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CONVERGENCE OF SCALED RANDOM SAMPLES IN $\mathbb{R}^d$

BY K. KINOSHITA[1] AND SIDNEY I. RESNICK[2]

Let $\{\mathbf{X}_j,\ 1 \le j \le n\}$ be a sequence of iid random vectors in $\mathbb{R}^d$ and $S_n = \{\mathbf{X}_j/b_n,\ 1 \le j \le n\}$. When do there exist scaling constants $b_n \to \infty$ such that S_n converges to some compact set S in $\mathbb{R}^d$ almost surely (in probability)? We show that a limit set S is star-shaped (i.e., $\mathbf{x} \in S$ implies $t\mathbf{x} \in S$, for $0 \le t \le 1$) so that after a polar coordinate transformation the limit set is the hypograph of an upper semicontinuous function. We specify necessary and sufficient conditions for convergence to a particular limit set. Some examples are also given.

1. Introduction. Let $\{\mathbf{X}_n,\ n \ge 1\}$ be an independent and identically distributed (iid) sequence of random vectors in $\mathbb{R}^d$. For a (nonrandom) compact set S in $\mathbb{R}^d$, when does there exist a sequence of real constants $b_n \to \infty$ such that, as $n \to \infty$,

$$S_n := \{\mathbf{X}_n/b_n,\ 1 \le j \le n\} \to S \quad \text{a.s.}$$

with respect to the Hausdorff metric? What is the class of possible limit sets S? Note that, if $\{S_n\}$ has an almost sure limit, this limit by the Hewitt–Savage 0–1 law must be almost surely constant.

A related but more restricted problem was first considered by Fisher (1966, 1969), who assumed that $d = 2$ and that the components X_{n1} and X_{n2} of $\mathbf{X}_n$ are nonnegative, independent and identically distributed. In this setting, Fisher showed that limit sets were of the form

$$S = \{(x, y) \ge \mathbf{0}\colon x^\alpha + y^\alpha \le 1\} \quad \text{for some } 0 \le \alpha \le \infty,$$

and that a necessary and sufficient condition for convergence to such a set was regular variation of the function $-\log P\{X_{n1} > x\}$. This result is partially generalized by Davis, Mulrow and Resnick (1988), whose result is stated in Section 5. Eddy (1982) considers a related problem of limits in the sense of convergence in probability for scaled unions of iid random sets and derives a convergence criterion based on the inclusion functional.

In this paper, we give a complete solution to our problem. We characterize the class of limit sets S as being transforms of hypographs of upper semicontinuous functions and give necessary and sufficient conditions for convergence to a particular limit set.

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Section 2 discusses some preliminaries and Section 3 states some results on relative stability which will be needed in Section 4.

In Section 4 we first show that limit sets are star-shaped (i.e., $\mathbf{x} \in S$ implies $\varepsilon\mathbf{x} \in S$ for $0 \le \varepsilon \le 1$), which suggests that a polar coordinate transformation is a natural transformation, and we give necessary and sufficient conditions for almost sure convergence in the transformed coordinates. Our condition can be modified to cover convergence in probability as well.

In Section 5 we give a construction which shows that the class of limit sets in polar coordinates is the class of hypographs of upper semicontinuous functions on the unit sphere. We give some additional examples and show that the log-regular variation condition used in Davis, Mulrow and Resnick (1988) can be checked to be sufficient by showing that it implies our sufficient condition.

Based on our conditions, it is easy to construct examples of samples which converge in probability but not almost surely, something not seen in the literature until now.

2. Preliminaries. Let $\mathscr{F}_d$ be the closed subsets of $\mathbb{R}^d$ topologized by the usual topology as discussed in Matheron (1975) or Vervaat (1988). Let $\mathscr{K}_d$ denote the collection of nonempty compact subsets of $\mathbb{R}^d$ and $C(\mathscr{K}_d)$ the collection of nonempty convex compact subsets of $\mathbb{R}^d$. Suppose that d is a metric on $\mathbb{R}^d$ and, for S, $T \in \mathscr{K}_d$, define the Hausdorff metric [Matheron (1975)] $D: \mathscr{K}_d \times \mathscr{K}_d \to \mathbb{R}_+$ by

$$D(S,T) = \inf\{\delta \colon S \subset T^\delta, T \subset S^\delta\},$$

where

$$S^\delta = \{x \colon d(x,y) < \delta \text{ for some } y \in S\}.$$

The following convergence criteria are useful [cf. Matheron (1975), page 6, Theorem 1-2-2].

LEMMA 2.1. *For $F_n, F \in \mathscr{F}_d$, $n \ge 1$, $F_n \to F$ as $n \to \infty$ if and only if the following two conditions hold.*

(2.1) *For any $\mathbf{y} \in F$, there exists, for all large n, $\mathbf{y}_n \in F_n$ such that $d(\mathbf{y}_n, \mathbf{y}) \to 0$ as $n \to \infty$.*

(2.2) *For any subsequence $\{n_k\}$, if $\mathbf{y}_{n_k} \in F_{n_k}$ converges, then $\lim_{k\to\infty} \mathbf{y}_{n_k} \in F$.*

Furthermore, convergence of sets $S_n \to S$ in $\mathscr{K}_d$ is equivalent to the analogues of (2.1) *and* (2.2) *holding as well as $\sup_{n\ge1} \sup\{\|\mathbf{x}\| \colon \mathbf{x} \in S_n\} < \infty$ for some norm $\|\cdot\|$ on $\mathbb{R}^d$.*

Note that if the sets S_n are random elements of $\mathscr{K}_d$ and $S \in \mathscr{K}_d$ is nonrandom, then Lemma 2.1 can be used to characterize almost sure convergence or convergence in probability.

A measurable function U: $\mathbb{R}_+ \to \mathbb{R}_+$ is called regularly varying at ∞ with index $\rho \in [-\infty, \infty]$ [written $U \in RV(\rho)$] if, for $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho}.$$

(Interpret x^{∞} by $x^{\infty} = 0$ if $x < 1$, $x^{\infty} = 1$ if $x = 1$ and $x^{\infty} = \infty$ if $x > 1$. A similar interpretation applies to $x^{-\infty}$.) We call ρ the exponent of variation. If $\rho = 0$, U is slowly varying. A frequently used property of regular varying functions [cf. Resnick (1987), page 23, Proposition 0.8(v)] is that if U is nondecreasing, $U(\infty) = \infty$ and $U \in RV(\rho)$, $0 \le \rho \le \infty$, then

$$U^{\leftarrow} \in RV(\rho^{-1}), \tag{2.3}$$

where $U^{\leftarrow}(y) = \inf\{x\colon U(x) \ge y\}$.

Let $\{\mathbf{X}_n, n \ge 1\}$ be iid random variables with common distribution function $F(\cdot)$, $F(x) < 1$ for all x, and set $M_n = \bigvee_{j=1}^{n} X_j$. The sequence $\{M_n\}$ is almost surely relatively stable (relatively stable in probability) if there exist constants b_n such that

$$\frac{M_n}{b_n} \to 1 \quad \text{almost surely (in probability) as } n \to \infty.$$

It is known that if $\{M_n\}$ is relatively stable in probability then

$$b_n \sim F^{\leftarrow}(1 - 1/n)$$

[cf. Resnick and Tomkins (1973)]. Necessary and sufficient conditions for relative stability are given as follows [cf. Barndorff-Nielson (1963), Resnick and Tomkins (1973), Gnedenko (1943) and de Haan (1970)].

LEMMA 2.2. *$\{M_n\}$ is a.s. relatively stable if and only if*

$$\int_1^{\infty} \frac{dF(x)}{1 - F(\varepsilon x)} < \infty, \tag{2.4}$$

for all $0 < \varepsilon < 1$. $\{M_n\}$ is relatively stable in probability if and only if

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\infty}. \tag{2.5}$$

The normal and exponential distributions satisfy (2.4) and M_n is a.s. relatively stable. The Cauchy and Pareto distributions do not satisfy (2.5). An interesting example is when F is given by

$$-\log(1 - F(x)) = (\log l)^{-1} \log x \log\log x \quad \text{for } x > e,\ l > 1 \tag{2.6}$$

[Resnick and Tomkins (1973)]. This F satisfies (2.5) but not (2.4) and hence $\{M_n\}$ is relatively stable in probability but not a.s. relatively stable. (For this example, in fact, $\limsup_{n\to\infty} M_n/b_n = l$ a.s.)

Note that if $\{M_n\}$ is relatively stable in probability, then it follows from (2.5) that $1/(1-F) \in RV(\infty)$. Since $b_n \sim F^{\leftarrow}(1-1/n) = (1/(1-F))^{\leftarrow}(n)$, we have from (2.3) that

$$\left(\frac{1}{1-F}\right)^{\leftarrow} \in RV(0) \tag{2.7}$$

and, furthermore [Resnick (1987), page 17, Proposition 0.5],

$$\lim_{n\to\infty} \frac{b_{[nt]}}{b_n} = 1 \quad \text{locally uniformly on } (0,\infty). \tag{2.8}$$

For a nonnegative function f defined on a topological space E, we define the hypograph of f by

$$\text{hypo}(f) := \{(t,x) \in E \times (0,\infty): 0 < x \le f(t)\}.$$

[$E \times (0,\infty)$ has the relative topology inherited from $E \times [0,\infty)$.] We call f upper semicontinuous (USC) if, for all $t \in E$,

$$f(t) = \bigwedge_{G \in \mathscr{G}_t} \bigvee_{u \in G} f(u),$$

where $\mathscr{G}_t$ is the collection of all open sets containing t. In practice, the collection $\mathscr{G}_t$ can be thinned considerably. The following lemma provides characterizations of USC functions and hypographs.

LEMMA 2.3 [Vervaat (1988)]. *The following are equivalent*:

(i) $f \in \text{USC}$;
(ii) $\text{hypo}(f)$ *is closed in* $E \times (0,\infty)$;
(iii) $f^{\leftarrow}[0,x) = \{t \in E,\ f(t) < x\}$ *is open in E for all* $0 < x \le 1$.

If there exists some metric d on E, then f is USC iff $\limsup_{n\to\infty} f(t_n) \le f(t)$ *for all* $t_n, t \in E$ *such that* $d(t_n,t) \to 0$ *as* $n \to \infty$.

3. Further results on relative stability. Throughout this section, we assume that $\{X_n,\ n \ge 1\}$ and $\{X_n^{(i)},\ n \ge 1\}$ are sequences of iid random variables from common d.f.'s F and F_i, $i = 1,2$, respectively, and that $\{X_n^{(1)},\ n \ge 1\}$ and $\{X_n^{(2)},\ n \ge 1\}$ are independent. For convenience, we also assume that all the d.f.'s have support $\mathbb{R}_+$. Set

$$M_n = \bigvee_{j=1}^{n} X_j, \qquad M_n^{(i)} = \bigvee_{j=1}^{n} X_j^{(i)}.$$

Whether or not relative stability is assumed, we always define

$$b_n = F^{\leftarrow}(1-1/n), \qquad b_n^{(i)} = F_i^{\leftarrow}(1-1/n), \qquad i = 1,2.$$

Finally, define $\overline{F} = 1 - F$, $\overline{F}_i = 1 - F_i$.

We first show that relative stability in $\mathbb{R}$ is equivalent to the scaled sample converging to the unit interval.

PROPOSITION 3.1. *M_n is a.s. relatively stable if and only if*

$$S_n := \left\{ \frac{X_j}{b_n}, 1 \le j \le n \right\} \to [0,1] =: S$$

a.s. with respect to the Hausdorff metric on $\mathscr{K}_d$.

PROOF. We first show that if $\{M_n\}$ is a.s. relatively stable, then for any $p \in [0,1]$ there exists a sequence $k_n = k_n(p)$ such that $1 \le k_n \le n$ and

$$\frac{M_{k_n}}{b_n} \to p \quad \text{a.s.} \tag{3.1}$$

If $p = 0$, then (3.1) trivially holds with $k_n = 1$, so assume $p > 0$. Since

$$\frac{M_{k_n}}{b_n} = \frac{M_{k_n}}{b_{k_n}} \frac{b_{k_n}}{b_n},$$

it suffices to show that there exists a sequence $k_n \to \infty$ such that

$$\frac{b_{k_n}}{b_n} \to p \quad \text{as } n \to \infty. \tag{3.2}$$

This follows in a standard way from the fact that $\{b_n\}$ is a slowly varying sequence [Loeve (1977), page 335].

Now we are ready to prove Proposition 3.1. The sufficiency is obvious, so assume $\{M_n\}$ is a.s. relatively stable. To prove $S_n \to S$, it suffices to show (2.1) and (2.2). Pick $y \in S$ and it follows from (3.1) that there exists $k_n \to \infty$ such that $M_{k_n}/b_n \to y$ a.s. as $n \to \infty$. Since $M_{k_n}/b_n \in S_n$, (2.1) is satisfied.

Now suppose there exists $y_{n_j} \in S_{n_j}$ such that $y_{n_j} \to y_0$ a.s. and $y_0 > 1$. Then

$$\frac{M_{n_j}}{b_{n_j}} = \bigvee_{i=1}^{n_j} \frac{X_i}{b_{n_j}} \ge_{\text{a.s.}} y_{n_j} \to_{\text{a.s.}} y_0 > 1 \quad \text{as } n \to \infty,$$

which violates relative stability, so that (2.2) follows. □

We now discuss preservation of relative stability under asymptotic equivalence. Recall $b_n^{(i)} = F_i^{\leftarrow}(1 - n^{-1})$, $i = 1, 2$.

PROPOSITION 3.2. *Suppose $\{M_n^{(1)}\}$ is a.s. relatively stable and $l b_n^{(1)} \sim b_n^{(2)}$ for some $l > 0$. Then $\{M_n^{(2)}\}$ is also a.s. relatively stable.*

PROOF. We need to show, for any convenient $A > 0$,

$$\int_A^\infty \frac{F_2(dx)}{\overline{F}_2(\varepsilon x)} = \int_{F_2(A)}^1 \frac{ds}{\overline{F}_2(\varepsilon F_2^{\leftarrow}(s))} < \infty,$$

for $0 < \varepsilon < 1$. Since by assumption $F_2^{\leftarrow}(s)/F_1^{\leftarrow}(s) \to l$, $s \to 1$, it suffices to prove

$$\int_{A_1}^{1} \frac{ds}{\overline{F}_2(\varepsilon(l+\varepsilon_1)F_1^{\leftarrow}(s))} = \int_{A_2}^{\infty} \frac{F_1(dx)}{\overline{F}_2(\varepsilon(l+\varepsilon_1)x)} < \infty, \tag{3.3}$$

for $0 \le A_1 \le 1$, $0 \le A_2 < \infty$ and $\varepsilon_1 > 0$ arbitrarily small. From the integration by parts formula

$$\int \frac{dF_1(y)}{\overline{F}_2(ky)} = \int \frac{\overline{F}_1(y/k)}{\overline{F}_2^2(y)}\, dF_2(y), \tag{3.4}$$

a change of variable and a second use of $F_2^{\leftarrow}(s)/F_1^{\leftarrow}(s) \to l$, we get that finiteness of the integral on the right-hand side of (3.3) is equivalent to

$$\int_0^1 \overline{F}_1\left(\frac{(l-\varepsilon_2)}{\varepsilon(l+\varepsilon_1)} F_1^{\leftarrow}(s)\right) \frac{ds}{(1-s)^2} = \int_{F_1^{\leftarrow}(0)}^{\infty} \overline{F}_1\left(\frac{(l-\varepsilon_2)x}{\varepsilon(l+\varepsilon_1)}\right) \frac{dF_1(x)}{\overline{F}_1^2(x)} < \infty,$$

where $\varepsilon_2 > 0$ can be chosen arbitrarily small. Since $0 < \varepsilon < 1$ and $\varepsilon_1, \varepsilon_2$ can be chosen arbitrarily small, we may assume $C := (l - \varepsilon_2)/(\varepsilon(l + \varepsilon_1)) > 1$ and from the integration by parts formula (3.4) the finiteness follows from the assumed stability of F_1. □

The next sequence of comparison results is needed in Section 4.

LEMMA 3.3. *$M_n^{(1)}/M_n^{(2)} \to 0$ a.s. as $n \to \infty$ if and only if, for any $t > 0$,*

$$\int_0^{\infty} \frac{dF_1(x)}{1 - F_2(tx)} < \infty. \tag{3.5}$$

PROOF. For $\varepsilon > 0$, let $E_n = \{X_n^{(1)}/M_n^{(2)} > \varepsilon\}$. Suppose $M_n^{(1)}/M_n^{(2)} \to 0$ a.s. Then $P(E_n \text{ i.o.}) = 0$ and, using the independence of $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$, we have, for $m < n$,

$$\begin{aligned} P(E_m E_n) &= P\left\{\frac{X_m^{(1)}}{M_m^{(2)}} > \varepsilon, \frac{X_n^{(1)}}{M_n^{(2)}} > \varepsilon\right\} \\ &\le P\left\{\frac{X_m^{(1)}}{M_m^{(2)}} > \varepsilon, \frac{X_n^{(1)}}{\bigvee_{j=m+1}^{n} X_j^{(2)}} > \varepsilon\right\} \\ &= P\left\{\frac{X_m^{(1)}}{M_m^{(2)}} > \varepsilon\right\} P\left\{\frac{X_n^{(1)}}{\bigvee_{j=m+1}^{n} X_j^{(2)}} > \varepsilon\right\} \\ &= P(E_m)P(E_{n-m}) \end{aligned}$$

and, from an extension of the Borel–Cantelli lemma [Spitzer (1964)], we deduce that

$$\infty > \sum_n PE_n = \sum_n \int_0^{\infty} F_2^n(\varepsilon^{-1}x)\, dF_1(x) = \int_0^{\infty} \frac{dF_1(x)}{1 - F_2(\varepsilon^{-1}x)}$$

and we have the desired result.

Conversely, if (3.5) holds, then $\sum P(E_n) < \infty$ and it follows from the Borel–Cantelli lemma that $P(E_n \text{ i.o.}) = 0$, which implies

$$\frac{X_n^{(1)}}{M_n^{(2)}} \to 0 \quad \text{a.s. as } n \to \infty.$$

Define $L_n^{(1)} = \inf\{k \le n;\ X_k^{(1)} = \bigvee_{j=1}^{n} X_j^{(1)}\}$. Since $L_n^{(1)} \to \infty$ a.s., it follows from the preceding relation that

$$\frac{M_n^{(1)}}{M_n^{(2)}} = \frac{X_{L_n^{(1)}}^{(1)}}{M_{L_n^{(1)}}^{(2)}} \frac{M_{L_n^{(1)}}^{(2)}}{M_n^{(2)}} \le \frac{X_{L_n^{(1)}}^{(1)}}{M_{L_n^{(1)}}^{(2)}} \to 0 \quad \text{a.s.} \qquad \square$$

PROPOSITION 3.4. *Suppose $M_n^{(2)}$ is a.s. relatively stable. Then $M_n^{(1)}/b_n^{(2)} \to 0$ a.s. if and only if $b_n^{(1)}/b_n^{(2)} \to 0$ as $n \to \infty$.*

PROOF. If $M_n^{(1)}/b_n^{(2)} \to 0$ a.s., then, for $\delta > 0$, $P\{M_n^{(1)}/b_n^{(2)} < \delta\} \to 1$, that is

$$F_1^n\big(\delta b_n^{(2)}\big) \to 1 \quad \text{as } n \to \infty.$$

Taking logarithms on both sides and using the relation $\log(1-u) \sim -u$ as $u \downarrow 0$, we have

$$n\big(1 - F_1(\delta b_n^{(2)})\big) \to 0,$$

which is equivalent to

$$\frac{1 - F_1\big(\delta b_n^{(2)}\big)}{1 - F_2\big(b_n^{(2)}\big)} \to 0 \quad \text{as } n \to \infty.$$

This implies that, for any $\varepsilon > 0$,

$$\frac{1}{1 - F_2(u)} \le \frac{\varepsilon}{1 - F_1(\delta u)} \quad \text{for } u > u_0(\varepsilon).$$

Taking inverses we get

$$\frac{b^{(1)}(t/\varepsilon)}{b^{(2)}(t)} \le \delta \quad \text{for } t \ge t_0(\varepsilon). \tag{3.6}$$

Since $b^{(2)}(\cdot)$ is slowly varying, $b^{(2)}(t) \sim b^{(2)}(t/\varepsilon)$, and we get from (3.6) that

$$\limsup_{t \to \infty} \frac{b^{(1)}(t)}{b^{(2)}(t)} \le \delta.$$

Since $\delta > 0$ is arbitrary, the result follows.

Conversely, assume $b_n^{(1)}/b_n^{(2)} \to 0$ as $n \to \infty$. Then, for $\delta > 0$, we have

$$0 < \frac{b_n^{(1)}}{b_n^{(2)}} < \delta \quad \text{for } n \ge n_0(\delta). \tag{3.7}$$

For the desired result, by Lemma 3.3 it suffices to show that, for $t > 0$,

$$J_1 := \int_0^\infty \frac{dF_1(y)}{1 - F_2(ty)} = \int_0^1 \frac{ds}{1 - F_2(tF_1^{\leftarrow}(s))} < \infty.$$

Because of (3.7), $J_1 < \infty$ if

$$J_2 := \int_0^1 \frac{ds}{1 - F_2(t\delta F_2^{\leftarrow}(s))} < \infty.$$

However, $J_2 < \infty$ because we assume $\{M_n^{(2)}\}$ is a.s. relatively stable. □

The following proposition translates asymptotic equivalence of inverse functions into a more readily verified condition on distribution tails.

PROPOSITION 3.5. *Suppose $\{M_n^{(2)}\}$ is relatively stable in probability. Then*

$$\frac{b_n^{(1)}}{b_n^{(2)}} \to l \geq 0 \quad \text{as } n \to \infty$$

if and only if

$$\lim_{t\to\infty} \frac{1 - F_1(tx)}{1 - F_2(t)} = \begin{cases} 0, & \text{if } x > l, \\ \infty, & \text{if } x < l. \end{cases} \tag{3.8}$$

PROOF. We concentrate on the case $l > 0$; the case when $l = 0$ will be clear. Suppose $b_n^{(1)}/b_n^{(2)} \to l > 0$. Then, by the relative stability of $M_n^{(2)}$, we have

$$\frac{b_n^{(1)}}{M_n^{(2)}} = \frac{b_n^{(2)}}{M_n^{(2)}} \frac{b_n^{(1)}}{b_n^{(2)}} \to_P 1l. \tag{3.9}$$

Thus, for $\varepsilon > 0$,

$$P\left\{ \frac{b_n^{(1)}}{M_n^{(2)}} > l + \varepsilon \right\} = F_2^n\left[\frac{b_n^{(1)}}{l + \varepsilon} \right] \to 0 \quad \text{as } n \to \infty.$$

Remembering that $\log(1 - x) \sim -x$ as $x \downarrow 0$, we have from the preceding that

$$n\left(1 - F_2\left(\frac{b_n^{(1)}}{l + \varepsilon} \right)\right) \to \infty \quad \text{as } n \to \infty,$$

but this is equivalent to

$$\frac{1 - F_2(b_n^{(1)}/(l + \varepsilon))}{1 - F_1(b_n^{(1)})} \to \infty \quad \text{as } n \to \infty,$$

which gives one half of (3.8). The other half is similar.

To show the converse, we use the following lemma from Resnick (1971): For $0 \le L \le \infty$, we have

$$\lim_{t\to\infty} \frac{1 - F_1(t)}{1 - F_2(t)} = L$$

if and only if

$$\lim_{n\to\infty} \int_0^\infty F_1^n(x)\, dF_2^n(x) = \frac{1}{1+L}. \tag{3.10}$$

If (3.8) holds, then

$$\lim_{t\to\infty} \frac{1 - F_2(t/(l+\varepsilon))}{1 - F_1(T)} = \infty,$$

whence, from (3.10),

$$\lim_{n\to\infty} \int_0^\infty F_2^n\left(\frac{x}{l+\varepsilon}\right) dF_1^n(x) = 0.$$

Since $M_n^{(1)}$ and $M_n^{(2)}$ are independent, the preceding is the same as

$$\lim_{n\to\infty} P\left\{\frac{M_n^{(1)}}{M_n^{(2)}} > l + \varepsilon\right\} = 0. \tag{3.11}$$

A reverse inequality follows in the same way, yielding $M_n^{(1)}/M_n^{(2)} \to_P l$. Noting that

$$\frac{M_n^{(1)}}{M_n^{(2)}} = \frac{M_n^{(1)}}{b_n^{(2)}} \frac{b_n^{(2)}}{M_n^{(2)}}$$

and $M_n^{(2)}$ is relatively stable, we have that $M_n^{(1)}/b_n^{(2)} \to_P l$. Thus $M_n^{(1)}$ is relatively stable in probability and therefore $b_n^{(1)} \sim b_n^{(2)} l$. □

4. Limit sets of scaled random samples. From now on, we assume $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is a sequence of iid random vectors in $\mathbb{R}^d$. When do there exist scaling constants $b_n > 0$ such that the sequence of random sets $S_n := \{\mathbf{X}_j/b_n, 1 \le j \le n\}$ converges to some nonempty compact set S in $\mathbb{R}^d$? Since $S = \{\mathbf{0}\}$ is a degenerate case (the sequence $\{b_n\}$ grows too fast), we exclude this from consideration. Thus $S \ne \{\mathbf{0}\}$ and, without loss of generality, we may assume

$$\sup_{\mathbf{x}\in S} \|\mathbf{x}\| = 1, \tag{4.1}$$

where $\|\mathbf{x}\|$ is a norm on $\mathbb{R}^d$; any convenient norm suffices for our purposes. The relation between the scaling constants $\{b_n\}$ used in the definition of S_n and those used to scale maxima in Section 3 will soon become apparent.

We say that a set $S \subset \mathbb{R}^d$ is star-shaped if $\mathbf{x} \in S$ implies $\varepsilon\mathbf{x} \in S$, for any $0 \le \varepsilon < 1$. We will show later that any possible limit set is always star-shaped. This is an important property of limit sets and suggests that our problem is better understood in polar coordinates rather than in the usual Cartesian coordinates. The relevance of star-shaped sets is noted in Eddy (1982) as well.

PROPOSITION 4.1. *Let $\{\mathbf{X}_j, j \geq 1\}$ be iid in $\mathbb{R}^d$. If there exist $b_n \to \infty$ and a nonempty compact set S such that*

$$S_n := \left\{ \frac{\mathbf{X}_j}{b_n}, j \leq n \right\} \to S$$

then

(i) $\{M_n\} := \{\bigvee_{i=1}^{n} \|\mathbf{X}_i\|\}$ *is a.s. relatively stable and*
(ii) S *is star-shaped.*

PROOF. (i) Since $S_n \subset S^\varepsilon$ for large n we get eventually

$$\bigvee_{j=1}^{n} \|b_n^{-1}\mathbf{X}_j\| = \frac{M_n}{b_n} \leq \sup_{\mathbf{x}\in S} \|\mathbf{x}\| + \delta(\varepsilon) = 1 + \delta(\varepsilon),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. A reverse inequality comes from the fact that eventually $S \subset S_n^\varepsilon$.

(ii) Pick $s \in S$. Since $S_n \to S$, there exist $l(n) \leq n$ with $\mathbf{X}_{l(n)}/b_n \to s$. From (3.2) there exists $\{k_n\}$ such that, for given ε, $0 < \varepsilon < 1$, we have as $n \to \infty$

$$\frac{b_{k_n}}{b_n} \to \varepsilon$$

and $k_n \to \infty$ since $\varepsilon > 0$. Therefore,

$$S_n \ni \frac{\mathbf{X}_{l(k_n)}}{b_n} = \frac{\mathbf{X}_{l(k_n)}}{b_{k_n}} \frac{b_{k_n}}{b_n} \to \varepsilon s,$$

which by (2.2) shows $\varepsilon s \in S$. Hence S is star-shaped. □

In order to pass to more convenient representations of our sets, we need the following result, which is a variant of Proposition 3.18 in Resnick (1987).

Consistent with previous notation, let $\mathscr{F}(E)$ be the closed subsets of a topological space E and $\mathscr{K}(E)$ be the compact subsets.

PROPOSITION 4.2. *Let E_1, E_2 be locally compact with countable bases and suppose $T: E_1 \to E_2$ is continuous. For $S \subset E_1$, define $T^*S = \{T(s), s \in S\} \subset E_2$.*

(i) *If*

$$T^{-1}(\mathscr{K}(E_2)) \subset \mathscr{K}(E_1), \tag{4.2}$$

then $T^*: \mathscr{K}(E_1) \to \mathscr{K}(E_2)$ is continuous: If $S_n \in \mathscr{K}(E_1)$, $1 \leq n \leq \infty$, and $S_n \to S_\infty$ in $\mathscr{K}(E_1)$, then $T^*S_n \to T^*S_\infty$ in $\mathscr{K}(E_2)$.

(ii) *If T is a homeomorphism, then T^*: $\mathscr{F}(E_1) \to \mathscr{F}(E_2)$ is continuous*: *If $S_n \in \mathscr{F}(E_1)$, $1 \le n \le \infty$, and $S_n \to S_\infty$ in $\mathscr{F}(E_1)$, then $T^*S_n \to T^*S_\infty$ in $\mathscr{F}(E_2)$.*

REMARK. (4.2) holds if T is a homeomorphism. Variant formulations are possible. For instance, if the domain D of T is compact, T is continuous and $S_n \subset D$, $1 \le n \le \infty$, then the conclusion holds since (4.2) is automatic.

PROOF OF PROPOSITION 4.2. We only check (i), as (ii) is almost the same. One can proceed by Lemma 2.1 or alternatively by showing inverse images of open sets are open. Basis sets in $\mathscr{K}(E_2)$ are of the form ($K_2 \in \mathscr{K}(E_2)$, G_2 open in E_2)

$$\{F_2 \in \mathscr{K}(E_2): F_2 \cap K_2 = \varnothing\}, \qquad \{F_2 \in \mathscr{K}(E_2): F_2 \cap G_2 \ne \varnothing\}.$$

Now

$$(T^*)^{-1}\{F_2 \in \mathscr{K}(E_2): F_2 \cap K_2 = \varnothing\} = \{F_1 \in \mathscr{K}(E_1): T^*F_1 \cap K_2 = \varnothing\}.$$

Since $K_2 = T^*(T^{-1}K_2)$ and, from (4.2), $T^{-1}K_2 \in \mathscr{K}(E_1)$, we have this inverse image equal to

$$\{F_1 \in \mathscr{K}(E_1): T^*(F_1 \cap T^{-1}K_2) = \varnothing\} = \{F_1 \in \mathscr{K}(E_1): F_1 \cap T^{-1}K_2 = \varnothing\},$$

which is open in $\mathscr{K}(E_1)$. Likewise,

$$(T^*)^{-1}\{F_2 \in \mathscr{K}(E_2): F_2 \cap G_2 \ne \varnothing\} = \{F_1 \in \mathscr{K}(E_1): F_1 \cap T^{-1}G_2 \ne \varnothing\},$$

which is open in $\mathscr{K}(E_1)$ since $T^{-1}G_2$ is open in E_1 as a consequence of the continuity of T. □

Pick a convenient norm $\|\mathbf{x}\|$ on $\mathbb{R}^d$ and let $\Theta = \{\mathbf{x} \in \mathbb{R}^d: \|\mathbf{x}\| = 1\}$ be the unit sphere in $\mathbb{R}^d$ with respect to this norm, and define a polar coordinate transformation T: $\mathbb{R}^d \setminus \{\mathbf{0}\} \to \Theta \times (0, \infty)$ by

$$\begin{aligned} T\mathbf{x} &= (\boldsymbol{\theta}, r) = (\boldsymbol{\theta}(\mathbf{x}), r(\mathbf{x})) \\ &= (\mathbf{x}/\|\mathbf{x}\|, \|\mathbf{x}\|) \end{aligned}$$

for $\mathbf{x} \ne \mathbf{0}$. Sometimes we write $T\mathbf{x} = \mathbf{x}^*$. If T is applied to the random vector $\mathbf{X}_i \ne \mathbf{0}$ we write $T(\mathbf{X}_i) = (\boldsymbol{\theta}_i, \|\mathbf{X}_i\|)$. Note that, for $b > 0$, $\mathbf{x} \ne \mathbf{0}$,

$$T\left(\frac{\mathbf{x}}{b}\right) = \left(\frac{\mathbf{x}/b}{\|\mathbf{x}/b\|}, \frac{\|\mathbf{x}\|}{b}\right) = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \frac{\|\mathbf{x}\|}{b}\right) \tag{4.3}$$

and this suggests our problem is simpler if phrased in polar coordinates. The fact that T is not defined at $\mathbf{0}$ necessitates working in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ with the relative topology specified by

$$\mathscr{F}(\mathbb{R}^d \setminus \{\mathbf{0}\}) = \{F^\# \cap \{\mathbf{0}\}^c: F^\# \in \mathscr{F}(\mathbb{R}^d)\}.$$

T: $\mathbb{R}^d \setminus \{\mathbf{0}\} \to \Theta \times (0, \infty)$ is a homeomorphism provided we specify $\Theta \times (0, \infty)$

has the product topology and

$$\mathscr{F}(0,\infty) = \{F^{\#} \cap (0,\infty) : F^{\#} \in \mathscr{F}([0,\infty))\}.$$

LEMMA 4.3. *Suppose* $\{\mathbf{X}_j, j \geq 1\}$ *are iid in* $\mathbb{R}^d$. *If* $S \in \mathscr{K}_d$ *is star-shaped and* $S_n \in \mathscr{K}_d$, *then*

$$S_n \to S \quad in\ \mathscr{K}_d$$

if and only if

$$S_n \cap \{\mathbf{0}\}^c \to S \cap \{\mathbf{0}\}^c \quad in\ \mathscr{F}(\mathbb{R}^d \setminus \{\mathbf{0}\})$$

and

$$\sup_n \sup\{\|\mathbf{s}\| : \mathbf{s} \in S_n\} < \infty.$$

PROOF. Let $S_n \to S$. If $s \in S \cap \{\mathbf{0}\}^c$, then there exist $s_n \in S_n$, $s_n \to s$. Eventually $s_n \neq 0$, so $s_n \in S_n \cap \{\mathbf{0}\}^c$ and (2.1) holds. If $s_{n'} \in S_{n'} \cap \{\mathbf{0}\}^c$ and $\lim_{n' \to \infty} s_{n'} = s_\infty$ exists in $\mathbb{R}^d \setminus \{\mathbf{0}\}$, then $s_\infty \neq 0$, $s_\infty \in S$, so $s_\infty \in S \cap \{\mathbf{0}\}^c$ as required for (2.2).

Conversely, suppose $S_n \cap \{\mathbf{0}\}^c \to S \cap \{\mathbf{0}\}^c$. If $s \neq \mathbf{0}$ there exist $s_n \neq \mathbf{0}$, $s_n \in S_n$, $s_n \to s$ as required by (2.1). If $s = 0$, pick $s^{\#} \in S$. Since S is star-shaped, for every positive integer k there exist $s_n^{(k)} \in S_n \cap \{\mathbf{0}\}^c$ such that $s_n^{(k)} \to s^{\#}/k$. There exists a function $f(j) \to \infty$ as $j \to \infty$ with

We have $s_j^{(f(j))} \in S_j$ and

$$d\left(s_j^{(f(j))}, s^{\#}/f(j)\right) < 1/f(j).$$

$$\begin{aligned} d\left(s_j^{(f(j))}, \mathbf{0}\right) &\leq d\left(s_j^{(f(j))}, s^{\#}/f(j)\right) + d\left(s^{\#}/f(j), \mathbf{0}\right) \\ &\leq 1/f(j) + d\left(s^{\#}/f(j), \mathbf{0}\right) \to 0. \end{aligned}$$

So (2.1) is satisfied in this case as well.

Suppose $s_{n'} \in S_{n'}$ and $\lim_{n' \to \infty} s_{n'} = s_\infty$ exist. If $s_\infty = 0$, then (2.2) is satisfied since $\mathbf{0}$ is in the closed, star-shaped set S. If $s_\infty \neq \mathbf{0}$, then, since we assume $S_n \cap \{\mathbf{0}\}^c \to S \cap \{\mathbf{0}\}^c$, we get from (2.2) that $s_\infty \in S \cap \{\mathbf{0}\}^c \subset S$ as required. □

As before, let T be the polar coordinate transformation. Define

$$T^* : \mathscr{F}(\mathbb{R}^d \setminus \{\mathbf{0}\}) \to \mathscr{F}(\Theta \times (0,\infty))$$

by

$$T^* S = \{Ts, s \in S, s \neq 0\} = S^*.$$

We now show S is a limit set for a scaled random sample if and only if after a polar coordinate transformation $S^* = \text{hypo}\, l$, where $l \in \text{USC}$. This characterizes the class of limit sets.

THEOREM 4.4. *There exists* $b_n \to \infty$ *such that in* $\mathscr{K}_d$

$$S_n = \{\mathbf{X}_i / b_n, 1 \leq i \leq n\} \to S,$$

with S compact and $\bigvee_{\mathbf{x}\in S}\|\mathbf{x}\| = 1$, if and only if there exists $l\colon \Theta \to (0,\infty)$ with l USC and $\bigvee_{\boldsymbol{\theta}\in\Theta} l(\boldsymbol{\theta}) = 1$ and in $\mathscr{F}(\Theta \times (0,\infty))$,

$$S_n^* = \{(\boldsymbol{\theta}_i, \|\mathbf{X}_i\|/b_n) : 1 \le i \le n, \|\mathbf{X}_i\| \ne 0\}$$
$$\to \text{hypo}(l) = S^* = \{(\boldsymbol{\theta}(\mathbf{x}), r(\mathbf{x})) : \mathbf{x} \in S, \mathbf{x} \ne \mathbf{0}\}.$$

Furthermore, for any $S \in \mathscr{K}_d$ such that $T^*S = \text{hypo}\, l$ for some $l \in USC$, there exists an iid sequence $\{\mathbf{X}_j, j \ge 1\}$ such that, for appropriate scaling constants $b_n \to \infty$,

$$\{\mathbf{X}_j/b_n, 1 \le j \le n\} \to S$$

a.s. in $\mathscr{K}_d$.

PROOF. Since $S_n^* = T^*(S_n \cap \{\mathbf{0}\}^c)$ and $S^* = T^*(S \cap \{\mathbf{0}\}^c)$ and since T is a homeomorphism, we conclude $S_n \to S$ if and only if $S_n^* \to S^*$ from Propositions 4.1, 4.2 and Lemma 4.3. The star-shaped property of S means, for $x \in S$ and $0 < t \le 1$,

$$(\boldsymbol{\theta}, tr) \in S^* \quad \text{if } (\boldsymbol{\theta}, r) \in S^*.$$

Let $\Theta' = \{\boldsymbol{\theta}(\mathbf{x}), \mathbf{x} \in S, \mathbf{x} \ne 0\}$. Define $l\colon \Theta \to [0,\infty)$ by

$$l(\boldsymbol{\theta}) = \begin{cases} \sup\{r : (\boldsymbol{\theta}, r) \in S^*\}, & \text{if } \boldsymbol{\theta} \in \Theta', \\ 0, & \text{if } \boldsymbol{\theta} \in \Theta \setminus \Theta'. \end{cases}$$

Then

$$S^* = \text{hypo}(l).$$

Since S^* is closed, l is USC from Lemma 2.3.

Given any USC function l, we explain how to construct a scaled sample S_n^* converging to hypo(l). This will show that the class of limit sets in polar coordinates is indeed {hypo(l), $l \in$ USC}.

Given $l(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$, where Θ is the unit sphere in $\mathbb{R}^d$ with respect to the norm $\|\mathbf{x}\|$. Then hypo(l) is closed in $\Theta \times (0,\infty)$. Assume $\sup_{\boldsymbol{\theta}\in\Theta} l(\Theta) = 1$. We first construct a distribution μ on $\Theta \times (0,\infty)$ whose support is hypo(l).

Fix $\varepsilon > 0$ and let $(\text{hypo}(l))^\varepsilon$ be the ε-swelling of hypo(l), and define π: $(\text{hypo}(l))^\varepsilon \to \text{hypo}(l)$ by

$$\pi(\boldsymbol{\theta}, r) = \text{closest point of hypo}(l) \text{ to } (\boldsymbol{\theta}, r).$$

So π is the identity on hypo(l). On $(\text{hypo}(l))^\varepsilon \setminus \text{hypo}(l)$, π maps points to the closest element of hypo(l). If there is more than one closest element, choose one arbitrarily (but measurably). Let L be Lebesgue measure on $(\text{hypo}(l))^\varepsilon$ and define μ by $\mu = L \circ \pi^{-1}$.

We now verify that the support of μ is hypo(l). Let $(\boldsymbol{\theta}_0, r_0) \in \text{hypo}(l)$ and suppose $G \subset \Theta \times (0,\infty)$ is open and $(\boldsymbol{\theta}_0, r_0) \in G$. We must show $\mu(G) > 0$. From the definition of μ, it suffices to show $\pi^{-1}(G)$ has positive Lebesgue measure; for this it is enough if we prove $\pi^{-1}G$ contains an open set. We seek an open $U \subset (\text{hypo}(l))^\varepsilon$ such that $\pi(U) \subset G \cap (\text{hypo}(l))$. We may set

$$U = (\text{hypo}(l))^\varepsilon \cap \{(\boldsymbol{\theta}, r) \in G : d((\boldsymbol{\theta}, r), (\boldsymbol{\theta}_0, r_0)) < (1/2) d((\boldsymbol{\theta}_0, r_0), G^c)\}.$$

If $(\boldsymbol{\theta}, r) \in U$, then because π maps $(\boldsymbol{\theta}, r)$ to the closest point of hypo(l),

$$d(\pi(\boldsymbol{\theta}, r), (\boldsymbol{\theta}_0, r_0)) \le d(\pi(\boldsymbol{\theta}, r), (\boldsymbol{\theta}, r)) + d((\boldsymbol{\theta}, r), (\boldsymbol{\theta}_0, r_0))$$
$$\le 2d((\boldsymbol{\theta}, r), (\boldsymbol{\theta}_0, r_0)) < d((\boldsymbol{\theta}_0, r_0), G^c)$$

from the definition of U. Thus $\pi(\boldsymbol{\theta}, r) \in G$, whence $\pi(U) \subset G$. Since U is open and nonempty, the verification that support μ = hypo(l) is complete.

Now let $\{(\boldsymbol{\theta}_i, R_i), i \ge 1\}$ be iid with distribution μ. A sample from any distribution will converge to the support of the distribution so

$$\{(\boldsymbol{\theta}_i, R_i), i \le n\} \to \text{hypo}(l)$$

in $\mathscr{F}(\Theta \times (0,\infty))$. [If G is open and $G \cap \text{hypo}(l) \ne \varnothing$, then $\mu(G) > 0$ and

$$P\Big[\liminf_{n\to\infty}(G \cap \{(\boldsymbol{\theta}_i, R_i), i \le n\} \ne \varnothing)\Big] = 1.$$

If K is compact in $\Theta \times (0,\infty)$ and $K \cap \text{hypo}(l) = \varnothing$, then $P[\{(\boldsymbol{\theta}_i, R_i), i \le n\} \cap K = \varnothing \; \forall \; n] = 1$.]

Finally, let $\{E_i, i \ge 1\}$ be iid unit exponential random variables independent of $\{(\boldsymbol{\theta}_i, R_i), i \ge 1\}$. Then we claim

$$\left\{\boldsymbol{\theta}_i, \frac{R_i E_i}{\log n}, i \le n\right\} \to \text{hypo}(l)$$

as $n \to \infty$. To check this suppose K is compact and $K \subset ((\text{hypo}(l))^{\varepsilon})^c$. Since

$$P\big[\{(\boldsymbol{\theta}_i, R_i), i \le n\} \cap K \text{ i.o.}\big] = 0$$

and

$$\bigvee_1^n \frac{E_i}{\log n} \to 1,$$

we get, for all large n,

$$\left\{\left(\boldsymbol{\theta}_i, \frac{R_i E_i}{\log n}\right), i \le n\right\} \cap K = \varnothing.$$

Also let G be open with $G \cap \text{hypo}(l) \ne \varnothing$. We may suppose G is of the form

$$G = I_1 \times I_2,$$

where $I_1 \subset \Theta$ is open and $I_2 = (r_1, r_2)$, $0 < r_1 < r_2$. We need to show

$$\left\{\left(\boldsymbol{\theta}_i, \frac{R_i E_i}{\log n}\right), i \le n\right\} \cap G \ne \varnothing.$$

for all large n. Pick δ small so that $I_2' = (r_1 + \delta_1, r_2 - \delta) \subset I_2$. Then $G' = I_1 \times I_2' \cap \text{hypo}(l) \ne \varnothing$. There exist indices $K(n)$ such that $(\boldsymbol{\theta}_{K(n)}, R_{K(n)}) \in G'$ and $K(n) \sim n/\mu(G')$. [Let $K(1) = \inf\{j \ge 1: (\boldsymbol{\theta}_j, R_j) \in G'\}$, $K(2) = \inf\{j > K(1): (\boldsymbol{\theta}_j, R_j) \in G'\}$, etc.] Then $\{K_n\}$ is independent of $\{E_i\}$ and $\{E_{K(n)}\} = \{E_n'\}$ is iid unit exponential. Let $L(n)$ be the record value times of the sequence $\{E_n'\}$ so that $E'_{L(n)} \sim n$ and $\log L(n) \sim n$ [Resnick (1987)]. Then

$$\log K(L(n)) \sim \log L(n) \sim n.$$

Since $E'_{L(n)} \sim n$, for all n large

$$\left(\boldsymbol{\theta}_{K(L(n))}, R_{K(L(n))}\frac{E'_{L(n)}}{n}\right) \in G.$$

If

$$K(L(n)) \le m < K(L(n+1)),$$

then

$$\log K(L(n)) \le \log m \le \log K(L(n+1))$$

so that

$$\log K(L(n)) \sim n \sim \log m$$

as $n \to \infty$. Thus for all large m,

$$\left\{\left(\boldsymbol{\theta}_i, \frac{R_i E_i}{\log m}\right), i \le m\right\} \cap G \ne \varnothing$$

since if $K(L(n)) \le m < K(L(n+1))$,

$$\left(\boldsymbol{\theta}_{K(L(n))}, R_{K(L(n))}\frac{E'_{L(n)}}{\log m}\right) = \left(\boldsymbol{\theta}_{K(L(n))}, R_{K(L(n))}\frac{E'_{L(n)}}{n}\frac{n}{\log m}\right) \in G.$$

This proves the claim and completes the construction. □

Having identified the class of limits, we turn our attention to criteria for convergence to a specified limit. We continue to work in polar coordinates. Define open and closed balls in Θ as follows: For $\boldsymbol{\theta} \in \Theta$, $\varepsilon > 0$ and $d(\cdot, \cdot)$, the metric on $\mathbb{R}^d$, set

$$B(\boldsymbol{\theta}, \varepsilon) := \{\boldsymbol{\omega} \in \Theta: d(\boldsymbol{\theta}, \boldsymbol{\omega}) < \varepsilon\},$$

$$B(\boldsymbol{\theta}, \varepsilon+) := \{\boldsymbol{\omega} \in \Theta: d(\boldsymbol{\theta}, \boldsymbol{\omega}) \le \varepsilon\}.$$

For $B \subset \Theta$ and $S \subset \mathbb{R}^d$, the restriction of S to B is defined by

$$S(B) := \{\mathbf{x} \in S \cap \{\mathbf{0}\}^c: \boldsymbol{\theta}(\mathbf{x}) \in B\}$$

and in polar coordinates we have

$$S^*(B) := T^*S(B).$$

Finally, for $B \subset \Theta$ and a function $l: \Theta \to \mathbb{R}_+$, we define the sup measure $l^\vee$: $2^\Theta \to \mathbb{R}_+$ by

$$l^\vee(B) := \sup_{\boldsymbol{\theta} \in B} l(\theta)$$

and

$$l^\vee(\phi) \equiv 0.$$

The next result reduces our set convergence problem to convergence of compact subsets of $\mathbb{R}_+$.

PROPOSITION 4.5 [cf. Vervaat (1988), Section 10]. *Let $l\colon \Theta \to \mathbb{R}_+$ be an USC function with $l^\vee(\Theta) = 1$. Set $S^* = \text{hypo}(l)$, $S = (T^*)^{-1}(S^*) \cup \{\mathbf{0}\}$. If*

$$S_n^* := \{(\boldsymbol{\theta}_{n,j}, r_{n,j}), 1 \le j \le j_n\}$$

is a set of points in $\Theta \times (0,\infty)$, $j_n \to \infty$, then

$$S_n := (T^*)^{-1}(S_n^*) \to S$$

in $\mathscr{K}_d$ if and only if

$$\begin{aligned} K_n = K_n(B(\boldsymbol{\theta},t)) &:= \{\|\mathbf{x}\|: \mathbf{x} \in S_n(B(\boldsymbol{\theta},t))\} \\ &\to K = K(B(\boldsymbol{\theta},t)) := [0, l^\vee(B(\boldsymbol{\theta},t))] \end{aligned} \tag{4.4}$$

in $\mathscr{K}[0,\infty)$ for any $\boldsymbol{\theta} \in \Theta$ and $t > 0$ such that

$$l^\vee(B(\boldsymbol{\theta},t)) = l^\vee(B(\boldsymbol{\theta},t+)). \tag{4.5}$$

One might expect that $S_n \to S$ implies $S_n(B) \to S(B)$ as $n \to \infty$ provided (4.5) holds. However, this is not true in general.

EXAMPLE. In $\mathbb{R}^2$, let Θ be the unit circle, which we parametrize by $\Theta = \{\theta\colon 0 \le \theta < 2\pi\}$. Let

$$l_n(\theta) = 1_{\{\pi/2, \pi+1/n\}}(\theta), \qquad l(\theta) = 1_{\{\pi/2, \pi\}}(\theta),$$

and $S_n^* = \text{hypo}(l_n)$, and $S^* = \text{hypo}(l)$. Then (4.5) is satisfied with $\theta = t = \pi/2$ and one checks readily that $S_n^* \to S^*$ as $n \to \infty$. But $S_n^*(B(\pi/2, (\pi/2)+))$ does not converge to $S^*(B(\pi/2,(\pi/2)+))$ since

$$S_n^*\left(B\left(\frac{\pi}{2}, \frac{\pi}{2}+\right)\right) = \{(r,\theta): 0 < r \le 1_{\{\pi/2\}}(\theta), 0 \le \theta \le \pi\}$$

and

$$S^*\left(B\left(\frac{\pi}{2}, \frac{\pi}{2}+\right)\right) = \{(r,\theta): 0 < r \le 1_{\{\pi/2,\pi\}}(\theta), 0 \le \theta \le \pi\}.$$

Combining Proposition 4.5 and the results in the previous section, we have the following criterion for the convergence of scaled random samples to a limiting compact set. Recall that $\{\mathbf{X}_n\}$ are i.i.d. in $\mathbb{R}^d$. Set $q(B) = P[\boldsymbol{\theta}(\mathbf{X}_1) \in B]$.

THEOREM 4.6. *Let $l\colon \Theta \to [0,1]$ be a USC function with $l^\vee(\Theta) = 1$, $S^* = \text{hypo}(l)$, $S = (T^*)^{-1}(S^*) \cup \{\mathbf{0}\}$. Then there exists a sequence $b_n \to \infty$ such that*

$$S_n := \left\{\frac{\mathbf{X}_j}{b_n}, 1 \le j \le n\right\} \to S \quad a.s. \text{ as } n \to \infty$$

in $\mathscr{K}_d$ if and only if the following two conditions hold:

(i) $\{M_n\} = \{\bigvee_{j=1}^{n} \|\mathbf{X}_j\|\}$ *is a.s. relatively stable.*

(ii) *For any $B(\boldsymbol{\theta}, t)$ satisfying* (4.5), *if $q(B(\boldsymbol{\theta}, t)) = 0$, then $l^{\vee}(B(\boldsymbol{\theta}, t)) = 0$ and if $q(B(\boldsymbol{\theta}, t)) > 0$, then*

$$(4.6)\quad \lim_{s\to\infty} \frac{P\{\|\mathbf{X}_1\| > sx,\ \theta(\mathbf{X}_1) \in B(\boldsymbol{\theta}, t)\}}{P\{\|\mathbf{X}_1\| > s\}} = \begin{cases} 0, & \text{if } x > l^{\vee}(B(\boldsymbol{\theta}, t)), \\ \infty, & \text{if } x < l^{\vee}(B(\boldsymbol{\theta}, t)). \end{cases}$$

PROOF (Necessity). Condition (i) follows from Proposition 4.1. Assume that $B(\boldsymbol{\theta}, t)$ satisfies (4.5). If $q(B(\boldsymbol{\theta}, t)) = 0$, then clearly $l^{\vee}(B(\boldsymbol{\theta}, t)) = 0$, so assume $q(B(\boldsymbol{\theta}, t)) > 0$. Observing Proposition 3.5, we have that (4.6) is equivalent to

$$(4.7)\qquad \lim_{n\to\infty} \frac{b_n^{\#}}{b_n} = l^{\vee}(B(\boldsymbol{\theta}, t)),$$

where $b_n^{\#} = G^{\leftarrow}(1 - 1/n)$ and

$$(4.8)\qquad G(x) = P(\|\mathbf{X}_1\| \le x | \boldsymbol{\theta}(\mathbf{X}_1) \in B(\boldsymbol{\theta}, t)).$$

Thus it suffices to show that (4.7) holds. Since $S_n \to S$ a.s. we have from Lemma 4.3 and Proposition 4.5 that

$$(4.9)\qquad \left\{\frac{1}{b_n}\|\mathbf{X}_j\|: \boldsymbol{\theta}(\mathbf{X}_j) \in B(\boldsymbol{\theta}, t),\ 1 \le j \le n\right\} \to \left[0, l^{\vee}(B(\boldsymbol{\theta}, t))\right] \quad \text{a.s. as } n \to \infty.$$

Define random indices as follows:

$$K_0 = 0,$$
$$K_i = \inf\{j > K_{i-1}: \boldsymbol{\theta}(\mathbf{X}_j) \in B(\boldsymbol{\theta}, t)\},$$
$$L_n = \sup\{j: K_j \le n\},$$
$$N_n = \sum_{j=1}^{n} 1_{[\boldsymbol{\theta}(\mathbf{X}_j) \in B(\boldsymbol{\theta}, t)]}$$

and

$$\mathbf{X}_1^{\#} = \mathbf{X}_{K_1}, \mathbf{X}_2^{\#} = \mathbf{X}_{K_2}, \dots, \mathbf{X}_{N_n}^{\#} = \mathbf{X}_{L_n}.$$

The "découpage de Lévy" [see, e.g., Resnick (1987)] says $\{\mathbf{X}_i^{\#}, i \ge 1\}$ are iid with distribution $P[\mathbf{X}_1 \in \cdot | \boldsymbol{\theta}(\mathbf{X}_1) \in B(\boldsymbol{\theta}, t)]$ and (4.9) is equivalent to

$$(4.10)\qquad \left\{\frac{1}{b_n}\|\mathbf{X}_j^{\#}\|,\ 1 \le j \le N_n\right\} \to \left[0, l^{\vee}(B(\boldsymbol{\theta}, t))\right] \quad \text{a.s.},$$

implying

$$(4.11)\qquad \frac{M_{N_n}^{\#}}{b_n} := \frac{1}{b_n} \bigvee_{j=1}^{N_n} \|\mathbf{X}_j^{\#}\| \to l^{\vee}(B(\boldsymbol{\theta}, t)) \quad \text{a.s. as } n \to \infty.$$

Since $N_n/n \to q(B(\theta, t)) > 0$ a.s. as $n \to \infty$ and $b_n \in RV(0)$, it follows from (4.11) that

$$\frac{M^{\#}_{N_n}}{b_{N_n}} = \frac{M^{\#}_{N_n}}{b_n} \frac{b_n}{b_{[nN_n/n]}} \to l^{\vee}(B(\theta, t)) \cdot 1 \quad \text{a.s. as } n \to \infty,$$

implying

$$\frac{M^{\#}_n}{b_n} \to l^{\vee}(B(\theta, t)) \quad \text{a.s. as } n \to \infty. \tag{4.12}$$

If $l^{\vee}(B(\theta, t)) > 0$, then (4.12) implies that $\{M^{\#}_n\}$ is a.s. relatively stable with normalizing constant $b_n l^{\vee}(B(\theta, t))$. However, since $\|\mathbf{X}^{\#}_j\|$ has the distribution given by (4.8), it follows that $b^{\#}_n \sim b_n l^{\vee}(B(\theta, t))$, which is (4.7). If $l^{\vee}(B(\theta, t)) = 0$, then (4.7) immediately follows from (4.12) and Proposition 3.4.

(Sufficiency.) It again follows from Proposition 4.5 that it suffices to show that (4.10) holds for $B(\theta, t)$ satisfying (4.5). We consider two cases.

(a) $l^{\vee}(B(\theta, t)) > 0$. Note that $l^{\vee}(B(\theta, t)) > 0$ clearly implies $q(B(\theta, t)) > 0$. Then we have from (4.7) and Proposition 3.2 that $\{M^{\#}_n\}$ is a.s. relatively stable, which is equivalent to

$$\frac{M^{\#}_n}{b_n} \to l^{\vee}(B(\theta, t)) \quad \text{a.s. as } n \to \infty.$$

From Proposition 3.1,

$$\left\{ \frac{\|\mathbf{X}^{\#}_j\|}{b_n l^{\vee}(B(\theta, t))}, 1 \le j \le n \right\} \to [0, 1],$$

whence

$$\left\{ \frac{\|\mathbf{X}^{\#}_j\|}{b_n}, 1 \le j \le n \right\} \to \left[0, l^{\vee}(B(\theta, t))\right].$$

Therefore, since $b_{[\cdot]}$ is slowly varying,

$$\left\{ \frac{\|\mathbf{X}^{\#}_j\|}{b_n}, 1 \le j \le N_n \right\} = \left(\frac{b_{N_n}}{b_n} \right) \left\{ \frac{\|\mathbf{X}^{\#}_j\|}{b_{N_n}}, 1 \le j \le N_n \right\}$$
$$\to 1\left[0, l^{\vee}(B(\theta, t))\right] \quad \text{a.s.}$$

and we get (4.10) as required.

(b) $l^{\vee}(B(\theta, t)) = 0$. If $q(B(\theta, t)) = 0$, then $S_n(B(\theta, t)) = \varnothing$ a.s. and hence $S(B(\theta, t)) = \varnothing$, so $l = 0$ on $B(\theta, t)$. Assume $q(B(\theta, t)) > 0$. Then it follows from (4.7) and Proposition 3.4 that

$$\frac{M^{\#}_n}{b_n} \to 0 \quad \text{a.s. as } n \to \infty,$$

implying

$$\left\{\frac{1}{b_n}\|\mathbf{X}_j^{\#}\|,\ 1 \le j \le N_n\right\} \to \{0\} \quad \text{a.s.},$$

which is (4.8). □

As a special case of the previous proposition, we have the following.

COROLLARY 4.7. *Suppose* $\mathbf{X}_1$ *is spherically symmetric so that* $\boldsymbol{\theta}_1$, $\|\mathbf{X}_1\|$ *are independent and* $\boldsymbol{\theta}_1$ *is uniformly distributed on* Θ. *Then*

$$S_n \to S = \{(\boldsymbol{\theta}, r): r \le 1\} \quad a.s.$$

iff $\{M_n\} = \{\bigvee_{j=1}^{n} \|\mathbf{X}_j\|\}$ *is a.s. relatively stable.*

5. Complements and examples. We first show in this section that the condition of Theorem 4.6 is implied by the log-regular variation condition in Davis, Mulrow and Resnick (1988).

PROPOSITION 5.1 [Davis, Mulrow and Resnick (1988)]. *Let* $\mathbf{X}_1, \mathbf{X}_2, \dots$ *be an iid sequence of random vectors in* $\mathbb{R}^d$ *having a common joint distribution which has support in* $\{\mathbf{x} \ge 0\}$. *Assume that the components are identically distributed.*

Suppose for $\mathbf{x} \in [0,\infty)^d \setminus \{\mathbf{0}\} =: C$, *as* $t \to \infty$,

$$f(t,\mathbf{x}) := \frac{-\log P\{\mathbf{X}_1 \ge t\mathbf{x}\}}{-\log P\{\mathbf{X}_1 \ge t\mathbf{1}\}} \to \lambda(\mathbf{x}), \tag{5.1}$$

where $\lambda(\mathbf{x})$ *satisfies for* $t > 0$, $\mathbf{x} \in C$, *and some* $\alpha > 0$,

$$\lambda(t\mathbf{x}) = t^{\alpha}\lambda(\mathbf{x}). \tag{5.2}$$

Assume $\lambda(\mathbf{x})$ *is strictly increasing on* C, *that is,*

$$\lambda(\mathbf{x}) < \lambda(\mathbf{y}) \quad \textit{when } \mathbf{x} \le \mathbf{y} \textit{ and } \mathbf{x} \ne \mathbf{y}.$$

Then in $\mathscr{K}_d$, *as* $n \to \infty$, *for some* $b_n > 0$,

$$S_n := \left\{\frac{\mathbf{X}_j}{b_n},\ 1 \le j \le n\right\} \to S := \{\mathbf{x} \ge 0: \lambda(\mathbf{x}) \le 1\} \quad a.s.$$

We now verify that the assumptions in Proposition 5.1 are sufficient to guarantee that the limit set is $[\lambda \le 1]$ by showing that (4.6) holds. We restrict attention to the case $d = 2$ and show that (4.6) holds with the norm $\|(x, y)\| =$

$x \vee y$. Assume, for example, that $B(\boldsymbol{\theta}, \varepsilon)$ is of the form

$$B(a,b) := \{(1,y): a < y < b\} \quad \text{for } 0 \le a < b \le 1. \tag{5.3}$$

From (5.2),

$$\begin{aligned} S &= \{(x,y): x \ge y \ge 0,\, \lambda((x,y)) \le 1\} \cup \{(x,y): y > x \ge 0,\, \lambda((x,y)) \le 1\} \\ &= \left\{(x,y): x \ge y \ge 0,\, x^{\alpha}\lambda\left(\left(1, \frac{y}{x}\right)\right) \le 1\right\} \\ &\quad \cup \left\{(x,y): y > x \ge 0,\, y^{\alpha}\lambda\left(\left(\frac{x}{y}, 1\right)\right) \le 1\right\}, \end{aligned}$$

which is expressed in polar coordinates as

$$S^* = \{(\boldsymbol{\theta}, r): \boldsymbol{\theta} \in \{1\} \times [0,1] \cup [0,1] \times \{1\},\, r^{\alpha}\lambda(\boldsymbol{\theta}) \le 1\}.$$

This identifies $l(\boldsymbol{\theta})$ as

$$l(\boldsymbol{\theta}) = \left(\frac{\lambda((1,0))}{\lambda(\boldsymbol{\theta})}\right)^{1/\alpha}. \tag{5.4}$$

Since λ is strictly increasing, it follows from (5.3) and (5.4) that

$$l^{\vee}(B(a,b)) = \left(\frac{\lambda((1,0))}{\lambda((1,a))}\right)^{1/\alpha}.$$

We also have that

$$\begin{aligned} \{\|(x,y)\| > tu,\, \boldsymbol{\theta}((x,y)) \in B(a,b)\} &= \left\{x \vee y > tu,\, \frac{(x,y)}{(x \vee y)} \in B(a,b)\right\} \\ &= \left\{x \vee y > tu,\, \left(1, \frac{y}{x}\right) \in B(a,b)\right\} \\ &= \left\{x > tu,\, a < \frac{y}{x} < bx\right\}. \end{aligned}$$

Because (5.1) implies

$$\lim_{t \to \infty} \frac{P\{X \vee Y > t\}}{P\{X > t\}} = 1,$$

it suffices to show

$$\lim_{t \to \infty} \frac{P\{X > ut,\, aX < Y < bX\}}{P\{X > t\}} = \begin{cases} 0, & \text{if } u > \lambda_0, \\ \infty, & \text{if } u < \lambda_0, \end{cases}$$

where

$$\lambda_0 = \left(\frac{\lambda((1,0))}{\lambda((1,a))}\right)^{1/\alpha}.$$

This is a fairly straightforward consequence of the assumptions.

For the rest of the section, we examine some examples of Theorem 4.6 for the case $d = 2$. We denote again by S_n and S the set of sample points and a limit set in the Cartesian coordinates and by S_n^* and S^* those in the polar coordinates. Note that Proposition 5.1 does not apply to the following example since λ is not strictly increasing [cf. Davis, Mulrow and Resnick (1988), Example 3.6].

EXAMPLE 5.2. Suppose the distribution is given by $P\{X > x,\ Y > y\} = (e^x + e^y - 1)^{-1}$. Then we shall show that with $b_n = \log n$, $S_n \to S$ a.s., where

$$S = \left\{(x,y): 0 \le x, y \le 1, 2x - 1 \le y \le \frac{x+1}{2}\right\}.$$

As in Proposition 5.1, choosing $\|(x,y)\| = x \vee y$ as the norm and using the symmetry, we need to show that

$$T_n^* \to T^* \quad \text{a.s. as } n \to \infty, \tag{5.5}$$

where

$$T_n^* = \{(\boldsymbol{\theta}(x,y), r(x,y)): (x,y) \in S_n, x \ge y\}$$
$$= \left\{\left(\left(1, \frac{Y_j}{X_j}\right), \frac{X_j}{b_n}\right): X_j \ge Y_j, 1 \le j \le n\right\}$$

and

$$T^* = \{(\boldsymbol{\theta}(x,y), r(x,y)): (x,y) \in S, x \ge y\}$$
$$= \left\{\left(\left(1, \frac{y}{x}\right), x\right): 0 \le y \le x \le \frac{y+1}{2}\right\}.$$

Since (5.5) is equivalent to

$$U_n^* \to U^* \quad \text{a.s. as } n \to \infty, \tag{5.6}$$

where

$$U_n^* = \left\{\left(\frac{Y_j}{X_j}, \frac{X_j}{b_n}\right): X_j \ge Y_j, 1 \le j \le n\right\}$$

and

$$U^* = \left\{(\theta, r): 0 \le \theta \le 1, 0 < r \le \frac{1}{2 - \theta}\right\},$$

we apply Theorem 4.6 to (5.6). Observe that since

$$\begin{aligned} P\{X \vee Y > t] &= P[X > t] + P[Y > t] - P[X > t, Y > t] \\ &= 2e^{-t} - (2e^t - 1)^{-1} \sim \tfrac{3}{2}e^{-t}, \end{aligned} \tag{5.7}$$

we have that $\{\bigvee_{i=1}^n \|(X_i, Y_i)\|,\ n \ge 1\}$ is relatively stable and hence the first condition in Theorem 4.6 is satisfied. Thus, since (X, Y) has support $\mathbb{R}_+^2$, we only need to show that (4.8) holds. Now the corresponding USC function

$l(\theta) = 1/(2-\theta)$ is strictly increasing, and therefore (4.8) is equivalent to $(0 \le a < b \le 1)$

$$\lim_{t\to\infty} f(t,a,b,x) := \lim_{t\to\infty} \frac{P\{X > tx,\, a < Y/X < b\}}{P\{X \vee Y > t\}}$$

$$(5.8) \qquad = \begin{cases} 0, & \text{if } x > \dfrac{1}{2-b}, \\ \infty, & \text{if } x < \dfrac{1}{2-b}. \end{cases}$$

Since the Jacobian of the transformation $(x,y) \to (r = x,\ \theta = y/x)$ is $J = r$, we have

$$(5.9) \qquad P\left\{X > tx,\, a < \frac{Y}{X} < b\right\} = \int_{r=tx}^{\infty}\int_{\theta=a}^{b} \frac{2re^{r(1+\theta)}}{(e^r + e^{r\theta} - 1)^3}\, d\theta\, dr.$$

For $x > 1/(2-b)$, it follows from (5.7) and (5.8) that, as $t \to \infty$,

$$\begin{aligned} f(t,a,b,x) &\le \frac{2}{3}e^t \int_{r=tx}^{\infty} 2e^{-2r}\, dr \int_{\theta=a}^{b} re^{r\theta}\, d\theta \\ &= \frac{4}{3}e^t \int_{tx}^{\infty} e^{-2r}(e^{br} - e^{ar})\, dr \\ &\le \frac{4}{3}e^t \int_{tx}^{\infty} e^{-(2-b)r}\, dr \\ &= \frac{4}{3(2-b)} e^{-((2-b)x-1)t} \\ &\to 0, \end{aligned}$$

which is one half of (5.8). For $x < 1/(2-b)$, similar calculations show

$$f(t,a,b,x) \to \infty,$$

which is the other half of (5.8). □

For the following examples, we make the usual polar coordinate transformation $\|\mathbf{X}\| = (X^2 + Y^2)^{1/2}$, $\theta(\mathbf{X}) = \arctan(Y/X)$.

EXAMPLE 5.3. Suppose the distribution of $(\theta(\mathbf{X}), \|\mathbf{X}\|)$ is

$$P\{\theta(\mathbf{X}) \in d\theta,\, \|\mathbf{X}\| \in dr\} = F(dr)\frac{d\theta}{2\pi},$$

where F is specified by (2.6). Then since $\{M_n\}$ is not a.s. relatively stable, S_n does not converge a.s. However, since $\{M_n\}$ is relatively stable in probability, we have that

$$S_n \to_P S = \{(x,y): x^2 + y^2 \leq 1\} \quad \text{as } n \to \infty. \qquad \square$$

EXAMPLE 5.4. Let B_n be a random variable and $\mathbf{Y}_n^* = (\theta(\mathbf{Y}_n), \|\mathbf{Y}_n\|)$ and $\mathbf{Z}_n^* = (\theta(\mathbf{Z}_n), \|\mathbf{Z}_n\|)$ be random vectors. Suppose $P(B_n = 0) = P(B_n = 1) = \frac{1}{2}$, $\theta(\mathbf{Y}_n)$ and $\theta(\mathbf{Z}_n)$ are uniformly distributed on $[0, \pi)$ and $[\pi, 2\pi)$, respectively, $P(\|\mathbf{Y}_n\| > r) = P(\|\mathbf{Z}_n\| > r/2) = e^{-r}$, and these are all independent. Then define $\mathbf{X}_n^*$ as $\mathbf{X}_n^* = (\theta(\mathbf{X}_n), \|\mathbf{X}_n\|) := B_n \mathbf{Y}_n^* + (1 - B_n)\mathbf{Z}_n^*$. The distribution of $\mathbf{X}_n^*$ clearly satisfies the conditions of Theorem 4.6 with $b_n = \log n$ and the limit set is the union of two rectangles. It can be written

$$S^* = \text{hypo}\, l,$$

where

$$l(\theta) = \begin{cases} 1, & 0 \leq \theta \leq \pi, \\ 2^{-1}, & \pi < \theta < 2\pi. \end{cases}$$

For this example, the d.f. of $\theta(\mathbf{X}_n)$ is continuous since $\theta(\mathbf{X}_n)$ is uniformly distributed in $[0, 2\pi)$, but the corresponding USC function is discontinuous at $\theta = 0, \pi$.

EXAMPLE 5.5. As in the previous example, set

$$\mathbf{X}_n^* = B_n \mathbf{Y}_n^* + (1 - B_n)\mathbf{Z}_n^*,$$

where B_n is the same as before, $\theta(\mathbf{Y}_n)$ is uniformly distributed in $[0, 2\pi)$, $P(\|\mathbf{Y}_n\| > r) = e^{-r}$ and $\mathbf{Z}_n$ is degenerate at $(\pi, 0)$, that is, $P(\theta(\mathbf{Z}_n) = \pi, \|\mathbf{Z}_n\| = 0) = 1$. Then we see that with $b_n = \log n$, as $n \to \infty$,

$$S_n^* \to S^* = \{(\theta, r): 1_{[0, 2\pi)}(\theta) \cdot 1_{(0,1]}(r)\}.$$

Now the d.f. of $\theta(\mathbf{X}_n)$ is discontinuous at $\theta = \pi$ even though the USC $l(\theta) = 1$ is continuous.

From the last two examples, we see that the continuity of the d.f. of $\theta(\mathbf{X}_n)$ has nothing to do with that of the corresponding USC function.

EXAMPLE 5.6. Let C be the Cantor set and $F_C(\cdot)$ be the Cantor distribution. For the constructions of C and F_C, see, for example, Billingsley (1979). F_C can also be defined as the d.f. corresponding to the random variable $Z = \sum_{n=1}^{\infty} B_n / 3^n$, where B_n are iid and $P(B_n = 0) = P(B_n = 2) = \frac{1}{2}$.

Now suppose $P(\|\mathbf{X}_n\| > r) = e^{-r}$, $\theta(\mathbf{X}_n)$ has the distribution F_C and they are independent. Then it follows that with $b_n = \log n$, as $n \to \infty$,

$$S_n^* \to_{\text{a.s.}} S^* = \{(\theta, r): 0 \leq \theta \leq 1, 0 < r \leq 1_C(\theta)\}.$$

Note that since C is closed, $1_C(\theta)$ is USC and hence by Lemma 2.3, hypo(1_C) is closed. Since $\{M_n\}$ is clearly a.s. relatively stable, we only need to check the second condition in Theorem 4.6. For $0 \leq a < b \leq 1$, let $B(a,b) = \{\theta: a < \theta < b\}$ and $B[a,b] = \{\theta: a \leq \theta \leq b\}$. It is seen that $P\{\theta(\mathbf{X}_n) \in B(a,b)\} > 0$ and

$l^{\vee}(B(a,b)) = 1$ whenever (4.5) is satisfied. Thus we must show that (4.6) holds. However, since $\|\mathbf{X}_n\|$ and $\theta(\mathbf{X}_n)$ are independent, we have as $t \to \infty$,

$$\frac{P\{\|\mathbf{X}_n\| > tx, \theta(\mathbf{X}_n) \in B(a,b)\}}{P\{\|\mathbf{X}_n\| > t\}} = P\{\theta(\mathbf{X}_n) \in B(a,b)\} \frac{P\{\|\mathbf{X}_n\| > tx\}}{P\{\|\mathbf{X}_n\| > t\}}$$
$$\to \begin{cases} 0, & \text{if } x > 1, \\ \infty, & \text{if } x < 1, \end{cases}$$

as desired.

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1086-1, TAKAYA
SAKAIDE, KAGAWA, 762
JAPAN

SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
ETC BUILDING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853