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SHARP INEQUALITIES FOR THE CONDITIONAL SQUARE FUNCTION OF A MARTINGALE

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Let f be a real martingale and $s(f)$ its conditional square function. Then the following inequalities are sharp:

$$\|f\|_p \le \sqrt{\frac{2}{p}}\,\|s(f)\|_p, \qquad 0 < p \le 2,$$

$$\sqrt{\frac{2}{p}}\,\|s(f)\|_p \le \|f\|_p, \qquad p \ge 2.$$

The second inequality is still sharp if f is replaced by the maximal function f^*. Let $S(f)$ denote the square function of f. Then the following inequalities are also sharp:

$$\|S(f)\|_p \le \sqrt{\frac{2}{p}}\,\|s(f)\|_p, \qquad 0 < p \le 2,$$

$$\sqrt{\frac{2}{p}}\,\|s(f)\|_p \le \|S(f)\|_p, \qquad p \ge 2.$$

These inequalities hold for Hilbert-space-valued martingales and are strict inequalities in all of the nontrivial cases.

1. Introduction and summary of the results. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space with a nondecreasing sequence of σ-fields

$$\{\Omega, \phi\} = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \cdots \subset \mathscr{F}_n \subset \cdots \subset \mathscr{F}.$$

Let H be a real or complex Hilbert space with norm $|\cdot|$. A sequence of H-valued strongly integrable functions $(f_n)_{n \ge 1}$ is a martingale if for each $n \ge 1$, f_n is strongly measurable relative to $\mathscr{F}_n$, and for $n \ge 2$,

$$E(d_n | \mathscr{F}_{n-1}) = 0 \quad \text{a.e.}$$

Here the difference sequence $(d_n)_{n \ge 1}$ is defined by $f_n = \sum_{i=1}^n d_i$, $n \ge 1$. Let

$$s^2(f) = \sum_{n=1}^{\infty} E\left(|d_n|^2 | \mathscr{F}_{n-1}\right)$$

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denote the conditional square function of f and

$$S^2(f) = \sum_{n=1}^{\infty} |d_n|^2$$

the square function of f. We shall also use the notation $s_n^2(f) = \sum_{i=1}^n E(|d_i|^2 | \mathscr{F}_{i-1})$, $S_n^2 = \sum_{i=1}^n |d_i|^2$ and $f^* = \sup_{n \geq 1} |f_n|$.

Let $\| f_n \|_p = (E|f_n|^p)^{1/p}$ and $\| f \|_p = \sup_{n \geq 1} \| f_n \|_p$. Burkholder and Gundy [7] proved that there exist positive constants α_p and β_p such that

$$(1) \qquad \| f \|_p \leq \alpha_p \| s(f) \|_p, \qquad 0 < p \leq 2,$$

$$(2) \qquad \alpha_p \| s(f) \|_p \leq \| f \|_p, \qquad p \geq 2,$$

$$(3) \qquad \| S(f) \|_p \leq \beta_p \| s(f) \|_p, \qquad 0 < p \leq 2,$$

$$(4) \qquad \beta_p \| s(f) \|_p \leq \| S(f) \|_p, \qquad p \geq 2,$$

for all real martingales. The reverse directions of the above inequalities do not hold in general except for $p = 2$.

Garsia [9] showed that the following inequalities are satisfied by the best possible constants α_p and β_p: If $0 < p \leq 2$, then $\alpha_p \leq 4\sqrt{2/p}$ and $\beta_p \leq \sqrt{2/p}$; if $p \geq 2$, then $\alpha_p \geq \sqrt{2/p}$ and $\beta_p \geq \sqrt{2/p}$. We can prove that $\alpha_p = \beta_p = \sqrt{2/p}$ for all $p > 0$.

THEOREM 1. *Let f be an H-valued martingale. Then*

$$(5) \qquad \| f \|_p \leq \sqrt{\frac{2}{p}} \| s(f) \|_p, \qquad 0 < p \leq 2,$$

$$(6) \qquad \sqrt{\frac{2}{p}} \| s(f) \|_p \leq \| f \|_p, \qquad p \geq 2,$$

$$(6') \qquad \sqrt{\frac{2}{p}} \| s(f) \|_p \leq \| f^* \|_p, \qquad p \geq 2,$$

$$(7) \qquad \| S(f) \|_p \leq \sqrt{\frac{2}{p}} \| s(f) \|_p, \qquad 0 < p \leq 2,$$

$$(8) \qquad \sqrt{\frac{2}{p}} \| s(f) \|_p \leq \| S(f) \|_p, \qquad p \geq 2.$$

The constant $\sqrt{2/p}$ is best possible in each of these inequalities and is already best possible in the special case of real conditionally symmetric martingales. Furthermore, strict inequality holds for all of the mentioned cases except for $p = 2$ or for $\| f \|_p = \{0, \infty\}$.

For example, if $0 < \|f\|_p < \infty$ and $0 < p < 2$, then strict inequality holds in (5).

Notice that if in (6), f is replaced by f^* to obtain (6′), the best constant does not change.

A martingale f is conditionally symmetric if for all $n \geq 1$, d_n and $-d_n$ have the same conditional distribution on the σ-field generated by $f_1, \dots, f_{n-1}$, $\sigma(f_1, \dots, f_{n-1})$. All dyadic martingales are conditionally symmetric. A martingale is simple, if for all $n \geq 1$, f_n is a simple function and there exists an integer N such that when $n \geq N$, $f_n = f_N$. If $\hat{\mathscr{F}}_n = \sigma(f_1, \dots, f_{n-1}, |d_n|)$, then for a simple conditionally symmetric martingale, $E(d_n | \hat{\mathscr{F}}_{n-1}) = 0$ for all $n \geq 1$ (see Hitczenko [10] or Wang [11]). By using the standard approximation method with a little more care (see Wang [11]) this shows $E(d_n | \hat{\mathscr{F}}_{n-1}) = 0$ for a general conditionally symmetric martingale. This means $f = (f_n, \hat{\mathscr{F}}_n)_{n \geq 1}$ is a martingale. For this choice of σ-fields $\{\hat{\mathscr{F}}_n\}_{n \geq 1}$, $\hat{s}_n^2(f) = \sum_{i=1}^n E(|d_i|^2 | \hat{\mathscr{F}}_{i-1}) = S_n^2(f)$ for all $n \geq 1$. Hence, if z_p denotes the smallest positive zero of the confluent hypergeometric function $M(-p/2, 1/2, z^2/2)$ (see Abramowitz and Stegun [1]), then

$$\|f\|_p \leq z_p \|\hat{s}(f)\|_p, \qquad 0 < p \leq 2,$$

$$z_p \|\hat{s}(f)\|_p \leq \|f\|_p, \qquad p \geq 2,$$

and z_p is best possible for this choice of $\hat{\mathscr{F}}_n$ (see Davis [8] and Wang [12]). However, if we put no restriction on the σ-fields $\{\mathscr{F}_n\}_{n \geq 1}$, then as shown in Theorem 1, z_p is not best possible. As an interesting consequence, we prove

$$z_p \leq \sqrt{\frac{2}{p}}, \qquad 0 < p \leq 2,$$

$$\sqrt{\frac{2}{p}} \leq z_p, \qquad 2 \leq p.$$

Computer calculation shows that strict inequality holds except for $p = 2$.

Note that for dyadic martingales, the conclusion of Theorem 1 is not true. In fact, since $\|s(f)\|_p = \|S(f)\|_p$, the best constants in (5) and (6) are z_p and the best constants in (7) and (8) are 1. This is different from the situation in the square-function inequalities, in which case the best constant for conditionally symmetric martingales is the same for dyadic martingales (see [8] and [12]).

Even for conditionally symmetric martingales, the reverse directions of (5)–(8) do not hold except for $p = 2$. For example, if $f_1 = \pm 1$ with probability $a/2$ each and 0 otherwise, then $f = \{(0, \mathscr{F}_0), (f_1, \mathscr{F})\}$ is a conditionally symmetric martingale. It is easy to see that (5)–(8) cannot be reversed as $a \to 0$ for this choice of f.

THEOREM 2. *Let $\{e_n\}_{n\geq 0}$ be a sequence of nonnegative random variables. The following inequalities are sharp and strict in all nontrivial cases*

$$\frac{1}{p}\left\|\sum_{n=0}^{\infty} E(e_n|\mathscr{F}_n)\right\|_p \leq \left\|\sum_{n=0}^{\infty} e_n\right\|_p, \qquad p\geq 1, \tag{9}$$

$$\left\|\sum_{n=0}^{\infty} e_n\right\|_p \leq \frac{1}{p}\left\|\sum_{n=0}^{\infty} E(e_n|\mathscr{F}_n)\right\|_p, \qquad p\leq 1. \tag{10}$$

Burkholder, Davis and Gundy [6] proved an inequality for convex functions that implies (9) with some constant γ_p. Garsia [9] showed that the best constant γ_p satisfies $\gamma_p \geq 1/p$. Burkholder [2] proved an inequality for concave functions that implies (10) with a constant γ_p and Garsia showed that $\gamma_p \leq 1/p$. We can prove that the best constant $\gamma_p = 1/p$ for all $p > 0$.

2. Proofs. The proofs of inequalities (5)–(10) are based upon the following elementary lemma.

LEMMA. *If x and d are nonnegative numbers and $y > 0$, then*

$$(y+d)^{p/2}\left(\frac{x+d}{y+d} - \frac{2}{p}\right) \leq y^{p/2}\left(\frac{x}{y} - \frac{2}{p}\right), \qquad 0 < p \leq 2,$$

$$(y+d)^{p/2}\left(\frac{2}{p} - \frac{x+d}{y+d}\right) \leq y^{p/2}\left(\frac{2}{p} - \frac{x}{y}\right), \qquad p \geq 2.$$

PROOF. We prove the case $0 < p \leq 2$ only. If $p \geq 2$, the proof is similar.

$$\begin{aligned}(y+d)^{p/2}&\left(\frac{x+d}{y+d} - \frac{2}{p}\right) - y^{p/2}\left(\frac{x}{y} - \frac{2}{p}\right)\\ &= x\left[(y+d)^{p/2-1} - y^{p/2-1}\right]\\ &\quad + \left\{d(y+d)^{p/2-1} - \frac{2}{p}\left[(y+d)^{p/2} - y^{p/2}\right]\right\}.\end{aligned}$$

Since $p \leq 2$, the first term is nonpositive, and the second term is nonpositive by the mean value theorem. □

To show (5), define

$$W(x,y) = \frac{p}{2}t^{p/2-1}(x^2 - t), \quad \text{where } t = \frac{2}{p}y.$$

By the above lemma, for $x, d \geq 0$ and $y > 0$, $W(\sqrt{x+d}, y+d) \leq W(\sqrt{x}, y)$. Then, by the mean value theorem, when $y > 0$,

$$|x|^p - \left(\frac{2}{p}y\right)^{p/2} \leq W(x,y). \tag{11}$$

Hence for $n \geq 1$ and $\delta > 0$, by (11)

$$\begin{aligned} E&\left(|f_{n+1}|^p - \left(\frac{2}{p}\left[s_{n+1}^2(f) + \delta^2\right]\right)^{p/2}\right) \\ &\leq EW\left(f_{n+1}, s_{n+1}^2(f) + \delta^2\right) \\ &= EW\left(f_n + d_{n+1}, \left[s_n^2(f) + \delta^2\right] + E\left(|d_{n+1}|^2|\mathscr{F}_n\right)\right) \\ &= E\left\{E\left(W\left(f_n + d_{n+1}, \left[s_n^2(f) + \delta^2\right] + E\left(|d_{n+1}|^2|\mathscr{F}_n\right)\right)\Big|\mathscr{F}_n\right)\right\} \\ &= EW\left(\left[f_n^2 + E\left(|d_{n+1}|^2|\mathscr{F}_n\right)\right]^{1/2}, \left[s_n^2(f) + \delta^2\right] + E|d_{n+1}|^2|\mathscr{F}_n\right)\right) \\ &\leq EW\left(f_n, s_n^2(f) + \delta^2\right). \end{aligned}$$

The last equality is from the fact $E(d_{n+1}|\mathscr{F}_n) = 0$ and the definition of $W(x, y)$, and the last inequality comes from the inequality preceding (11), setting $x = f_n^2$, $y = s_n^2 + \delta^2$, $d = E(|d_{n+1}|^2|\mathscr{F}_n)$. Repeating this argument n times, we have

$$(12)\quad E\left(|f_{n+1}|^p - \left(\frac{2}{p}\left[s_{n+1}^2(f) + \delta^2\right]\right)^{p/2}\right) \leq EW\left(f_1, s_1^2(f) + \delta^2\right) \leq 0.$$

This proves (5) by letting $\delta \to 0$.

Using elementary inequalities

$$\left(\frac{2}{p}y\right)^{p/2} - |x|^p \leq \frac{p}{2}\left(\frac{2}{p}y\right)^{p/2-1}\left(\frac{2}{p}y - x^2\right), \qquad p \geq 2,$$

$$x^p - \left(\frac{1}{p}y\right)^p \leq p\left(\frac{1}{p}y\right)^{p-1}\left(x - \frac{1}{p}y\right), \qquad 0 < p \leq 1,$$

$$\left(\frac{1}{p}y\right)^p - x^p \leq p\left(\frac{1}{p}y\right)^{p-1}\left(\frac{1}{p}y - x\right), \qquad p \geq 1,$$

and the lemma, we can similarly show (6) and (9) and (10). (6′) is a consequence of (6). Finally, (7) and (8) are obtained by letting $\{e_n\}_{n \geq 0} = \{|d_{n+1}|^2\}_{n \geq 0}$ in (9) and (10).

To see that inequality (5) is strict if $0 < p < 2$ and $0 < \|f\|_p < \infty$, we assume without loss of generality that $Es_1^2(f) = Ef_1^2 > 0$. Then as $\delta \to 0$, (12) yields

$$\begin{aligned} E\left(|f_{n+1}|^p - \left(\frac{2}{p}s_{n+1}^2(f)\right)^{p/2}\right) &\leq EW\left(f_1, s_1^2(f)\right) \\ &= E\left(\frac{p}{2}s_1^2(f)\right)^{p/2-1}\left(\frac{p}{2} - 1\right)Es_1^2(f) < 0. \end{aligned}$$

This shows

$$E\|f\|_p < \sqrt{\frac{2}{p}}\, E\|s(f)\|_p.$$

The strictness of the other inequalities follows similarly.

We now prove the constants in the inequalities (5)–(8) are sharp for real conditionally symmetric martingales. More precisely, we shall prove that if f is a real conditionally symmetric martingale and $\mathscr{F}_n = \sigma(f_1, \dots, f_n)$ for $n \geq 1$, then constants α_p and β_p in the inequalities (1)–(4) satisfy the following inequalities: $\alpha_p \geq \sqrt{2/p}$ and $\beta_p \geq \sqrt{2/p}$ when $0 < p \leq 2$; and $\alpha_p \leq \sqrt{2/p}$ and $\beta_p \leq \sqrt{2/p}$ when $p \geq 2$. We shall also show that the constant in the inequality

$$\alpha'_p\|s(f)\|_p \leq \|f^*\|_p, \qquad p \geq 2, \tag{13}$$

satisfies $\alpha'_p \leq \sqrt{2/p}$. Therefore, combining the first half of the proofs, we prove that the best possible constants α_p and β_p satisfy $\alpha_p = \alpha'_p = \beta_p = \sqrt{2/p}$. This will prove Theorem 1. Since (7) and (8) are consequences of (9) and (10), the sharpness of (7) and (8) implies the sharpness of (9) and (10). Hence, it will prove Theorem 2 as well.

Because all the proofs are similar, we prove only $\alpha_p \geq \sqrt{2/p}$ when $0 < p \leq 2$ and indicate how to show the rest without giving details.

Denote by μ the Lebesgue measure on R. Let $\mathscr{M}$ be the set of all simple real conditionally symmetric martingales $f = (f_n, \mathscr{F}_n)_{n \geq 1}$ in the probability space $([0,1), \mathscr{B}[0,1), \mu)$, where $\mathscr{B}[0,1)$ is the Borel σ-field on $[0,1)$. Moreover, we let $\{\mathscr{F}_n\}_{n\geq 1}$ be $\{\sigma(f_1, \dots, f_n)\}_{n \geq 1}$. Since f is a simple martingale, we can define $f_\infty = \lim_{n\to\infty} f_n$.

When $0 < p \leq 2$, let α_p be a constant such that

$$\|f\|_p \leq \alpha_p\|s(f)\|_p$$

for $f \in \mathscr{M}$. For $x \in R$, $y \geq 0$, define

$$U_1(x,y) = \sup_{f\in\mathscr{M}}\left\{E|f_\infty + x|_p^p - \alpha_p^p E|s^2(f) + y|^{p/2}\right\}.$$

Working from the definition, we can show (see Burkholder [3] and [4]) that $U_1(x,y)$ has the following property:

$$V_p(x,y) = |x|^p - \alpha_p^p y^{p/2} \leq U_1(x,y),$$

$$U_1(\lambda x, \lambda^2 y) = |\lambda|^p U_1(x,y),$$

$$\sum_{i=1}^n a_i\left(U_1\left(x + d_i, y + 2\sum_{i=1}^n a_i d_i^2\right) + U_1\left(x - d_i, y + 2\sum_{i=1}^n a_i d_i^2\right)\right) \leq U_1(x,y), \tag{14}$$

where n is any positive integer, $\sum_{i=1}^{n} a_i = 1/2$ and $a_i \geq 0$ for all i in the last inequality.

From the definition, we see

$$U_1(0,1) \leq 0. \tag{15}$$

Let $b > \alpha_p$. Take $n = 2$, $x = 0$, $y = 1/ab^2 - 1$, $a_1 = a/2$, $a_2 = (1-a)/2$, $d_1 = 1/\sqrt{a}$ and $d_2 = 0$, where $a \in (0, \min(1, 1/b^2))$. Then by (14),

$$\begin{aligned} &\frac{a}{2}\left\{U_1\left(\frac{1}{\sqrt{a}}, \frac{1}{ab^2}\right) + U_1\left(-\frac{1}{\sqrt{a}}, \frac{1}{ab^2}\right)\right\} \\ &\quad \leq U_1\left(0, \frac{1}{ab^2} - 1\right) - (1-a)U_1\left(0, \frac{1}{ab^2}\right). \end{aligned} \tag{16}$$

Also by (14),

$$U_1\left(\frac{1}{\sqrt{a}}, \frac{1}{ab^2}\right) = U_1\left(-\frac{1}{\sqrt{a}}, \frac{1}{ab^2}\right) \geq V_p\left(\frac{1}{\sqrt{a}}, \frac{1}{ab^2}\right) > 0$$

and $U_1(0, \lambda) = |\lambda|^{p/2} U_1(0,1)$. Thus (16) implies

$$\left[(1-ab^2)^{p/2} - (1-a)\right] U_1(0,1) > 0.$$

By (15), this means

$$g(a) = (1-ab^2)^{p/2} - (1-a) < 0 \tag{17}$$

for $a \in (0, \min(1, 1/b^2))$. Since $g(0) = 0$, then $g'(0) < 0$ or

$$-\frac{p}{2}b^2 + 1 < 0.$$

Thus $b > \sqrt{2/p}$, which implies $\alpha_p \geq \sqrt{2/p}$. This completes the proof.

To give a brief idea of how to show the rest, we define

$$U_2(x,t,y) = \sup_{f \in \mathscr{M}} \left\{\alpha_p'^p E(s^2(f) + y)^{p/2} - E(|f_\infty + x| \vee t)^p\right\}$$

on $x \in R$, $t \geq 0$ and $y \geq 0$, when $p \geq 2$;

$$U_3(x,y) = \sup_{f \in \mathscr{M}} \left\{E(S^2(f) + x)^{p/2} - \beta_p^p E(s^2(f) + y)^{p/2}\right\}$$

on $x \geq 0$ and $y \geq 0$, when $0 < p \leq 2$; and

$$U_3(x,y) = \sup_{f \in \mathscr{M}} \left\{\beta_p^p E(s^2(f) + y)^{p/2} - E(S^2(f) + x)^{p/2}\right\}$$

on $x \geq 0$ and $y \geq 0$, when $p \geq 2$.

Here $a \vee b = \max(a,b)$ and constants α_p' and β_p are those such that (3), (4) and (13) hold. Then we can prove $\alpha_p' \leq \sqrt{2/p}$ by working with function U_2. Similarly, inequalities $\beta_p \geq \sqrt{2/p}$ when $0 < p \leq 2$ and $\beta_p \leq \sqrt{2/p}$ when $p \geq 2$ can be proven by function U_3.

For example, function $U_2(x,t,y)$ satisfies

$$\alpha_p'^p y^{p/2} - |x \vee t|^p \le U_2(x,t,y),$$
$$U_2(\lambda x, \lambda t, \lambda^2 y) = |\lambda|^p U_2(x,t,y),$$
$$\sum_{i=1}^n a_i \left\{ U_2\left(x + d_i, t, y + 2\sum_{i=1}^n a_i d_i^2\right) + U_2\left(x - d_i, t, y + 2\sum_{i=1}^n a_i d_i^2\right)\right\} \le U_2(x,t,y)$$

if $\sum_{i=1}^n a_i = 1/2$, $a_i \ge 0$ for all i and $t \ge |x|$ in the last inequality.

By using the above properties and taking $n = 2$, $x = t = 0$, $y = 1/a\alpha_p'^2 - 1$, $a_1 = a/2$, $a_2 = (1-a)/2$, $d_1 = 1/\sqrt{a}$ and $d_2 = 0$, this implies, when a is small,

$$\left(1 - a\alpha_p'^2\right)^{p/2} - (1-a) \ge 0,$$

and hence $\alpha_p' \le \sqrt{2/p}$. Unlike the previous case, we can take $b = \alpha_p'$ since

$$U_2(0,0,1) > 0$$

from the definition.

REMARK 1. Burkholder used this method to get the lower bound of the constants for martingale transform inequalities. He also used this method to give a new proof of the sharpness of Doob's maximal inequality for martingales without using examples (see [5] for details).

REMARK 2. Let $\mathscr{M}'$ be the set of all simple martingales f in $([0,1), \mathscr{B}[0,1), \mu)$ and c_p be a constant such that if $f \in \mathscr{M}'$,

$$\|f\|_p \le c_p \|S(f)\|_p, \qquad p \ge 2. \tag{18}$$

Define on $x \in R$, $y \ge 0$,

$$U(x,y) = \sup_{f \in \mathscr{M}'} \left\{ E|f_\infty + x|^p - c_p^p E\left(S^2(f) + y\right)^{p/2}\right\}, \qquad p \ge 2.$$

We can show

$$\overline{V}_p(x,y) \le U(x,y),$$
$$U(\lambda x, \lambda^2 y) = |\lambda|^p U(x,y), \tag{19}$$
$$\sum_{i=1}^n a_i U(x + d_i, y + d_i^2) \le U(x,y)$$

if $\sum_{i=1}^n a_i d_i = 0$, $\sum_{i=1}^n a_i = 1$ and $a_i \ge 0$ for all i, where $\overline{V}_p(x,y) = |x|^p - c_p^p y^{p/2}$. By choosing $n = 2$, $d_1 = 2x/(x^2-1)$, $d_2 < 0$ such that $(x + d_2)^2 = c_p^2(1 + d_2^2)$, $y = 1$, $a_1 = -d_2/(d_1 - d_2)$, $a_2 = d_1/(d_1 - d_2)$ and $x > c_p$, (19) implies

$$c_p \geq p - 1$$

as $x \to \infty$. Combining this inequality with Burkholder's inequality [4], it shows that $p - 1$ is the best constant in (18) without giving examples. Burkholder in his original proof constructed examples to show $p - 1$ is the best.

We also can apply this same method to show without giving examples that there exists no constant c_p such that

$$\|f\|_p \leq c_p \|S(f)\|_p, \qquad 0 < p < 1,$$

for general martingales.

REMARK 3. Examples which show inequalities (5)–(10) are sharp can also be found. They come naturally from the second half of the proofs. For instance, let $0 < p < 2$. Take $1 > b' > b > \sqrt{p/2}$, and let $a_0 = 1$, $a_n = 1 - b^2/n$ for $n \geq 1$. On $[0, 1)$, define a conditionally symmetric martingale f with difference sequence $(d_n)_{n \geq 1}$ by $d_n = \sqrt{n}$ on $[\prod_{i=0}^{n} a_i, ((1 + a_n)/2)\prod_{i=0}^{n-1} a_i)$, $-\sqrt{n}$ on $[((1 + a_n)/2)\prod_{i=0}^{n-1} a_i, \prod_{i=0}^{n-1} a_i)$ and 0 elsewhere. Then when n is large enough, we can show

$$b' \|f_n\|_p > \|s_n(f)\|_p.$$

This proves by example that $\sqrt{2/p}$ is sharp in (5).

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