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STRONG LAWS FOR SMALL INCREMENTS OF RENEWAL PROCESSES

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Let $\{N(t), t \geq 0\}$ be the (generalized) renewal process associated with an i.i.d. sequence $X_1, X_2, \dots$ of random variables having finite moment generating function on some left-sided neighborhood of the origin. Some strong limiting results are proved for the maximal increments $\sup_{0 \leq t \leq T-K} (N(t+K) - N(t))$, where $K = K_T$ is a function of T such that $K_T \uparrow \infty$, but $K_T/\log T \downarrow 0$ as $T \to \infty$. These provide analogs to a recent extension due to Mason (1989) of the Erdős–Rényi strong law of large numbers for partial sums.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of independent random variables with common distribution function F such that:

(i) X_1 is nondegenerate with $0 < \mu = EX_1 < \infty$ and
(ii) $(-\infty \leq) s_0 = \inf\{s: \varphi(s) = E\exp(sX_1) < \infty\} < 0$.

Let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, for $n \geq 1$, and consider the corresponding (generalized) renewal process $\{N(t), t \geq 0\}$ defined by

$$N(t) = \max\{n \geq 0: S_0, \dots, S_n \leq t\} \quad \text{for } t \geq 0.$$

Note that in the case of nonnegative $X_1, X_2, \dots,$ the process $\{N(t), t \geq 0\}$ coincides with the ordinary renewal process given by

$$N(t) = \max\{n \geq 0: S_n \leq t\} \quad \text{for } t \geq 0.$$

The purpose of this paper is to study the strong limiting behavior of the maximal increments of $\{N(t), t \geq 0\}$ over small subintervals of size K_T. *Here and throughout the sequel* $\{K_T, T \geq 0\}$ *will denote a positive function satisfying* $K_T \uparrow \infty$, *but* $K_T/\log T \downarrow 0$ *as* $T \to \infty$. Define

$$D(T, K) = \sup_{0 \leq t \leq T-K} (N(t+K) - N(t)). \tag{1.1}$$

Several results have been obtained concerning the limiting behavior of $D(T, K_T)$ for functions $\{K_T, T \geq 0\}$ of "large" size ($K_T/\log T \to \infty$) or "medium" size ($K_T \sim C \log T$: "Erdős–Rényi" case); see, for example, Deheuvels and Steinebach (1989) and the references mentioned therein. In the latter paper it was shown that, via a duality between events, it is often possible to deduce the strong laws for $D(T, K_T)$ from the knowledge of the correspond-

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ing minimal increments of partial sums,

$$M(n, b_n) = \min_{0 \le k \le n - b_n} (S_{k+b_n} - S_k), \tag{1.2}$$

for suitable choices of $1 \le b_n \le n$. But also in the partial sum situation, the cases $b_n/\log n \to \infty$ or $b_n \sim c \log n$ have received much interest [cf. Deheuvels, Devroye and Lynch (1986) for further references], whereas little work has been done on the small increments [see, e.g., Book (1976), de Acosta and Kuelbs (1983), Steinebach (1983) and Huse and Steinebach (1985)]. Recently, Mason (1989) has achieved remarkable progress in this field by providing suitable conditions for an extension of the Erdős–Rényi law to the case $b_n/\log n \to 0$. The following theorem is a simple consequence of his main result [cf. Deheuvels and Steinebach (1989)].

THEOREM A. *Assume, in addition to* (i), *that*

(iii) $\alpha = \operatorname{ess\,inf} X_1 > 0$.

Then, for all $\{K_T, T \ge 0\}$ *such that* $K_T \uparrow \infty$ *and* $K_T/\log T \downarrow 0$ *as* $T \to \infty$, *we have*

$$\lim_{T \to \infty} D(T, K_T)/K_T = 1/\alpha \quad a.s. \tag{1.3}$$

A direct duality argument, however, is not applicable if $\operatorname{ess\,inf} X_1 = 0$ or if $P(X_1 < 0) > 0$. In the latter case, the following result has been obtained in Steinebach (1979).

THEOREM B. *Assume, in addition to* (i) *and* (ii), *that*

(iv) $\varphi'(s^*) = 0$ *for some* $s^* \in (s_0, 0)$.

Put $\rho^* = \inf_s \varphi(s) = \varphi(s^*)$, *and let* $\{K_T, T \ge 0\}$ *denote a function satisfying* $K_T \uparrow \infty$, $K_T/\log T \downarrow 0$ *as* $T \to 0$ *and with* $T_j = \sup\{T\colon K_T \le j\}$, *assume*:

(v) $\lim_{j \to \infty} \log T_{j+1}/\log T_j = 1$.

Then

$$\lim_{T \to \infty} D(T, K_T)/\log T = 1/\log(1/\rho^*) \quad a.s. \tag{1.4}$$

A comparison of Theorems A and B also shows that the proper normalization of $D(T, K_T)$ obviously depends upon properties of the underlying distribution F.

In Section 2 we will discuss the limiting behavior of $D(T, K_T)$ when $\alpha = \operatorname{ess\,inf} X_1 = 0$. The two cases $P(X_1 = 0) = p > 0$ or $= 0$ are treated separately. This is in correspondence with Mason's (1989) results for partial sums where the asymptotics have to be distinguished according to whether $\omega = \operatorname{ess\,sup} X_1 < \infty$ or $= \infty$.

2. Statement of the results. Similar to the partial sum situation studied by Mason (1989), the following functions play an essential role in describing the limiting behavior of $D(T, K_T)$ as given in (1.1). Set for $0 < x \le \mu$ and $1/\mu \le u < \infty$, respectively,

$$\lambda(x) = -\log \rho(x) = \sup_{s \le 0} \{sx - \log \varphi(s)\},$$

$$\kappa(u) = u\lambda(1/u) = \sup_{s \le 0} \{s - u \log \varphi(s)\}.$$

It is well known or easily checked via the derivatives [see, e.g., Deheuvels, Devroye and Lynch (1986), Section 2] that

$\lambda(x)$ is strictly convex and decreasing on $(0, \mu]$ with

$$\lambda(\mu) = 0, \qquad \lambda(0+) = \log(1/p), \qquad p = P(X_1 = 0), \qquad 1/0 := \infty;$$

$\kappa(u)$ is strictly convex and increasing on $[1/\mu, \infty)$ with

$$\kappa(1/\mu) = 0, \qquad \kappa(\infty) = \infty.$$

For any $0 \le y < \log(1/p)$ and $0 \le v < \infty$ set

$$\gamma(y) = (\operatorname{inv} \lambda)(y), \qquad \beta(v) = (\operatorname{inv} \kappa)(v),$$

where inv denotes the inverse function. From the properties of λ and κ, it is readily seen that

γ is strictly convex and decreasing on $[0, \log(1/p))$ with

$$\gamma(0) = \mu, \qquad \gamma(\log(1/p)) = 0;$$

β is strictly concave and increasing on $[0, \infty)$ with

$$\beta(0) = 1/\mu, \qquad \beta(\infty) = \infty.$$

Our first result provides an analog of Theorem B for the ordinary renewal process:

THEOREM 1. *Let $X_1, X_2, \dots$ be an i.i.d. sequence with distribution function F satisfying* (i) *and*

(iii′) $\alpha = \operatorname{ess\,inf} X_1 = 0$, $P(X_1 = 0) = p > 0$.

Then for all functions $\{K_T,\ T \ge 0\}$ such that $K_T \uparrow \infty$, but $K_T/\log T \downarrow 0$ as $T \to \infty$,

$$\lim_{T \to \infty} D(T, K_T)/\log T = 1/\log(1/p) \quad a.s. \tag{2.1}$$

REMARK 1. Since $p = P(X = 0) = \inf_s \varphi(s) = \rho^*$, Theorem B thus immediately extends to the ordinary renewal process. A close look at the proof of Theorem B shows that the regularity assumption (v) on $\{K_T,\ T \ge 0\}$ there could also be removed.

On observing that as $T \to \infty$,

$$K_T \beta\left(K_T^{-1} \log T\right) \sim (\log T)/\log(1/p),$$

assertion (2.1) can be reformulated as

$$\lim_{T \to \infty} D(T, K_T)/K_T \beta\left(K_T^{-1} \log T\right) = 1 \quad \text{a.s.} \tag{2.1$'$}$$

Under additional assumptions, the latter assertion also extends to the case of $P(X_1 = 0) = 0$.

THEOREM 2. *Let $X_1, X_2, \dots$ be an i.i.d. sequence with distribution function F satisfying* (i) *and*

(iii$''$) $\alpha = \operatorname{ess\,inf} X_1 = 0$, $P(X_1 = 0) = 0$.

Then for all functions $\{K_T,\ T \geq 0\}$ such that $K_T \uparrow \infty$, but $K_T/\log T \downarrow 0$ as $T \to \infty$,

$$\limsup_{T \to \infty} D(T, K_T)/K_T \beta\left(K_T^{-1} \log T\right) = 1 \quad a.s. \tag{2.2}$$

Moreover, the lim sup *in* (2.2) *can be replaced by* lim *for all such functions if*

$$\lim_{x \downarrow 0} \frac{\gamma(-\log F(x))}{x} = 1. \tag{2.3}$$

3. Proofs of Theorems 1 and 2. The proofs make use of the duality arguments applied in Deheuvels and Steinebach (1989) in combination with similar ideas as developed by Mason (1989) for partial sums. Although there is an obvious correspondence between Theorems 1 and 2 of this article and Mason's (1989) results, an immediate inversion technique does not seem to work here.

Before presenting the proofs, let us introduce some additional notation. Consider $D_T = D(T, K_T)$ and $M_n = M(n, b_n)$ as defined in (1.1) and (1.2), and set, for $0 < \varepsilon < 1$,

$$A_T(\pm\varepsilon) = \left\{D_T > (1 \pm \varepsilon) K_T \beta\left(K_T^{-1} \log T\right)\right\}.$$

Then, (2.1) or (2.1$'$) is equivalent to showing that for any $\varepsilon > 0$,

$$P\left(A_T(\varepsilon) \text{ i.o.}\right) = 0 \quad \text{and} \quad P\left(A_T^c(-\varepsilon) \text{ i.o.}\right) = 0, \tag{3.1}$$

where $\{A_T(\varepsilon) \text{ i.o.}\}$ denotes the event that, for any $T_1 > 0$, there exists $T_2 > T_1$ such that $A_{T_2}(\varepsilon)$ holds.

Likewise, (2.2) holds iff, for any $\varepsilon > 0$,

$$P\left(A_T(\varepsilon) \text{ i.o.}\right) = 0 \quad \text{and} \quad P\left(A_T(-\varepsilon) \text{ i.o.}\right) = 1. \tag{3.2}$$

Moreover, lim sup in (2.2) can be replaced by lim if (3.1) can be verified under the assumption (2.3).

Now, in order to apply the desired duality argument, observe that for $0 < h < u$ and $b \in \mathbb{N}$,

$$\{D(u,h) > b\} = \{M(N(u),b) < h\}. \tag{3.3}$$

[Note that $\sup_{0 \le t \le u-h}\{N(t+h) - N(t)\}$ attains an integer maximum at some $t = 0, S_1, \dots, S_{N(u)}$. For similar arguments see, for example, Deheuvels and Steinebach (1989), Lemma 7.]

LEMMA 1. *Under the assumptions of Theorems* 1 *or* 2, *we have*

$$\limsup_{T\to\infty} D(T, K_T)/K_T\beta(K_T^{-1}\log T) \le 1 \quad a.s. \tag{3.4}$$

PROOF. Define for $T = Rn$, R suitably chosen below and $\varepsilon > 0$,

$$b_n = b_n(\varepsilon, R) = \left[(1+2\varepsilon)K_T\beta(K_T^{-1}\log T)\right],$$

where $[\cdots]$ denotes "integer part of." Note that $K_{Rn} \sim K_{R(n+1)}$ by the monotonicity of K_T and $K_T/\log T$. We first prove

$$P\big(M(n,b_n) < K_{R(n+1)} \text{ i.o.}\big) = 0. \tag{3.5}$$

With $b = b_n$, $K = K_{R(n+1)}$ and $\beta = \beta(K_{Rn}^{-1}\log Rn)$, we have for large n, by making use of Markov's inequality,

$$\begin{aligned} P(S_b < K) &\le \exp\{b\log\rho(K/b)\} = \exp\{-b\lambda(K/b)\} \\ &\le \exp\{-(1+\varepsilon)K_{Rn}\beta\lambda(1/(1+\varepsilon)\beta)\}. \end{aligned}$$

Convexity of $\kappa(u) = u\lambda(1/u)$ on $[1/\mu, \infty)$ with $\kappa(1/u) = 0$ implies $\kappa((1+\varepsilon)\beta) \ge (1+\varepsilon)\kappa(\beta)$. Hence it follows that

$$P(M(n,b) < K) \le n\exp\{-(1+\varepsilon)\log(Rn)\}. \tag{3.6}$$

Let $n_j = \max\{n\colon b_n = j\}$. Note that $b_{n+1} - b_n = o(1)$, so that n_j is well defined for all j sufficiently large. The latter relation can be verified as follows, setting $v_n = K_{Rn}^{-1}\log Rn$:

$$\begin{aligned} 0 &\le K_{R(n+1)}\beta(v_{n+1}) - K_{Rn}\beta(v_n) \\ &= (\log R(n+1))\beta(v_{n+1})/v_{n+1} - (\log Rn)\beta(v_n)/v_n \\ &\le (\log R(n+1) - \log Rn)\beta(v_n)/v_n = o(1), \end{aligned}$$

since $\beta(v)/v$ is decreasing by the concavity of β and $\beta(0) \ge 0$. Moreover, for some positive constant δ^{-1},

$$j = b_{n_j} \le (1+2\varepsilon)(\log Rn_j)\beta(v_{n_j})/v_{n_j} \le \delta^{-1}\log n_j,$$

that is, $n_j \ge \exp(\delta j)$.

Now, for $n_{j-1} < n \le n_j$ and j sufficiently large,

$$\begin{gathered} b_n = b_{n_j} = j, \\ M(n,b_n) \ge M(n_j, j), \\ K_{R(n+1)} \le K_{R(n_j+1)}. \end{gathered}$$

By (3.6) and $n_j \geq \exp(\delta j)$ for large j, the probabilities $P(M(n_j, j) < K_{R(n_j+1)})$ are summable (in j). Hence the Borel–Cantelli lemma yields (3.5).

Choose $0 < R < \mu$. By the strong law of large numbers, for $Rn \leq T \leq R(n+1)$ and n sufficiently large,

$$N(T) \leq n \quad \text{and} \quad K_T\beta(K_T^{-1}\log T) \geq K_{Rn}\beta(K_{Rn}^{-1}\log Rn). \tag{3.7}$$

In view of (3.3) and (3.7), assertion (3.5) thus implies

$$P(D(T, K_T) > (1+2\varepsilon)K_T\beta(K_T^{-1}\log T)\text{ i.o.}) = 0$$

for any $\varepsilon > 0$, which proves (3.4). □

Remark 2. It is obvious from the proof of Lemma 1 that the monotonicity assumption $K_T/\log T \downarrow$ could be replaced by $K_{R(n+1)} \sim K_{Rn}$ as $n \to \infty$ for some $R < \mu$.

Lemma 1 gives the upper halves of assertions (2.1) [or (2.1′)] and (2.2), respectively. The proof of Theorem 1 can now be completed as follows.

Lemma 2. *Under the assumptions of Theorem* 1, *we have*

$$\liminf_{T\to\infty} D(T, K_T)/\log T \geq 1/\log(1/p) \quad a.s. \tag{3.8}$$

Proof. For $R(n-1) \leq T \leq Rn$, $R > \mu$ and $0 < \varepsilon < \frac{1}{2}$, set

$$b'_n = b'_n(\varepsilon, R) = [(1-2\varepsilon)(\log Rn)/\log(1/p)] + 1.$$

Then, by the duality (3.3),

$$\{D(T, K_T) \leq (1-2\varepsilon)(\log T)/\log(1/p)\} \subset \{M(N(T), b'_n) \geq K_{R(n-1)}\}.$$

With $b = b'_n$ and $K = K_{R(n-1)}$, we have for large n

$$\begin{aligned} P(M(n,b) \geq K) &\leq \{1 - P(S_b < K)\}^{[n/b]} \\ &\leq \exp\{-[n/b]p^b\} \leq \exp\{-cn^{\varepsilon}\} \end{aligned}$$

for some positive constant c, $p = P(X = 0)$. Hence, applying the Borel–Cantelli lemma results in

$$P(M(n, b'_n) \geq K_{R(n-1)}\text{ i.o.}) = 0.$$

Observing that $N(T) \geq n$, for $R(n-1) \leq T \leq Rn$ and n sufficiently large, completes the proof. □

Proof of Theorem 1. Combine Lemmas 1 and 2. □

The proof of the lower half of Theorem 2 (with lim sup or lim, resp.) makes use of arguments similar to those in Mason [(1989), Lemma 2.5]. The following facts about λ are helpful.

LEMMA 3. *Let* $H = \{h\colon h$ *is convex and nonincreasing on* $[0, \infty)$ *with* $E\exp(h(X_1)) < \infty\}$ *and set for* $0 < x \le \mu$

$$\lambda^*(x) = \sup_{h \in H} \{h(x) - \log E\exp(h(X_1))\}.$$

Then

$$\lambda^*(x) = \lambda(x) \quad \text{on } (0, \mu].$$

Lemma 3 is used in the proof of the next lemma.

LEMMA 4. *Whenever* $\operatorname{ess\,inf} X_1 = 0$ *and* $P(X_1 = 0) = 0$,

$$\liminf_{x \downarrow 0} (-\log F(x))/\lambda(x) = 1. \tag{3.9}$$

The proofs of Lemmas 3 and 4 are similar to those of Lemmas 2.1 and 2.2 in Mason (1989) and can be omitted here.

We are now prepared to prove Theorem 2.

LEMMA 5. *Under the assumptions of Theorem* 2, *we have*

$$\limsup_{T \to \infty} D(T, K_T)/K_T \beta(K_T^{-1} \log T) \ge 1 \quad a.s. \tag{3.10}$$

PROOF. For $0 < \varepsilon < 1$ and $T = Rn$, $R > \mu$, set

$$b''_n = b''_n(\varepsilon, R) = \left[(1-\varepsilon)^3 K_{Rn} \beta(K_{Rn}^{-1} \log Rn)\right] + 1.$$

By the duality in (3.3),

$$\begin{aligned} &\{D(T, K_T) \ge (1-\varepsilon)^3 K_T \beta(K_T^{-1} \log T)\} \\ &\qquad \supset \{M(N(T), b''_n) < K_{Rn}\}. \end{aligned} \tag{3.11}$$

We show that

$$P\big(M(n_i, b''_{n_i}) < K_{Rn_i} \text{ i.o.}\big) = 1 \tag{3.12}$$

for a suitable subsequence $\{n_i,\ i = 1, 2, \dots\}$. With $b = b''_n$, $K = K_{Rn}$ and $\beta = \beta(K_{Rn}^{-1} \log Rn)$, we have for large n, by estimating $(1-\varepsilon)K/b$ from below and b from above,

$$\begin{aligned} &P(M(n, b) > (1-\varepsilon)K) \\ &\quad \le \{1 - P(S_b \le (1-\varepsilon)K)\}^{[n/b]} \\ &\quad \le \exp\big(-[n/b]\{P(X_1 \le (1-\varepsilon)K/b)\}^b\big) \\ &\quad \le \exp\big(-[n/b]\exp\{(1-\varepsilon)^2 K\beta \log F(1/(1-\varepsilon)\beta)\}\big) \\ &\quad =: p_n. \end{aligned} \tag{3.13}$$

By Lemma 4 we can find a strictly decreasing sequence $\{x_i\}$ such that $x_i \le 1/i$ for all integers i and

$$\log F(x_i) \ge -\lambda(x_i)/(1-\varepsilon).$$

Let

$$n_i = \max\{n\colon \beta(K_{Rn}^{-1}\log Rn) \le 1/(1-\varepsilon)x_i\}.$$

Note that necessarily

$$\beta(K_{R(n_i+1)}^{-1}\log R(n_i+1)) > 1/(1-\varepsilon)x_i.$$

Since β is concave, that is, $\beta(v)/v$ is bounded, $K_{Rn}^{-1}\log Rn = o(n)$ and $K_{Rn} \sim K_{R(n+1)}$ as $n \to \infty$, the latter inequality implies that $n_i \ge i$ for large i.

By continuity of β there exists

$$v_i = (1+\varepsilon_i)K_{Rn_i}^{-1}\log n_i$$

with $0 \le \varepsilon_i \to 0$ as $i \to \infty$ and

$$\beta(v_i) = 1/(1-\varepsilon)x_i.$$

Since $\beta(v)$ is increasing and $\log F < 0$,

$$\beta = \beta(K_{Rn_i}^{-1}\log Rn_i) \le \beta(v_i),$$

$$\beta\log F(1/(1-\varepsilon)\beta) \ge \beta(v_i)\log F(1/(1-\varepsilon)\beta(v_i)).$$

Hence for $n = n_i$ and i sufficiently large, the estimate in (3.13) continues as

$$p_{n_i} \le \exp(-[n_i/b''_{n_i}]\exp\{-(1-\varepsilon)K\beta(v_i)\lambda(1/(1-\varepsilon)\beta(v_i))\}).$$

By convexity of $\kappa(u) = u\lambda(1/u)$, $\kappa((1-\varepsilon)u) \le (1-\varepsilon)\kappa(u)$, which implies

$$(1-\varepsilon)\beta(v_i)\lambda(1/(1-\varepsilon)\beta(v_i)) \le (1-\varepsilon)v_i.$$

Thus we have, for i sufficiently large,

$$p_{n_i} \le \exp(-[n_i/b''_{n_i}]n_i^{-(1-\varepsilon)(1+\varepsilon_i)}),$$

which is summable in i, since $n_i \ge i$, $\varepsilon_i = o(1)$ and $b''_{n_i} = o(\log n_i)$. Hence the Borel–Cantelli lemma gives (3.12). Via the duality of (3.11), this suffices to prove (3.10), since $N(Rn_i) \ge n_i$ for i sufficiently large. □

It only remains to prove that the lim sup in (2.2) can be replaced by lim, if condition (2.3) holds.

LEMMA 6. *Under the assumptions of Theorem* 2 *including* (2.3), *we have*

(3.14) $$\liminf_{T\to\infty} D(T, K_T)/K_T\beta(K_T^{-1}\log T) \ge 1 \quad a.s.$$

PROOF. With the same estimates as in the proof of Lemma 5, we have for $0 < \varepsilon < 1$ and $R(n-1) \le T \le Rn$, $R > \mu$,

(3.15) $$\{D(T, K_T) \le (1-\varepsilon)^3 K_T\beta(K_T^{-1}\log T)\} \subset \{M(N(T), b''_n) \ge K_{R(n-1)}\}$$

and with $b = b''_n$, $K = K_{Rn}$, $\beta = \beta(K_{Rn}^{-1}\log Rn)$ and n large,

(3.16) $$\begin{aligned} p_n &= P(M(n,b) > (1-\varepsilon)K_{R(n-1)}) \le P(M(n,b) > (1-\varepsilon)^{3/2}K) \\ &\le \exp(-[n/b]\exp\{(1-\varepsilon)K\beta\log F(1/(1-\varepsilon)\beta)\}). \end{aligned}$$

By (2.3), for all sufficiently large n,

$$\gamma(-\log F(1/(1-\varepsilon)\beta)) \geq (1-\varepsilon)/(1-\varepsilon)\beta,$$

that is,

$$\log F(1/(1-\varepsilon)\beta) \geq -\lambda(1/\beta),$$

since γ is decreasing. Thus we have

$$p_n \leq \exp\left(-[n/b](Rn)^{-1+\varepsilon}\right),$$

which is summable in n. The rest of the proof now proceeds as that of Lemma 5. □

PROOF OF THEOREM 2. Assertion (2.2) follows from a combination of Lemmas 1 and 5. The replacement of lim sup in (2.2) by lim under (2.3) is justified by Lemma 6. □

REFERENCES

BOOK, S. A. (1976). Large deviation probabilities and the Erdős–Rényi law of large numbers. *Canad. J. Statist.* **4** 185–210.

DE ACOSTA, A. and KUELBS, J. (1983). Limit theorems for moving averages of independent random vectors. *Z. Wahrsch. Verw. Gebiete* **64** 67–123.

DEHEUVELS, P., DEVROYE, L. and LYNCH, J. (1986). Exact convergence rate in the limit theorems of Erdős–Rényi and Shepp. *Ann. Probab.* **14** 209–223.

DEHEUVELS, P. and STEINEBACH, J. (1989). Sharp rates for increments of renewal processes. *Ann. Probab.* **17** 700–722.

HUSE, V. R. and STEINEBACH, J. (1985). On an improved Erdős–Rényi-type law for increments of partial sums. *Canad. J. Statist.* **13** 311–315.

MASON, D. M. (1989). An extended version of the Erdős–Rényi strong law of large numbers. *Ann. Probab.* **17** 257–265.

STEINEBACH, J. (1979). Erdős–Rényi–Zuwächse bei Erneuerungsprozessen und Partialsummen auf Gittern. Habilitationsschrift, Universität Düsseldorf.

STEINEBACH, J. (1983). On the increments of partial sum processes with multidimensional indices. *Z. Wahrsch. Verw. Gebiete* **63** 59–70.

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