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DIFFUSION APPROXIMATION FOR A CLASS OF TRANSPORT PROCESSES WITH PHYSICAL REFLECTION BOUNDARY CONDITIONS

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Consider a stochastic process consisting of the pair of a position and a velocity, in a piecewise $\mathscr{C}^1$ d-dimensional domain. In the interior of the domain the dynamics are assigned by a potential and by random changes of the velocity occurring at exponentially distributed times, according to a probability distribution which may depend on the current position and velocity. On the boundary the process reflects physically (the angle of reflection equals the angle of incidence). First it is shown that the process is well defined for all times. Then, when the coefficients depend on a diverging parameter N, in particular such that the speed and the jump rate of the velocity go to ∞ with order $\sqrt{N}$ and at least N respectively, a diffusion approximation is sought. The position process is represented as a solution of a Skorohod reflection equation: A skewing effect on the boundary results from the interaction between the dynamics and the reflection law, so that the direction of reflection is in general oblique. The assumption that the mean change of the velocity in the interior is linear in the current velocity, up to order at least $1/2$ in $1/N$, ensures that the cone of directions of reflection is independent of N. The continuity properties of the Skorohod oblique reflection problem enable one to show tightness of the position processes without having to estimate explicitly local times (boundary layers) and, together with a suitable law of large numbers for the velocity, allow one to identify the limit stochastic differential equation with oblique reflection. The theory is illustrated by several applications, in particular one to a mechanical model of Brownian motion.

Introduction. This paper is devoted to studying a class of transport processes subject to physical reflection.

The typical process can be viewed as the pair of the position and the velocity of a particle moving in a piecewise smooth domain of $\mathbb{R}^d$: In the interior the particle moves under the effect of a potential U; at random exponentially distributed times it changes velocity, according to a probability distribution P^N which may depend on both the current position and velocity (these random changes of the velocity in the interior can be thought of, for instance, as the result of collisions with other particles); when the particle hits the boundary it reflects physically, that is, the angle of reflection equals the angle of incidence. The evolution of the motion is described in the interior of the domain by an

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infinitesimal operator of the form

$$L^N f(x,v) = \beta^N [\nabla_x f(x,v) \cdot v + \nabla_v f(x,v) \cdot \nabla_x U] \tag{1}$$
$$+ \mu^N(x,v) \int [f(x,v') - f(x,v)] P^N(x,v;dv'),$$

where N varies in $\mathbb{N}$ and on the boundary by the equations of conservation of energy and momentum:

$$v' = v - 2(v \cdot \nu(x))\nu(x) \tag{2}$$

[where $\nu(x)$ denotes the unit inward normal vector]. Incidentally, note that, even when μ^N and P^N do not depend on x, the process is a random evolution with feedback, due to the reflection on the boundary.

The main goal of this work is to prove a diffusion approximation for the stochastic process defined by (1) and (2) as the speed β^N diverges with order $\sqrt{N}$ and the jump rate of the velocity in the interior μ^N diverges with order at least N (or, equivalently, if the motion is observed on time and space scales, respectively, $\sqrt{N}$ times and at least N times coarser than the ones the particle "sees"). Of course, the question makes sense only if, at least in the limit, the velocity is centered. Note that P^N is allowed to depend on N. It turns out that, if the velocity process has sufficient ergodic properties, in the limit the dependence of the position on the velocity averages out, and the position approaches a reflecting diffusion process. It is worth noting that even though the transport process reflects physically, so that the angle (with respect to the normal direction) at which the particle leaves the boundary depends on the angle at which it arrived there and does not depend on the particular location, in the limit one obtains a diffusion with oblique reflection (with direction determined by μ^N and P^N), for which one may say that the angle of reflection (with respect to the normal direction) depends on the particular location and not on the "angle" at which the path arrived there. In fact, the scaling is such that the reflection operator defined by (2) and the internal dynamics defined by (1) interact in the limit.

There is a vast literature on random evolutions [for a review of the area see Hersh (1974)]. General limit theorems have been proved by Kurtz (1972, 1973), by means of an abstract theorem on convergence of perturbed operator semigroups which has partly suggested the approach taken here. More specifically, transport processes. have been studied, in smooth domains and with compact velocity state space, by Bensoussans, Lions and Papanicolaou (1979). In their work, both absorbing and reflecting boundary operators are considered (a reflecting process now being more in general one which spends zero time on the boundary). The reflecting process changes velocity on the boundary according to a given probability distribution; however, physical reflection is not allowed, since a certain uniformity condition on the angle at which the particle takes off from the boundary and a form of Doeblin's condition are required [(3.6.7) and (3.6.8) in Bensoussans, Lions and Papanicolaou (1979)]. These conditions are used both to ensure that the process is well defined for all times and to obtain a diffusion approximation. The diffusion approximation is based on asymptotic expansions in $1/N$ of both the interior and the boundary operators, combined with martingale problem techniques.

The approach proposed here is a different one. Existence for all times of the reflecting transport process is proved by using a result in the theory of deterministic dynamical systems due to Marchioro, Pellegrinotti, Presutti and Pulvirenti [(1976), Section 2]. A diffusion approximation is obtained for the class of processes such that the mean jump of the velocity in the interior is linear with respect to the current velocity, up to order at least $1/2$ in $1/N$; namely,

$$\mu^N(x,v)\int(v'-v)P^N(x,v;dv') = -NQ(x)v + \sqrt{N}q^N(x,v). \tag{3}$$

The central idea is to represent the position process as the first component of a solution to a Skorohod reflection problem, driven by the proper semimartingale, in order to exploit the continuity properties of the Skorohod reflection problem (Section 3) to derive tightness of the position processes from tightness of the driving semimartingales, without having to estimate explicitly the local times, that is, the boundary layers (Section 4). Equation (3) simplifies the problem by ensuring that all possible directions of reflection belong to one cone, independent of N. Note that this method does not require any localization argument, and enables one to deal, in many cases, with domains with corners. A suitable law of large numbers for the velocity process (nontrivial because of the dependence of the velocity on the position process), together with the martingale central limit theorem and a recent result by Jakubowski, Memin and Pages (1989) on the continuity of the stochastic integral, allow one to identify the limiting stochastic differential equation with oblique reflection (Section 5). Although the restriction imposed by (3) seems not to be intrinsic to the techniques employed here, the general case needs additional work. However, many models of physical interest satisfy (3) (cf. Section 6). In particular, the results of Sections 2 to 5 apply to a transport process model of physical Brownian motion proposed by Dürr, Goldstein and Lebowitz (1981) and shown by them to be asymptotically equivalent to the mechanical model.

1. Formulation of the problem. Let D be a domain in $\mathbb{R}^d$ with piecewise $\mathscr{C}_b^1$ boundary, that is,

$$D = \bigcap_{i=1}^{h} D_i,$$

where D_i, $i = 1,\dots,h$, are defined by

$$D_i = \{x: \psi_i(x) > 0\}, \qquad \partial D_i = \{x: \psi_i(x) = 0\},$$
$$\psi_i \in \mathscr{C}_b^1(\mathbb{R}^d,\mathbb{R}), \qquad \inf_{x\in\partial D_i}|\nabla\psi_i(x)| > 0.$$

Denote by $\partial_r D$ and $\partial_s D$ the regular and singular part of the boundary of D

respectively, that is,

$$\partial_r D = \dot{\bigcup_{i=1}^{h}} \partial_i D, \qquad \partial_i D = \partial D_i \cap \bigcap_{k \neq i} D_k, \qquad \partial_s D = \partial D - \partial_r D.$$

Throughout this paper it will be assumed that D satisfies the uniform exterior sphere condition [cf. Lions and Sznitman (1984), Section 1, (1) and Remark 1.2] and the following condition [cf. Saisho (1987), condition (B) and Remark 1.2]:

$$(1.1) \qquad \begin{aligned} &\lim_{\rho \to 0} \inf_{x \in \partial D} a(x, \rho) = a > 0, \\ &a(x, \rho) = \max_{|u|=1} \min_{y \in \partial D \cap \overline{B_\rho(x)}} \min_{|\nu|=1,\, \nu \in N(y)} \nu \cdot u, \end{aligned}$$

where $B_\rho(x)$ denotes the open ball of radius ρ centered at x and $N(x)$, for $x \in \partial D$, is the closed convex cone of inward normal vectors to D at x [cf. Costantini (1990), Remark 2.1]. Note that both the uniform exterior sphere condition and (1.1) are verified if ∂D is of class $\mathscr{C}_b^2$. If D is convex it always satisfies the uniform exterior sphere condition and if it is also bounded, or if $d = 2$, it always satisfies (1.1) as well [cf. Tanaka (1979), page 170].

Let U be a nonpositive function in $\mathscr{C}_b^2(\overline{D}, \mathbb{R})$ and E: $E(x, v) = \frac{1}{2}|v|^2 - U(x)$, $x \in \overline{D}$, $v \in \mathbb{R}^d$. All results in this work apply both when the state space is taken to be $O = D \times \mathbb{R}^d$ or $O = \{(x, v) \in D \times \mathbb{R}^d\colon E(x, v) = c\}$, for some positive constant $c > \|U\|$. Also let $\Delta_x = \{v \in \mathbb{R}^d\colon (x, v) \in \overline{O}\}$ [note that $\Delta_x = \mathbb{R}^d$ or $\Delta_x = S_{\sqrt{2(c + U(x))}}(0)$, where S_r is the ball of radius r centered at the origin].

Here and in the sequel $|\cdot|$ denotes the norm of a vector or a matrix, while $\|\cdot\|$ denotes the supremum norm of an $\mathbb{R}$-valued ($\mathbb{R}^d$-valued, $\mathbb{R}^{d\times d}$-valued) function. The symbol ∂f, for f of class $\mathscr{C}^1$, will be used indiscriminately for the gradient and the Jacobian; the subscripts x and v will be used to distinguish between differentiation with respect to the first and second variable, respectively.

For $f \in \mathscr{C}_b(\overline{O}, \mathbb{R})$ and for $f \in \mathscr{C}_b^1(\overline{D} \times \mathbb{R}^d, \mathbb{R})$, respectively, define

$$(1.2) \qquad \begin{aligned} Af(x, v) &= \mu(x, v) \int_{\Delta x} [f(x, v') - f(x, v)] P(x, v; dv'), \\ Lf(x, v) &= \beta[\partial_x f(x, v) \cdot v + \partial_v f(x, v)\, \partial U(x)] + Af(x, v), \end{aligned}$$

where β is a positive constant, μ is a nonnegative measurable function, $P(x, v; \cdot)$ is a probability measure on Δ_x, for every $v \in \Delta_x$, $x \in \overline{D}$, and $P(\cdot, \cdot\,; B)$ is a measurable function, for every Borel set B. It will be assumed that

$$(1.3) \qquad \sup_{x, v} \mu(x, v) = \mu < \infty$$

and that A maps $\mathscr{C}_b(\overline{O}, \mathbb{R})$ into itself and is continuous with respect to the topology of uniform convergence over compact sets (u.c. topology); this holds, for instance, if μ is continuous and P is continuous as a function from $\overline{O}$ into the space of probability measures on $\mathbb{R}^d$, endowed with the topology of weak convergence.

Obviously, (1.2) extends immediately to

$$f \in \mathscr{C}_b(\overline{O}, \mathbb{R}^{d_1 \times d_2}) \quad \text{and} \quad f \in \mathscr{C}_b^1(\overline{D} \times \mathbb{R}^d, \mathbb{R}^{d_1 \times d_2}),$$

respectively, by setting

$$(Af)_{i,j} = Af_{i,j}, \qquad (Lf)_{i,j} = Lf_{i,j}, \qquad i = 1, \dots, d_1,\ j = 1, \dots, d_2.$$

REMARK 1.1. Without loss of generality we can always suppose

$$\mu(x, v) = \mu, \qquad \forall\, (x, v) \in \overline{O}.$$

In fact A can always be represented as

$$Af(x,v) = \mu \int_{\Delta_x} [f(x,v') - f(x,v)]$$
$$\times \left[\frac{\mu(x,v)}{\mu} P(x, v; dv') + \left(1 - \frac{\mu(x,v)}{\mu}\right) \delta_v(dv') \right],$$

where $\delta_v(\cdot)$ denotes the Dirac measure on v.

Let m denote the Lebesgue measure on O, as well as on D, and let $m(x, \cdot)$ denote the Lebesgue measure on Δ_x [the surface Lebesgue measure on $\overline{O}$ and Δ_x respectively, if $O = \{(x,v) \in D \times \mathbb{R}^d \colon E(x,v) = c\}$].

As anticipated in the Introduction, the first goal of this work is to construct, for any given initial distribution P_o on $\overline{O}$, a transport process (X, V) whose behavior is assigned in D by an infinitesimal generator of the form L, and on ∂D by the laws of physical reflection. This is achieved in Section 2 by showing that if:

(H_1) P_o is absolutely continuous with respect to m on the Borel σ-algebra, and, for every nonnegative $p \in L^1(m(x, \cdot))$,

$$\int_{\Delta_x} P(x, v; \cdot) p(v) m(x, dv)$$

is absolutely continuous w.r.t. $m(x, \cdot)$ on the Borel σ-algebra, for m-almost all $x \in \overline{D}$, then there exists a unique solution, well defined for all $t \in \mathbb{R}^+$ with probability 1, to the following system of pathwise stochastic equations:

$$
(1.4)\qquad
\begin{aligned}
X(t) &= X_o + \beta\int_0^t V(s)\,ds,\\
V(t) &= V_o + \beta\int_0^t \partial U(X(s))\,ds\\
&\quad + \int_0^{t^+}\left[F\left(X(s),V(s^-);\xi_{Y^\mu(s)}\right) - V(s^-)\right] dY^\mu(s)\\
&\quad + \int_0^{t^+}\nu(X(s))\,d\Lambda(s),\\
\Lambda(t) &= \sum_{s\le t}\left[-2V(s^-)\cdot\nu(X(s))\right] I_{\partial D}(X(s)),\\
&\qquad X(t)\in D\cup\partial_r D, \qquad \forall\, t\in\mathbb{R}^+,
\end{aligned}
$$

where Y^μ is a Poisson process with parameter μ, $\{\xi_n\}$, $\xi_n =_{\mathscr{D}} \xi$, is a sequence of i.i.d. $[0,1]$-valued uniform random variables, $F\colon \overline{O}\times[0,1]\to\mathbb{R}^d$ is a measurable function such that $F(x,v;\xi)$ has the probability distribution $P(x,v;\cdot\,)$ (as it is well known, such a function always exists) and (X_o, V_o) is an $\overline{O}$-valued random variable with probability distribution P_o; (X_o,V_o), $\{\xi_n\}$, Y^μ are realized on the same complete probability space $(\Omega,\mathscr{F},P)$ and are mutually independent.

REMARK 1.2. Note that the assumption made earlier that A maps $\mathscr{C}_b(S,\mathbb{R}^d)$ into itself and is continuous in the u.c. topology is satisfied if, for instance, $F(\cdot,\cdot\,;u)$ is continuous for each $u\in[0,1]$.

REMARK 1.3. Condition (H_1) holds in particular in the following two cases:

(i) when $P(x,v;\cdot\,)$ is absolutely continuous with respect to $m(x,\cdot\,)$ for m-almost all $(x,v)\in\overline{O}$;

(ii) when $F(x,\cdot\,;u)$ belongs to $\mathscr{C}^1(\Delta_x,\mathbb{R}^d)$ and $|\det(\partial_v F(x,v;u))|$ is strictly positive for every $v\in\Delta_x$, for m-almost all $x\in\overline{D}$, for every $u\in[0,1]$ [here differentiation is meant with respect to the local coordinates on Δ_x, if $\Delta_x = S_{\sqrt{2(c+U(x))}}(O)$].

Suppose now that P, β and μ depend on the index N in such a way that

$$
\beta^N = O(\sqrt{N}),\qquad \mu^N\int_{\Delta_x}(v'-v)P^N(x,v;dv') = O(N)\quad\text{and}
$$

$$
\mu^N\int_{\Delta_x}|v'-v|^2 P^N(x,v;dv') = O(N).
$$

Typically this is the case when (X^N,V^N) is a central limit theorem space–time rescaling of a fixed process (X, V), that is, with our notation,

$$P^N = P, \qquad \beta^N = O(\sqrt{N}) \quad \text{and} \quad \mu^N = O(N)$$

(the potential U is supposed to vary on the macroscopic space scale). Here a more general formulation has been chosen which allows a unified treatment of many applications, in particular some which do not fit in the classical central limit theorem space–time rescaling set up [see, for instance, examples (A) and (B2) in Section 6]. Then, if, as $N \to \infty$, V^N oscillates averaging to 0, it is possible to study the limit behavior of the process X^N. In this context, setting for simplicity

$$\beta^N = \sqrt{N},$$

the second goal of this work is to obtain a diffusion approximation for the class of processes such that

$$(1.5)\qquad \begin{aligned} \mu^N \int_{\Delta_x} (v' - v) P^N(x, v; dv') &= \mu^N E[\![F^N(x, v; \xi) - v]\!] \\ &= -NQ(x)v + \sqrt{N} q^N(x, v), \\ |q^N(x, v)| &\le c_o(1 + |v|), \end{aligned}$$

for all (x, v) in $\overline{O}$, for some positive constant c_o. Then (1.4) can be written as

$$(1.6)\qquad \begin{aligned} X^N(t) &= X_o + \sqrt{N} \int_0^t V^N(s)\, ds, \\ V^N(t) &= V_o + \sqrt{N} \int_0^t \partial U(X(s))\, ds - N \int_0^t Q(X^N(s)) V^N(s)\, ds \\ &\quad + \sqrt{N} \int_0^t q^N(X^N(s), V^N(s))\, ds \\ &\quad + \sqrt{N} M^N(t) + \sqrt{N} \int_0^{t^+} \nu(X^N(s))\, d\Lambda^N(s), \\ \Lambda^N(t) &= \frac{1}{\sqrt{N}} \sum_{s \le t} [-2V^N(s^-) \cdot \nu(X^N(s))] I_{\partial D}(X^N(s)), \\ &\qquad X^N(t) \in D \cup \partial_r D, \qquad \forall\, t \in \mathbb{R}^+, \end{aligned}$$

with

$$(1.7)\qquad \begin{aligned} M^N(t) &= \frac{1}{\sqrt{N}} \int_0^{t^+} \left\{ F^N\left(X^N(s), V^N(s^-), \xi_{Y^{\mu^N}(s)}\right) - V^N(s^-) \right\} dY^{\mu^N}(s) \\ &\quad + \int_0^t \left\{ \sqrt{N} Q(X^N(s)) V^N(s) - q^N(X^N(s), V^N(s)) \right\} ds. \end{aligned}$$

A diffusion approximation for the process X^N defined by (1.6) and (1.7) will be derived in Sections 4 and 5 under the following assumptions:

$$(\mathrm{H}_2) \qquad Q \in \mathscr{C}_b^1(\overline{D}, \mathbb{R}^{d\times d}), \qquad \inf_{x\in\overline{D}} \min_{|u|=1} u^t Q(x) u = q_o > 0;$$

$$(\mathrm{H}_3) \qquad E[\![|V_o|^4]\!] < \infty;$$

(H_4) $P^N(x, v; \cdot\,)$ has finite fourth moment and, setting

$$H^N(x,v) = \frac{\mu^N}{N}\int_{\Delta_x} (v'-v)(v'-v)^t P^N(x,v;dv'),$$

$$h^N(x,v) = \mathrm{tr}(H^N(x,v)) = \frac{\mu^N}{N}\int_{\Delta_x} |v'-v|^2 P^N(x,v;dv'),$$

$$g^N(x,v) = \frac{\mu^N}{N}\int_{\Delta_x} |v'-v|^4 P^N(x,v;dv'),$$

one has

$$\sup_N \sup_{x,v} h^N(x,v) < \infty$$

and

$$\sup_N g^N(x,v) \le g_o(|v|^2+1), \qquad \forall\,(x,v)\in\overline{O},$$

for some positive constant g_o.

$$(\mathrm{H}_5) \qquad \overline{\lim_{N\to\infty}} \frac{N}{\mu^N} < \infty.$$

REMARK 1.4. Let $\mathscr{C}_\alpha(\overline{O},\mathbb{R})$ $[\mathscr{C}_\alpha(\overline{O},\mathbb{R}^d), \mathscr{C}_\alpha(\overline{O},\mathbb{R}^{d\times d})]$ denote the set of continuous functions f such that

$$|f(x,v)| \le f_o(1+|v|^\alpha), \qquad \forall\,(x,v)\in\overline{O}.$$

Note that every $f\in\mathscr{C}_\alpha(\overline{O},\mathbb{R})[\mathscr{C}_\alpha(\overline{O},\mathbb{R}^d), \mathscr{C}_\alpha(\overline{O},\mathbb{R}^{d\times d})]$ can be approximated in the u.c. topology by a sequence $\{f^n\}\subseteq \mathscr{C}_o^\infty(\overline{O},\mathbb{R})[\mathscr{C}_o^\infty(\overline{O},\mathbb{R}^d), \mathscr{C}_o^\infty(\overline{O},\mathbb{R}^{d\times d})]$ such that $\{f^n(x,v)/(1+|v|^\alpha)\}$ is uniformly bounded. Then, if (H_4) holds, A^N can be extended to a continuous operator in the u.c. topology from $\mathscr{C}_\alpha(\overline{O},\mathbb{R})$ $[\mathscr{C}_\alpha(\overline{O},\mathbb{R}^d), \mathscr{C}_\alpha(\overline{O},\mathbb{R}^{d\times d})]$ into itself, for every $\alpha < 4$. Moreover, for every sequence $\{f^n\}\subseteq \mathscr{C}_\alpha(\overline{O},\mathbb{R})[\mathscr{C}_\alpha(\overline{O},\mathbb{R}^d), \mathscr{C}_\alpha(\overline{O},\mathbb{R}^{d\times d})]$ converging to zero in the u.c. topology and such that $\{f^n(x,v)/(1+|v|^\alpha)\}$ is uniformly bounded, $\{A^N f^n(x,v)/\mu^N\}$ goes to zero in the u.c. topology uniformly in N, and $\{A^N f^n(x,v)/[\mu^N(1+|v|^\alpha)]\}$ is bounded, uniformly in n and N.

Let $b\colon \overline{O}\to\mathbb{R}^d$ be defined by

$$b_i(x,v) = \sum_{j,k=1}^{d} \frac{\partial\big((Q^{-1})\big)_{i,k}(x)}{\partial x_j} v_k v_j. \tag{1.8}$$

Note that by (H_2),

$$|b(x,v)| \le b_o(1+|v|^2), \qquad \forall\,(x,v) \in \overline{O},$$

for some positive constant b_o.

(H_6) There exist functions

$$\bar{b}, \bar{q} \in \mathscr{C}_b(\overline{D}, \mathbb{R}^d), \qquad \overline{H} \in \mathscr{C}_b(\overline{D}, \mathbb{R}^{d\times d}),$$

and sequences of functions

$$\{\bar{b}^n\}, \{\bar{q}^n\} \subseteq \bigcup_{\alpha<4} \mathscr{C}_\alpha(\overline{O}, \mathbb{R}^d), \qquad \{\overline{H}^n\} \subseteq \bigcup_{\alpha<4} \mathscr{C}_\alpha(\overline{O}, \mathbb{R}^{d\times d}),$$

such that, for every compact set $K \subseteq \overline{O}$,

$$\lim_{n\to\infty} \overline{\lim_{N\to\infty}} \sup_{(x,v)\in K} \left| b(x,v) - \bar{b}(x) - \frac{A^N}{\mu^N}\bar{b}^n(x,v)\right| = 0,$$

$$\lim_{n\to\infty} \overline{\lim_{N\to\infty}} \sup_{(x,v)\in K} \left| q^N(x,v) - \bar{q}(x) - \frac{A^N}{\mu^N}\bar{q}^n(x,v)\right| = 0,$$

$$\lim_{n\to\infty} \overline{\lim_{N\to\infty}} \sup_{(x,v)\in K} \left| H^N(x,v) - \overline{H}(x) - \frac{A^N}{\mu^N}\overline{H}^n(x,v)\right| = 0,$$

and, for some $\alpha < 4$, for every compact set $K \subseteq \overline{D}$,

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{x\in K} \sup_{v} \frac{\left|\frac{A^N}{\mu^N}\bar{b}^n(x,v)\right|}{1+|v|^\alpha} < \infty,$$

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{x\in K} \sup_{v} \frac{\left|\frac{A^N}{\mu^N}\bar{q}^n(x,v)\right|}{1+|v|^\alpha} < \infty,$$

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{x\in K} \sup_{v} \frac{\left|\frac{A^N}{\mu^N}\overline{H}^n(x,v)\right|}{1+|v|^\alpha} < \infty.$$

(H_7) $\overline{H}(x)$ is strictly positive definite, uniformly in x.

Despite its complicated appearance due to the general formulation of the problem that has been adopted here, (H_6) is essentially an ergodicity condition which ensures that in the limit the dependence on the velocity averages out, as appears clearly from the following remark.

REMARK 1.5. In the case when $\mu^N/N \to \mu_o$ and the kernels $P^N(x,v;\cdot)$ of A^N/μ^N converge weakly, uniformly over compact subsets of $\overline{O}$, to a probability measure $P(x,v;\cdot)$, (H_6) can be easily verified as follows: For $f \in \mathscr{C}_\alpha(\overline{O}, \mathbb{R})$

$[\mathscr{C}_\alpha(\overline{O}, \mathbb{R}^d), \mathscr{C}_\alpha(\overline{O}, \mathbb{R}^{d\times d}]$, $\alpha < 4$, define

$$Af(x,v) = \int_{\Delta_x} [f(x,v') - f(x,v)] P(x,v;dv')$$

and assume that, for every compact set $K \subseteq \overline{D}$,

$$\sup_{f \in \mathscr{C}_b(\overline{O},\mathbb{R})} \overline{\lim_{N\to\infty}} \sup_{x\in K} \sup_{v} \frac{\left| \frac{A^N}{\mu^N} f(x,v) - Af(x,v) \right|}{1 + |Af(x,v)|} < \infty. \tag{1.9}$$

By (H_4) we know that there exists $H \in \mathscr{C}_b(\overline{O}, \mathbb{R}^{d\times d})$ such that

$$H^N \xrightarrow[N\to\infty]{\text{u.c.}} H.$$

Now suppose there exists also a function q such that

$$q^N \xrightarrow[N\to\infty]{\text{u.c.}} q$$

[in which case q belongs to $\mathscr{C}_1(\overline{O}, \mathbb{R}^d)$].

Assume that, for every x, the family of transition probabilities $P(x,\cdot\,;\cdot\,)$ admits a unique invariant probability measure $\pi(x,\cdot\,)$, with finite moment of order $\alpha_o > 2$, uniformly bounded in x, and define, for $f \in \mathscr{C}_\alpha(\overline{O}, \mathbb{R})$ $[\mathscr{C}_\alpha(\overline{O}, \mathbb{R}^d), \mathscr{C}_\alpha(\overline{O}, \mathbb{R}^{d\times d})]$, $\alpha < \alpha_o$,

$$\Pi f(x) = \int_{\Delta_x} f(x,v')\pi(x,dv').$$

Then (H_6) holds with $\bar{b} = \Pi b$, $\bar{q} = \Pi q$, $\overline{H} = \Pi H$ if

$$b - \Pi b, q - \Pi q, H - \Pi H \in \mathscr{R}(A), \tag{1.10}$$

or if

$$\forall f \in \operatorname{Ker}(\Pi | \mathscr{C}_b(\overline{O},\mathbb{R})), \qquad \exists \{f^n\} \subseteq \mathscr{C}_b(\overline{O},\mathbb{R}) \text{ s.t.:}$$

$$Af^n \xrightarrow[n\to\infty]{\text{u.c.}} f, \qquad \sup_{f\in \operatorname{Ker}(\Pi|\mathscr{C}_b(\overline{O},\mathbb{R}))} \sup_n \frac{\|Af^n\|}{\|f\|} < \infty. \tag{1.11}$$

A sufficient condition for (1.11), for a family of transition probabilities $P(x,\cdot\,;\cdot\,)$ of period $d(x)$, $d(\cdot)$ bounded over compact sets, is

$$\frac{1}{d(x)} \sum_{k=0}^{d(x)-1} \int_{\Delta_x} f(x,v')P^{h+k}(x,v;dv') \xrightarrow[h\to\infty]{\text{u.c.}} \int_{\Delta_x} f(x,v')\pi(x,dv'), \tag{1.12}$$

where $P^h(x,\cdot\,;\cdot\,)$ denotes the hth power of $P(x,\cdot\,;\cdot\,)$ with respect to convolution. Equation (1.12) holds if $P(x,\cdot\,;\cdot\,)$ is uniformly positive recurrent (in a suitable sense) over compact sets. For instance, (1.12) holds if for every $x \in \overline{D}$, $P(x,\cdot\,;\cdot\,)$ is φ_x-recurrent, for some positive σ-finite positive measure φ_x, and for every compact set $K_1 \subseteq \overline{D}$ for every compact set $K \subseteq \overline{O}$ and for every

$\varepsilon > 0$ there exist a set $B_o \subseteq \mathbb{R}^d$ and a positive integer h_o such that

$$\varphi_x(B_o \cap \Delta_x) > 0, \qquad \forall\, x \in K_1,$$

$$\sup_{x \in K_1} \sup_{v \in \Delta_x \cap B_o} \sup_{B \subseteq \Delta_x} \left| \frac{1}{d(x)} \sum_{i=0}^{d(x)-1} P^{h+i}(x, v; B) - \pi(x, B) \right| < \varepsilon, \qquad \forall\, h \geq h_o,$$

$$\lim_{h \to \infty} \sup_{(x,v) \in K} \sum_{i=h+1}^{\infty} P^i_{B_o \cap \Delta_x}(x, v; B_o \cap \Delta_x) = 0, \tag{1.13}$$

$$\Bigg(P^i_{B_o \cap \Delta_x}(x, v; B_o \cap \Delta_x) = \int_{\Delta_x - B_o} P(x, v; du^1) \times \cdots \int_{\Delta_x - B_o} P(x, u^{i-2}; du^{i-1}) P\big(x, u^{i-1}; B_o \cap \Delta_x\big) \Bigg).$$

In the case when Δ_x and $P(x, \cdot\,; \cdot\,)$ are independent of x, (1.13) is implied by the "uniform over compact sets" φ-recurrence condition; namely,

$$\lim_{h \to \infty} \sup_{v \in K} \sum_{i=h+1}^{\infty} P^i_B(v; B) = 0 \quad \text{for every compact set } K, \tag{1.14}$$

for every Borel set B such that $\varphi(B) > 0$.

Finally, if ∂D is not smooth enough, namely if ∂D is not of class $\mathscr{C}^1_b$ with $\nu(\cdot)$ uniformly continuous, at least one of the following assumptions on D and Q will be required to hold.

For every $x \in \partial D$ let

$$\Gamma(x) = \{Q^{-1}(x)\nu,\ \nu \in N(x)\}. \tag{1.15}$$

$$(\mathrm{H}_8\mathrm{i}) \qquad \frac{\inf_{|u|=1} u^t Q^{-1}(x) u}{|Q^{-1}(x)|} > \frac{2}{\sqrt{a^2 + 4}} \wedge \sqrt{\frac{\sqrt{1 + 4a^4} - 1}{2a^4}}$$

where a is defined in (1.1).

$$(\mathrm{H}_8\mathrm{ii}) \qquad \lim_{\rho \to 0} \sup_{x \in \partial D} \frac{\sqrt{1 - \beta(x, \rho)^2}}{\alpha(x, \rho)^2 \vee [\alpha(x, \rho)/2]} < 1,$$

where

$$\alpha(x, \rho) = \max_{|u|=1} \inf_{y \in \partial D \cap \overline{B_\rho(x)}} \min_{\gamma \in \Gamma(y),\, |\gamma|=1} \gamma \cdot u$$

and

$$\beta(x, \rho) = \inf_{y \in \partial D \cap \overline{B_\rho(x)}} \min_{\gamma \in \Gamma(y),\, |\gamma|=1} \max_{\nu \in N(y),\, |\nu|=1} ((\nu \cdot \gamma) \vee 0).$$

(H_8iii) Q is symmetric and D satisfies the following admissibility condition: There exists a sequence $\{D_h\}$ of bounded smooth open sets in $\mathbb{R}^d$ such that D_h

satisfies the uniform exterior sphere condition for all h, uniformly in h, and if $x_h \in \overline{D}_h$, $x_h \to_{h\to\infty} x$, then $x \in \overline{D}$ and if K is compact, $K \subseteq D$, then $K \subseteq O_h$, for h large enough.

REMARK 1.6. (H_8i) implies (H_8ii) [cf. Costantini (1990), example of Section 3].

REMARK 1.7. (H_8i) is trivially satisfied if $Q(x)$ is proportional to the identity matrix, at each $x \in \partial D$.

REMARK 1.8. If ∂D is of class $\mathscr{C}_b^1$ and $\nu(\cdot)$ is uniformly continuous, then (H_8ii) is satisfied [cf. (3.13), (2.21), (2.20) and Proposition 2.5(ii) in Costantini (1990)].

REMARK 1.9. The admissibility condition for D in (H_8iii) is satisfied if ∂D is piecewise smooth or D is convex [cf. Lions and Sznitman (1984), Remark 4.3 and Remark 2.4].

2. Construction of physically reflecting transport processes. In this section, existence and uniqueness of the solution to (1.4) is shown. As the jump rate of the velocity in the interior is constant (see Remark 1.1), this reduces to showing that the boundary of D is never hit with tangential velocity or in its singular part, and that the boundary hitting times do not accumulate. The proof which is presented here (Theorem 2.2) relies on the observation that between two random changes of the velocity the solution to (1.4) evolves deterministically according to a special flow. This allows one to apply a result by Marchioro, Pellegrinotti, Presutti and Pulvirenti (1976), for deterministic dynamical systems, which exploits the fact that in the phase space of position and velocity the Lebesgue measure is preserved both by the potential and by physical reflection, to prove that the set of "bad" initial positions and velocities has Lebesgue measure 0.

The theorem by Marchioro, Pellegrinotti, Presutti and Pulvirenti (1976) is stated below in a suitable form to be applied to (1.4).

THEOREM 2.1 [Marchioro, Pellegrinotti, Presutti and Pulvirenti (1976)]. *Let*

$$\partial_+ O = \{(x,v) \in \partial O \colon x \in \partial_r D,\ v \cdot \nu(x) > 0\},$$

$$\partial_- O = \{(x,v) \in \partial O \colon x \in \partial_r D,\ v \cdot \nu(x) < 0\},$$

$$\partial_s O = \{(x,v) \in \partial O \colon x \in \partial_s D \text{ or } x \in \partial_r D,\ v \cdot \nu(x) = 0\}.$$

Consider the system of ordinary differential equations,

$$\dot{x}_{x,v}(t) = v_{x,v}(t), \qquad x_{x,v}(0) = x,$$

$$\dot{v}_{x,v}(t) = \partial U(x_{x,v}(t)), \qquad v_{x,v}(0) = v,$$

and define

$$S_t\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d, \qquad S_t(x,v) = (x_{x,v}(t), v_{x,v}(t)),$$
$$\sigma\colon \overline{O} \to \mathbb{R}^+ \cup \{+\infty\}, \qquad \sigma(x,v) = \inf\{t > 0\colon S_t(x,v) \in D^c \times \mathbb{R}^d\},$$
$$S\colon \{(x,v) \in \overline{O}\colon \sigma(x,v) < +\infty\} \to \partial O, \qquad S(x,v) = S_{\sigma(x,v)}(x,v),$$
$$R\colon \partial_- O \to \partial_+ O, \qquad R(x,v) = (x, v - 2(v \cdot \nu(x))\nu(x)),$$
$$T\colon \{(x,v) \in \overline{O}\colon \sigma(x,v) < +\infty,\ S(x,v) \in \partial_- O\} \to \partial_+ O,$$
$$T(x,v) = R(S(x,v)).$$

Let T^k denote the kth power of T, $\mathscr{D}(T^k)$ its domain and T^o the identity map on $\overline{O}$. For $k \in \mathbb{N}$, define

$$\sigma_k\colon \mathscr{D}(T^{k-1}) \to \mathbb{R}^+ \cup \{+\infty\},$$
$$\sigma_k(x,v) = \begin{cases} \sum_{h=0}^{k-1} \sigma(T^h(x,v)) & \text{if } \sigma(T^{k-1}(x,v)) < +\infty, \\ +\infty & \text{if } \sigma(T^{k-1}(x,v)) = +\infty. \end{cases}$$

Then the set O_o

$$O_o = \left\{(x,v) \in \bigcap_{k=1}^{\infty} \mathscr{D}(T^k)\colon \lim_{k\to\infty} \sigma_k(x,v) < +\infty\right\} \tag{2.1}$$
$$\cup \bigcup_{k=0}^{\infty} \{(x,v) \in \mathscr{D}(T^k)\colon \sigma_{k+1}(x,v) < +\infty,\ S(T^k(x,v)) \in \partial_s O\}$$

is measurable and

$$m(O_o) = 0$$

[m being the Lebesgue measure on $\overline{O}$ or the surface Lebesgue measure on $\overline{O}$ if $O = \{(x,v) \in D \times \mathbb{R}^d\colon E(x,v) = c\}$].

For every $(x,v) \in \overline{O} - O_o$,

$$T_t(x,v) = \begin{cases} S_{t-\sigma_k(x,v)}(T^k(x,v)), & \sigma_k(x,v) \le t < \sigma_{k+1}(x,v), \\ & \forall\, k \text{ s.t. } \sigma_k(x,v) < +\infty, \\ S_t(x,v), & 0 \le t < \sigma_1(x,v) \text{ for } (x,v) \in \overline{O} - \partial_- O \end{cases}$$

is defined for all $t \in \mathbb{R}^+$. For every $t \in \mathbb{R}^+$, T_t is a one-to-one measurable map of $\overline{O} - O_o$ into itself; as a map from $\mathbb{R}^+ \times (\overline{O} - O_o) \to \overline{O}$, it is jointly measurable. The measure m is invariant under T_t, for every $t \in \mathbb{R}^+$.

REMARK 2.1. Theorem 2.1 is a special case of Theorems 2.4 and 2.5 in Marchioro, Pellegrinotti, Presutti and Pulvirenti (1976) [see also Section 4 in Marchioro, Pellegrinotti, Presutti and Pulvirenti for the case $O = \{(x,v) \in D \times \mathbb{R}^d\colon E(x,v) = c\}$].

In Marchioro, Pellegrinotti, Presutti and Pulvirenti (1976), D is supposed to be bounded, but the result extends to unbounded domains by a standard localization argument.

THEOREM 2.2. *If* (H_1) *holds, then there exists a measurable set* $\Omega_o \subseteq \Omega$, *with* $P(\Omega_o) = 0$, *such that, for every* $\omega \in \Omega - \Omega_o$ *there exists a unique solution to* (1.4), *defined for all* $t \in \mathbb{R}^+$; *for every* $\omega \in \Omega - \Omega_o$ *the counting measures* dY^μ *and* $d\Lambda$ *are mutually singular.*

PROOF. Equation (1.4) is pathwise equivalent to

$$(2.2)\qquad \begin{aligned} &(X(0), V(0)) = (X_o, V_o),\\ &(X(t), V(t)) = T_{t-\tau_{n-1}}(X(\tau_{n-1}), V(\tau_{n-1})), \qquad \tau_{n-1} \le t < \tau_n,\\ &(X(\tau_n), V(\tau_n)) = \tilde{F}\big(T_{\tau_n - \tau_{n-1}}(X(\tau_{n-1}), V(\tau_{n-1})); \xi_n\big), \end{aligned}$$

where $\{\tau_n\}$ is the sequence of the jump times of the Poisson process Y^μ, $\tau_o = 0$ and $\tilde{F}(x, v; u) = (x, F(x, v; u))$.

Let $\tilde{O}_o$ be a Borel set such that $\tilde{O}_o \supseteq O_o$ and $m(\tilde{O}_o) = 0$ (O_o itself is not necessarily a Borel set, but the existence of $\tilde{O}_o$ is ensured by the regularity of m),

$$\Omega_o^o = \{\omega: (X(\tau_o), V(\tau_o)) \in \tilde{O}_o\},$$

and, for $\omega \in \Omega_o^{n-1}$,

$$\Omega_o^n = \Omega_o^{n-1} \cup \{\omega \in \Omega - \Omega_{n-1}^o: (X(\tau_n), V(\tau_n)) \in \tilde{O}_o \cup \partial O\}.$$

The claim is that

$$P(\Omega_o^n) = 0, \qquad \forall\, n \in \mathbb{Z}^+.$$

This can be easily seen by induction; in fact,

$$P(\Omega_o^o) = P_o(\tilde{O}_o) = 0,$$

because $m(\tilde{O}_o) = 0$ by Theorem 2.1, and P_o is absolutely continuous with respect to m.

Moreover, denoting by P_n the probability distribution of $(X(\tau_n), V(\tau_n))$, for every $n \in \mathbb{Z}^+$, it follows from (H_1), by the independence of ξ_n, $\tau_n - \tau_{n-1}$, $(X(\tau_{n-1}), V(\tau_{n-1}))$, and by the fact that T_t preserves the Lebesgue measure m for every t, that if P_{n-1} is absolutely continuous with respect to m, so also is P_n. Therefore P_n is absolutely continuous with respect to m for all $n \in \mathbb{Z}^+$, and

$$P(\Omega_o^n) = P(\Omega_o^{n-1}) + P_n(\tilde{O}_o \cup \partial O) = P(\Omega_o^{n-1}).$$

By setting $\Omega_o = \bigcup_{n=0}^\infty \Omega_o^n$, the assertion follows. □

3. The Skorohod reflection problem: some continuity properties of solutions. The derivation of a diffusion approximation for the processes X^N which will be presented in Sections 4 and 5 relies on some results about solutions of the Skorohod reflection problem, contained in Costantini (1990) and Lions and Sznitman (1984). The consequences of these results which are relevant to our present purposes are stated in this section.

Let D be as in Section 1 and Γ be defined by (1.15), where Q is defined by (1.5) and satisfies (H_2).

DEFINITION 3.1. Let w be a function in $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $w(0) \in \overline{D}$. A solution to the Skorohod problem for (D, Γ, w) is a pair (x, ϕ), $x \in \mathscr{D}(\mathbb{R}^+, \overline{D})$, $\phi \in \mathscr{BV}(\mathbb{R}^+, \mathbb{R}^d)$, such that

$$x(t) = w(t) + \phi(t),$$

(3.1)

$$\phi(t) = \int_0^t \gamma(s)\, d\lambda(s), \qquad \gamma(s) \in \Gamma(x(s)), \qquad |\gamma(s)| = 1, \qquad d\lambda\text{-a.e.},$$

$$d\lambda(\{t: x(t) \in D\}) = 0,$$

where λ denotes the total variation of ϕ. λ will be called the local time of the solution. Equation (3.1) will sometimes be referred to as a Skorohod reflection equation.

THEOREM 3.1. *Let $\mathscr{W}$ be a compact subset of $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^d)$ in the Skorohod topology, such that $w(0) \in \overline{D}$ for every $w \in \mathscr{W}$, and let $\mathscr{S}$ be the set of four-tuples $(w, x, \phi, \lambda) \in \mathscr{D}(\mathbb{R}^+, \mathbb{R}^d) \times \mathscr{D}(\mathbb{R}^+, \overline{D}) \times \mathscr{BV}(\mathbb{R}^+, \mathbb{R}^d) \times \mathscr{BV}(\mathbb{R}^+, \mathbb{R}^+)$ such that (x, ϕ) is a solution to the Skorohod problem for (D, Γ, w) for some $w \in \mathscr{W}$, λ is the total variation of φ and x is continuous.*

Assume (H_8i) *or* (H_8ii) *holds. Then $\mathscr{S}$ is a relatively compact subset of $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$ in the Skorohod topology and, for every accumulation point (w, x, φ, λ) of $\mathscr{S}$, (x, φ) is a solution to the Skorohod problem for (D, Γ, w).*

PROOF. We are in a special case of the setup considered in the example of Section 3 of Costantini (1990) [note the change of notation: $a(x, \rho)$ and $b(x, \rho)$ in Costantini (1990) are denoted here by $\alpha(x, \rho)$ and $\beta(x, \rho)$, respectively, and conversely $\alpha(x, \rho)$ is denoted here by $a(x, \rho)$]. (H_8i) corresponds to (3.15) in Costantini (1990), while (H_8ii) corresponds to (2.22) in Costantini (1990). For every $(w, x, \phi) \in \mathscr{S}$, x is continuous. Therefore, by Proposition 2.5 in Costantini (1990), Theorems 2.4 and 3.2 in Costantini (1990) apply, for any value of the constant ρ_o. □

THEOREM 3.2. *Let $\mathscr{W}$ and $\mathscr{S}$ be as in Theorem* 3.1. *If* (H_8iii) *is satisfied, then the same assertion as in Theorem* 3.1 *holds.*

PROOF. The assertion follows by Theorem 4.2 in Lions and Sznitman (1984). The original result is stated for $\mathscr{W} \subseteq \mathscr{C}(\mathbb{R}^+, \mathbb{R}^d)$, but can be extended to $\mathscr{W} \subseteq \mathscr{D}(\mathbb{R}^+, \mathbb{R}^d)$ by standard arguments. □

REMARK 3.1. A compactness result for the Skorohod problem of the type of Theorems 3.1 and 3.2 was first proved by Tanaka (1979), in the case when $\Gamma(\cdot)$ is the normal cone $N(\cdot)$ and D is convex.

4. Diffusion approximation: Tightness. This section and the next one are devoted to the study of the asymptotic behavior of the process X^N as N diverges.

This section contains all the estimates and the technical result which are needed in the proof of the tightness of the family $\{X^N\}$. The proofs refer to the realization of X^N given by (1.6) and (1.7); however, the final result concerns the distribution of the process and therefore holds independently of the choice of the realization. The main idea is to represent X^N as the first component of a solution to a Skorohod problem for $(D, \Gamma, B^N + Z^N)$, where Γ is defined by (1.15) and $B^N + Z^N$ is the proper semimartingale. The family $\{B^N + Z^N\}$ turns out to be tight and this enables one to exploit the results recalled in Section 3 to conclude that the family $\{X^N\}$ is tight as well.

Consider (1.6) and (1.7). Clearly we can always assume that there is a unique underlying complete probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_o$ denote the sub-σ-algebra of $\mathscr{F}$ of sets of P-measure 0 or 1 and let

$$\mathscr{F}_t^N = \mathscr{F}_o \vee \sigma\big((X_o, V_o), Y^{\mu^N}(s), \xi_{Y^{\mu^N}}(s), s \le t\big).$$

Then $\{\mathscr{F}_t^N\}$ is a complete right-continuous filtration, the stochastic processes X^N, V^N, M^N, Λ^N defined by (1.6) and (1.7) are $\{\mathscr{F}_t^N\}$-progressive and M^N is an $\{\mathscr{F}_t^N\}$-square integrable martingale with

$$\langle M^N \rangle(t) = \int_0^t H^N(X^N(s), V^N(s))\, ds. \tag{4.1}$$

Let q^N and Q be defined by (1.5). Throughout this section it will be assumed that Q satisfies (H_2) and that (H_1), (H_3) and (H_4) hold.

Let b and Γ be defined by (1.8) and (1.15), respectively.

PROPOSITION 4.1. *For every N the pair of stochastic processes (X^N, Φ^N), where*

$$\Phi^N(t) = \int_0^{t^+} Q^{-1}(X^N(s))\nu(X^N(s))\, d\Lambda^N(s) \tag{4.2}$$

is, almost surely, a solution to the Skorohod problem for $(D, \Gamma, B^N + Z^N)$, where

$$\begin{aligned} B^N(t) &= X_o + \frac{Q^{-1}(X_o)V_o - Q^{-1}(X^N(t))V^N(t)}{\sqrt{N}} \\ &\quad + \int_0^t b(X^N(s), V^N(s))\, ds \\ &\quad + \int_0^t Q^{-1}(X^N(s))\, \partial U(X^N(s))\, ds \\ &\quad + \int_0^t Q^{-1}(X^N(s)) q^N(X^N(s), V^N(s))\, ds, \\ Z^N(t) &= \int_0^{t^+} Q^{-1}(X^N(s))\, dM^N(s). \end{aligned} \tag{4.3}$$

PROOF. Consider (1.6) and (1.7). By applying the change of variable formula for semimartingales to the function $f(x,v) = Q^{-1}(x)v$, substituting $\sqrt{N}(X^N(t) - X_o)$ for the term $N\int_0^t V^N(s)\, ds$ and solving for $X^N(t)$, we obtain

$$X^N(t) = B^N(t) + Z^N(t) + \Phi^N(t).$$

Then it can be easily checked that (X^N, Φ^N) verifies Definition 3.1 with probability 1. □

The following inequality will be used several times in the next lemmas. Let x and g belong to $\mathscr{D}([0,T],\mathbb{R})$, $T>0$, $g(0)=0$, and let c be a positive constant. If

$$x(t) \le x(s) - c\int_s^t x(r)\,dr + g(t) - g(s), \qquad 0 \le s \le t \le T, \tag{4.4}$$

then

$$x(t) \le e^{-ct}[x(0) + g(t)] + c\int_0^t e^{-c(t-r)}[g(t) - g(r)]\,dr, \qquad 0 \le t \le T. \tag{4.5}$$

If the opposite inequality holds in (4.4) then it also holds in (4.5).

LEMMA 4.2. *For every* $T > 0$,

$$\sup_N \sup_{t\le T} E[\![|V^N(t)|^2]\!] < \infty.$$

PROOF. Consider (1.6) and (1.7). The change of variable formula for the function $f(x,v)=|v|^2$ yields, for every pair of stopping times σ and τ such that $0\le\sigma\le\tau\le T$ almost surely,

$$\begin{aligned}|V^N(\tau)|^2 = {}& |V^N(\sigma)|^2 + 2\sqrt{N}\int_\sigma^\tau V^N(r)\cdot\partial U(X^N(r))\,dr\\ &+ 2\sqrt{N}\int_\sigma^\tau V^N(r)\cdot q^N(X^N(r),V^N(r))\,dr\\ &- 2N\int_\sigma^\tau (V^N(r))^t Q(X^N(r))V^N(r)\,dr\\ &+ 2\sqrt{N}\int_{\sigma^+}^{\tau^+} V^N(r^-)\cdot dM^N(r) + N\,\mathrm{tr}([M^N](\tau) - [M^N](\sigma)).\end{aligned} \tag{4.6}$$

Note that the local time Λ^N does not appear in the right-hand side of (4.6) because under physical reflection the speed does not change on the boundary.

By applying (4.6) to a localizing sequence of stopping times and making use of (1.5), (H_2), (H_3), (H_4) and Gronwall's inequality, one can see that $E[\![|V^N(t)|^2]\!]$ is finite for every t and bounded over $[0,T]$ for each N. Then by again applying (4.6) to the deterministic times $\sigma=s$, $\tau=t$, $0\le s\le t\le T$, it follows by (1.5), (H_2), (H_3), (H_4) and (4.5) that, for N sufficiently large,

$$\sup_{t\le T} E[\![|V^N(t)|^2]\!] \le E[\![|V_o|^2]\!] + \frac{2\sqrt{N}(\|\partial U\| + c_o) + N\|h^N\|}{2(Nq_o - \sqrt{N}(\|\partial U\| + 2c_o))}. \qquad \square$$

LEMMA 4.3. *For every* $T > 0$,

$$E\left[\!\!\left[\sup_{t\le T}\left|[M^N](t) - \langle M^N\rangle(t)\right|^2\right]\!\!\right] \xrightarrow[N\to\infty]{} 0.$$

PROOF. By the properties of the quadratic variation process we have

$$E\left[\!\!\left[\sum_{i,j}[[M^N]_{i,j} - \langle M^N\rangle_{i,j}](T)\right]\!\!\right] = E\left[\!\!\left[\sum_{t\le T}|M^N(t) - M^N(t^-)|^4\right]\!\!\right]$$

and, by (1.6) and (H_4),

$$= \frac{1}{N^2} E\left[\!\!\left[\int_0^T |V^N(t) - V^N(t^-)|^4 \, dY^{\mu^N}(t) \right]\!\!\right]$$

$$= \frac{1}{N} E\left[\!\!\left[\int_0^T g^N(X^N(t), V^N(t)) \, dt \right]\!\!\right]$$

$$\leq \frac{g_o}{N} \int_0^T \left(1 + E[\![|V^N(t)|^2]\!]\right) dt.$$

Therefore the assertion follows by Doob's inequality and Lemma 4.2. □

LEMMA 4.4. *For every* $T > 0$,

$$E\left[\!\!\left[\sup_{t \leq T} \frac{|V^N(t)|^2}{N} \right]\!\!\right] \xrightarrow[N \to \infty]{} 0.$$

PROOF. Consider (4.6) for the deterministic times $\sigma = s$, $\tau = t$, $0 \leq s \leq t \leq T$. Making use of (4.5) and then dividing by N, one finds, by (1.5), (H_2) and (H_3),

$$E\left[\!\!\left[\sup_{t \leq T} \frac{|V^N(t)|^2}{N} \right]\!\!\right]$$

$$\leq \frac{E[\![|V_o|^2]\!]}{N} + \frac{2}{\sqrt{N}} (\|\partial U\| + c_o) \int_0^T E[\![|V^N(t)|]\!] \, dt$$

$$+ \frac{2}{\sqrt{N}} E\left[\!\!\left[\sup_{t \leq T} \left| \int_0^{t^+} V^N(r^-) \cdot dM^N(r) \right| \right]\!\!\right]$$

$$+ E\left[\!\!\left[\sup_{t \leq T} \left\{ e^{-2(Nq_o - \sqrt{N} c_o)t} \operatorname{tr}[M^N](t) \right\} \right]\!\!\right]$$

$$+ \frac{2}{\sqrt{N}} (\|\partial U\| + c_o) E\left[\!\!\left[\sup_{t \leq T} \left\{ 2(Nq_o - \sqrt{N} c_o) \int_0^t e^{-2(Nq_o - \sqrt{N} c_o)(t-s)} \times \int_s^t |V^N(r)| \, dr \, ds \right\} \right]\!\!\right]$$

$$+ \frac{2}{\sqrt{N}} E\left[\!\!\left[\sup_{t \leq T} \left| 2(Nq_o - \sqrt{N} c_o) \int_0^t e^{-2(Nq_o - \sqrt{N} c_o)(t-s)} \times \int_{s^+}^{t^+} V^N(r^-) \cdot dM^N(r) \, ds \right| \right]\!\!\right]$$

$$+ 2(Nq_o - \sqrt{N} c_o) E\left[\!\!\left[\sup_{t \leq T} \int_0^t e^{-2(Nq_o - \sqrt{N} c_o)(t-s)} \times \operatorname{tr}([M^N](t) - [M^N](s)) \, ds \right]\!\!\right].$$

The first three summands on the right-hand side converge to 0 by (H_3), Lemma 4.2, Doob's inequality and (H_4). The fourth summand is bounded by

$$E\left[\!\left[\sup_{t\le T}\left|\operatorname{tr}([M^N](t)-\langle M^N\rangle(t))\right|\right]\!\right]$$

$$+\,E\left[\!\left[\sup_{t\le T}\left\{e^{-2(Nq_o-\sqrt{N}c_o)t}\operatorname{tr}\langle M^N\rangle(t)\right\}\right]\!\right],$$

which converges to 0 by Lemma 4.3 and (H_4). By exchanging the order of integration the fifth summand can be immediately seen to be bounded by

$$\frac{2}{\sqrt{N}}(\|\partial U\|+c_o)\int_0^T E[\![|V^N(r)|]\!]\,dr.$$

The sixth one is dominated by

$$\frac{2}{\sqrt{N}}E\left[\!\left[\sup_{t\le T}\left[1-e^{-2(Nq_o-\sqrt{N}c_o)t}\right]\times 2\sup_{t\le T}\left|\int_0^{t^+}V^N(s^-)\cdot dM^N(s)\right|\right]\!\right],$$

which goes to 0 by Doob's inequality, Lemma 4.2 again, and (H_4). Finally, the last summand tends to 0 by Lemma 4.3 and (H_4). □

LEMMA 4.5. *For every* $T>0$,

$$\sup_N E\left[\!\left[\int_0^T |V^N(t)|^4\,dt\right]\!\right]<\infty.$$

PROOF. Let

$$\tau_n^N=\inf\{t\ge 0: |V^N(t)|>n\}.$$

By applying the change of variable formula to the function $f(u)=u^2$ and to the semimartingale $|V^N|^2$ with stochastic differential given by (4.6), taking into account (1.5), (H_2) and the fact that

$$\sum_{t\le T\wedge\tau_n^N}\left\{|V^N(t)|^4-|V^N(t^-)|^4-2|V^N(t^-)|^2\left[|V^N(t)|^2-|V^N(t^-)|^2\right]\right\}$$

$$=\sum_{t\le T\wedge\tau_n^N}\{2\sqrt{N}V^N(t^-)\cdot(M^N(t)-M^N(t^-))$$

$$+N\operatorname{tr}([M^N](t)-[M^N](t^-))\}^2,$$

one obtains, for N sufficiently large,

$$\begin{aligned}|V^N(T\wedge\tau_n^N)|^4 \le |V_o|^4 &+ 4\sqrt{N}(\|\partial U\| + c_o)\int_0^{(T\wedge\tau_n^N)^+}|V^N(t)|^2\,dt\\ &- 4\big(Nq_o - 2\sqrt{N}c_o - \sqrt{N}\|\partial U\|\big)\int_0^{(T\wedge\tau_n^N)^+}|V^N(t)|^4\,dt\\ &+ 4\sqrt{N}\int_0^{(T\wedge\tau_n^N)^+}|V^N(t^-)|^2 V^N(t^-)\cdot dM^N(t)\\ &+ 10N\int_0^{(T\wedge\tau_n^N)^+}|V^N(t^-)|^2\,d(\mathrm{tr}[M^N])(t)\\ &+ 2\int_0^{(T\wedge\tau_n^N)^+}|V^N(t) - V^N(t^-)|^4\,dY^{\mu^N}(t),\end{aligned}$$

which implies, by (H_4),

$$\begin{aligned}E\bigg[\!\!\bigg[\int_0^{T\wedge\tau_n^N}|V^N(t)|^4\,dt\bigg]\!\!\bigg] &\le \frac{E[\![|V_0|^4]\!]}{4(Nq_o - 2\sqrt{N}c_o - \sqrt{N}\|\partial U\|)}\\ &\quad+\frac{\big[4\sqrt{N}(\|\partial U\| + c_o) + 10N\|h^N\| + 2Ng_o\big]\int_0^T E[\![|V^N(t)|^2]\!]\,dt}{4(Nq_o - 2\sqrt{N}c_o - \sqrt{N}\|\partial U\|)}\\ &\quad+\frac{2Ng_oT}{4(Nq_o - 2\sqrt{N}c_o - \sqrt{N}\|\partial U\|)},\end{aligned}$$

where the right-hand side is bounded uniformly in N by Lemma 4.2. □

THEOREM 4.6. *Under hypotheses* (H_1) *to* (H_4) *and* (H_8), *the family of stochastic processes* $\{(X^N, \Phi^N, \Lambda^N, B^N, Z^N)\}$ *is relatively compact in* $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^{4d+1})$.

PROOF. Consider (4.3). Since Z^N is an $\mathscr{F}_t^N$-square integrable martingale with

$$\langle Z^N\rangle(t) = \int_0^t Q^{-1}(X^N(s))H^N(X^N(s), V^N(s))\big(Q^{-1}(X^N(s))\big)^t\,ds,$$

by (H_2) and (H_4) the family $\{Z^N\}$ is relatively compact [see, e.g., Ethier and Kurtz (1986), Theorem 8.6, Chapter 3]. Moreover, by Lemma 4.3 all the accumulation points of $\{Z^N\}$ are continuous [see, e.g., Ethier and Kurtz (1986), Theorem 10.2, Chapter 3]. As far as $\{B^N\}$ is concerned, we have, by (H_2), (1.5), (1.8) and Lemma 4.4, for every $T > 0$,

$$\begin{aligned}\overline{\lim_{N\to\infty}}\, E\Big[\!\!\Big[\sup_{t\le T}|B^N(t)|\Big]\!\!\Big] &\le \Big\{b_o + \|Q^{-1}\|\,\|\partial U\| + 2\|Q^{-1}\|c_o\\ &\qquad+ (b_o + \|Q^{-1}\|c_o)\sup_N\sup_{t\le T}E[\![|V^N(t)|^2]\!]\Big\}T\end{aligned}$$

and

$$\overline{\lim_{N\to\infty}} E\left[\left[\sup_{\substack{|t-s|<\delta \\ 0\le s\le t\le T}} |B^N(t) - B^N(s)| \right]\right]$$

$$\le \left(b_o + \|Q^{-1}\| \, \|\partial U\| + 2\|Q^{-1}\| c_o\right)\delta$$

$$+ \left(b_o + \|Q^{-1}\| c_o\right)\left\{\delta \sup_N E\left[\left[\int_0^T |V^N(t)|^4 \, dt\right]\right]\right\}^{1/2},$$

so that, by Lemmas 4.2 and 4.5, $\{B^N\}$ is relatively compact and all its accumulation points are continuous.

Therefore $\{B^N + Z^N\}$ is relatively compact and the assertion follows by Theorem 3.1 [if (H_8i) or (H_8ii) holds] or by Theorem 3.2 [if (H_8iii) holds]. □

5. Diffusion approximation: Identification of the limit. This section is devoted to the identification of a stochastic differential equation with reflection which describes the behavior of X^N in the limit as $N \to \infty$. The starting point is Proposition 4.1: It is first shown that the "explicit" dependence of B^N and Z^N on V^N averages out, as N diverges (Lemmas 5.1, 5.2, and 5.3); then the "implicit" dependence of Φ^N on V^N is taken care of (Theorem 5.5) by exploiting the results of Section 3, the martingale central limit theorem and a result by Jakubowski, Memin and Pages [(1989), Theorem 5.4].

Throughout this section all the hypotheses made in Section 1, in particular (H_5) to (H_8), will be assumed to hold.

LEMMA 5.1. *For $f \in \mathscr{C}_\alpha(\overline{O}, \mathbb{R})$, $\alpha < 4$, $T > 0$ and for every $\varepsilon > 0$:*

$$\lim_{N\to\infty} P\left\{ \sup_{t\le T} \left| \int_0^t \frac{A^N}{\mu^N} f(X^N(s), V^N(s)) \, ds \right| \ge \varepsilon \right\} = 0.$$

PROOF. Consider first a function $f \in \mathscr{C}_b^1(\overline{O}, \mathbb{R})$. By the change of variable formula, we have

$$f(X^N(t), V^N(t))$$

$$= f(X_o, V_o) + \sqrt{N} \int_0^t \partial_x f(X^N(s), V^N(s)) \cdot V^N(s) \, ds$$

$$+ \sqrt{N} \int_0^t \partial_v f(X^N(s), V^N(s)) \cdot \partial U(X^N(s)) \, ds$$

$$+ \int_0^t A^N f(X^N(s), V^N(s)) \, ds + \sqrt{\mu^N} M_f^N(t)$$

$$+ \sqrt{N} \int_0^{t^+} \frac{\left[f(X^N(s), V^N(s)) - f(X^N(s), V^N(s^-)) \right]}{|V^N(s) - V^N(s^-)|} \, d\Lambda^N(s),$$

where

$$M_f^N(t) = \frac{1}{\sqrt{\mu^N}} \int_0^{t^+} [f(X^N(s), V^N(s)) - f(X^N(s), V^N(s^-))] \, dY^{\mu^N}(s)$$
$$- \frac{1}{\sqrt{\mu^N}} \int_0^t A^N f(X^N(s), V^N(s)) \, ds.$$

By solving for $\int_0^t (A^N/\mu^N) f(X^N(s), V^N(s)) \, ds$, we obtain

$$\overline{\lim_{N\to\infty}} P\left\{ \sup_{t\le T} \left| \int_0^t \frac{A^N}{\mu^N} f(X^N(s), V^N(s)) \, ds \right| \ge \varepsilon \right\}$$
$$\le \overline{\lim_{K\to\infty}} \; \overline{\lim_{N\to\infty}} \frac{1}{\varepsilon} \left\{ \frac{2\|f\|}{\mu^N} + \frac{N}{\mu^N} T \|\partial_x f\| E\left[\sup_{t\le T} \frac{|V^N(t)|}{\sqrt{N}} \right] \right.$$
$$+ \frac{1}{\sqrt{\mu^N}} E\left[\sup_{t\le T} \left| M_f^N(t) \right| \right]$$
$$\left. + \frac{\sqrt{N}}{\mu^N} \|\partial_v f\| (K + T\|\partial U\|) \right\}$$
$$+ \overline{\lim_{K\to\infty}} \; \overline{\lim_{N\to\infty}} P\{\Lambda^N(T) > K\},$$

where the right-hand side vanishes by (4.2), (H_2), Theorem 4.6, (H_5), Lemma 4.4 and Doob's inequality.

Then the assertion follows by (H_4) and Remark 1.4. □

REMARK 5.1. Note that, by (1.5), if N and μ^N are of the same order of magnitude, Lemma 5.1 implies

$$\lim_{N\to\infty} P\left\{ \sup_{t\le T} \left| \int_0^t V^N(s) \, ds \ge \varepsilon \right. \right\} = 0, \tag{5.1}$$

which gives a precise meaning to the intuitive idea that in the limit V^N oscillates averaging to 0.

REMARK 5.2. In the setup of Remark 1.5, if (1.10) or (1.11) holds, Lemma 5.1 yields the following law of large numbers:

$$\forall f \in \mathscr{C}_\alpha(\overline{O}, \mathbb{R}), \qquad \alpha < \min(\alpha_o, 4), \qquad \forall \varepsilon, T > 0,$$

$$\lim_{N\to\infty} P\left\{ \sup_{t\le T} \left| \int_0^t [f(X^N(s), V^N(s)) - \Pi f(X^N(s))] \, ds \right| \ge \varepsilon \right\} = 0. \tag{5.2}$$

LEMMA 5.2. *Let $\bar{b}$, $\bar{q}$ and $\bar{H}$ be the functions in* (H_6). *Then, for every $T > 0$, for every $\varepsilon > 0$*:

$$\lim_{N\to\infty} P\left\{ \sup_{t\le T} \left| \int_0^t \left[b(X^N(s), V^N(s)) - \bar{b}(X^N(s)) \right] ds \right| \ge \varepsilon \right\} = 0,$$

$$\lim_{N\to\infty} P\left\{ \sup_{t\le T} \left| \int_0^t \left[q^N(X^N(s), V^N(s)) - \bar{q}(X^N(s)) \right] ds \right| \ge \varepsilon \right\} = 0,$$

$$\lim_{N\to\infty} P\left\{ \sup_{t\le T} \left| \int_0^t \left[H^N(X^N(s), V^N(s)) - \bar{H}(X^N(s)) \right] ds \right| \ge \varepsilon \right\} = 0.$$

PROOF. The assertion follows immediately by Lemma 5.1, (H_6), Lemma 4.5 and Theorem 4.6. □

LEMMA 5.3. *Let $\bar{H}$ be the $\mathbb{R}^{d\times d}$-valued function in* (H_6) *and let $\Sigma(x)$, $x \in \bar{D}$, be the symmetric positive definite matrix such that $\Sigma(x)^2 = \bar{H}(x)$, $x \in \bar{D}$. Define*

$$W^N(t) = \int_0^t \Sigma^{-1}(X^N(s))\, dM^N(s), \qquad t \in \mathbb{R}^+. \tag{5.3}$$

Then $\{W^N\}$ converges in distribution as $N \to \infty$ to a standard Brownian motion.

PROOF. By (H_7), $\Sigma^{-1}(x)$ is bounded, uniformly for $x \in \bar{D}$; hence, for every N, W^N is an $\{\mathscr{F}_t^N\}$-square integrable martingale, and

$$\langle W^N \rangle(t) = \int_0^t \Sigma^{-1}(X^N(s)) H^N(X^N(s), V^N(s)) \Sigma^{-1}(X^N(s))\, ds.$$

Therefore the assertion follows from Lemma 5.2 and 4.3 by the martingale central limit theorem [see, e.g., Ethier and Kurtz (1986), Chapter 7, Theorem 1.4]. □

In order to complete the proof of the convergence of $\{(X^N, \Phi^N)\}$, we need a lemma on the convergence of stochastic integrals, due to Jakubowski, Memin and Pages (1989) which, for convenience, is stated next, in a form suitable for our purposes.

THEOREM 5.4 [Jakubowski, Memin and Pages (1989)]. *Let $\{(K^n, W^n)\}$ be a sequence of stochastic processes on a probability space $(\Omega, \mathscr{F}, P)$, with paths in $\mathscr{D}([0,\infty), \mathbb{R}^{2d})$ and such that, for every $n \in \mathbb{N}$, W^n is a local martingale with respect to the filtration $\mathscr{F}_t^n$: $\mathscr{F}_t^n = \bigcap_{s>t} \sigma((K^n(r), W^n(r)), r \le s)$ and, for every $T > 0$,*

$$\sup_n E\left[\sup_{t\le T} |W^n(t) - W^n(t^-)| \right] < +\infty.$$

Assume $\{(K^n, W^n)\}$ converges almost surely to (K, W).

Then W is a semimartingale with respect to the filtration $\{\mathscr{T}_t\}$:

$$\mathscr{T}_t = \bigcap_{s>t} \sigma((K(r), W(r)), r \le s)$$

and

$$\left\{ \int_0^{\cdot +} K^n(s^-)\, dW^n(s) \right\}$$

converges in probability to $\int_0^{\cdot +} K(s^-)\, dW(s)$.

PROOF. Cf. the proofs of Proposition 3.2 and Theorems 2.1 and 2.6 in Jakubowski, Memin and Pages (1989). □

THEOREM 5.5. *Let* $\{(X^N, \Phi^N)\}$ *be defined by* (1.6), (1.7) *and* (4.2).

Then, under hypotheses (H_1) *to* (H_8i) [*or* (H_8ii), (H_8iii)], *the family* $\{(X^N, \Phi^N)\}$ *is relatively compact in* $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ *and every accumulation point is a solution to the stochastic differential equation with reflection*

$$X(t) = X_0 + \int_0^t \{\bar{b}(X(s)) + Q^{-1}(X(s))[\partial U(X(s)) + \bar{q}(X(s))]\}\, ds$$

$$+ \int_0^t Q^{-1}(X(s)) \Sigma(X(s))\, dW(s) + \Phi(t),$$

$$\Phi(t) = \int_0^t \gamma(s)\, d|\Phi|(s), \qquad \gamma(s) \in \Gamma(X(s)), \quad |\gamma(s)| = 1, \quad d|\Phi|\text{-}a.e., \tag{5.4}$$

$$d|\Phi|(\{t: X(t) \in D\}) = 0,$$

where Γ *is defined by* (1.15), $\bar{q}$ *and* $\bar{b}$ *are the functions in* (H_6) *and* Σ *is defined in Lemma* 5.3.

If (5.4) *has a unique solution* (X, Φ) [*see, for instance, Theorem* 5.7 *in Stroock and Varadhan* (1971), *Theorem* 4.1 *in Tanaka* (1979), *Theorem* 4.4 *in Lions and Sznitman* (1984), *Theorems* 3.4 *and* 3.10 *in Varadhan and Williams* (1985), *Theorem* 5.1 *in Costantini* (1987), *etc.*], $\{(X^N, \Phi^N)\}$ *converges in distribution to* (X, Φ).

PROOF. Consider the five-tuple $((X^N, \Phi^N, B^N, Z^N, W^N))$, where B^N, Z^N and W^N are defined by (4.3) and (5.3) respectively. Since each component of the five-tuple is relatively compact in $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^d)$ and admits only continuous accumulation points, the family $\{((X^N, \Phi^N, B^N, Z^N, W^N))\}$ is relatively compact in $\mathscr{D}(\mathbb{R}^+, \mathbb{R}^{5d})$. Let N_n be a diverging nondecreasing sequence of values of N such that $\{(X^{N_n}, \Phi^{N_n}, B^{N_n}, Z^{N_n}, W^{N_n})\}$ converges in distribution to a stochastic process (X, Φ, B, Z, W). By the Skorohod representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{T}}, \tilde{P})$ and realizations $(\tilde{X}^{N_n}, \tilde{\Phi}^{N_n}, \tilde{B}^{N_n}, \tilde{Z}^{N_n}, \tilde{W}^{N_n})$ and $(\tilde{X}, \tilde{\Phi}, \tilde{B}, \tilde{Z}, \tilde{W})$ of $(X^{N_n}, \Phi^{N_n}, B^{N_n}, Z^{N_n}, W^{N_n})$ and (X, Φ, B, Z, W) such that $\{(\tilde{X}^{N_n}, \tilde{\Phi}^{N_n}, \tilde{B}^{N_n}, \tilde{Z}^{N_n}, \tilde{W}^{N_n})\}$ converges $\tilde{P}$-almost surely to $(\tilde{X}, \tilde{\Phi}, \tilde{B}, \tilde{Z}, \tilde{W})$. Therefore, by Theorem 3.1 [if (H_8i) or (H_8ii) holds] or Theorem 3.2 [if (H_8iii) holds], $(\tilde{X}, \tilde{\Phi})$ is a solution to the Skorohod problem for $(D, \Gamma, \tilde{B} + \tilde{Z})$, $\tilde{P}$-almost surely (note that $\tilde{B} + \tilde{Z}$ has continuous paths by Lemmas 4.3 and 4.4).

On the other hand, by Lemma 5.2 and the convergence of $\{\tilde{X}^{N_n}\}$ to $\tilde{X}$, $\tilde{B}$ must be given by

$$\tilde{B}(t) = \int_0^t \left[\bar{b}(\tilde{X}(s)) + Q^{-1}(\tilde{X}(s))\left(\partial U(\tilde{X}(s)) + \bar{q}(\tilde{X}(s))\right)\right] ds, \qquad \tilde{P}\text{-a.s.}$$

Moreover,

$$\tilde{Z}^{N_n}(t) = \int_0^t Q(\tilde{X}^{N_n}(s))\Sigma(\tilde{X}^{N_n}(s))\, d\tilde{W}^{N_n}(s), \qquad \tilde{P}\text{-a.s.},$$

where $\tilde{W}^{N_n}$ is a martingale with respect to the filtration

$$\tilde{\mathscr{F}}_t^n = \bigcap_{s>t} \sigma(\tilde{X}^{N_n}(r), \tilde{W}^{N_n}(r), r \le s),$$

because W^{N_n} is a martingale with respect to the filtration $\{\mathscr{F}_t^{N_n}\}$ and

$$\mathscr{F}_t^{N_n} \supseteq \bigcap_{s>t} \sigma(X^{N_n}(r), W^{N_n}(r), r \le s).$$

Therefore, by Lemma 5.4,

$$\tilde{Z}(t) = \int_0^t Q^{-1}(\tilde{X}(s))\Sigma(\tilde{X}(s))\, d\tilde{W}(s). \qquad \square$$

REMARK 5.3. In particular, if $P^N(x, v; \cdot)$ is independent of x, and U is linear, any accumulation point of the family $\{(X^N, \Phi^N)\}$ is an obliquely reflecting Brownian motion, with covariance matrix $t \cdot Q^{-1}\overline{H}(Q^{-1})^t$ and constant drift $Q^{-1}(\partial U + \bar{q})$.

6. An application to a mechanical model of Brownian motion and other examples. The results of Sections 2 to 5 can be used to derive diffusion approximations for a variety of transport process models with physical reflection. Two applications are presented here: to a model of physical Brownian motion proposed by Dürr, Goldstein and Lebowitz (1981) [(A)] and to a "random billiard" [(B)].

The diffusion approximation of model (A) is of independent interest. Model (A) also shows how in some cases localization arguments can be employed to extend the results of Sections 2 to 5 to processes which satisfy the assumptions of Section 1 only locally.

The most interesting characteristic of model (B1) is that the limit is an obliquely reflecting Brownian motion. Model (B2) is an example of how a nontrivial drift term can appear in the limit. Models (B2) and (A) are instances of models which are not classical central limit theorem space–time rescalings of a given stochastic process.

(A). Dürr, Goldstein and Lebowitz (1981) presented a mechanical model for the motion in $\mathbb{R}^3$ of a heavy particle in an ideal gas of point particles of much smaller mass: A spherical particle of radius r and mass M is immersed in a Poisson bath of point particles (atoms) of mass m, with position distributed with density $\mu/\sqrt{m}$ and velocity distributed according to the probability density $m^{3/2}\psi(\sqrt{m}\ \cdot)$, where ψ is a rotationally invariant probability density having at least five moments (for instance, the density of the Maxwell distribution); the particle moves with constant velocity V except that it undergoes collisions with the atoms; each collision changes its velocity according to the laws of elastic reflection; namely,

$$V' = V - \frac{2m}{m+M}(V - v)\cdot \nu(z)\nu(z),$$

where v is the velocity of the atom, z is the point on the surface of the particle where the collision occurs and $\nu(z)$ is the unit inward normal at z. The stochastic process of the position and velocity of the particle, (X'^m, V'^m) is then non-Markovian, since the particle can collide with atoms which it has previously met and which hence carry information of the past. However, Dürr, Goldstein and Lebowitz construct a Markov process (X^m, V^m) and show that there exists a probability space $(\Omega, \mathscr{F}, P^m)$, on which both (X^m, V^m) and (X'^m, V'^m) can be realized, such that, for every $T > 0$, $\varepsilon > 0$,

$$\lim_{m\to 0} P^m\Big\{\sup_{t\le T}|V'^m(t) - V^m(t)| \ge \varepsilon\Big\} = 0,$$

$$\lim_{m\to 0} P^m\Big\{\sup_{t\le T}|X'^m(t) - X^m(t)| \ge \varepsilon\Big\} = 0.$$

The velocity process V^m is itself a Markov process with infinitesimal generator A^m of the form

$$\begin{aligned} A^m f(V) &= \int_{-\infty}^{+\infty}\int_{S_1(0)}\left[f\left(V - \frac{2m}{m+M}(V\cdot\nu(z) - v_n)\nu(z)\right) - f(V)\right] \\ &\qquad \times p^m(V; z, v_n)\, dz\, dv_n, \end{aligned} \tag{6.1}$$

$$p^m(V; z, v_n) = \mu r^2 |(V\cdot\nu(z) - v_n)\wedge 0|\psi_1(\sqrt{m}\, v_n) I_{\{v_n \ge m^{-\beta}\}},$$

where ψ_1 is the one-dimensional marginal density of ψ and $0 < \beta < \frac{1}{4}$; $p^m(\cdot\,; \cdot\,, \cdot\,)$ determines the jump rate μ^m of V^m via

$$\mu^m(V) = \int_{-\infty}^{+\infty}\int_{S_1(0)} p^m(V; z, v_n)\, dz\, dv_n. \tag{6.2}$$

Suppose now that the motion of the particle takes place in a piecewise $\mathscr{C}_b^1$ domain D, satisfying the uniform exterior sphere condition and (1.1), and that the particle reflects (physically) on the boundary. Then it is shown below that the results of Sections 2 to 5 allow one to say that the process (X^m, V^m) is still well defined for all time and to study the asymptotic behavior of X^m as m

goes to 0, in the case when

$$M = m^{\alpha}, \qquad 0 < \alpha < \beta. \tag{6.3}$$

First of all note that A^m is invariant under rotations and is independent of the position x, so that $|V^m|$ is itself a Markov process and has the same law whether X^m is unrestrained or reflects in D. Therefore a standard localization argument allows one to extend the results of Section 2 to (X^m, V^m), even though the jump rate of V^m is not bounded.

Next observe that, for $|V| \leq m^{-\beta}$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{S_1(0)}\left[-\frac{2m}{m+m^{\alpha}}\left(V\cdot\nu(z)-v_n\right)\right]\nu(z)p^m(V;z,v_n)\,dz\,dv_n \\ = (m^{-\alpha}c_o + q^m)V, \\ c_o = \frac{16\pi r^2\mu}{3}\int_0^{+\infty} v_n\psi_1(v_n)\,dv_n, \qquad \lim_{m\to 0} q^m = 0,\end{aligned} \tag{6.4}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{S_1(0)}\left(\frac{2m}{m+m^{\alpha}}\right)^2\left(V\cdot\nu(z)-v_n\right)^2\nu(z)\nu(z)^t p^m(v;z,v_n)\,dz\,dv_n \\ = m^{-2\alpha}H^m(V), \\ \lim_{m\to 0}\sup_{|V|\leq m^{-\beta}}\left|H^m(V)-\sigma^2 I\right| = 0, \qquad \sigma^2 = \frac{16\pi r^2\mu}{3}\int_0^{+\infty} v_n^3\psi_1(v_n)\,dv_n\end{aligned} \tag{6.5}$$

and

$$\int_{-\infty}^{+\infty}\int_{S_1(0)}\left(\frac{2m}{m+m^{\alpha}}\right)^4\left(V\cdot\nu(z)-v_n\right)^4 p^m(V;z,v_n)\,dz\,dv_n \leq g_o m^{1-4\alpha},$$

$$\lim_{m\to 0}\frac{m^{-\alpha}}{\sup_{|V|\leq m^{-\beta}}\mu^m(V)} = 0.$$

Then if we set

$$\tilde{V}^m(t) = m^{\alpha/2}V^m(t), \qquad N = m^{-\alpha},$$

and if we assume that $V^m(0)$ is of the form $m^{-\alpha/2}V_o$, with V_o having a finite fourth moment, the family of processes $\{\tilde{V}^m\}$ satisfies (H_2) through (H_8) for $|V| \leq m^{-\beta}$, and it holds that

$$X^m(t) = X_o + m^{-\alpha/2}\int_0^t \tilde{V}^m(s)\,ds.$$

Therefore, defining

$$\tau^m = \inf\left\{t \geq 0 \colon |\tilde{V}^m(t)| > m^{\alpha/2-\beta}\right\},$$

it can be easily verified that Lemmas 4.2 through 4.5 and Lemmas 5.1 and 5.2 hold for $\{\tilde{V}^m(\cdot \wedge \tau^m)\}$, replacing T by $T \wedge \tau^m$. On the other hand, Lemma 4.4

and (6.3) ensure that, for every $T > 0$,

$$\lim_{m \to 0} P\{\tau^m \le T\} = 0,$$

which enables one to conclude, by the same arguments as in Theorem 4.6, Lemma 5.3 and Theorem 5.5, that $\{X^m\}$ converges in distribution to a Brownian motion X with covariance matrix $(\sigma^2/c_o^2)I$, and normal reflection on the boundary.

(B1). Let $d = 2$, $D = (0,1) \times (0,1)$, $U = 0$, $O = \{(x,v) \in D \times \mathbb{R}^2 \colon E(x,v) = 1/2\} = D \times S_1(0)$ and

$$\mu^N = N, \qquad F^N(x,v;\xi) = R(\xi)v, \qquad \overline{R} = E[\![R(\xi)]\!], \tag{6.6}$$

where $R(\xi)$ is a $SO(2)$-valued random variable, that is a random rotation, with probability distribution P_R not concentrated on $\{I\}$.

The operator A^N corresponding to (6.6) is continuous on $\mathscr{C}_b(\overline{O}, \mathbb{R})$ in the u.c. topology, by Remark 1.2. Moreover we have

$$\int_{\Delta_x} (v' - v) P^N(x, v, dv') = E\big[F^N(x,v;\xi) - v\big] = -(I - \overline{R})v, \tag{6.7}$$

so that (1.5) holds with $q^N = 0$ and

$$Q = (I - \overline{R}) = \bar{r}\begin{bmatrix} \cos\bar{\theta} & -\sin\bar{\theta} \\ \sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix}, \tag{6.8}$$

for some angle $\bar{\theta}$, $|\bar{\theta}| < \pi/2$, and $\bar{r} = \sqrt{\det(I - \overline{R})}$.

Let m_H denote the normalized Hahr measure on $SO(2)$ (i.e., the unique probability measure invariant under multiplication).

Assume that

(i) P_o is absolutely continuous with respect to the Lebesgue measure m;

(ii) P_R is nonsingular with respect to m_H and, denoting by P_R^c its absolutely continuous component with respect to m_H,

$$\frac{dP_R^c}{dm_H} \ge \lambda > 0;$$

(iii)

$$|\bar{\theta}| < \frac{\pi}{6}.$$

The various hypotheses made in Section 1 can be easily verified. In particular, as far as (H_6) is concerned, we are in the situation considered in Remark 1.5. In fact in this case P^N is actually equal to P and independent of x, and q^N is identically 0. Moreover, by (ii) the family of transition probabilities $\{P(\cdot\,;\,\cdot\,)\}$ is aperiodic and uniformly Harris recurrent, with invariant probability distribution the uniform distribution on $S_1(0)$, $P_u(\cdot)$, so that (1.14) holds.

The limit stochastic differential equation with reflection (5.4)–(1.15) in this case reduces to

$$X(t) = X_o + \sqrt{\bar{r}\cos\bar{\theta}}\,(I - \bar{R})^{-1} W(t) + \Phi(t),$$

$$\Phi(t) = \int_0^t \gamma(s)\, d|\Phi|(s), \quad \gamma(s) \in \Gamma(X(s)), \quad \gamma(s)| = 1, \quad d|\Phi|\text{-a.e.},$$

$$(6.9) \qquad d|\Phi|(\{t\colon X(t) \in D\}) = 0,$$

$$\Gamma(x) = \begin{bmatrix} \cos\bar{\theta} & \sin\bar{\theta} \\ -\sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix} N(x),$$

which has a unique solution, by (ii) and by Corollary 4.4 and Theorem 5.1, Chapter 1 in Costantini (1987). X is the reflecting Brownian motion with covariance matrix $t(\cos\bar{\theta}/\bar{r})I$ and direction of reflection given by a rotation by an angle $-\bar{\theta}$ of the normal cone.

(B2). Let $d = 2$, $D = (0,1) \times (0,1)$, $U = -x_2$, $O = \{(x,v) \in D \times \mathbb{R}^2\colon E(x,v) = 9/2\}$, $\Delta_x = S_{\sqrt{9-2x_2}}(0)$ and

$$(6.10) \quad A^N f(x) = N\left[\mu_o + \frac{1}{\sqrt{N}}\mu(v)\right]\int_{\Delta_x} [f(x,v') - f(x,v)]\, P_u(x, dv'),$$

where $P_u(x,\cdot)$ denotes the uniform distribution on Δ_x. Assume μ is a continuous function on $\overline{B_3(0)}$. By Remark 1.1, (6.10) can be written as

$$A^N f(x) = \mu^N \int_{\Delta_x} [f(x,v') - f(x,v)]\, P^N(x,v;dv'),$$

$$(6.11) \qquad \mu^N = N\left[\mu_o + \frac{1}{\sqrt{N}}\mu_1\right], \qquad \mu_1 = \sup_{\sqrt{7} \le |v| \le 3} \mu(v),$$

$$P^N(x,v;\cdot) = \frac{N\mu_o + \sqrt{N}\mu(v)}{\mu^N} P_u(x,\cdot) + \left[1 - \frac{N\mu_o + \sqrt{N}\mu(v)}{\mu^N}\right]\delta_v(\cdot).$$

The operator A^N defined by (6.11) is continuous on $\mathscr{C}_b(\bar{O},\mathbb{R})$ in the u.c. topology by Remark 1.2. Moreover (1.5) holds with

$$Q = \mu_o I, \qquad q^N(x,v) = -\mu(v)v.$$

It can be easily checked that the assumptions of Section 1 are satisfied. In particular for (H_1) we only need to observe that, for every $p \in L^1(m(x,\cdot))$ and for any Borel set $B \subseteq \Delta_x$, we have

$$\int_{\Delta_x} P^N(x,v;B) p(v) m(x,dv)$$

$$= P_u(x,B)\int_{\Delta_x} \frac{N\mu_o + \sqrt{N}\mu(v)}{\mu^N} p(v) m(x,dv)$$

$$+ \int_B \left[1 - \frac{N\mu_o + \sqrt{N}\mu(v)}{\mu^N}\right] p(v) m(x,dv),$$

which vanishes whenever $m(x,B) = 0$.

As far as (H_6) is concerned, we are in the setup of Remark 1.5 with

$$Af(x) = \int_{\Delta_x} [f(x,v') - f(x,v)] P_u(x, dv'),$$

$$q(x,v) = -\mu(v)v \quad \text{and} \quad H(x,v) = \mu_o\left[vv^t + \frac{9 - 2x_2}{2} I\right], \qquad v \in \Delta_x.$$

(1.9) holds because μ is bounded by μ_1. Moreover (1.12) is trivially satisfied with $d(x) = 1$ and $\pi(x, \cdot) = P_u(x, \cdot)$.

The limiting stochastic differential equation with reflection (5.4) in this case is

$$\begin{aligned} X(t) &= X_o + \frac{1}{\mu_o}\int_0^t \left\{\bar{q}(X(s)) - \begin{bmatrix}0\\1\end{bmatrix}\right\} ds \\ &\quad + \int_0^t \left\{\frac{9 - 2X_2(s)}{\mu_o}\right\}^{1/2} dW(s) + \int_0^t \nu(s)\, d|\Phi|(s), \end{aligned}$$

$$\nu(s) \in N(X(s)), \qquad |\nu(s)| = 1, \qquad d|\Phi|\text{-a.e.}, \tag{6.12}$$

$$d|\Phi|(\{t: X(t) \in D\}) = 0, \quad \bar{q}(x) = \int_{\Delta_x} \mu(v) v P_u(x, dv),$$

which has a unique solution by Theorem 4.1 in Tanaka (1979), if we assume that μ is Lipschitz.

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