A(s, \cdot)$ in the space variable. Its elementary properties follow from (2.3) and the remarks preceding it: it is infinite if $s \le t$ and finite if $s > t$. Indeed, for $s > t$, $m_s \le \psi_s(t) < M_s$, where m and M are the minimum and maximum of the Brownian path. As a process in s for fixed t, it is adapted, decreasing and continuous on (t, ∞), with $\psi_{s+}(s) = M_s$ and $\lim_{s \to \infty} \psi_s(t) = -\infty$. In particular, $B_s - \psi_s(t)$ is a semimartingale, so it has a local time at zero.

The set $\mathscr{R}_t$ can be conveniently expressed as $(\psi_0, \psi_t] \equiv \{(s, x): \psi_s(0) < x \le \psi_s(t)\}$. Figure 1 illustrates its appearance.

Let H_{ab} be the horizontal strip $H_{ab} = (0, \infty) \times (a, b]$.

THEOREM 4.1. *Fix $t > 0$ and $a \in \mathbf{R}$. Then for $x \ge a$,*

$$\tilde{L}(t,x) + 2x^- = \tilde{L}(t,a) + 2a^- + X_t(x) - C_t(x), \tag{4.1}$$

where $\{X_t(x), \mathscr{E}_x, x \ge a\}$ is a martingale and $\{C_t(x), x \ge a\}$ is a continuous increasing $\mathscr{E}_x$-adapted process. Moreover, for $x > a$, if we define the curve

$$\phi(s) = \begin{cases} \psi_s(t) & \text{if } \tau(t,x) < s \le \tau(t,a), \\ \infty & \text{otherwise,} \end{cases}$$

then

$$X_t(x) = \iint_{\mathscr{R}_t \cap H_{ax}} dL, \tag{4.2}$$

$$C_t(x) = \int_\phi \partial_s L \tag{4.3}$$

$$= \lambda(\phi) \tag{4.4}$$

(4.5) $=$ *the local time at zero of*

$$\{B_s - \psi_s(t), \tau(t,x) < s \le \tau(t,a)\}.$$

PROOF. Fix t, x and a such that $x > a$, let $T = \tau(t, a)$ and define the curve ϕ as above. Then

$$\begin{aligned}(\phi, \infty) &= \{(s, y) : \tau(t, x) < s \le \tau(t, a),\ A(s, y) > t\} \\ &= \{(s, y) : A(s, y) > t\} \cap \{(s, y) : A(s, y) > 0, s \le T, y > a\} \\ &\quad \cap [\Re - (\tau(0, x), \tau(t, x)] \times (x, \infty)] \\ &= \Lambda_1 \cap \Lambda_2 \cap \Lambda_3.\end{aligned}$$

This is identifiable, being the intersection of identifiable sets. (Identifiability is immediate for Λ_1 and Λ_3 and it follows for Λ_2 by Lemma 2.8.) Moreover, $(s, y) \mapsto I_{\{\phi(s) < y \le x\}}$ is in $L^2(\mathscr{E})$, again from Lemma 2.8. From the definition of local time along ϕ, then

$$\begin{aligned}\int_\phi \partial_s L &= \int I_{\{\phi(s) < x\}} L(ds, x) - \int\int I_{(\phi, x]}\, dL \\ &= L(T, x) - \tilde{L}(t, x) - \int\int I_{(\phi, x]}\, dL.\end{aligned}$$

After rearranging terms, we see that (4.2)–(4.5) follow from Lemma 2.8 and Theorem 3.3. □

Let us look at the continuity properties of X, C and ψ. We already know that X is continuous in the space variable x, for it is an $\mathscr{E}_x$-martingale and hence continuous by Theorem 2.1. But the situation is better than that: X and C are jointly continuous in (t, x) and $\psi_s(t)$ is even differentiable in t.

THEOREM 4.2. *$C_t(x)$ and $X_t(x)$ have versions which are almost surely jointly continuous in x and t. $\psi_s(t)$ is jointly continuous in (s, t) and differentiable in t on the set $\{(s, t)$: $s > t > 0\}$. Moreover,*

$$\frac{\partial}{\partial t}\psi_s(t) = \frac{1}{L(s, \psi_s(t))}, \qquad s > t > 0. \tag{4.6}$$

PROOF. $\tilde{L}$ is continuous, so by (4.1) it is enough to show that X is continuous. This will follow from Kolmogorov's criterion once we establish the following inequality. Let $s < t$ and $x > a$. Then we claim that for $p \ge 2$,

$$E\left\{\left|\int\int_{(\mathscr{R}_t - \mathscr{R}_s) \cap H_{ax}} dL\right|^2\right\} \le c_p |x - a|^{p/2} |t - s|^{p/4}. \tag{4.7}$$

To prove this, use Burkholder's inequality, (2.10) and Hölder's inequality in turn to see that

$$E\left\{\left|\iint_{(\mathscr{R}_t-\mathscr{R}_s)\cap H_{ax}} dL\right|^p\right\} \le C_p E\left\{\left|\int_0^\infty I_{H_{ax}} I_{(\mathscr{R}_t-\mathscr{R}_s)}(u,B_u)\,du\right|^{p/2}\right\}$$

$$= C_p E\left\{\left|\int_a^x \big(L(\tau(t,y),y) - L(\tau(s,y),y)\big)\,dy\right|^{p/2}\right\}$$

$$\le C_p |x-a|^{p/2-1}\int_a^x E\{|L(\tau(t,y),y) - L(\tau(s,y),y)|^{p/2}\}\,dy.$$

Define stopping times by $S = \inf\{u > \tau(s,y)\colon\ B_u = y\}$, $T = \inf\{u > S\colon A(u,y) - A(S,y) > t - s\}$. Then $L(S,y) = L(\tau(s,y),y)$ and $T \ge \tau(t,y)$, so

$$L(\tau(t,y),y) - L(\tau(s,y),y) \le L(T,y) - L(S,y).$$

Apply the strong Markov property at S. This last quantity has the same distribution as $L(\tau(t-s,0),0)$. By Brownian scaling this has the same distribution as $\sqrt{t-s}\,L(\tau(1,0),0)$, which is exponential and has moments of all orders. If we let $c_p = E\{L(\tau(1,0),0)^{p/2}\}C_p$, we get (4.7).

The rest of the proof is standard. Set $d(t,x;s,y) = ((y-x)^2 + (t-s)^2)^{1/2}$. Suppose s, t, x, and y are bounded in absolute value by, say, N, and suppose without loss of generality that $s < t$. Then

$$E\left\{\left|\iint_{\mathscr{R}_t\cap H_{ax}} dL - \iint_{\mathscr{R}_s\cap H_{ay}} dL\right|^p\right\}^{1/p}$$

$$\le \left\|\iint_{(\mathscr{R}_t-\mathscr{R}_s)\cap H_{a,x\vee y}} dL\right\|_{L^p} + \left\|\iint_{\mathscr{R}_t\cap H_{x\wedge y,x\vee y}} dL\right\|_{L^p}.$$

Apply (4.7) to both terms on the right-hand side (take $s = 0$ in the second term) and do a little algebra to see that this is

$$\le K d(x,t;y,s)^{1/4},$$

where K depends on p and N. Since p can be chosen as large as we please. Kolmogorov's criterion applies to tell us that the integral has a continuous version.

Turning to ψ, remark that (4.6) follows from fact 3 preceding (2.3).

To prove the joint continuity, let $0 < t < s$ and suppose $s_n \mapsto s$ and $t_n \mapsto t$. Let $x_n = \psi_{s_n}(t_n)$. Note that $t_n = A(s_n, x_n)$. If some subsequence, say x_n itself, converges to a limit x, then by the continuity of A, t_n converges to $A(s,x) = t$. Since $A(s,x) < s$, $x < M_s$, so $A(s,\cdot\,)$ is strictly increasing at x by (2.3) and we can conclude that $\psi_s(t) = x = \lim \psi_{s_n}(t_n)$. This proves continuity. □

REMARK. The characterisation (4.5) of $C_t(x)$ can also be proved directly by using Tanaka's formula; this proof is given in the companion paper Rogers and Walsh (1991).

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STATISTICAL LABORATORY
16 MILL LANE
CAMBRIDGE CB2 1SB
ENGLAND

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BRITISH COLUMBIA
CANADA V6T 1Y4