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ON THE ALMOST SURE BEHAVIOR OF SUMS OF IID RANDOM VARIABLES IN HILBERT SPACE

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We study the almost sure behavior of sums of iid random variables satisfying the bounded LIL in Hilbert space. We show that the almost sure behavior is different from the Gaussian case, whenever the second strong moments are infinite. A law of the k times iterated logarithm is established which refines the bounded LIL. The interesting feature here is that contrary to the known conditions for the bounded LIL, one needs not only moment type conditions but also a nice structure of the covariance operator.

1. Introduction. One of the classical results of probability is the Hartman–Wintner law of the iterated logarithm (LIL) which states that one has for any sequence of independent, identically distributed (iid) random variables (r.v.'s) $\{X_n\}$ with $E[X_1] = 0$ and $\sigma^2 := E[X_1^2] < \infty$ almost surely (a.s.),

$$\limsup_{n\to\infty} \pm \sum_{j=1}^{n} X_j/\sqrt{2nL_2 n} = \sigma, \tag{1.1}$$

where $Lt := \log(t \vee e)$, $L_2 t := L(Lt)$, $t \geq 0$.

From (1.1), it immediately follows that

$$\limsup_{n\to\infty} \left| \sum_{j=1}^{n} X_j \right| / \sqrt{2nL_2 n} = \sigma \quad a.s. \tag{1.2}$$

Moreover, it is known that the conditions $E[X] = 0$ and $E[X^2] < \infty$ are also necessary for (1.2) to hold.

Considering this so-called two-sided version of the LIL, it is natural to ask whether a generalization to multidimensional random vectors, or even to random variables taking values in infinite-dimensional Hilbert spaces is possible.

From now on let $X, X_1, X_2, \ldots$ be iid r.v.'s taking values in a separable Hilbert space H with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$. Further, set $S_n := \sum_{j=1}^{n} X_j$, $n \in \mathbb{N}$.

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The following basic result is due to de Acosta and Kuelbs [3].

THEOREM. *Suppose that X is an H-valued r.v. satisfying*

$$E[X] = 0, \qquad E\left[\|X\|^2/L_2\|X\|\right] < \infty \tag{1.3}$$

and

$$E\left[(X,y)^2\right] < \infty, \qquad y \in H. \tag{1.4}$$

Then we have

$$\limsup_{n\to\infty} \|S_n\|/\sqrt{2nL_2n} = \sigma, \tag{1.5}$$

where $\sigma^2 := \sup\{E[(X,y)^2]: \|y\| \le 1\}$.

Using the aforementioned converse to the Hartman–Wintner LIL, it is easy to see that the conditions (1.3) and (1.4) are also necessary for (1.5) to hold. This means that the previous theorem is the exact analogue of the Hartman–Wintner LIL in Hilbert space.

The Hartman–Wintner LIL, however, has one shortcoming: It gives only an incomplete description of the almost sure behavior of sums of iid r.v.'s. A thorough investigation of the almost sure behavior requires the determination of lower and upper classes. This concept is due to Lévy and we now give the relevant definitions for sums of H-valued random variables. The basic observation is that one has for any sequence $a_n \uparrow \infty$ almost surely,

$$\|S_n\| \le a_n \quad \text{eventually} \tag{1.6}$$

or

$$\|S_n\| > a_n \quad \text{infinitely often (i.o.)}. \tag{1.7}$$

This can be easily seen when using the 0–1 law of Hewitt–Savage. In the first case one says that $\{a_n\}$ belongs to the upper class, whereas in the second case $\{a_n\}$ belongs to the lower class.

From (1.5) it now follows that $\{\sigma(anL_2n)^{1/2}\}$ belongs to the upper class (lower class) whenever $a > 2$ ($a < 2$). But it is not clear at all what type of sequence, for instance, $\{\sigma(2nL_2n)^{1/2}\}$ is.

Feller [10] established an integral test for lower and upper classes, which applies to any sequence of iid real-valued r.v.'s with mean zero and finite variance, thereby substantially refining the Hartman–Wintner LIL. From his work it follows that sums of iid r.v.'s behave like sums of standard normal r.v.'s whenever the following condition is fulfilled:

$$E\left[X^2 1\{|X| \ge t\}\right] = O\left((L_2t)^{-1}\right) \quad \text{as } t \to \infty. \tag{1.8}$$

Moreover, the almost sure behavior is different if one has

$$L_2tE\left[X^2 1\{|X| \ge t\}\right] \to \infty \quad \text{as } t \to \infty.$$

This means that (1.8) provides a sharp condition.

The first result on lower and upper classes for sums of iid H-valued r.v.'s is due to Kuelbs [15]. From his work it follows that one has if X is a *Gaussian* mean zero r.v., for any eventually nondecreasing function ϕ: $(0,\infty) \to (0,\infty)$:

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi(n) \text{ i.o.}\} = 0 \quad \text{or} \quad = 1$$

$$\text{according as } \sum_{n=1}^{\infty} \frac{\phi(n)^d}{n} \exp\left(-\frac{\phi(n)^2}{2}\right) < \infty \quad \text{or} \quad = \infty, \tag{1.9}$$

where d is the dimension of the eigenspace of cov(X) (= covariance operator of X) corresponding to σ^2 which is an eigenvalue of cov(X) in this case.

A typical application of (1.9) is the following refinement of (1.5), the exact formulation of which requires some additional notation.

We define recursively for $t \geq 0$ and $k \geq 3$, $L_k t := L(L_{k-1}t)$. Further set $\alpha_2 := 2$, $\alpha_3 := d + 2$, $\alpha_k = 2$, $k \geq 4$ and put for $a \in \mathbb{R}$ and $k \geq 3$,

$$\phi_{a,k}(t) := \left(\sum_{j=2}^{k-1} \alpha_j L_j t + a L_k t\right)^{1/2}, \qquad t \geq 0.$$

Then, we can infer from (1.9) for all $k \geq 3$,

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi_{a,k}(n) \text{ i.o.}\} = 0 \quad \text{or} \quad = 1 \\ \text{according as } a > \alpha_k \quad \text{or } a < \alpha_k. \tag{1.10}$$

In a recent paper ([6]), it has been shown that (1.9) and consequently (1.10) remain true for any H-valued mean zero r.v. satisfying the conditions

$$E\left[\|X\|^2\right] < \infty \tag{1.11}$$

and

$$E\left[(X,y)^2 1\{|(X,y)| \geq t\}\right] = O\left((L_2 t)^{-1}\right) \quad \text{as } t \to \infty, y \in V, \tag{1.12}$$

where V is the d-dimensional eigenspace corresponding to the (largest) eigenvalue σ^2 of cov(X).

Condition (1.12) is the Hilbert space analogue of condition (1.8) and it is clear from Feller's work that this is an essential requirement for sums of iid r.v.'s behaving like sums of iid Gaussian r.v.'s. The importance of (1.11), however, is less clear, and it is the purpose of the present paper to investigate the almost sure behavior of sums of iid r.v.'s with $E[\|X\|^2] = \infty$. Before we state our results, we still would like to mention a phenomenon in connection with infinite-dimensional LIL results which will be of fundamental importance for our work. It is clear that we have now to consider the covariance operator instead of the variance. From condition (1.4), which is necessary for the LIL in Hilbert space, it follows that it exists as a *bounded* self-adjoint, positive semidefinite operator. If it is even a *compact* operator, one can prove a compact LIL which refines the bounded LIL (1.5). A necessary and sufficient condition for the covariance operator being compact is

$$\{(X,y)^2: \|y\| \le 1\} \text{ is a uniformly integrable family of random variables.} \tag{1.13}$$

It is obvious from (1.13) that assumption (1.11) also implies that cov(X) is compact. This means that this assumption has two effects: First, it enables us to control the tails of the distributions of the random variables $\{X_n\}$ which is essential for any result on almost sure behavior. Second, it guarantees that the covariance operators behave nicely. This last property will be essential in the present setting. It will turn out that moment type conditions are not sufficient in order to obtain a nice almost sure behavior for sums of iid r.v.'s with $E[\|X\|^2] = \infty$. One also needs that, at least to a certain extent, the covariance operators behave like compact ones.

2. The main results. Let now X be an H-valued r.v. satisfying

$$E[X] = 0, \qquad E\big[\|X\|^2/L_2\|X\|\big] < \infty \tag{2.1}$$

and

$$E\big[(X,y)^2\big] < \infty, \qquad y \in H. \tag{2.2}$$

Set $\sigma^2 := \sup\{E[(X,y)^2]: \|y\| \le 1\}$.
Suppose that

$$\sigma^2 \text{ is an eigenvalue of } \mathrm{cov}(X), \tag{2.3}$$

and that

$$\dim(V) =: d < \infty, \tag{2.4}$$

where V is the eigenspace of cov(X) determined by the eigenvalue σ^2. Further, let W be the orthogonal complement of V and suppose that there exists a $\rho \in [0,1)$ such that

$$E\big[(X,y)^2\big] \le \rho\sigma^2, \qquad \|y\| \le 1, \qquad y \in W. \tag{2.5}$$

Note that conditions (2.3)–(2.5) are always satisfied when cov(X) is a compact operator. Finally, assume again that for any $y \in V$,

$$E\big[(X,y)^2 1\{|(X,y)| \ge t\}\big] = O\big((L_2 t)^{-1}\big) \quad \text{as } t \to \infty. \tag{2.6}$$

We then have the following extension of (1.10) which might be called the law of the k times iterated logarithm.

THEOREM 1. *Let X be an H-valued r.v. satisfying conditions* (2.1)–(2.6) *and let $k \ge 3$. Suppose that*

$$E\big[\|X\|^2 1\{\|X\| \le t\}\big] = o(L_k t) \quad \textit{as } t \to \infty. \tag{2.7}$$

Then we have

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi_{a,k}(n)\ i.o.\} = 0 \quad or \ = 1$$

according as $a > \alpha_k$ *or* $a < \alpha_k$.

It is easy to see that condition (2.7) holds true whenever

$$E\left[\|X\|^2/L_k\|X\|\right] < \infty. \tag{2.8}$$

The next result shows that condition (2.7) is sharp. Namely, we have:

THEOREM 2. *Let X be an H-valued r.v. satisfying conditions* (2.1)–(2.6) *and let $k \geq 3$. Suppose that*

$$\liminf_{t\to\infty} E\left[\|X\|^2 1\{\|X\| \leq t\}\right]/L_k t \geq \beta > 0.$$

Then we have

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi_{a,k}\ i.o.\} = 1,$$

whenever $a < \alpha_k + C\beta$, where C is a positive constant depending on the distribution of X.

We note that if the covariance operator of X is compact, a possible choice of C is given by $C = 1/\sigma^2$. Moreover, it is immediate from Theorem 2 that we have for *any* $a > 0$,

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi_{a,k}(n)\ \text{i.o.}\} = 1$$

if $E[\|X\|^2 1\{\|X\| \leq t\}]/L_k t \to \infty$ as $t \to \infty$.

Before we state our last result, we still have to introduce some notation.

Let $\Sigma_n\colon H \to H$, $n \in \mathbb{N}$ be a sequence of self-adjoint positive semidefinite operators which is defined by the relation

$$(x, \Sigma_n y) = E\left[(x, X_n')(y, X_n')\right], \qquad x, y \in H, \tag{2.9}$$

where $X_n' := X_n 1\{\|X_n\| \leq c_n\}$, $c_n := \sqrt{n}/(L_2 n)^5$, $n \in \mathbb{N}$.

Since the random variables X_n' are bounded, it follows that the operators Σ_n have finite traces which implies that they are compact.

Let $\sigma_{1,n}^2 \geq \sigma_{2,n}^2 \geq \cdots$ be the positive eigenvalues of Σ_n arranged in a nonincreasing order and taking into account the multiplicities. If there are only finitely many positive eigenvalues, we set $\sigma_{i,n}^2 = 0$ for large enough i.

Put

$$\gamma_n := \left[\frac{1}{2}\sum_{i=d+1}^{\infty} \log\left(\frac{\sigma^2}{\sigma^2 - \sigma_{i,n}^2}\right)\right] \vee 1, \qquad n \in \mathbb{N},$$

and note that it is well defined since the first term is

$$\leq \frac{1}{2}\sum_{i=d+1}^{\infty}\frac{\sigma_{i,n}^2}{\sigma^2-\sigma_{i,n}^2} \leq \frac{1}{2\sigma^2(1-\rho)}\sum_{i=d+1}^{\infty}\sigma_{i,n}^2.$$

The last series is finite, since Σ_n has a finite trace.

We are now in a position to formulate Theorem 3, which comes very close to an integral test.

THEOREM 3. *Let X be an H-valued r.v. satisfying conditions* (2.1)–(2.5). *Then we have for any eventually nondecreasing function* ϕ: $(0,\infty) \to (0,\infty)$:

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi(n) \text{ i.o.}\} = 0,$$

(A)
$$\text{whenever } \sum_{n=1}^{\infty}\frac{\phi(n)^d}{n}\exp\left(-\frac{1}{2}\phi(n)^2+\gamma_n\right) < \infty.$$

Moreover, under the additional assumption (2.6) *it follows that*

$$P\{\|S_n\| > \sigma\sqrt{n}\,\phi(n) \text{ i.o.}\} = 1,$$

(B)
$$\text{whenever } \sum_{n=1}^{\infty}\frac{\phi(n)^d}{n\gamma_n}\exp\left(-\frac{1}{2}\phi(n)^2+\gamma_n\right) = \infty.$$

Theorems 1 and 2 can be easily inferred from Theorem 3 so that we have only to prove the latter result. The proof of the upper class part (A) of Theorem 3 is very similar to that in [6]. In order to prove the lower class part (B), however, we have to use an entirely different method. Whereas it was possible in [6] to bypass the difficulties which one encounters when giving a direct proof for lower class results by using a strong approximation argument, we must now return to the classical approach developed by Erdös [8] and Feller [9, 10]. This method is based on a combination of sharp exponential inequalities with "direct" calculations of the probabilities involved.

In order to extend this approach to H-valued random variables, we first derive precise bounds for probabilities of the type $P\{\|Y\| \geq t\}$, where Y is a Gaussian mean zero r.v. (see Section 3). Then we show that the almost sure behavior of the sums S_n is equivalent to the behavior of sums of appropriately defined bounded r.v.'s which we obtain from the original sequence $\{X_n\}$ by truncation (see Section 5). Using the central limit theorem in Hilbert space, we then can apply the results of Section 3 in the proof of both the lower class part (B) (see Section 6) and the upper class part (A) (see Section 7). In order to guarantee that one can disregard the error terms in the normal approximation, we need some auxiliary results which are detailed in Section 4. In Section 8, we finally give an example demonstrating that moment type conditions are not sufficient for a nice almost sure behavior if $E[\|X\|^2] = \infty$. Some results on the behavior of eigenvalues of covariance operators which we decisively need in Sections 6 and 7 can be found in the Appendix to this paper.

3. Some results on Gaussian random variables. Let Y be a nondegenerate Gaussian mean zero r.v. with covariance operator Σ, say. Denote the largest eigenvalue of Σ by σ^2 and let d be the dimension of the corresponding eigenspace. Let σ_i^2, $1 \le i < d'$ be the positive eigenvalues of Σ arranged in a nonincreasing order and taking into account the multiplicities. Further, if $d' < \infty$, put $\sigma_i^2 = 0$, $i \ge d'$. Note that we always have $\sigma_i^2 = \sigma^2$, $1 \le i \le d$ and $\sigma_i^2 < \sigma^2$, $i > d$.

Denote the density of $\|Y\|^2$ by g. Then it follows from [17] that

$$\lim_{z\to\infty} \left(g(z)/z^{d/2-1}\exp(-z/2\sigma^2)\right) = (2\sigma^2)^{-d/2}\Gamma(d/2)^{-1}K(\Sigma), \tag{3.1}$$

where

$$K(\Sigma) := \prod_{i=d+1}^{\infty} \left(1 - \sigma_i^2/\sigma^2\right)^{-1/2}. \tag{3.2}$$

The main purpose of this section is to refine (3.1) so as to obtain explicit bounds for g. The method used here is similar to that of Hoeffding [13].

Let $\{e_i\}$ be a sequence of orthonormal eigenvectors corresponding to the eigenvalues $\{\sigma_i^2\}$. Then we have with probability 1,

$$Y = \sum_{i=1}^{\infty} (Y, e_i)e_i =: \sum_{i=1}^{\infty} \sigma_i \eta_i e_i, \tag{3.3}$$

where $\{\eta_i\}$ is a sequence of independent standard normal r.v.'s.

Put

$$Z := \sigma^2 \sum_{i=1}^{d} \eta_i^2, \qquad \bar{Z} := \sum_{i=d+1}^{\infty} \sigma_i^2 \eta_i^2.$$

Then $Z, \bar{Z}$ are *independent* r.v.'s such that

$$\|Y\|^2 = Z + \bar{Z} \quad \text{a.s.} \tag{3.4}$$

Denoting the densities of Z and $\bar{Z}$ by f_d and $\bar{f}$, respectively, it is clear that

$$f_d(z) = (2\sigma^2)^{-d/2}\Gamma(d/2)^{-1}z^{d/2-1}\exp(-z/2\sigma^2), \qquad z > 0, \tag{3.5}$$

and also that

$$g(z) = \int_0^z f_d(z-y)\bar{f}(y)\,dy, \qquad z > 0. \tag{3.6}$$

We infer

$$g(z) = f_d(z)\int_0^z (1 - y/z)^{d/2-1}\exp(y/2\sigma^2)\bar{f}(y)\,dy. \tag{3.7}$$

We first consider the case where the multiplicity d of the eigenvalue σ^2 is at least 2.

LEMMA 1. *If $d \ge 2$, we have*

(a) $g(z) \le K(\Sigma) f_d(z)$, $z > 0$.
(b) $g(z) \ge \frac{1}{4} K(\Sigma) f_d(z)$, $z \ge 2\, dE[\|Y\|^2]/(1 - \sigma_{d+1}^2/\sigma^2)$.

PROOF. Using the independence of the η_i's and the assumption $d \ge 2$, we immediately get from (3.7)

$$\begin{aligned} g(z) &\le f_d(z) \int_0^\infty \exp(y/2\sigma^2)\, \bar{f}(y)\, dy \\ &= f_d(z) E\left[\prod_{i=d+1}^\infty \exp\left(\sigma_i^2 \eta_i^2 / 2\sigma^2\right)\right] \\ &= f_d(z) \cdot K(\Sigma). \end{aligned}$$

In order to establish the lower bound, we first note that

$$\int_t^\infty \exp(y/2\sigma^2)\, \bar{f}(y)\, dy \le t^{-1} E\left[\bar{Z} \exp(\bar{Z}/2\sigma^2)\right].$$

From the definition of $\bar{Z}$, it is easy to see that

$$\begin{aligned} &E\left[\bar{Z}\exp(\bar{Z}/2\sigma^2)\right] \\ &\quad = \sum_{i=d+1}^\infty E\left[\sigma_i^2\eta_i^2 \exp\left(\sigma_i^2\eta_i^2/2\sigma^2\right)\right] \prod_{\substack{j=d+1 \\ j \ne i}}^\infty E\left[\exp\left(\sigma_j^2\eta_j^2/2\sigma^2\right)\right] \\ &\quad = K(\Sigma) \sum_{i=d+1}^\infty \sigma_i^2 \left(1 - \sigma_i^2/\sigma^2\right)^{-1} \\ &\quad \le K(\Sigma) E\left[\|Y\|^2\right]\left(1 - \sigma_{d+1}^2/\sigma^2\right)^{-1}. \end{aligned}$$

We thus have shown for $t > 0$,

$$\int_t^\infty \exp(y/2\sigma^2)\, \bar{f}(y)\, dy \le K(\Sigma) E\left[\|Y\|^2\right]\left(1 - \sigma_{d+1}^2/\sigma^2\right)^{-1} t^{-1}. \tag{3.8}$$

Next observe that

$$\begin{aligned} &\int_0^{z/d} \exp(y/2\sigma^2)\, \bar{f}(y)\, dy - \int_0^{z/d} (1 - y/z)^{d/2-1} \exp(y/2\sigma^2)\, \bar{f}(y)\, dy \\ &\quad \le \frac{d}{2} \int_0^{z/d} \frac{y}{z} \exp(y/2\sigma^2)\, \bar{f}(y)\, dy \\ &\quad \le \frac{1}{2} \int_0^{z/d} \exp(y/2\sigma^2)\, \bar{f}(y)\, dy. \end{aligned}$$

We infer that

$$\int_0^{z/d} (1 - y/z)^{d/2-1} \exp(y/2\sigma^2)\, \bar{f}(y)\, dy \ge \tfrac{1}{2} \int_0^{z/d} \exp(y/2\sigma^2)\, \bar{f}(y)\, dy,$$

which on account of (3.8) is

$$\geq \tfrac{1}{4}K(\Sigma)$$

provided $z/d \geq 2E[\|Y\|^2](1 - \sigma_{d+1}^2/\sigma^2)^{-1}$. The last bound in conjunction with (3.7) implies (b). □

We now want to prove a related result if $d = 1$. For that purpose, we need a further lemma.

LEMMA 2. *Let $\{\eta_i\}$ be a sequence of independent standard normal r.v.'s and let $\{\sigma_i^2\}$ be a sequence of nonnegative real numbers.*
Set

$$Z_1 := \sum_{i=1}^{\infty} \sigma_i^2 \eta_i^2, \qquad Z_2 := Z_1 + a\eta_m^2, \quad \text{where } a \geq 0,\ m \in \mathbb{N}.$$

Let g_i be the density of Z_i, $i = 1, 2$.
Then we have:

(a) $g_1(z) \leq (1 + a/\sigma_m^2)^{1/2} g_2(z)$, $z > 0$.
(b) $g_1(z) \geq g_2(z)\exp(-az/2\sigma_m^4)$, $z > 0$.

PROOF. Denoting the density of $Z_1 - \sigma_m^2\eta_m^2$ by h we have

$$g_1(z) = \int_0^z h(z-x)\sigma_m^{-1}x^{-1/2}\exp(-x/2\sigma_m^2)\,dx/\sqrt{2\pi} \tag{3.9}$$

and

$$g_2(z) = \int_0^z h(z-x)\tilde{\sigma}_m^{-1}x^{-1/2}\exp(-x/2\tilde{\sigma}_m^2)\,dx/\sqrt{2\pi}, \tag{3.10}$$

where $\tilde{\sigma}_m^2 = \sigma_m^2 + a$. Comparing (3.9) and (3.10), we immediately obtain the assertion. □

LEMMA 3. *If $d = 1$, we have*:

(a) $g(z) \leq 18K(\Sigma)(\sigma/\sigma_3)^2 f_1(z)$, $z \geq 16E[\|Y\|^2](1 - \sigma_2^2/\sigma^2)^{-2}$.
(b) $g(z) \geq \frac{1}{2}K(\Sigma) f_1(z)$, $z \geq 2E[\|Y\|^2](1 - \sigma_2^2/\sigma^2)^{-1}$.

PROOF. Using the representation (3.7), we get

$$g(z)/f_1(z) \leq 2K(\Sigma) + \int_{3z/4}^{z} (1 - y/z)^{-1/2}\exp(y/2\sigma^2)\tilde{f}(y)\,dy. \tag{3.11}$$

Set

$$\tilde{Y} := \left((\sigma^2 + \sigma_2^2)/2\right)^{1/2}(\eta_2 e_2 + \eta_3 e_3) + \sum_{i=4}^{\infty} \sigma_i \eta_i e_i,$$

and denote the density of $\|\tilde{Y}\|^2$ by $\tilde{f}$.

Then it follows from Lemma 2(a) that

$$\bar{f}(y) \le (\sigma/\sigma_3)^2 \tilde{f}(y), \qquad y > 0. \tag{3.12}$$

Applying Lemma 1(a), we obtain

$$\tilde{f}(y) \le 2K(\tilde{\Sigma})\sigma^{-2}\exp\big(-y/(\sigma^2+\sigma_2^2)\big), \qquad y > 0, \tag{3.13}$$

where $K(\tilde{\Sigma}) = \prod_{i=4}^{\infty}(1 - 2\sigma_i^2/(\sigma^2+\sigma_2^2))^{-1/2}$.

Combining (3.12) and (3.13), we find that

$$\begin{aligned}
&\int_{3z/4}^{z} (1 - y/z)^{-1/2} \exp(y/2\sigma^2)\bar{f}(y)\,dy \\
&\quad \le 2K(\tilde{\Sigma})\sigma_3^{-2}\int_{3z/4}^{z}(1-y/z)^{-1/2}\exp\big(-(\sigma^2-\sigma_2^2)y/4\sigma^4\big)\,dy \\
&\quad \le 2K(\tilde{\Sigma})\sigma_3^{-2}\exp\big(-3(\sigma^2-\sigma_2^2)z/16\sigma^4\big)z.
\end{aligned}$$

It is easily checked that

$$\begin{aligned}
K(\tilde{\Sigma}) &\le \exp\left(\sum_{i=4}^{\infty}\sigma_i^2/(\sigma^2-\sigma_i^2)\right) \\
&\le \exp\big(E\big[\|Y\|^2\big]/(\sigma^2-\sigma_2^2)\big).
\end{aligned}$$

We infer

$$\begin{aligned}
&K(\tilde{\Sigma})\exp\big(-3(\sigma^2-\sigma_2^2)z/16\sigma^4\big)z \\
&\quad \le \exp\big(-(\sigma^2-\sigma_2^2)z/8\sigma^4\big)z \quad \text{if } z \ge 16E\big[\|Y\|^2\big]\big(1-\sigma_2^2/\sigma^2\big)^{-2}.
\end{aligned}$$

Using the trivial inequality $\exp(-t) \le 2/t^2$, we see that the last term is

$$\le 8\sigma^2 \quad \text{if } z \ge 16E\big[\|Y\|^2\big]\big(1-\sigma_2^2/\sigma^2\big)^{-2} \ge 16\sigma^6\big(\sigma^2-\sigma_2^2\big)^{-2}.$$

Since $K(\Sigma) \ge 1$, the last inequality in conjunction with (3.11) implies (a). To prove (b), we first infer from (3.7),

$$g(z) \ge \left(\int_0^z \exp(y/2\sigma^2)\bar{f}(y)\,dy\right) f_1(z). \tag{3.14}$$

Noticing that (3.8) remains true if $d = 1$, we find that

$$\int_z^{\infty}\exp(y/2\sigma^2)\bar{f}(y)\,dy \le \tfrac{1}{2}K(\Sigma), \tag{3.15}$$

provided $z \ge 2E[\|Y\|^2](1-\sigma_2^2/\sigma^2)^{-1}$.

Combining (3.14) and (3.15), we obtain (b). □

Lemma 1 and Lemma 3 imply the following theorem which is less accurate, but still precise enough for the purpose of proving Theorem 3.

THEOREM 4. *We have for* $z \geq 16\, dE[\|Y\|^2](1 - \sigma_{d+1}^2/\sigma^2)^{-2}$

$$\tfrac{1}{4}K(\Sigma) \leq g(z)/f_d(z) \leq 18K(\Sigma)\sigma^2/\sigma_3^2.$$

From Theorem 4 we now infer:

THEOREM 5. *We have for* $x \geq 5\sqrt{d}\,(E[\|Y\|^2])^{1/2}/(1 - \sigma_{d+1}^2/\sigma^2)$:

(a) $P\{\|Y\| \geq x\} \leq C_1(\sigma/\sigma_3)^2 K(\Sigma)(x/\sigma)^{d-2}\exp(-x^2/2\sigma^2)$.

(b) $P\{\|Y\| \geq x\} \geq C_2 K(\Sigma)(x/\sigma)^{d-2}\exp(-x^2/2\sigma^2)$, *where* C_i, $i = 1, 2$ *are positive constants depending on* d.

(c) $P\{\|Y\| \geq x + \beta/x\} \leq \frac{1}{2}P\{\|Y\| \geq x\}$, *whenever* $\beta \geq 4\sigma^2 \log(12\sigma/\sigma_3)$.

(d) $P\{\|Y\| \geq x - t\} \leq 144(\sigma/\sigma_3)^2 \exp(2tx/\sigma^2)P\{\|Y\| \geq x\}$, $0 < t \leq \sigma$.

PROOF. Parts (a) and (b) follow from an integration of the inequalities of Theorem 4 and the known fact that

$$C_3 t^{d-2}\exp(-t^2/2) \geq \int_{t^2}^{\infty} x^{d/2-1}\exp(-x/2)\,dx \geq C_4 t^{d-2}\exp(-t^2/2), \qquad t \geq 1,$$

where C_3, C_4 are constants depending on d.

To prove (c), observe that

$$\begin{aligned} &P\{\|Y\| \geq x + \beta/x\} \\ &\quad \leq P\{\|Y\|^2 \geq x^2 + 2\beta\} = \int_{x^2}^{\infty} g(t + 2\beta)\,dt. \end{aligned}$$

Using the inequalities of Theorem 4, we obtain

$$\begin{aligned} g(t + 2\beta)/g(t) &\leq 72(\sigma/\sigma_3)^2 \exp(-\beta/\sigma^2)(1 + 2\beta/t)^{d/2} \\ &\leq 72(\sigma/\sigma_3)^2 \exp(-\beta/\sigma^2)\exp(d\beta/t) \\ &\leq 72(\sigma/\sigma_3)^2 \exp(-\beta/2\sigma^2), \qquad t \geq x^2 \geq 2\,d\sigma^2. \end{aligned}$$

The last term is less than $\frac{1}{2}$ whenever $\beta > 2\sigma^2 \log(144\sigma^2/\sigma_3^2)$. Integrating the previous inequality, we obtain (c). The proof of (d) is similar. □

We finally note the following exponential inequality, which is not so precise as that of Theorem 5(a). It will come in handy since it is easier to apply.

LEMMA 4. *We have for* $t \geq 2(E[\|Y\|^2])^{1/2}$

$$P\{\|Y\| \geq t\} \leq \exp(-t^2/8\sigma^2).$$

PROOF. Without loss of generality, we assume $\sigma^2 = 1$. Then using the representation (3.3), we obtain

$$\begin{aligned} P\{\|Y\| \geq t\} &\leq E\left[\exp\left(\tfrac{1}{4}\sum_{i=1}^{\infty} \sigma_i^2 \eta_i^2\right)\right]\exp\left(-\tfrac{1}{4}t^2\right) \\ &= \prod_{i=1}^{\infty} \left(1 - \sigma_i^2/2\right)^{-1/2} \exp\left(-t^2/4\right) \\ &\leq \exp\left(\left\{\sum_{i=1}^{\infty} \sigma_i^2/4\left(1 - \sigma_i^2/2\right)\right\} - t^2/4\right) \\ &\leq \exp\left(\left(2E\left[\|Y\|^2\right] - t^2\right)/4\right), \qquad t \geq 0, \end{aligned}$$

from which the assertion easily follows. □

4. Auxiliary results. We need the following inequalities for sums of independent random variables with finite third moments which can be inferred from the proof of Theorem 2.1(A), [16].

LEMMA 5. *Let $X_1, \dots, X_n$ be independent H-valued r.v.'s with $E[X_j] = 0$, $E[\|X_j\|^3] < \infty$ and let $Y_1, \dots, Y_n$ be independent Gaussian mean zero r.v.'s with $\mathrm{cov}(X_j) = \mathrm{cov}(Y_j)$, $j = 1, \dots, n$. Then we have for $s, t > 0$*

$$\text{(a)} \quad P\left\{\left\|\sum_{j=1}^{n} X_j\right\| \geq s\right\} \leq P\left\{\left\|\sum_{j=1}^{n} Y_j\right\| \geq s - t\right\} + At^{-3}\sum_{j=1}^{n} E\left[\|X_j\|^3\right],$$

$$\text{(b)} \quad P\left\{\left\|\sum_{j=1}^{n} X_j\right\| \geq s\right\} \geq P\left\{\left\|\sum_{j=1}^{n} Y_j\right\| \geq s + t\right\} - At^{-3}\sum_{j=1}^{n} E\left[\|X_j\|^3\right],$$

where A is a universal constant.

For the proof of part (a), refer to Lemma A, [6]. The proof of (b) is similar.

Combining Lemma 4 and Lemma 5, we obtain the following Fuk–Nagaev-type inequality in Hilbert space. A related inequality has been proved in [4]. Lemma 6, however, is sharper since Λ_n is defined in terms of the *weak* second moments.

LEMMA 6. *Let $X_1, \dots, X_n$ be independent H-valued mean zero r.v.'s with $E[\|X_j\|^3] < \infty$, $j = 1, \dots, n$.*

Then we have for $s \geq 4(\sum_{j=1}^{n} E[\|X_j\|^2])^{1/2}$

$$P\left\{\left\|\sum_{j=1}^{n} X_j\right\| \geq s\right\} \leq \exp\left(-s^2/32\Lambda_n\right) + A's^{-3}\sum_{j=1}^{n} E\left[\|X_j\|^3\right],$$

where $\Lambda_n := \sup\{\sum_{j=1}^{n} E[(X_j, y)^2] : \|y\| \leq 1\}$ and A' is a universal constant.

PROOF. Let $Y_1, \dots, Y_n$ be independent Gaussian mean zero r.v.'s with $\operatorname{cov}(Y_j) = \operatorname{cov}(X_j)$, $j = 1, \dots, n$. Applying Lemma 5 with $t = s/2$, we get

$$P\left\{\left\| \sum_{j=1}^{n} X_j \right\| \ge s\right\} \le P\left\{\left\| \sum_{j=1}^{n} Y_j \right\| \ge s/2\right\} + 8As^{-3} \sum_{j=1}^{n} E\left[\|X_j\|^3\right].$$

Using Lemma 4 we obtain for

$$s/2 \ge 2\left(E\left[\left\| \sum_{j=1}^{n} Y_j \right\|^2\right]\right)^{1/2} = 2\left(\sum_{j=1}^{n} E\left[\|X_j\|^2\right]\right)^{1/2}:$$

$$P\left\{\left\| \sum_{j=1}^{n} Y_j \right\| \ge s/2\right\} \le \exp\left(-s^2/32\Lambda_n\right).$$

Combining the last two inequalities, we obtain the assertion. □

The next two lemmas will be frequently used in the remaining part of the paper and we include them in this section for the sake of easy reference.

LEMMA 7. *Let X be an H-valued mean zero r.v. satisfying the assumptions* (2.1) *and* (2.6). *Then we have*:

(a) $E[\|X\|^2 1\{\|X\| \le t\}] = o(L_2 t)$ *as* $t \to \infty$.
(b) $E[(X, y)^2 1\{\|X\| \ge t\}] = O((L_2 t)^{-1})$ *as* $t \to \infty$, $y \in V$.

PROOF. Since $E[\|X\|^2/L_2\|X\|] < \infty$, there exists a nondecreasing function $g\colon [0, \infty) \to [1, \infty)$ such that

$$g(t) = o(L_2 t) \text{ as } t \to \infty \tag{4.1}$$

and

$$E\left[\|X\|^2/g(\|X\|)\right] < \infty. \tag{4.2}$$

Using (4.1) and (4.2), we find that

$$E\left[\|X\|^2 1\{\|X\| \le t\}\right] \le E\left[\|X\|^2/g\|X\|\right]g(t) = o(L_2 t)$$

as $t \to \infty$, thereby proving (a).

To establish (b), we note that

$$\begin{aligned} &E\left[(X, y)^2 1\{\|X\| \ge t\}\right] \\ &\quad \le E\left[(X, y)^2 1\{\|X\| \ge t, |(X, y)| \le t/L_2 t\}\right] \\ &\qquad + E\left[(X, y)^2 1\{\|X\| \ge t, |(X, y)| \ge t/L_2 t\}\right] \\ &\quad \le P\{\|X\| \ge t\} t^2/(L_2 t)^2 + E\left[(X, y)^2 1\{|(X, y)| \ge t/L_2 t\}\right], \end{aligned}$$

where both terms are $O((L_2 t)^{-1})$ on account of (2.1) and (2.6). □

LEMMA 8. *Let X be an H-valued r.v. satisfying* (2.1). *Let $\{n_j\}$ be a subsequence of $\mathbb{N}$ such that for an appropriate constant $a > 0$ and large enough j,*

$$\exp\left(a/(Lj)^2\right) \le n_j/n_{j-1} \le 4. \tag{4.3}$$

Then we have

$$\sum_{j=1}^{\infty} (Lj)^{3/2} E\left[\|X\|^3 1\{\|X\| \le c(n_j)\}\right]/\sqrt{n_j} < \infty,$$

where $c(t) := \sqrt{t}/(L_2 t)^5$, $t \ge 0$.

PROOF. To simplify our notation, we set $p_i = P\{c(n_{i-1}) < \|X\| \le c(n_i)\}$, $i \in \mathbb{N}$, where $n_0 := 0$.

Then it is easy to see that

$$\begin{aligned}\sum_{j=1}^{\infty} &(Lj)^{3/2} E\left[\|X\|^3 1\{\|X\| \le c(n_j)\}\right]/\sqrt{n_j} \\ &\le \sum_{j=1}^{\infty} (Lj)^{3/2} n_j^{-1/2} \sum_{i=1}^{j} c(n_i)^3 p_i \\ &= \sum_{i=1}^{\infty} c(n_i)^3 p_i \sum_{j=i}^{\infty} (Lj)^{3/2} n_j^{-1/2}.\end{aligned}$$

For large enough i we have

$$\sum_{j=i}^{\infty} (Lj)^{3/2} n_j^{-1/2} \frac{n_j - n_{j-1}}{n_j - n_{j-1}} \le \frac{5}{a} \sum_{j=i}^{\infty} (Lj)^{7/2} n_j^{-3/2} (n_j - n_{j-1}),$$

where we use (4.3).

Noticing that $L_2 n_j \sim Lj$ as $j \to \infty$, which follows from (4.3), we see that the last term is

$$\le K_1 \int_{n_{i-1}}^{\infty} (L_2 x)^{7/2} x^{-3/2}\, dx,$$

where $K_1 > 0$. Using the fact that

$$\int_t^{\infty} (L_2 x)^{7/2} x^{-3/2}\, dx = O\left((L_2 t)^{7/2} t^{-1/2}\right) \quad \text{as } t \to \infty,$$

we obtain for some constant $K_2 > 0$,

$$\begin{aligned}\sum_{j=1}^{\infty} &(Lj)^{3/2} n_j^{-1/2} E\left[\|X\|^3 1\{\|X\| \le c(n_j)\}\right] \\ &\le K_2 \sum_{i=1}^{\infty} c(n_i)^3 p_i n_i^{-1/2} (L_2 n_i)^{7/2} \\ &\le K_2 \sum_{i=1}^{\infty} c(n_i)^2 p_i / L_2 n_i,\end{aligned}$$

where the last series is finite on account of (2.1). □

5. A truncation argument. We now introduce a sequence $\{\bar{X}'_n\}$ of independent bounded mean zero r.v.'s as follows.

We set $X'_n := X_n 1\{\|X_n\| \le c(n)\}$, $\bar{X}'_n := X'_n - E[X'_n]$, $n \in \mathbb{N}$, where $c(t) := \sqrt{t}/(L_2 t)^5$, $t \ge 0$. We denote the corresponding partial sums by S'_n, $n \in \mathbb{N}$ and $\bar{S}'_n$, $n \in \mathbb{N}$.

The purpose of this section is to show that, in a certain sense, the almost sure behavior of the sums S_n is equivalent to that of the (bounded) sums $\bar{S}'_n$, $n \in \mathbb{N}$.

The proof would be much easier if we could set $c(t) = \sqrt{tL_2 t}$, $t \ge 0$. But this is not possible since Lemma 8 would not be true any longer. This lemma, however, will be decisive in Sections 6 and 7, where it enables us to control the error terms when using the normal approximation.

In order to make sure that the partial sum sequence $\{\bar{S}'_n\}$ has the desired property, we shall employ a double truncation argument which goes back to Feller [10], where it was given for symmetric real-valued r.v.'s. The extension thereof to (not necessarily symmetric) H-valued r.v.'s can be found in [6]. We now have to show that Feller's argument can also be utilized for r.v.'s satisfying only (2.1) and (2.2), but not necessarily $E[\|X\|^2] < \infty$. Since the proof is similar to that of [7], it will be enough to only indicate the main arguments.

We set

$$X'''_n := X_n 1\{\|X_n\| > d(n)\}, \qquad \bar{X}'''_n = X'''_n - E[X'''_n],$$

$$X''_n := X_n - X'_n - X'''_n, \qquad \bar{X}''_n := X''_n - E[X''_n], \qquad n \in \mathbb{N},$$

where $d(t) := \sqrt{tL_2 t}$, $t \ge 0$. Let S'''_n, $\bar{S}'''_n$, S''_n, $\bar{S}''_n$, $n \in \mathbb{N}$ be the corresponding partial sums.

We first note:

LEMMA 9. *We have*

$$(5.1) \qquad \sum_{n=1}^{\infty} E[\|X\| 1\{\|X\| > c(n)\}] n^{-1/2} (L_2 n)^{-6} < \infty.$$

In particular,

$$(5.2) \qquad \sum_{j=1}^{n} E[\|X\| 1\{\|X\| > c(j)\}] = o\left(\sqrt{n}\,(L_2 n)^6\right) \quad \text{as } n \to \infty.$$

PROOF. Using integration by parts, we obtain from assumption (2.1),

$$(5.3) \qquad \int_0^{\infty} E[\|X\| 1\{\|X\| > u\}] (L_2 u)^{-1}\, du < \infty.$$

We infer

$$\sum_{n=1}^{\infty} (c(n) - c(n-1))E[\|X\|1\{\|X\| > c(n)\}]/L_2c(n) < \infty,$$

from which (5.1) easily follows.

A straightforward application of the Kronecker lemma yields (5.2). □

We set

$$\mu_n := \sum_{j=1}^{n} E[\|X\|1\{\|X\| > c(j)\}].$$

Using exactly the same arguments as in the proof of Lemma 3, [7], we obtain:

LEMMA 10. *We have*

$$\sum_{n=1}^{\infty} \mu_n n^{-3/2}(L_2n)^{-6} < \infty.$$

Next we need:

LEMMA 11. *We have with probability* 1

$$S_n - \bar{S}'_n = o\left(\sqrt{nL_2n}\right) \quad \text{as } n \to \infty.$$

PROOF. Applying the Borel–Cantelli lemma, it is readily seen that

$$S'''_n = O(1) \quad \text{a.s.} \tag{5.4}$$

Moreover, it follows from Lemmas 2 and 3, [5] that

$$E[S'''_n] = o\left(\sqrt{nL_2n}\right) \quad \text{as } n \to \infty. \tag{5.5}$$

[Note that $P\{\|X\| > u\} = o(L_2u/u^2)$ as $u \to \infty$ on account of (2.1).] Thus, it suffices to show

$$\bar{S}''_n = o\left(\sqrt{nL_2n}\right) \quad \text{a.s.} \tag{5.6}$$

Employing Lemma 7(a), we find that

$$\begin{aligned} E\left[\|\bar{S}''_n\|\right] &\le E\left[\|\bar{S}''_n\|^2\right]^{1/2} \\ &\le \left(nE\left[\|X\|^2 1\{\|X\| \le d(n)\}\right]\right)^{1/2} \\ &= o\left((nL_2n)^{1/2}\right) \quad \text{as } n \to \infty. \end{aligned}$$

Using standard arguments, it is readily seen that (5.6) would follow if we were able to show for any $\varepsilon > 0$,

$$\sum_{j=2}^{\infty} P\left\{\|\bar{S}''_{n_j} - \bar{S}''_{n_{j-1}}\| > \varepsilon 2^{j/2}\sqrt{Lj}\right\} < \infty, \tag{5.7}$$

where $n_j := 2^j - 1$, $j \ge 1$.

Since we have $E[\|X_i''\|^3] \le 8E[\|X\|^3 1\{\|X\| \le d(i)\}]$, $i \in \mathbb{N}$, it follows from Lemma 6 that it is enough to show

$$\sum_{i=1}^{\infty} E\big[\|X\|^3 1\{\|X\| \le d(i)\}\big]/d(i)^3 < \infty \tag{5.8}$$

and

$$\sum_{j=1}^{\infty} \exp\left(-\varepsilon^2 2^j Lj/32 \sum_{n_{j-1}+1}^{n_j} \tilde{\sigma}_i^2\right) < \infty, \tag{5.9}$$

where $\tilde{\sigma}_i^2 := \sup\{E[(X_i'', y)^2]: \|y\| \le 1\}$.

(5.8) is immediate from Lemma 1, [5], so that it remains to show (5.9). Using the trivial inequalities $\tilde{\sigma}_i^2 \le \sigma^2$ and

$$\tilde{\sigma}_i^2 \le E\big[\|X_i''\|^2\big] \le E\big[\|X\|^3 1\{\|X\| \le d(i)\}\big]/c(n_{j-1}), \qquad n_{j-1} < i,$$

we obtain

$$\sum_{j=2}^{\infty} \exp\left(-\varepsilon^2 2^j Lj/32 \sum_{n_{j-1}+1}^{n_j} \tilde{\sigma}_i^2\right) \le 64\varepsilon^{-2} \sum_{j=2}^{\infty} d(n_j)^{-2} c(n_{j-1})^{-1} \times \sum_{n_{j-1}+1}^{n_j} E\big[\|X\|^3 1\{\|X\| \le d(i)\}\big] j^{-\varepsilon^2/64\sigma^2},$$

which is convergent on account of (5.8). □

We are now in a position to show:

LEMMA 12. *We have*

$$P\Big\{\|S_n - \bar{S}_n'\| > \sqrt{n}/L_2 n,\ \|\bar{S}_n'\| > \sigma\sqrt{nL_2 n}\ i.o.\Big\} = 0.$$

PROOF. From (5.2) and Lemma 11, it follows that

$$S_n - S_n' = o\big(\sqrt{n}\,(L_2 n)^6\big) \quad \text{a.s.} \tag{5.10}$$

Set $m(n) := [n/(L_2 n)^{14}]$, $n \in \mathbb{N}$, $n_k := 2^k - 1$, $k \in \mathbb{N}$,

$$\mathbb{N}_1 := \big\{k \in \mathbb{N}: E[\|S_n''\|] \le \sqrt{n}/2L_2 n \text{ for all } n \in (n_{k-1}, n_k]\big\},$$

$$\mathbb{N}_2 := \mathbb{N} - \mathbb{N}_1,$$

$$G_k := \bigcup_{n_{k-1}+1}^{n_k} \big(\tilde{A}_n \cap C_n\big), \qquad H_k := \bigcup_{n_{k-1}+1}^{n_k} C_n,$$

where $\tilde{A}_n := \bigcup_{k=m(n)}^{n} \{X_k \ne X_k'\}$, $C_n := \{\|\bar{S}_n'\| > \sigma\sqrt{nL_2 n}\}$.

Arguing as in [7], one can prove that it suffices to show

$$\sum_{k \in \mathbb{N}_1} P(G_k) < \infty \tag{5.11}$$

and

$$\sum_{k \in \mathbb{N}_2} P(H_k) < \infty. \tag{5.12}$$

Using the same arguments as in [7], we obtain for large enough k,

$$P(G_k) \le \sum_{l=m_k}^{n_k} P\{X_l \neq X_l'\} P\left\{ \max_{1 \le n \le n_k} \|\bar{S}_n'\| > \frac{\sigma}{4} 2^{k/2}\sqrt{Lk}\right\},$$

where $m_k := m(2^{k-1})$.

Since $E[\|\bar{S}_n'\|] = o(\sqrt{nL_2 n})$ as $n \to \infty$, we can use a standard symmetrization argument to show that for large k,

$$P\left\{ \max_{1 \le n \le n_k} \|\bar{S}_n'\| > \frac{\sigma}{4} 2^{k/2}\sqrt{Lk}\right\} \le 4P\left\{ \max_{1 \le n \le n_k} \|S_n^*\| > \frac{\sigma}{16} 2^{k/2}\sqrt{Lk}\right\},$$

where $S_n^* := \sum_{j=1}^n \varepsilon_j \bar{X}_j'$ and $\{\varepsilon_j\}$ is a Rademacher sequence independent of the sequence $\{\bar{X}_j'\}$.

Applying the Hoffmann–Jørgensen inequality ([14], (3.3), page 164) and the Lévy inequality for symmetric r.v.'s, we obtain for large enough k,

$$\begin{aligned} P&\left\{ \max_{1 \le n \le n_k} \|S_n^*\| > \frac{1}{16} 2^{k/2}\sqrt{Lk}\right\} \\ &\le 2P\left\{ \|S_{n_k}^*\| > \frac{1}{16} 2^{k/2}\sqrt{Lk}\right\} \\ &\le 32P\left\{ \|S_{n_k}^*\| > \frac{1}{144} 2^{k/2}\sqrt{Lk}\right\}^4. \end{aligned}$$

Lemma 6 now implies for large enough k,

$$\begin{aligned} P&\left\{ \|S_{n_k}^*\| > \frac{1}{144} 2^{k/2}\sqrt{Lk}\right\} \\ &\le k^{-\delta} + \tilde{A}E\left[\|X\|^3 1\{\|X\| \le c(n_k)\}\right] n_k^{-1/2}(Lk)^{-3/2}, \end{aligned}$$

where δ and $\tilde{A}$ are positive constants.

Recalling Lemma 7(a), it is easy to see that

$$E\left[\|X\|^3 1\{\|X\| \le c(n_k)\}\right] = o(c(n_k)Lk) = o\left(n_k^{1/2}(Lk)^{-4}\right) \quad \text{as } k \to \infty.$$

Thus it suffices to show, in order to prove (5.11),

$$\sum_{k=1}^{\infty} (Lk)^{-22} \sum_{l=m_k}^{n_k} P\{X_l'' \neq 0\} < \infty \tag{5.13}$$

and

$$\sum_{k=1}^{\infty} (Lk)^{-22} \sum_{l=m_k}^{n_k} P\{X_l''' \neq 0\} < \infty. \tag{5.14}$$

[Note that $P\{X_l \neq X_l'\} \leq P\{X_l'' \neq 0\} + P\{X_l''' \neq 0\}$.]

Using the inequality $P\{X_l'' \neq 0\} \leq E[\|X_l''\|^3]/c(l)^3$, we get for some $k_0 \in \mathbb{N}$:

$$\sum_{k=k_0}^{\infty} (Lk)^{-22} \sum_{l=m_k}^{n_k} P\{X_l'' \neq 0\}$$

$$\leq \sum_{k=k_0}^{\infty} (Lk)^{-5} \sum_{l=m_k}^{n_k} E\big[\|X_l''\|^3\big]/\sqrt{lL_2 l}^{\,3}$$

$$\leq \sum_{n=1}^{\infty} \left(\sum_{k_1(n)}^{k_2(n)} (Lk)^{-5} \right) E\big[\|X_n''\|^3\big]/\sqrt{nL_2 n}^{\,3},$$

where $k_1(n) := [Ln/\log(2)]$, $k_2(n) := \max\{k\colon m_k \leq n\}$.

Since $k_2(n) - k_1(n) = O(L_2 n)$ as $n \to \infty$, it easily follows from Lemma 1, [5] that the last series is convergent, thereby establishing (5.13).

The proof of (5.14) is similar. To prove (5.12), we first infer from Lemma 10 that

$$\sum_{k \in \mathbb{N}_2} (Lk)^{-7} < \infty. \tag{5.15}$$

Moreover, we have for large enough k,

$$P(H_k) \leq P\left\{ \max_{1 \leq n \leq n_k} \|\bar{S}_n'\| > \frac{\sigma}{4} 2^{k/2} \sqrt{Lk} \right\} \leq (Lk)^{-22}.$$

It is now clear from (5.15) that (5.12) is true and our proof of Lemma 12 is complete. □

Combining the two previous lemmas, we finally get:

LEMMA 13. *We have*

$$P\left\{ \|S_n - \bar{S}_n'\| \geq \sqrt{n}/L_2 n,\ \|S_n\| > \tfrac{5}{4}\sigma\sqrt{nL_2 n}\ i.o. \right\} = 0.$$

6. The lower class part. It is enough to prove this part for the special case where $\sigma^2 = 1$. We first define a subsequence $\{n_j\}$ by the following recursion:

$$n_1 := 1,$$

$$n_j := \min\{n\colon n \geq n_{j-1} \exp(\alpha\tilde{\gamma}_n/Lj)\}, \qquad j \geq 2,$$

where $\tilde{\gamma}_n := E[\|X\|^2 1\{\|X\| \leq c(n)\}] \vee 1$ and $\alpha > 2$ is a constant which will be specified later.

The function c is defined as in the previous sections, that is, $c(t) = \sqrt{t}/(L_2 t)^5$, $t \geq 0$.

Note that $\{n_j\}$ is well defined since we have on account of Lemma 7(a) for any $C > 0$, $n/\exp(C\tilde{\gamma}_n) \to \infty$ as $n \to \infty$. From the above definition it follows that

$$\exp\left(\alpha\tilde{\gamma}_{n_j}/Lj\right) \le n_j/n_{j-1} \le \exp\left(2\alpha\tilde{\gamma}_{n_j}/Lj\right), \qquad j \ge 2. \tag{6.1}$$

To see this observe that

$$n_j - 1 < n_{j-1}\exp\left(\alpha\tilde{\gamma}_{n_j-1}/Lj\right),$$

which implies that

$$\begin{aligned} n_j &= (n_j - 1)\left(1 + 1/(n_j - 1)\right) \le n_{j-1}\exp\left(\alpha\tilde{\gamma}_{n_j}(Lj)^{-1} + (j-1)^{-1}\right) \\ &\le n_{j-1}\exp\left(2\alpha\tilde{\gamma}_{n_j}/Lj\right). \end{aligned}$$

We use the trivial inequalities $n_j \ge j$ and $Lj \le j - 1$, $j \ge 2$.

We now need an upper bound for n_j. First observe that on account of Lemma 7(a), there exists a j_0 such that

$$\tilde{\gamma}_n \le L_2 n/4\alpha, \qquad n \ge n_{j_0}. \tag{6.2}$$

Let $q \ge 4$ be an integer such that $n_j \le q^j$, $j \le j_0$. Then it follows that

$$n_j \le q^j \quad \text{for all } j \in \mathbb{N}. \tag{6.3}$$

To establish (6.3), we use an induction argument. We have to show that for any $j \ge j_0$,

$$n_{j+1} \le q^{j+1} \quad \text{if } n_j \le q^j.$$

Recalling the definition of our subsequence $\{n_j\}$, we see that it suffices to show

$$\exp\left(\alpha\tilde{\gamma}_{q^{j+1}}/L(j+1)\right) \le q. \tag{6.4}$$

This follows from (6.2) after some calculation. Combining (6.1), (6.3) with Lemma 7(a), we find that

$$n_{j+1}/n_j \to 1 \quad \text{as } j \to \infty. \tag{6.5}$$

Moreover, one can infer from (6.1) that

$$n_j \ge \exp(\alpha(j-1)/Lj) \tag{6.6}$$

and

$$L_2 n_j \ge \tfrac{1}{2}Lj, \qquad j \ge 1. \tag{6.7}$$

We now consider the r.v.'s $\{\overline{X}'_n\}$ which have been defined in Section 5. After the subsequence $\{n_j\}$ has been introduced, we are in a position to give the decisive portion of the proof of Theorem 3(B). It is divided into two parts. First we show that

$$\sum_{j=1}^{\infty} P(A_j) = \infty \Rightarrow P\left(\limsup_{j\in\mathbb{N}} A_j\right) \ge C_0 > 0, \tag{6.8}$$

where

$$A_j := \left\{ \sqrt{n_j}\,\psi(n_j) < \|\bar{S}'_{n_j}\| < \sqrt{n_j}\left(\psi(n_j) + \beta/\psi(n_j)\right) \right\}, \qquad j \in \mathbb{N},$$

and ψ is an eventually nondecreasing function such that for large enough t,

$$\tfrac{4}{3}\sqrt{L_2 t} \le \psi(t) \le 2\sqrt{L_2 t}. \tag{6.9}$$

β is a positive constant which will be specified later.

Second, we prove that

$$\sum_{n=1}^{\infty} \frac{\psi(n)^d}{n\gamma_n} \exp\left(-\frac{1}{2}\psi(n)^2 + \gamma_n \right) = \infty \tag{6.10}$$

implies

$$\sum_{j=1}^{\infty} P(A_j) = \infty. \tag{6.11}$$

Combining these two parts of the proof, we see that we have for any function ψ as before which satisfies (6.9) and (6.10),

$$P\left(\|\bar{S}'_n\| > \sqrt{n}\,\psi(n) \text{ i.o.}\right) \ge C_0. \tag{6.12}$$

Now let ϕ be an eventually nondecreasing function such that

$$\tfrac{4}{3}\sqrt{L_2 t} \le \phi(t) \le \tfrac{3}{2}\sqrt{L_2 t}, \qquad t \ge 0, \tag{6.13}$$

and

$$\sum_{n=1}^{\infty} \frac{\phi(n)^d}{n\gamma_n} \exp\left(-\frac{1}{2}\phi(n)^2 + \gamma_n \right) = \infty. \tag{6.14}$$

Arguing as in Section 5, [7], we obtain from Lemma 13

$$P\left(\|S_n\| > \sqrt{n}\,\phi(n) \text{ i.o.}\right) \ge P\left(\|\bar{S}'_n\| > \sqrt{n}\,\psi(n) \text{ i.o.}\right), \tag{6.15}$$

where $\psi(t) = \phi(t) + 1/\phi(t)$, $t \ge 0$.

It is now easily checked that this function ψ satisfies (6.9) and (6.10). We infer from (6.12) and (6.15)

$$P\left(\|S_n\| > \sqrt{n}\,\phi(n) \text{ i.o.}\right) \ge C_0. \tag{6.16}$$

But this means that this probability is different from 0, whence it has to be equal to 1. Thus, the assertion follows for all functions satisfying (6.13). Using a standard argument we can finally remove this additional assumption (see, for instance, Lemma 1, [10]).

It remains to show (6.8), and (6.9) $\Rightarrow$ (6.10). To simplify our notation, we set $M_j := E[\|X\|^3 1\{\|X\| \le c(n_j)\}]$, $j \in \mathbb{N}$.

PART 1 [Proof of (6.8)]. Let $k < l$ be two fixed integers. We first need a lower bound for $P(\bigcup_{i=k}^{l} A_i)$. To that end we note that

$$P\left(\bigcup_{i=k}^{l} A_i\right) = P\left(\bigcup_{i=k}^{l}\left(A_i - \bigcup_{j=i+1}^{l} A_j\right)\right)$$

$$\geq \sum_{i=k}^{l}\left\{P(A_i) - \sum_{j=i+1}^{l} P(A_i A_j)\right\}$$

$$\geq \sum_{i=k}^{l} P(A_i)\left(1 - \sum_{j=i+1}^{l} P(B_{j,i})\right),$$

where

$$B_{j,i} := \left\{\|\bar{S}'_{n_j} - \bar{S}'_{n_i}\| > \sqrt{n_j}\,\psi(n_j) - \sqrt{n_i}\left(\psi(n_i) + \beta/\psi(n_i)\right)\right\}.$$

We now derive upper bounds for $P(B_{j,i})$.

CASE 1. $n_j \leq 4n_i$.
Recalling (6.9) it is readily seen that

$$\psi(n_j)/\psi(n_i) \leq 2(L_2 n_j/L_2 n_i)^{1/2} \leq 4.$$

We infer

$$\sqrt{n_j}\,\psi(n_j) - \sqrt{n_i}\left(\psi(n_i) + \beta/\psi(n_i)\right)$$

$$\geq \left(\sqrt{n_j} - \sqrt{n_i}\right)\psi(n_j) - 4\sqrt{n_i}\,\beta/\psi(n_j)$$

$$\geq \tfrac{1}{2}\left(\sqrt{n_j} - \sqrt{n_i}\right)\psi(n_j)$$

provided $\alpha \geq 48\beta$.

The last inequality can be easily verified when using (6.7), (6.9) and

$$n_j/n_i - 1 \geq \alpha \sum_{l=i+1}^{j} \tilde{\gamma}_{n_l}/Lj, \tag{6.17}$$

which in turn follows from (6.1).

Since $n_j/n_i \leq 4$ we obtain, by again using (6.7) and (6.9),

$$\tfrac{1}{2}\left(\sqrt{n_j} - \sqrt{n_i}\right)\psi(n_j) \geq \tfrac{1}{12}\sqrt{n_j - n_i}\,(n_j/n_i - 1)^{1/2}\sqrt{Lj}.$$

A straightforward calculation shows that we have

$$4\left(\sum_{m=n_i+1}^{n_j} E\left[\|\bar{X}'_m\|^2\right]\right)^{1/2} \leq 4\sqrt{n_j - n_i}\,\tilde{\gamma}_{n_j}^{1/2} \leq \tfrac{1}{12}\sqrt{n_j - n_i}\,(n_j/n_i - 1)^{1/2}\sqrt{Lj},$$

whenever $\alpha \geq 48^2$. Therefore, it is possible to apply Lemma 6 and get

$$P(B_{j,i}) \leq \exp\left(-\frac{1}{32 \cdot 144}(n_j/n_i - 1)Lj\right) + A'M_j/\sqrt{n_j - n_i},$$

where we use the trivial inequality,

$$E\left[\|\bar{X}'_{n_j}\|^3\right] \leq 8M_j, \qquad j \in \mathbb{N},$$

which is immediate from the Hölder inequality. Note also that

$$\sqrt{n_j}\psi(n_j) - \sqrt{n_i}(\psi(n_i) + \beta/\psi(n_i)) \geq 4\sqrt{n_j - n_i}.$$

Using (6.17), we obtain for $k \leq i < j$ such that $n_j \leq 4n_i$,

$$P(B_{j,i}) \leq 3^{i-j} + A'M_j\sqrt{Lj}/\sqrt{n_j}, \tag{6.18}$$

provided $\alpha \geq 2\log(3)48^2 \vee 48\beta$.

CASE 2. $4n_i \leq n_j \leq 4n_i(Li)^3$.
In this case we have for large enough k,

$$\begin{aligned}\sqrt{n_j}\psi(n_j) - \sqrt{n_i}(\psi(n_i) + \beta/\psi(n_i)) &\geq \tfrac{1}{2}\sqrt{n_j}\left(\psi(n_j) - \beta/\psi(n_i)\right)\\ &\geq \tfrac{1}{4}\sqrt{n_j}\psi(n_j).\end{aligned}$$

Applying Lemma 6, (6.7) and (6.9), we obtain

$$\begin{aligned}P(B_{j,i}) &\leq P\left\{\|\bar{S}'_{n_j} - \bar{S}'_{n_i}\| \geq \tfrac{1}{4}\sqrt{n_j}\psi(n_j)\right\}\\ &\leq j^{-1/1024} + 512A'M_j/\sqrt{n_j},\end{aligned}$$

provided k has been chosen so large that

$$\psi(n_j) \geq 16\tilde{\gamma}_{n_j}^{1/2}, \qquad j \geq k.$$

[Such a k exists on account of Lemma 7(a).]

To simplify our notation we set

$$I(i) := \left\{j: 4n_i \leq n_j \leq 4n_i(Li)^3\right\}.$$

Using (6.1), it is easily seen that we have

$$\#I(i) \leq C(\alpha)LiL_2i,$$

where $C(\alpha)$ is a positive constant depending on α.

We get for $k \geq k_0$ (say) and $i \geq k$,

$$\sum_{j \in I(i)} P(B_{j,i}) \leq k^{-1/1025} + 512A' \sum_{j \in I(i)} M_j/\sqrt{n_j}. \tag{6.19}$$

CASE 3. $n_j > 4n_i(Li)^3$.

In this case we have for large enough k,

$$\sqrt{n_j}\psi(n_j) - \sqrt{n_i}\big(\psi(n_i) + \beta/\psi(n_i)\big) \geq \sqrt{n_j}\big(\psi(n_j) - 1/\psi(n_j)\big). \tag{6.20}$$

We now need the following lemma which is the most difficult part of the proof of (6.8).

LEMMA 14. *We have for large enough n, $0 \leq m \leq n$ and an appropriate choice of β,*

$$\begin{aligned} P\{\|\bar{S}'_n - \bar{S}'_m\| \geq \sqrt{n}\,(\psi(n) - 1/\psi(n))\} &\\ \leq K_1\Big(P\{\sqrt{n}\,\psi(n) \leq \|\bar{S}'_n\| \leq \sqrt{n}\,(\psi(n) + \beta/\psi(n))\} &\\ + E\big[\|X\|^3 1\{\|X\| \leq c(n)\}\big]\psi(n)^3/\sqrt{n}\Big),& \end{aligned}$$

where K_1 is a positive constant depending on the distribution of X.

PROOF. For technical reasons we prove Lemma 14 under the additional assumption

(6.21) $\operatorname{cov}(X)$ has at least three positive eigenvalues in case it is compact.

If (6.21) does not hold, it follows that $\operatorname{cov}(X)$ has a finite trace and the proof which would be much easier could be based on (3.1). So we can assume from now on w.l.o.g. that (6.21) holds.

We start the proof with an application of Lemma 5 which leads to

$$\begin{aligned} P\{\|\bar{S}'_n - \bar{S}'_m\| \geq \sqrt{n}\,(\psi(n) - 1/\psi(n))\} &\\ \leq P\left\{\left\|\sum_{j=m+1}^{n} \bar{Y}'_j\right\| \geq \sqrt{n}\,(\psi(n) - 2/\psi(n))\right\} &\\ + 8AE\big[\|X\|^3 1\{\|X\| \leq c(n)\}\big]\psi(n)^3/\sqrt{n},& \end{aligned} \tag{6.22}$$

where $\{\bar{Y}'_j\}$ is a sequence of independent Gaussian mean zero r.v.'s with $\operatorname{cov}(\bar{Y}'_j) = \operatorname{cov}(\bar{X}'_j)$, $j \in \mathbb{N}$.

Let $\mu'_{1,n} \geq \mu'_{2,n} \geq \mu'_{3,n} \geq \cdots$ be the eigenvalues of $\operatorname{cov}(\bar{Z}'_n)$, where $\bar{Z}'_n := \sum_{j=1}^{n}\bar{Y}'_j/\sqrt{n}$ and further let $\{e'_{1,n}; e'_{2,n}; e'_{3,n}; \dots\}$ be a corresponding sequence of orthonormal eigenvectors so that we have

$$\bar{Z}'_n = \sum_{i=1}^{\infty} (\mu'_{i,n})^{1/2}\eta'_{i,n}e'_{i,n} \quad \text{a.s,} \tag{6.23}$$

where $\eta'_{i,n}$, $i \in \mathbb{N}$ are independent standard normal r.v.'s.

Set

$$\tilde{Z}_n := \sum_{i=1}^{d} \eta'_{i,n}e'_{i,n} + \sum_{i=d+1}^{\infty} (\mu'_{i,n})^{1/2}\eta'_{i,n}e'_{i,n}.$$

Then it is easy to see that

$$\operatorname{cov}\left(\sum_{j=m+1}^{n} \bar{Y}_j'/\sqrt{n}\right) \le \operatorname{cov}(\bar{Z}_n') \le \operatorname{cov}(\tilde{Z}_n),$$

where we use the notation $\Gamma_1 \le \Gamma_2$ whenever Γ_1, Γ_2 are self-adjoint operators such that $\Gamma_2 - \Gamma_1$ is positive semidefinite.

Employing the inequality of Anderson [2], we get

$$(6.24)\qquad \begin{aligned} &P\left\{\left\|\sum_{j=m+1}^{n} \bar{Y}_j'\right\| \ge \sqrt{n}\,(\psi(n) - 2/\psi(n))\right\} \\ &\quad \le P\{\|\tilde{Z}_n\| \ge \psi(n) - 2/\psi(n)\}. \end{aligned}$$

Next observe that since $\operatorname{cov}(\bar{Z}_n') \le \Sigma_n \le \operatorname{cov}(X)$, we have on account of Lemma A.1b,

$$\mu'_{d+1,n} \le \rho < 1,$$

and also

$$E\left[\|\tilde{Z}_n\|^2\right] = d + \sum_{i=d+1}^{\infty} \mu'_{i,n} \le d + \sum_{i=d+1}^{\infty} \sigma_{i,n}^2 \le d + E\left[\|X\|^2 1\{\|X\| \le c(n)\}\right],$$

which on account of Lemma 7(a) is

$$o(L_2 n) \quad \text{as } n \to \infty.$$

Further note that for some constant $K_2 > 0$,

$$(6.25)\qquad \liminf_{n\to\infty} \mu'_{3,n} \ge K_2,$$

where we need assumption (6.21).

Using Theorem 5(d) we obtain for large enough n and some $K_3 > 0$,

$$(6.26)\qquad P\{\|\tilde{Z}_n\| \ge \psi(n) - 2/\psi(n)\} \le K_3 P\{\|\tilde{Z}_n\| \ge \psi(n) + 1/\psi(n)\}.$$

Moreover, Theorem 5(c) enables us to find a constant $\bar{\beta} > 1$ such that for large enough n,

$$(6.27)\qquad \begin{aligned} &P\{\|\tilde{Z}_n\| \ge \psi(n) + 1/\psi(n)\} \\ &\quad \le 2P\{\psi(n) + 1/\psi(n) \le \|\tilde{Z}_n\| \le \psi(n) + \bar{\beta}/\psi(n)\}. \end{aligned}$$

We now need a lower bound for $\mu'_{d,n}$. Set $\Gamma_n := (\Sigma_1 + \cdots + \Sigma_n)/n$, and let $\mu_{1,n} \ge \mu_{2,n} \ge \cdots$ be the eigenvalues of Γ_n.

Then it is easy to see that

$$\|\operatorname{cov}(\bar{Z}_n') - \Gamma_n\| \le \frac{1}{n}\sum_{j=1}^{n} \|\Sigma_j - \operatorname{cov}(\bar{X}_j')\| \le \frac{1}{n}\sum_{j=1}^{n} \|E[X_j']\|^2.$$

Using Lemma A.1(a) in conjunction with the subsequent Lemma 15, we obtain

$$(6.28)\qquad |\mu'_{d,n} - \mu_{d,n}| = o(1/L_2 n) \quad \text{as } n \to \infty.$$

Since the operator sequence Σ_n, $n \in \mathbb{N}$ is eventually increasing, it follows for large enough n that

$$\Gamma_n \geq \frac{1}{n} \sum_{j=m_n}^{n} \Sigma_j \geq (1 - 1/L_2 n)\Sigma_{m_n},$$

where $m_n := [n/L_2 n]$.

Applying Lemma A.1(a), we find that

$$\mu_{d,n} \geq (1 - 1/L_2 n)\sigma^2_{d,m_n}. \tag{6.29}$$

Lemma A.2(a) along with Lemma 7(b) finally implies for some $K_4 > 0$,

$$1 - \sigma^2_{d,m_n} \leq K_4/L_2 n. \tag{6.30}$$

Combining (6.28)–(6.30), we see that for some positive constant K_5,

$$\mu'_{d,n} \geq 1 - K_5/L_2 n. \tag{6.31}$$

We denote the densities of $\|\bar{Z}'_n\|^2$ and $\|\tilde{Z}_n\|^2$ by g_n and $\tilde{g}_n$, respectively.

Applying Lemma 2(b) d times we obtain for some positive constant K_6

$$\tilde{g}_n(z) \leq K_6 g_n(z), \qquad z \leq 5L_2 n. \tag{6.32}$$

An integration of this inequality shows that we have for large enough n,

$$\begin{aligned} &P\{\psi(n) + 1/\psi(n) \leq \|\tilde{Z}_n\| \leq \psi(n) + \bar{\beta}/\psi(n)\} \\ &\quad \leq K_6 P\{\psi(n) + 1/\psi(n) \leq \|\bar{Z}'_n\| \leq \psi(n) + \bar{\beta}/\psi(n)\}. \end{aligned} \tag{6.33}$$

A second application of Lemma 5 yields

$$\begin{aligned} &P\{\psi(n) + 1/\psi(n) \leq \|\bar{Z}'_n\| \leq \psi(n) + \bar{\beta}/\psi(n)\} \\ &\quad \leq P\{\sqrt{n}\,\psi(n) \leq \|\bar{S}'_n\| \leq \sqrt{n}\,(\psi(n) + \beta/\psi(n))\} \\ &\qquad + 16AE\big[\|X\|^3 1\{\|X\| \leq c(n)\}\big]\psi(n)^3/\sqrt{n}, \end{aligned} \tag{6.34}$$

where $\beta := \bar{\beta} + 1$. Combining (6.22), (6.24), (6.26), (6.27), (6.33) and (6.34) we obtain the assertion. □

LEMMA 15. *Let X be a mean zero r.v. with $E[\|X\|^2/L_2\|X\|] < \infty$. Then we have as $n \to \infty$,*

$$\sum_{j=1}^{n} \|E[X1\{\|X\| \leq c(j)\}]\|^2 = o\big(n/(L_2 n)^2\big).$$

PROOF. Set $m_n := [n/(L_2 n)^3]$. Using Lemma 7(a), it is easy to see that as $n \to \infty$,

$$\begin{aligned} \sum_{j=1}^{m_n} \|E[X1\{\|X\| \leq c(j)\}]\|^2 &\leq \sum_{j=1}^{m_n} E\big[\|X\|^2 1\{\|X\| \leq c(n)\}\big] \\ &= o(m_n L_2 n) = o\big(n/(L_2 n)^2\big). \end{aligned}$$

Moreover, we have on account of $E[X] = 0$,

$$\sum_{j=m_n}^{n} \|E[X1\{\|X\| \le c(j)\}]\|^2 \le nE[\|X\|1\{\|X\| > c(m_n)\}]^2$$

$$= o\big(n/(L_2 n)^2\big) \quad \text{as } n \to \infty,$$

since $E[\|X\|^2/L_2\|X\|] < \infty$. □

Using Lemma 14 and relation (6.20), we see that if k is large enough for $k \le i < j$ such that $n_j \ge 4n_i(Li)^3$ we have

$$P(B_{j,i}) \le K_1\big(P(A_j) + M_j\psi(n_j)^3/\sqrt{n_j}\big). \tag{6.35}$$

We are now in a position to conclude the proof of (6.8). We set $\alpha = 2\log(3)48^2(\beta + 1)$, where β has to be chosen in accordance with Lemma 14. Then we obtain from (6.18), (6.19) and (6.35) for large enough k and all $l \ge k$

$$P\left(\bigcup_{i=k}^{l} A_i\right) \ge \sum_{i=k}^{l} P(A_i)\left(\tfrac{1}{2} - k^{-1/(1025)} - K_0\left(\varepsilon_k + \sum_{i=k}^{l} P(A_i)\right)\right),$$

where $K_0 > 0$ and $\varepsilon_k := \sum_{j=k}^{\infty} M_j\psi(n_j)^3/\sqrt{n_j}$, $k \ge 1$.

From Lemma 8 in conjunction with (6.3) and (6.9), it follows that $\{\varepsilon_k\}$ is a null sequence. Thus, we have for $k \ge k_1$ (say) and $l \ge k$

$$P\left(\bigcup_{i=k}^{l} A_i\right) \ge \sum_{i=k}^{l} P(A_i)\left(\tfrac{1}{3} - K_0\sum_{i=k}^{l} P(A_i)\right). \tag{6.36}$$

It is easy to see that

$$P(A_k) \to 0 \quad \text{as } k \to \infty. \tag{6.37}$$

(Use, for instance, Lemma 6 and notice Lemma 8.) Since we have, at the same time, $\sum_{k=1}^{\infty} P(A_k) = \infty$, we can find for any $k \ge k_2$, (say) an $l_k \ge k$ such that

$$(12K_0)^{-1} \le \sum_{i=k}^{l_k} P(A_i) \le (6K_0)^{-1}. \tag{6.38}$$

Combining (6.36) and (6.38) we get for large enough k,

$$P\left(\bigcup_{i=k}^{\infty} A_i\right) \ge P\left(\bigcup_{i=k}^{l_k} A_i\right) \ge (72K_0)^{-1} =: C_0.$$

But this implies

$$P\Big(\limsup_{i\in\mathbb{N}} A_i\Big) = \lim_{k\to\infty} P\left(\bigcup_{i=k}^{\infty} A_i\right) \ge C_0,$$

thereby completing part 1 of our proof.

PART 2 [Proof of (6.10) ⇒ (6.11)]. (i) We now assume that (6.10) holds. Recalling Lemma 14 and Lemma 8, it is readily seen that it suffices to show

$$\sum_{j=1}^{\infty} P\Big\{\|\bar{S}'_{n_j}\| \geq \sqrt{n_j}\,\big(\psi(n_j) - 1/\psi(n_j)\big)\Big\} = \infty. \tag{6.39}$$

Using Lemmas 5 and 8, it is obvious that (6.39) would follow if we were able to show

$$\sum_{j=1}^{\infty} P\Big\{\|\bar{Z}'_{n_j}\| \geq \psi(n_j) - 1/2\psi(n_j)\Big\} = \infty, \tag{6.40}$$

where $\bar{Z}'_n$, $n \in \mathbb{N}$ is defined as in the proof of Lemma 14.

(ii) Let $\{\hat{Z}_n\}$ be a sequence of Gaussian mean zero random variables such that

$$\operatorname{cov}\big(\hat{Z}_n\big) = n^{-1} \sum_{i=1}^{n} \big\{\Sigma_i - \operatorname{cov}\big(\bar{X}'_i\big)\big\}, \qquad n \in \mathbb{N}.$$

Suppose also that $\hat{Z}_n$ is independent of $\bar{Z}'_n$ and set $Z'_n := \bar{Z}'_n + \hat{Z}_n$, $n \in \mathbb{N}$.

Then it is easy to see that

$$\begin{aligned} &P\Big\{\|Z'_{n_j}\| \geq \psi(n_j) - 1/4\psi(n_j)\Big\} \\ &\quad \leq P\Big\{\|\bar{Z}'_{n_j}\| \geq \psi(n_j) - 1/2\psi(n_j)\Big\} + P\Big\{\|\hat{Z}_{n_j}\| \geq 1/4\psi(n_j)\Big\}. \end{aligned} \tag{6.41}$$

Since $\|\operatorname{cov}(\bar{X}'_i) - \Sigma_i\| \leq \|E[X'_i]\|^2$, $i \in \mathbb{N}$, we obtain from Lemma 15

$$\|\operatorname{cov}\big(\hat{Z}_{n_j}\big)\| = o\big((Lj)^{-2}\big) \quad \text{as } j \to \infty.$$

Using this relation in conjunction with Lemma 4, we find that

$$\sum_{j=1}^{\infty} P\Big\{\|\hat{Z}_{n_j}\| \geq 1/4\psi(n_j)\Big\} < \infty.$$

In view of (6.41), it is enough to prove that

$$\sum_{j=1}^{\infty} P\Big\{\|Z'_{n_j}\| \geq \psi(n_j) - 1/4\psi(n_j)\Big\} = \infty. \tag{6.42}$$

(iii) Let $\{Y'_n\}$ be a sequence of independent Gaussian mean zero r.v.'s with $\operatorname{cov}(Y'_n) = \Sigma_n$, $n \in \mathbb{N}$. We want to show that (6.42) follows from

$$\sum_{j=1}^{\infty} P\Big\{\|Y'_{n_j}\| > \psi(n_j)\Big\} = \infty. \tag{6.43}$$

Note that we have

$$\operatorname{cov}(Z'_n) = n^{-1} \sum_{i=1}^{n} \Sigma_i \leq \Sigma_n = \operatorname{cov}(Y'_n).$$

Let $\{\hat{Y}_n\}$ be a sequence of Gaussian mean zero r.v.'s with $\operatorname{cov}(\hat{Y}_n) = \Sigma_n - \operatorname{cov}(Z'_n)$ and suppose that Z'_n and $\hat{Y}_n$ are independent, $n \in \mathbb{N}$. Then we have

$$\mathscr{L}\left(Z'_n + \hat{Y}_n\right) = \mathscr{L}(Y'_n), \qquad n \in \mathbb{N},$$

and it is easy to see that

$$\begin{aligned} P\left\{\|Y'_{n_j}\| \geq \psi(n_j)\right\} \leq P\left\{\|\hat{Y}_{n_j}\| \geq 1/4\psi(n_j)\right\} \\ +P\left\{\|Z'_{n_j}\| \geq \psi(n_j) - 1/4\psi(n_j)\right\}. \end{aligned} \tag{6.44}$$

Introduce two subsequences $\mathbb{N}_1, \mathbb{N}_2 \subseteq \mathbb{N}$ as follows:

$$\mathbb{N}_1 := \left\{j \in \mathbb{N}: E\left[\|\hat{Y}_{n_j}\|^2\right] \leq 1/(Lj)^4\right\},$$

$$\mathbb{N}_2 := \mathbb{N} - \mathbb{N}_1.$$

Using Lemma 4, (6.7) and (6.9), it is readily seen that

$$\sum_{j \in \mathbb{N}_1} P\left\{\|\hat{Y}_{n_j}\| \geq 1/4\psi(n_j)\right\} < \infty. \tag{6.45}$$

We now show that

$$\sum_{j \in \mathbb{N}_2} P\left\{\|Y'_{n_j}\| \geq \psi(n_j)\right\} < \infty. \tag{6.46}$$

In order to establish (6.46), we first prove

$$\sum_{j=1}^{\infty} E\left[\|\hat{Y}_{n_j}\|^2\right]/(Lj)^2 < \infty. \tag{6.47}$$

From the definition of $\hat{Y}_{n_j}$, it is clear that

$$n_j E\left[\|\hat{Y}_{n_j}\|^2\right] \leq \sum_{i=1}^{n_j} E\left[\|X\|^2 1\{c(i) \leq \|X\| \leq c(n_j)\}\right],$$

so that (6.47) follows from the subsequent Lemma 16.

Using relation (6.47) and the definition of our subsequence $\mathbb{N}_2$, we infer

$$\sum_{j \in \mathbb{N}_2} (Lj)^{-6} < \infty. \tag{6.48}$$

Employing Lemma 4, it is now easy to see that (6.46) is true. Combining (6.43) and (6.46), we find that

$$\sum_{j \in \mathbb{N}_1} P\left\{\|Y'_{n_j}\| \geq \psi(n_j)\right\} = \infty,$$

which by virtue of (6.44) and (6.45) implies

$$\sum_{j\in\mathbb{N}_1} P\{\|Z'_{n_j}\| \geq \psi(n_j) - 1/4\psi(n_j)\} = \infty.$$

Thus we have shown that (6.43) implies (6.42).

(iv) Let $\{e_{i,n}: i \in \mathbb{N}\}$ be a sequence of orthonormal eigenvectors of Σ_n corresponding to the eigenvalues $\{\sigma_{i,n}^2: i \in \mathbb{N}\}$ so that we have

$$Y'_n = \sum_{i=1}^{\infty} \sigma_{i,n}\eta_{i,n}e_{i,n} \quad \text{a.s.},$$

where $\eta_{i,n}$, $i \in \mathbb{N}$ is a sequence of independent standard normal r.v.'s.

Set

$$Y_n := \sum_{i=1}^{d} \eta_{i,n}e_{i,n} + \sum_{i=d+1}^{\infty} \sigma_{i,n}\eta_{i,n}e_{i,n}.$$

Denote the densities of $\|Y'_n\|^2$ and $\|Y_n\|^2$ by g_n and h_n, respectively.

Since we have on account of Lemma 7(b),

$$1 - \sigma_{i,n}^2 = O\big((L_2 n)^{-1}\big) \quad \text{as } n \to \infty, \qquad 1 \leq i \leq d,$$

we obtain from Lemma 2(b)

$$h_n(z) \leq C g_n(z), \qquad z \leq 16\psi(n)^2, \qquad n \in \mathbb{N}, \tag{6.49}$$

where C is a positive constant.

It is obvious from (6.49) that (6.43) and consequently the assertion would follow if we were able to show that (6.8) implies

$$\sum_{j=1}^{\infty} P\{\psi(n_j) \leq \|Y_{n_j}\| \leq 4\psi(n_j)\} = \infty. \tag{6.50}$$

(v) We first note that on account of Lemma 4, (6.7) and (6.9),

$$\sum_{j=1}^{\infty} P\{\|Y_{n_j}\| > 4\psi(n_j)\} < \infty$$

so that (6.50) is equivalent to

$$\sum_{j=1}^{\infty} P\{\|Y_{n_j}\| \geq \psi(n_j)\} = \infty. \tag{6.51}$$

From Theorem 5(b), it follows that we have for some constant $C_2 > 0$ and for large enough j,

$$P\{\|Y_{n_j}\| \geq \psi(n_j)\} \geq C_2\psi(n_j)^{d-2}\exp\big(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_j}\big). \tag{6.52}$$

Next observe that for an appropriate $j_0 \in \mathbb{N}$,

$$\sum_{n_{j_0}+1}^{\infty} \psi(n)^d \exp\left(-\tfrac{1}{2}\psi(n)^2 + \gamma_n\right)/n\gamma_n$$

$$= \sum_{j=j_0}^{\infty} \sum_{n=n_j+1}^{n_{j+1}} \psi(n)^d \exp\left(-\tfrac{1}{2}\psi(n)^2 + \gamma_n\right)/n\gamma_n$$

$$\leq \sum_{j=j_0}^{\infty} \psi(n_j)^d \exp\left(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_{j+1}}\right)\gamma_{n_{j+1}}^{-1} \sum_{n=n_j+1}^{n_{j+1}} 1/n$$

$$\leq \sum_{j=j_0}^{\infty} \psi(n_j)^d \exp\left(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_{j+1}}\right)\log(n_{j+1}/n_j)\gamma_{n_{j+1}}^{-1},$$

which is, on account of (6.1), (6.3), and (6.9),

$$\leq 10\alpha \sum_{j=j_0}^{\infty} \psi(n_j)^{d-2} \exp\left(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_{j+1}}\right)\tilde{\gamma}_{n_{j+1}}/\gamma_{n_{j+1}}.$$

It is easy to see that $\tilde{\gamma}_n/\gamma_n = O(1)$ as $n \to \infty$, so that we have

$$\sum_{j=1}^{\infty} \psi(n_j)^{d-2} \exp\left(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_{j+1}}\right) = \infty. \tag{6.53}$$

From the definition of γ_n, it immediately follows that

$$\gamma_{n_{j+1}} - \gamma_{n_j} \leq E\left[\|X\|^2 1\{c(n_j) < \|X\| \leq c(n_{j+1})\}\right]/2(1-\rho), \qquad j \in \mathbb{N}.$$

Using this inequality along with our assumption (2.1), we find that

$$\sum_{j=1}^{\infty} (\gamma_{n_{j+1}} - \gamma_{n_j})/Lj < \infty.$$

Employing a similar argument as in part (iii), we infer that

$$\sum_{j:\, \gamma_{n_{j+1}} - \gamma_{n_j} \geq 1} \psi(n_j)^{d-2} \exp\left(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_{j+1}}\right) < \infty.$$

This means, in view of (6.53),

$$\sum_{j=1}^{\infty} \psi(n_j)^{d-2} \exp\left(-\tfrac{1}{2}\psi(n_j)^2 + \gamma_{n_j}\right) = \infty.$$

Recalling (6.52), we see that (6.51) is true.

LEMMA 16. *Let X be an H-valued r.v. such that $E[\|X\|^2/L_2\|X\|] < \infty$. Then we have*

$$\sum_{j=1}^{\infty} n_j^{-1}(Lj)^{-2} \sum_{i=1}^{n_j} E\left[\|X\|^2 1\{c(i) < \|X\| \leq c(n_j)\}\right] < \infty.$$

PROOF. Set $p_k := P\{c(k-1) < \|X\| \le c(k)\}$, $k \in \mathbb{N}$.
Then it is easy to see that

$$\sum_{i=1}^{n_j} E\left[\|X\|^2 1\{c(i) < \|X\| \le c(n_j)\}\right] \le \sum_{k=1}^{n_j} kc(k)^2 p_k.$$

Moreover, we have

$$\begin{aligned}\sum_{j=1}^{\infty} &\left(\sum_{k=1}^{n_j} kc(k)^2 p_k\right)/n_j(Lj)^2 \\ &\le \sum_{k=1}^{\infty}\left(\sum_{j:\, n_j \ge k} n_j^{-1}(Lj)^{-2}\right) kc(k)^2 p_k \\ &=: \sum_{k=1}^{\infty} \delta_k kc(k)^2 p_k.\end{aligned}$$

Since we know that $E[\|X\|^2/L_2\|X\|] < \infty$, it now suffices to show

$$\delta_k = O\big((kL_2k)^{-1}\big) \quad \text{as } k \to \infty.$$

First note that $L_2k \le L_2n_j \le 2Lj$ if $n_j \ge k$ and k is large enough. From the definition of our subsequence $\{n_j\}$, it follows that

$$n_j - n_{j-1} \ge n_{j-1}/Lj \ge n_j/2Lj$$

for large enough j.

We infer

$$\begin{aligned}\delta_k &\le 4(L_2k)^{-1} \sum_{j:\, n_j \ge k} n_j^{-2}(n_j - n_{j-1}) \\ &\le 4(L_2k)^{-1} \int_{k/2}^{\infty} x^{-2}\,dx = 8(kL_2k)^{-1}\end{aligned}$$

provided k is large enough. □

REMARK. The reader may have noticed that one could improve Theorem 3 to an integral test if one were able to prove (6.8) for the subsequence $[\exp(\alpha j/Lj)]$ rather than the sequence $\{n_j\}$. However, our approach does not yield any useful bounds for $P(A_iA_j)$ in this case if n_j/n_i is small [see the proof of (6.18)].

7. The upper class part. Without loss of generality, we assume that $\sigma^2 = 1$ and

(7.1) cov(X) has at least three positive eigenvalues in case it is a compact operator.

Using the same arguments as in Section 2.2, [6], it is readily seen from our extension of Feller's truncation lemma to the present setting that it is enough to prove

$$P\{\|\bar{S}_n'\| \geq \sqrt{n}\,(\phi(n) - 1/L_2 n)\text{ i.o.}\} = 0, \tag{7.2}$$

whenever ϕ is an eventually nondecreasing function such that

$$\tfrac{4}{3}\sqrt{L_2 t} \leq \phi(t) \leq 2\sqrt{L_2 t}, \qquad t \geq 0, \tag{7.3}$$

and

$$\sum_{n=1}^{\infty} n^{-1}\phi(n)\exp\left(-\tfrac{1}{2}\phi(n)^2 + \gamma_n\right) < \infty. \tag{7.4}$$

We need the following Lévy-type inequality which is implicitly contained in Lemma 2.7, [1].

LEMMA 17. *Let $X_1, \dots, X_n$ be independent mean zero random variables in Hilbert space. Suppose that $E[(X_j, y)^2] \leq \sigma^2$, $\|y\| \leq 1$, $1 \leq j \leq n$. Then we have for $1 \leq m \leq n$ and $x > 0$,*

$$P\left\{\max_{m \leq k \leq n}\left\|\sum_1^k X_j\right\| > x\right\} \leq 2P\left\{\left\|\sum_1^n X_j\right\| > x - \sqrt{2(m-n)}\,\sigma\right\}.$$

Set $n_j := [\exp(j/Lj)]$, $j \geq 1$. In view of Borel–Cantelli, it is clear that in order to establish (7.2) it suffices to show

$$\sum_{j=1}^{\infty} P\left\{\max_{n_j \leq n \leq n_{j+1}}\|\bar{S}_n'\| \geq \sqrt{n_j}\,(\phi(n_j) - 1/L_2 n_{j+1})\right\} < \infty. \tag{7.5}$$

Using Lemma 17 and (7.3), we obtain for large enough j,

$$P\left\{\max_{n_j \leq n \leq n_{j+1}}\|\bar{S}_n'\| \geq \sqrt{n_j}\,(\phi(n_j) - 1/L_2 n_{j+1})\right\}$$
$$\leq 2P\left\{\|\bar{S}_{n_{j+1}}'\| \geq \sqrt{n_{j+1}}\,(\phi(n_j) - K_1/\phi(n_j))\right\},$$

where K_1 is a positive constant.

A straightforward application of Lemma 5 shows that it is enough to prove

$$\sum_{j=1}^{\infty} P\left\{\left\|\sum_{i=1}^{n_{j+1}} \bar{Y}_i'\right\| \geq \sqrt{n_{j+1}}\,(\phi(n_j) - K_2/\phi(n_j))\right\} < \infty \tag{7.6}$$

and

$$\sum_{j=1}^{\infty} \phi(n_j)^3 E\left[\|X\|^3 1\{\|X\| \leq c(n_{j+1})\}\right]/\sqrt{n_{j+1}} < \infty, \tag{7.7}$$

where $K_2 := K_1 + 1$ and $\{\bar{Y}_n'\}$ is a sequence of independent Gaussian mean zero r.v.'s with $\operatorname{cov}(\bar{Y}_n') = \operatorname{cov}(\bar{X}_n')$, $n \in \mathbb{N}$.

Let the sequence $\{Y_n\}$ be defined as in Section 6, Part 2(iv). Applying the inequality of Anderson [2], it follows that

$$P\left\{\left\|\sum_{i=1}^{n_{j+1}} \bar{Y}_i'\right\| \geq \sqrt{n_{j+1}}\left(\phi(n_j) - K_2/\phi(n_j)\right)\right\}$$
$$\leq P\left\{\|Y_{n_{j+1}}\| \geq \phi(n_j) - K_2/\phi(n_j)\right\},$$

which on account of Theorem 5(a) and (7.1) is

$$\leq K_3\phi(n_j)^{d-2}\exp\left(-\tfrac{1}{2}\phi(n_j)^2 + \gamma_{n_{j+1}}\right),$$

where K_3 is a positive constant.

Arguing as in Section 6, we find that

$$\sum_{j=1}^{\infty} \phi(n_j)^{d-2}\exp\left(-\tfrac{1}{2}\phi(n_j)^2 + \gamma_{n_{j+1}}\right) < \infty,$$

whenever (7.4) holds true.

This implies (7.6). Relation (7.7) follows immediately from Lemma 8.

8. An example. The purpose of this section is to provide the following.

EXAMPLE. Let g be a continuous nondecreasing function such that

(i) $g(t) \to \infty$ as $t \to \infty$

and

(ii) $g(t) = O(L_2 t)$ as $t \to \infty$.

Let H be an infinite-dimensional separable Hilbert space. There exists a mean zero r.v. X with $E[\|X\|^2/g(\|X\|)] < \infty$ and $E[(X,y)^2] = \|y\|^2$, $y \in H$ such that we have for any $a > 0$,

$$P\left\{\|S_n\| > \sqrt{n}\,\phi_{a,3}(n) \text{ i.o.}\right\} = 1.$$

PROOF. Our example is a modification of that one given in Theorem 7.1, [12]. Let $\{e_n\}$ be a complete sequence of orthonormal vectors in H.

We set

$$X = \sum_{k=1}^{\infty} \xi_k e_k, \tag{8.1}$$

where $\{\xi_k\}$ is a sequence of independent r.v.'s such that for $k \geq 1$,

$$\begin{aligned} P\left\{\xi_k = \pm\sqrt{a_k}\right\} &= 1/2a_k, \\ P\{\xi_k = 0\} &= 1 - 1/a_k \end{aligned} \tag{8.2}$$

and $a_k = h(ak^2)^2$, where $h\colon [g(0),\infty) \to [0,\infty)$ is the inverse function of g and $a := g(1)$.

Since the ξ_k's are independent r.v.'s, it follows that

$$E[(X,x)(X,y)] = (x,y), \qquad x,y \in H. \tag{8.3}$$

Further observe that

$$E\big[\|X\|^2/g(\|X\|)\big] \le \sum_{k=1}^{\infty} E\big[\xi_k^2/g(|\xi_k|)\big] = (1/a)\sum_{k=1}^{\infty} k^{-2} < \infty.$$

Set $V_N := \operatorname{span}\{e_1,\dots,e_N\}$ and let Π_N be the projection onto V_N.

Let $X^{(N)} = \Pi_N(X)$, $X_k^{(N)} = \Pi_N(X_k)$, $k \in \mathbb{N}$ and set $S_n^{(N)} := \sum_{k=1}^n X_k^{(N)}$, $n \in \mathbb{N}$.

It is easily checked that $X^{(N)}$ satisfies the assumptions of the theorem in [6] and we obtain on account of (8.3),

$$P\big\{\|S_n^{(N)}\| > \sqrt{n}\,\phi_{N,3}(n) \text{ i.o.}\big\} = 1. \tag{8.4}$$

Since $\|S_n^{(N)}\| \le \|S_n\|$ and N can be chosen arbitrarily large, the assertion is an immediate consequence of (8.4). □

It is clear that the law of the three times iterated logarithm fails in the previous example, since assumption (2.4) is not satisfied. We note that one can construct related examples when σ^2 is not attained as an eigenvalue, or when condition (2.5) is not satisfied. But since this is more difficult than above, it will be done somewhere else.

APPENDIX

LEMMA A.1. *Let Σ_j: $H \to H$ be compact, self-adjoint positive semidefinite operators, $j = 1,2$. Let $\sigma_{i,j}^2$, $1 \le i < d_j$, be the positive eigenvalues of Σ_j arranged in a nonincreasing order and taking into account the multiplicities. Further, if $d_j < \infty$, set $\sigma_{i,j}^2 = 0$, $d_j \le i < \infty$, $j = 1,2$. Then we have*:

(a) $|\sigma_{i,1}^2 - \sigma_{i,2}^2| \le \|\Sigma_1 - \Sigma_2\|$, $i \in \mathbb{N}$.

(b) *Under the additional assumption that $\Sigma_2 - \Sigma_1$ is positive semidefinite, it follows that $\sigma_{i,1}^2 \le \sigma_{i,2}^2$, $i \in \mathbb{N}$.*

PROOF. Let $\{e_{i,j}: i \in \mathbb{N}\}$ be an orthonormal system of eigenvectors corresponding to the eigenvalues $\{\sigma_{i,j}^2: i \in \mathbb{N}\}$ of Σ_j, $j = 1,2$.

Further put $V_{k,j} = \operatorname{span}\{e_{i,j}: 1 \le i \le k\}$ and let $W_{k,j}$ be its orthogonal complement, $k \in \mathbb{N}$, $j = 1,2$.

Then it is easy to see that for $j = 1,2$ and $k \ge 1$,

$$(x, \Sigma_j x) \ge \sigma_{k,j}^2 \|x\|^2, \qquad x \in V_{k,j} \tag{A.1}$$

and

$$(x, \Sigma_j x) \le \sigma_{k+1,j}^2 \|x\|^2, \qquad x \in W_{k,j}. \tag{A.2}$$

Let $k \ge 2$ be fixed and let Π_k: $H \to V_{k-1,2}$ be the orthogonal projection onto $V_{k-1,2}$. Consider the restriction

$$T_k := \Pi_k|_{V_{k,1}}: V_{k,1} \to V_{k-1,2}.$$

Since $k = \dim(V_{k,1}) > k - 1 = \dim(V_{k-1,2})$, we find by using a standard argument from linear algebra that

$$1 \le \dim\bigl(T_k^{-1}\{0\}\bigr) = \dim\bigl(V_{k,1} \cap W_{k-1,2}\bigr).$$

This means that there exists an $\bar{x} \in H$ such that

$$\bar{x} \in V_{k,1} \cap W_{k-1,2} \quad \text{and} \quad \|\bar{x}\| = 1. \tag{A.3}$$

Using relations (A.1) and (A.2), we find that

$$\sigma_{k,1}^2 - \sigma_{k,2}^2 \le \bigl(\bar{x}, (\Sigma_1 - \Sigma_2)\bar{x}\bigr) \le \|\Sigma_1 - \Sigma_2\|.$$

An obvious modification of the above argument shows

$$\sigma_{k,2}^2 - \sigma_{k,1}^2 \le \|\Sigma_2 - \Sigma_1\|, \qquad k \ge 2.$$

Thus, we have proven part (a) for $k \ge 2$.

Choosing $\bar{x}$ according to (A.3), we also see that

$$\sigma_{k,1}^2 - \sigma_{k,2}^2 \le \bigl(\bar{x}, (\Sigma_1 - \Sigma_2)\bar{x}\bigr) \le 0, \qquad k \ge 2,$$

whenever $\Sigma_2 - \Sigma_1$ is positive semidefinite.

If $k = 1$, the assertion easily follows from the known fact that the largest eigenvalue of a self-adjoint positive semidefinite operator is equal to the operator norm. □

A slight modification of the proof of Lemma A.1 yields:

LEMMA A.2. *Let Σ_j: $H \to H$, $j = 1, 2$ be bounded self-adjoint positive semidefinite operators, $j = 1, 2$. Suppose that Σ_1 is compact and that $\sigma^2 := \|\Sigma_2\|$ is an eigenvalue of Σ_2. If the corresponding eigenspace V of Σ_2 is finite-dimensional, we have for the eigenvalue sequence $\{\sigma_{i,1}^2\colon i \in \mathbb{N}\}$ of Σ_1,*

(a) $$|\sigma_{i,1}^2 - \sigma^2| \le \sup_{\substack{\|v\|=1 \\ v \in V}} |(v, (\Sigma_2 - \Sigma_1)v)|, \qquad i = 1, \dots, d,$$

where $d := \dim(V)$.

(b) *Under the additional assumption that $\Sigma_2 - \Sigma_1$ is positive semidefinite, we also have*

$$\sigma_{i,1}^2 \le \sup_{\substack{\|w\|=1 \\ w \in W}} (w, \Sigma_2 w), \qquad i \ge d + 1,$$

where W is the orthogonal complement of V.

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