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BROWNIAN EXCURSIONS, TREES AND MEASURE-VALUED BRANCHING PROCESSES

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We propose a trajectorial construction of a class of measure-valued Markov processes, called superprocesses or measure-valued branching processes, which have been studied extensively in the last few years. These processes were originally defined as weak limits of systems of branching particles. The basic idea of our construction is to use the branching structure of excursions of a linear Brownian motion to model the branching mechanism of the superprocess. Without any additional effort, our approach leads to the so-called historical process, which contains more information than the superprocess in the sense that it keeps track of the individual paths followed by the particles. We emphasize the relationship between the properties of the historical process and the corresponding results of excursion theory. We also give a description of the support of the superprocess at a fixed time, using a simple tree model. Finally, we use our construction to recover certain pathwise properties recently obtained by Perkins.

1. Introduction. The purpose of this work is to give a simple trajectorial construction of certain measure-valued Markov processes, called superprocesses by Dynkin, which have been recently studied by several authors; see, for example, Dawson, Iscoe and Perkins (1989), Dawson and Perkins (1989), Dynkin (1988, 1989a, b), El Karoui and Roelly-Coppoletta (1988), Fitzsimmons (1988), Fleischmann (1988), Iscoe (1986, 1988) and Perkins (1988, 1989, 1990). This construction relies on the Itô theory of excursions of linear Brownian motion and on the properties of the so-called excursion local times. It is also related to the random tree associated with a Brownian excursion [see Neveu and Pitman (1989a, b) and Le Gall (1989b)].

Let $(x_t, t \geq 0; P_x, x \in \mathbb{R}^d)$ be a diffusion process in $\mathbb{R}^d$ associated with the stochastic differential equation:

$$\text{(A)} \qquad dx_t = \sigma(x_t)\, dB_t + b(x_t)\, dt,$$

where B denotes a Brownian motion in $\mathbb{R}^d$ and the functions $\sigma\colon \mathbb{R}^d \to M_{dn}(\mathbb{R})$ and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz and uniformly bounded [$M_{dn}(\mathbb{R})$ denotes the set of $d \times n$ matrices with real coefficients]. We denote by (P_t) the semigroup of (x_t). We are interested in the construction of the (Dawson–Watanabe) superprocess associated with (x_t), which can be defined as follows. Let $\mathscr{M}(\mathbb{R}^d)$ be the space of all finite measures on $\mathbb{R}^d$, and let $\mu \in \mathscr{M}(\mathbb{R}^d)$. The superprocess

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associated with (x_t), starting from μ, is the $\mathscr{M}(\mathbb{R}^d)$-valued Markov process $(X_t, t \geq 0)$ whose law is uniquely determined by the relations

$$X_0 = \mu$$

and, for $0 \leq s \leq t$, for any bounded continuous function φ from $\mathbb{R}^d$ into $\mathbb{R}_+$,

$$E[\exp - \langle X_t, \varphi \rangle | X_s] = \exp - \langle X_s, V_{t-s}\varphi \rangle,$$

where $(t, z) \to V_t\varphi(z)$ is the unique continuous function from $\mathbb{R}_+ \times \mathbb{R}^d$ into $\mathbb{R}_+$ that solves the integral equation

$$V_t\varphi = P_t\varphi - 2\int_0^t P_s\left((V_{t-s}\varphi)^2\right) ds.$$

The choice of the multiplicative factor 2 is only for convenience. The case of a general constant $\lambda > 0$ can be reduced to this one by a suitable scaling.

These measure-valued processes were introduced by Jirina (1958) and then studied by Watanabe (1968), Dawson (1977, 1978) and more recently by many others. At least two distinct methods may be used to construct superprocesses. The first one actually motivated the study of these processes. It is based on an approximation by systems of branching diffusions. One considers a finite number of "particles" in $\mathbb{R}^d$ which move independently according to the law of the solutions of equation (A). The particles die independently at an exponential rate. When a particle dies, it gives rise to either zero or two new particles with probability 1/2. These new particles move independently starting from the final position of their "father," and so on.... One then introduces the measure-valued process whose value at time t is the sum of the Dirac masses at the positions of all particles alive at time t. If the initial number of particles tends to ∞, after a suitable scaling in both time and space, this measure-valued process converges in distribution toward the superprocess $(X_t, t \geq 0)$ [for this approach, see, e.g., Ethier and Kurtz (1986), page 400, and Roelly-Coppoletta (1986)].

Another possible approach is to define (X_t) from its semigroup. One shows that there exists a unique semigroup on the space $\mathscr{M}(\mathbb{R}^d)$, whose Laplace functional is given in terms of the nonlinear semigroup (V_t) by the previous formula. The superprocess is then the Markov process associated with this semigroup [see Dynkin (1989b) and Fitzsimmons (1988)].

We propose here a third approach, which allows us to give a direct trajectorial construction of the superprocess. This approach also leads to a probabilistic interpretation of the nonlinear semigroup (V_t). We hope that our method will have some applications to the sample path properties of (X_t), of the type considered by Perkins (1988, 1989, 1990) and Dawson, Iscoe and Perkins (1989). One such application is described in Section 9 of the present work.

Let us briefly describe our method in the special case when (x_t) is Brownian motion in $\mathbb{R}^d$ and $\mu = \delta_{(0)}$, the Dirac measure at 0. Let us first consider a continuous function f from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $f(0) = 0$. The kernel

$$K(s,t) = \inf_{[s,t]} f(u)$$

is easily checked to be nonnegative definite. Therefore, we may consider the Gaussian process $(W_t,\ t \geq 0)$ with mean 0 and covariance matrix

$$\mathrm{Cov}(W_s, W_t) = K(s,t) \quad \mathrm{Id},$$

where Id denotes the $d \times d$ identity matrix. If we further assume that f is Hölder continuous, a simple application of the Kolmogorov lemma yields the existence of a continuous version of (W_t). We denote by $\mathbb{Q}^f$ the law of this version, on the canonical space $C(\mathbb{R}_+, \mathbb{R}^d)$ of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$.

We then randomize f. We denote by $R(df)$ the law, on the canonical space $C(\mathbb{R}_+, \mathbb{R}_+)$, of reflecting Brownian motion [i.e., the law of $(|\beta_t|,\ t \geq 0)$, where β is a standard linear Brownian motion started at 0]. Under the probability $R(df)$, we may consider the local time process of f, that is, the two-parameter continuous process $(L_t^a(f),\ a \geq 0,\ t \geq 0)$ such that, $R(df)$ a.e.:

(i) for every $a \geq 0$, the function $t \to L_t^a(f)$ is nondecreasing and increases only when $f(t) = a$;

(ii) for every measurable φ: $\mathbb{R}_+ \to \mathbb{R}_+$ and every $s \geq 0$,

$$\int_0^s \varphi(f(u))\,du = \int_{\mathbb{R}_+} \varphi(a) L_s^a(f)\,da.$$

We set

$$\eta = \eta(f) = \inf\{t;\ L_t^0(f) > 1\}.$$

Notice that $\eta < \infty$, $R(df)$ a.e.

Let $\mathbb{P}$ be the probability measure on $C(\mathbb{R}_+, \mathbb{R}_+) \times C(\mathbb{R}_+, \mathbb{R}^d)$ defined by

$$\mathbb{P}(df\,d\omega) = R(df)\mathbb{Q}^f(d\omega)$$

and let $(X_t,\ t \geq 0)$ be the $\mathscr{M}(\mathbb{R}^d)$-valued process, such that

$$\langle X_t(f,\omega), \varphi\rangle = \int_0^{\eta(f)} d_s L_s^t(f) \varphi(\omega(s)),$$

where in the right-hand side the notation $d_s L_s^t(f)$ means that the integration is with respect to the nondecreasing function $s \to L_s^t(f)$.

The process $(X_t,\ t \geq 0)$ is then, under the probability measure $\mathbb{P}(df\,d\omega)$, a superprocess associated with Brownian motion in $\mathbb{R}^d$, which starts from $\delta_{(0)}$.

This somewhat surprising result can be better understood by investigating the properties of the laws $\mathbb{Q}^f$, which are not apparent here because of the special features of the Gaussian case. The reason for introducing these measures can be explained as follows. The probabilistic structure of the superprocess involves two phenomena, the spatial motion of the "particles" and the underlying branching mechanism. If the spatial motion is well understood, the situation is quite different for the branching mechanism. Indeed this is the reason why it is often convenient to use discrete approximations, for which the branching structure can be more easily described [see in particular Perkins (1988)]. The key idea of our construction is that the underlying branching mechanism of a superprocess can be described by the tree structure of a

Brownian excursion, or rather of a Poisson process of Brownian excursions. The role of f in the pair (f, ω) is thus to determine the branching structure of the superprocess. The choice of ω, which is performed under the probability $\mathbb{Q}^f$, then determines the spatial motion of the particles, but only once the branching structure has been fixed. The reader who is unfamiliar with the relationship between Brownian excursions and branching processes should have a look at the figures in Neveu and Pitman (1989b) or in Le Gall (1989b).

The construction and main properties of the measures $\mathbb{Q}^f$, in the general case of the superprocess associated with equation (A), are given in Sections 2, 3 and 4. In Section 2 we work in a deterministic setting, with a fixed function f satisfying certain assumptions. We introduce a tree structure related to the excursions of the function f above all positive levels. In Section 3 we define a process indexed by the tree of excursions of f, which roughly speaking moves along each branch according to the law of solutions of (A). We also state some basic properties of this process. Finally, in Section 4 we show how to construct the measures $\mathbb{Q}^f$ from the process indexed by the tree, by using a suitable density argument, and we prove that these measures satisfy a sort of "branching property" (Proposition 4.3), which plays a basic role in the next sections.

In Section 5 we introduce the measure-valued process and check that it has the desired distribution. In contrast to the special case described above, it turns out that it is more convenient to construct first the "canonical measure" associated with the superprocess [see Kallenberg (1983) for canonical measures associated with random measures, and Dawson and Perkins (1989) and El Karoui and Roelly-Coppoletta (1988) for the particular case of superprocesses]. In our setting, this means that instead of taking for f a path of reflecting Brownian motion, which indeed would not satisfy the desired assumptions, we let f be a Brownian excursion. In other words, we replace the measure $R(df)$ with the Itô measure of positive excursions of linear Brownian motion. This leads us to the canonical measure of the superprocess, from which it is then easy to construct the superprocess itself, just as linear Brownian motion can be reconstructed from its excursions from 0. One may also notice that many sample path properties can be read from the canonical measure as well as from the superprocess itself.

In Section 6 we show that, without any additional effort, our construction yields a process taking values in the set of measures on $C(\mathbb{R}_+, \mathbb{R}^d)$, which contains more information than the superprocess in the sense that it keeps track of the paths followed by the particles. This process is the so-called historical process of Dawson and Perkins (1989) or the grand superprocess of Dynkin (1991) [see also Zähle (1988) for a related work]. Heuristically, if X_t is interpreted as the empirical measure on the set of all particles alive at time t (this interpretation only makes sense in the discrete approximation), the historical process at time t gives the corresponding measure on the set of the paths (stopped at time t) of these particles. This heuristic interpretation is made precise by using the discrete approximation and passing to the limit. A basic tool in this connection is the Neveu–Pitman theorem on branching processes in Brownian excursions [see Neveu and Pitman (1989a, b) and Le Gall (1989b)].

In Section 7 we apply our construction to simple proofs of two important properties of the historical process. We emphasize the connection between well-known properties of Brownian excursions and the corresponding results for superprocesses. On one hand, the structure of Brownian excursions above a fixed level leads to the inhomogeneous Markov property of the historical process. On the other hand, the properties of excursions above the minimum process of a linear Brownian motion allow us to recover a description of the Palm measure of the historical process given by Dawson and Perkins (1989).

In Section 8 we use our approach to derive a representation of the law of the superprocess at a fixed time $t > 0$. This representation is based on a tree model inspired by Neveu (1986). It leads to a probabilistic interpretation of the nonlinear semigroup (V_t), which may be compared to the work of Sznitman (1988). These results are closely related to a theorem of Le Gall (1989b) and to some limit theorems for branching trees due to Fleischmann and Siegmund-Schultze (1977, 1978). They should also be compared to the results of Dawson and Hochberg (1982), Section 7, in the slightly different context of the Fleming–Viot measure-valued process.

Finally, in Section 9 we give an application of our construction to the discontinuities of the support process. We recover a theorem of Perkins (1990), which in our approach is seen to be related to the well-known property that the local maxima of a Brownian path are distinct.

One may wonder whether the use of Brownian excursions is really necessary to our approach. The basic object seems to be the infinite branching tree "hidden in a Brownian excursion," which models the branching mechanism of the superprocess [see Rogers (1984) and Le Gall (1986) for the tree associated with the excursion of discrete random walk]. Note, however, that the Brownian local times provide natural measures on the level sets of this tree, and that these measures play an essential role in our construction. It is also very plausible that our construction can be applied to superprocesses associated with more general Markov processes; see Le Jan (1991) for a different but related approach.

Some of the results of the present work have been announced in the note [Le Gall (1989a)].

2. The tree of excursions of a continuous function.

Throughout this section we consider a continuous function f from $\mathbb{R}_+$ into $\mathbb{R}_+$ which satisfies the following assumption:

(H1) $f(0) = 0$, and there exists some $\tau = \tau(f) \in (0, \infty)$ such that $f(t) > 0$ if and only if $t \in (0, \tau)$.

Let $\mathscr{L}_f$ denote the set of times that correspond to a local minimum of the function f on $[0, \tau)$:

$$\mathscr{L}_f = \left\{ u \in [0, \tau); \text{there exists } \varepsilon > 0 \text{ such that } f(u) = \inf_{(u-\varepsilon, u+\varepsilon)} f(s) \right\}.$$

We agree that 0 belongs to $\mathscr{L}_f$. We also assume the following:

(H2) If $u, v \in \mathscr{L}_f$ and $u \neq v$, then $f(u) \neq f(v)$.

In particular, this implies that the function f is nonconstant on any nontrivial subinterval of $(0, \tau)$.

By definition, a *raw excursion* of f is a pair $(x, (a, b))$, where $x \geq 0$, (a, b) is a nonempty subinterval of $(0, \tau)$ and

(i) $f(a) = f(b) = x$,
(ii) $f(u) > x$ for any $u \in (a, b)$.

The set of raw excursions is clearly uncountable. We will consider its quotient set for the following equivalence relation. By definition,

$$(x, (a, b)) \sim (x', (a', b'))$$

if and only if either $x \leq x'$, $(a, b) \supset (a', b')$ and

$$\sup_{[a,b]} f(u) = \sup_{[a',b']} f(u), \qquad \sup_{[a,a']} f(u) < \sup_{[a,b]} f(u);$$

or the same holds when we interchange $(x, (a, b))$ with $(x', (a', b'))$.

The condition $\sup_{[a,a']} f(u) < \sup_{[a,b]} f(u)$ is needed to ensure that the relation $\sim$ is transitive. Note, however, that this condition is superfluous if the local maxima of f are distinct, which is the case when f is a typical Brownian excursion.

We denote by E_f the set of equivalence classes of raw excursions, which are simply called *excursions*. The next lemma is elementary.

LEMMA 2.1. *Let $e \in E_f$. There exists a unique representative of e, denoted by $(x(e), (a(e), b(e)))$, such that, for any other representative $(x, (a, b))$, one has $x \geq x(e)$ and $(a, b) \subset (a(e), b(e))$. Moreover, $a(e)$ or $b(e)$ belongs to $\mathscr{L}_f$, and the map $e \to x(e)$ is a bijection from E_f onto the set $\mathscr{M}_f$ of all local minima of f.*

By assumption (H2), f induces a bijection between $\mathscr{L}_f$ and $\mathscr{M}_f$. For any $s \in \mathscr{L}_f$, we denote by $\Phi_f(s)$ the unique excursion such that $x(\Phi_f(s)) = f(s)$.

In the next sections, the space variable for the function f will become the time variable for a certain stochastic process. In this setting the number $x(e)$ is understood as the birth time of the excursion e. The death time of e is

$$y(e) = \sup_{[a(e), b(e)]} f(u),$$

and its height is $h(e) = y(e) - x(e)$. Finally, we set $e_0 = \Phi_f(0)$, the excursion having $(0, (0, \tau))$ as a representative.

We now introduce an order on E_f. By definition, $e < e'$ if and only if $(a(e), b(e)) \supset (a(e'), b(e'))$. This implies $x(e) \leq x(e')$, $y(e) \geq y(e')$ and thus $h(e) \geq h(e')$. We set $e \ll e'$ if $e < e'$, $e \neq e'$ and if whenever $e < e'' < e'$ we have $e'' = e$ or $e'' = e'$.

LEMMA 2.2. *Let $e \in E_f$. There exists an integer $n \in \mathbb{N}$ and a finite sequence $(e_1, \dots, e_n)$ of elements of E_f, such that*

$$e_0 \ll e_1 \ll \cdots \ll e_{n-1} \ll e_n = e.$$

Moreover, $e' < e$ if and only $e' = e_i$ for some $i \in \{0, 1, \dots, n\}$. In particular, n and the sequence $(e_1, \dots, e_n)$ are unique.

The proof is easy and left to the reader. The partially ordered set E_f may be interpreted as the tree of excursions of the function f. Each $e \in E_f$ represents a branch of this tree, of length $h(e)$. This branch originates from the branch represented by e' defined by $e' \ll e$. A subset F of E_f is called saturated if the relations $e \in F$ and $e' < e$ imply $e' \in F$. An element e of F is then called extreme if, for $e' \in F$, $e < e'$ implies $e = e'$.

3. The process indexed by the tree. We now propose to define a stochastic process indexed by the tree of excursions of the function f. More precisely, for every $e \in E_f$, we will define an $\mathbb{R}^d$-valued process $(X^e(t), t \geq 0)$, in such a way that X^e will be a solution, stopped at time $y(e)$, of equation (A) and that, if $e \ll e'$, X^e and $X^{e'}$ will coincide on the time interval $[0, x(e')]$.

We let σ, b be as in the Introduction and we fix $z \in \mathbb{R}^d$. We also denote by $(B^e, e \in E_f)$ a family, indexed by E_f, of independent Brownian motions in $\mathbb{R}^n$. The cardinality of a finite set F is denoted by $\mathrm{Card}(F)$.

PROPOSITION 3.1. *For every $e \in E_f$, let X^e be the $\mathbb{R}^d$-valued process defined by the stochastic integral equation*

$$X^e(t) = z + \sum_{i=0}^{n} \left(\int_0^t 1_{(x(e_i), x(e_{i+1})]}(s) \sigma(X^e(s))\, dB^{e_i}(s) + \int_0^t 1_{(x(e_i), x(e_{i+1})]}(s) b(X^e(s))\, ds \right),$$

where n and the family $(e_1, \dots, e_n)$ are as in Lemma 2.2, and by convention $x(e_{n+1}) = y(e)$.

Then X^e is a solution started from z, stopped at time $y(e)$, to equation (A).

Moreover, the law of $(X^e, e \in E_f)$ is characterized by the following two properties:

(i) *The process X^{e_0} is a solution started from z, stopped at time $y(e_0)$, to equation* (A).

(ii) *Let F be a finite saturated subset of E_f such that* $\mathrm{Card}(F) \geq 2$, *let e be an extreme element of F and let $e' \in F$ be determined by $e' \ll e$. Then $X^e(t) = X^{e'}(t)$ for any $t \leq x(e)$. Moreover, the process $(X^e(x(e) + t), t \geq 0)$ is independent of $(X^g, g \in F - \{e\})$ conditionally on $X^{e'}(x(e))$; its conditional law is that of a solution of* (A) *started from $X^{e'}(x(e))$, stopped at time $h(e)$.*

The proof of Proposition 3.1 is straightforward and will be left to the reader. For property (ii) one uses the fact that the process $(X^e(x(e)+t),\ t \geq 0)$ is a measurable functional of $X^e(x(e)) = X^{e'}(x(e))$ and of B^e, and is therefore independent of $(B^g,\ g \in F - \{e\})$ conditionally on $X^e(x(e))$.

The construction of the family (X^e) may clearly be extended to the case of a general Markov process. In this general situation, one argues by induction using property (ii).

For every $u \in \mathscr{L}_f$, we define a process $(\Gamma_u(t),\ t \geq 0)$ by setting

$$\Gamma_u(t) = X^e(t \wedge x(e)),$$

where $e = \Phi_f(u)$, and the mapping $\Phi_f\colon \mathscr{L}_f \to E_f$ was defined after Lemma 2.1. We will prove in Section 4 that, under suitable assumptions, the mapping $u \to \Gamma_u$ can be continuously extended to any $u \in [0, \tau]$. We first need to record some properties of this map.

PROPOSITION 3.2. (i) *For every* $u \in \mathscr{L}_f$, *the process* Γ_u *is a solution of equation* (A), *started from* z *and stopped at time* $f(u)$.

(ii) *Let* $u, u' \in \mathscr{L}_f$, $u \leq u'$, *and let* $u'' \in [u, u']$ *be the unique element of* $\mathscr{L}_f$, *such that*

$$f(u'') = \inf_{[u,u']} f(v).$$

Then $\Gamma_u(t) = \Gamma_{u'}(t) = \Gamma_{u''}(t)$ *for any* $t \in [0, f(u'')]$. *Moreover, the two processes* $(\Gamma_u(f(u'')+t),\ t \geq 0)$, $(\Gamma_{u'}(f(u'')+t),\ t \geq 0)$ *are independent conditionally given* $\Gamma_{u''}(f(u''))$.

PROOF. Property (i) is immediate, since by construction $f(u) = x(\Phi_f(u))$. For property (ii), let $e = \Phi_f(u)$, $e' = \Phi_f(u')$; we restrict our attention to the case when e, e' are not comparable (the other case is simpler). Set $e'' = \Phi_f(u'')$ and let $\tilde{e}$ be such that $\tilde{e} \ll e''$. Then $\tilde{e} < e$, $\tilde{e} < e'$. Moreover, we have either $e'' < e$ or $e'' < e'$, but not both. Assume that $e'' < e$, the alternative case being symmetric. There exist two integers $p \geq 1$, $q \geq 1$, and $p+q$ distinct elements of E_f, denoted by $e^1, \dots, e^p, e'^1, \dots, e'^q$, such that

$$\tilde{e} \ll e'' = e^1 \ll e^2 \ll \cdots \ll e^p = e,$$
$$\tilde{e} \ll e'^1 \ll \cdots \ll e'^q = e'.$$

Moreover, $x(e'^1) > x(e^1) = f(u'')$, by the definition of u''. From our construction, we have for any $t \leq x(e^1)$, $X^e(t) = X^{\tilde{e}}(t) = X^{e'}(t)$. On the other hand, $(X^e(f(u'')+t),\ t \geq 0)$ is a measurable functional of $X^e(f(u''))$ and the Brownian motions $B^{e^1}, \dots, B^{e^p}$, whereas $(X^{e'}(f(u'')+t),\ t \geq 0)$ is a functional of $X^e(f(u''))$ and the Brownian motions $B^{\tilde{e}}, B^{e'^1}, \dots, B^{e'^q}$. This gives the desired conditional independence. □

We shall need a characterization of the law of the family $(\Gamma_u,\ u \in \mathscr{L}_f)$, which is provided by the next proposition. A subset U of $\mathscr{L}_f$ is called *saturated* if whenever $u, u' \in U$, $u < u'$, the time $u'' \in [u, u']$ uniquely determined by $f(u'') = \inf_{[u,u']} f(v)$ also belongs to U.

PROPOSITION 3.3. *The law of* $(\Gamma_u,\ u \in \mathscr{L}_f)$ *is characterized by the following two properties*:

(i) $\Gamma_0(t) = z$ *for any* $t \geq 0$.

(ii) *Let* U *be a finite saturated subset of* $\mathscr{L}_f$, *with* $0 \in U$ *and* $\mathrm{Card}(U) \geq 2$, *and let* $u \in U$ *be defined by*

$$f(u) = \sup_{v \in U} f(v);$$

define $u', u'' \in U$ *by*

$$u' = \sup([0,u) \cap U), \qquad u'' = \inf((u,T) \cap U),$$

where by convention $\inf \varnothing = 0$. *Set* $\tilde{u} = u'$ *if* $f(u') \geq f(u'')$, $\tilde{u} = u''$ *otherwise. Then* $\Gamma_u(t) = \Gamma_{\tilde{u}}(t)$ *for any* $t \leq f(\tilde{u})$, *and the process* $(\Gamma_u(f(\tilde{u}) + t),\ t \geq 0)$ *is, conditionally given* $\Gamma_{\tilde{u}}(f(\tilde{u}))$, *independent of* $(\Gamma_v,\ v \in U - \{u\})$; *its conditional law is that of a solution of* (A) *started from* $\Gamma_{\tilde{u}}(f(\tilde{u}))$ *and stopped at the time* $f(u) - f(\tilde{u})$.

PROOF. Property (i) is obvious, whereas property (ii) can be derived through the same arguments as in the proof of Proposition 3.2. The point is that the process $(\Gamma_u(f(\tilde{u}) + t),\ t \geq 0)$ is a measurable functional of $\Gamma_{\tilde{u}}(f(\tilde{u}))$ and of a finite number of Brownian motions B^e independent of $(\Gamma_v,\ v \in U - \{u\})$. The fact that properties (i) and (ii) characterize the law of $(\Gamma_u,\ u \in \mathscr{L}_f)$ is then easily checked using induction on $\mathrm{Card}(U)$. □

4. The extended process and its properties. Our first goal in this section is to extend the mapping $u \to \Gamma_u$ to the whole interval $[0,\tau]$. We shall need the following additional assumptions on the function f:

(H3) $\mathscr{L}_f$ is dense in $[0,\tau]$;

(H4) f satisfies a Hölder condition with exponent $\delta > 0$.

PROPOSITION 4.1. *Under assumptions* (H1), (H2), (H3) *and* (H4), *the process* $(\Gamma_u,\ u \in \mathscr{L}_f)$ *has a unique continuous extension to the interval* $[0,\tau]$. *The extended process, denoted by* $(\Gamma_u,\ u \in [0,\tau])$, *still satisfies properties* (i) *and* (ii) *of Proposition* 3.2.

PROOF. Set $\|g\| = \sup_{t \in \mathbb{R}_+} |g(t)|$, for any bounded function g from $\mathbb{R}_+$ into $\mathbb{R}^d$. Let $u, u' \in \mathscr{L}_f$ such that $u \leq u'$. With the notation of Proposition 3.2(ii), we have, for any $p \in \mathbb{N}$, $p \geq 2$,

$$E[\|\Gamma_u - \Gamma_{u'}\|^p] \leq C_p\big(E[\|\Gamma_u - \Gamma_{u''}\|^p] + E[\|\Gamma_{u'} - \Gamma_{u''}\|^p]\big).$$

On the other hand,

$$(\Gamma_u - \Gamma_{u''})(t) = (\Gamma_{u'} - \Gamma_{u''})(t) = 0 \quad \text{if } t \leq f(u''),$$

and $(\Gamma_u(\mathscr{L}(u'') + t),\ t \geq 0)$, resp. $(\Gamma_{u'}(\mathscr{L}(u'') + t),\ t \geq 0)$, is distributed as a solution of (A) stopped at time $f(u) - f(u'')$, resp. $f(u') - f(u'')$. It easily follows that

$$E[\|\Gamma_u - \Gamma_{u''}\|^p] \le C'_p\big((f(u) - f(u''))^{1/2} + f(u) - f(u'')\big)^p,$$

$$E[\|\Gamma_{u'} - \Gamma_{u''}\|^p] \le C'_p\big((f(u') - f(u''))^{1/2} + f(u') - f(u'')\big)^p,$$

where the constant C'_p depends only on p and the bounds on σ and b. By assumption (H4),

$$\begin{aligned}\sup(f(u) - f(u''), f(u') - f(u'')) &\le C \sup(|u - u''|^\delta, |u' - u''|^\delta)\\ &\le C|u - u'|^\delta.\end{aligned}$$

Therefore, for any $p \ge 2$ there exists a constant C''_p, such that

$$E[\|\Gamma_u - \Gamma_{u'}\|^p] \le C''_p|u - u'|^{\delta p/2},$$

when $|u - u'| \le 1$. Standard arguments then imply that the process $(\Gamma_u, u \in \mathscr{L}_f)$ is a.s. uniformly continuous (even Hölder continuous). By assumption (H3), this process has a unique continuous extension to $[0, \tau]$. A straightforward passage to the limit shows that the extended process still satisfies properties (i) and (ii) of Proposition 3.2. □

We shall often use the following simple consequence of property (ii) of Proposition 3.2. If $(x, (a, b))$ is a raw excursion of f, then, for every $u \in [a, b]$, $t \in [0, x]$, $\Gamma_u(t) = \Gamma_a(t) = \Gamma_b(t)$. The continuity of Γ implies that this property holds simultaneously for all raw excursions of f outside a set of zero probability.

NOTATION. We agree that if $u \ge \tau$, $\Gamma_u(t) = z$ for any $t \ge 0$. The process $(\Gamma_u, u \ge 0)$ is then continuous. We denote by $\mathbb{Q}_z^f$ its law on the canonical space $C(\mathbb{R}_+, C(\mathbb{R}_+, \mathbb{R}^d))$. Finally, we denote by C^* the set of all functions $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ that satisfy assumptions (H1), (H2), (H3) and (H4). The set C^* is a Borel subset of $C(\mathbb{R}_+, \mathbb{R}_+)$.

PROPOSITION 4.2. *The mapping $(z, f) \to \mathbb{Q}_z^f$, defined on $\mathbb{R}^d \times C^*$ and with values in the space $\mathscr{M}_1(C(\mathbb{R}_+, C(\mathbb{R}_+, \mathbb{R}^d)))$ of all probability measures on $C(\mathbb{R}_+, C(\mathbb{R}_+, \mathbb{R}^d))$, is measurable.*

PROOF. For any $T > 0$, $\delta > 0$, $C > 0$, denote by $C^*_{T,\delta,C}$ the closed subset of C^* consisting of all functions f, such that

$$\mathrm{supp}(f) \subset [0, T],$$

and, for any $s, t \ge 0$,

$$|f(s) - f(t)| \le C|t - s|^\delta.$$

It is enough to prove that the mapping $(z, f) \to \mathbb{Q}_z^f$ is continuous when restricted to $\mathbb{R}^d \times C^*_{T,\delta,C}$. Let (z_n, f_n) be a convergent sequence in $\mathbb{R}^d \times C^*_{T,\delta,C}$, and let (z, f) be its limit. The bounds of the proof of Proposition 4.1 imply that the sequence $(\mathbb{Q}_{z_n}^{f_n})$ is tight. It is then enough to check that any convergent subsequence must converge toward $\mathbb{Q}_z^f$. Consider such a subsequence, still denoted by $(\mathbb{Q}_{z_n}^{f_n})$, and let $\tilde{\mathbb{Q}}$ be its limit.

To show that $\tilde{\mathbb{Q}} = \mathbb{Q}_z^f$, we use the characterization of $\mathbb{Q}_z^f$ given by properties (i) and (ii) of Proposition 3.3. We let $U = \{u_1, \dots, u_p\}$ be a saturated (with respect to the function f) subset of $\mathscr{L}_f$, such that $p = \mathrm{Card}(U) \geq 2$. Without loss of generality, we may assume

$$f(u_1) < f(u_2) < \cdots < f(u_p).$$

We may choose $\varepsilon > 0$ so that the intervals $[u_i - \varepsilon, u_i + \varepsilon]$ are disjoint and the restriction of f to $[u_i - \varepsilon, u_i + \varepsilon]$ has a unique global minimum at u_i. Then, for n large enough, for any $i \in \{1, \dots, p\}$, the restriction of f_n to $[u_i - \varepsilon, u_i + \varepsilon]$ must have a unique global minimum, attained at $v_i^n \in (u_i - \varepsilon, u_i + \varepsilon)$. Furthermore,

$$\lim_{n \to \infty} v_i^n = u_i,$$

so that, for n large, we have

$$f_n(v_1^n) < f_n(v_2^n) < \cdots < f_n(v_p^n).$$

Finally, a simple argument shows that, if n is large enough, the set $U_n = \{v_1^n, \dots, v_p^n\}$ is saturated with respect to f_n. By writing property (ii) of Proposition 3.3 for f_n and U_n, under the law $\mathbb{Q}_{z_n}^{f_n}$, we get in the limit that the same property holds for f and U under $\tilde{\mathbb{Q}}$. Since property (i) is immediate, this gives $\tilde{\mathbb{Q}} = \mathbb{Q}_z^f$. □

The next result will play an important role in Section 5. We fix $f \in C^*$, $z \in \mathbb{R}^d$, and we consider the associated process $(\Gamma_u, u \geq 0)$. Let $a > 0$ be fixed. Denote by $(a, (\alpha_i, \beta_i))$, $i \in I$, the raw excursions of f above level a. Here the set of indices I is at most countable (I is empty if a is large). For any $i \in I$ and $u \geq 0$, set

$$f_i(u) = f((\alpha_i + u) \wedge \beta_i) - a.$$

Observe that $f_i \in C^*$ for any $i \in I$. We also set

$$\sigma_{(a)}(u) = \inf\left\{s, \int_0^s 1_{[0,a]}(f(v))\,dv > u\right\}$$

and

$$f_{(a)}(u) = f(\sigma_{(a)}(u)).$$

Again, $f_{(a)} \in C^*$. Finally, for any $i \in I$, for any $u \geq 0$, $t \geq 0$, we set

$$\Gamma_u^i(t) = \Gamma_{(\alpha_i + u) \wedge \beta_i}(a + t), \qquad \Gamma_u^{(a)}(t) = \Gamma_{\sigma_{(a)}(u)}(t).$$

PROPOSITION 4.3. *The process $(\Gamma_u^{(a)}, u \geq 0)$ is continuous, and its law is $\mathbb{Q}_z^{f_{(a)}}$. Moreover, conditionally given $(\Gamma_{\alpha_i}(a), i \in I)$, the processes $(\Gamma_u^i, u \geq 0)$,*

$i \in I$, are independent and independent of $(\Gamma_u^{(a)}, u \geq 0)$. The conditional law of Γ^j given that $\Gamma_{\alpha_i}(a) = z_i$, $i \in I$, is $\mathbb{Q}_{z_j}^{f_j}$.

PROOF. The continuity of $\Gamma^{(a)}$ derives from the fact that $\Gamma_{\alpha_i} = \Gamma_{\beta_i}$ for any $i \in I$, a.s. The identification of the law of $\Gamma^{(a)}$ follows then easily from Proposition 3.3. Indeed, every finite saturated subset of $\mathscr{L}_{f_{(a)}}$ corresponds to a finite saturated subset of $\mathscr{L}_f$. By writing property (ii) of Proposition 3.3 for the latter, and using the definition of $\Gamma^{(a)}$, we get that the same property (ii) holds for $\Gamma^{(a)}$, with respect to the function $f_{(a)}$. Since property (i) is immediate, this suffices to show that the law of $\Gamma^{(a)}$ is $\mathbb{Q}^{f_{(a)}}$.

The other assertions follow from our construction of Γ. For any $i \in I$ denote by $e_{(i)}$ the excursion represented by $(a, (\alpha_i, \beta_i))$. Set

$$E_i = \{e \in E_f; (a(e), b(e)) \subset (\alpha_i, \beta_i), x(e) > a\},$$

$$E_{(a)} = \{e \in E_f; y(e) \leq a\}.$$

The process $\Gamma^{(a)}$ is measurable with respect to the σ-algebra generated by the Brownian motions B^e, $e \in E_{(a)}$, and by processes $(B^{e_{(i)}}(t \wedge a), t \geq 0)$, $i \in I$. Indeed, it is enough to check that, for any $u \in \mathscr{L}_f$ such that $f(u) < a$, the variable Γ_u is measurable with respect to this σ-algebra. If $e = \Phi_f(u)$, we have

$$e_0 \ll e_1 \ll \cdots \ll e_n = e,$$

where $e_0, e_1, \ldots, e_n \in E_{(a)} \cup \{e_{(i)}, i \in I\}$. The definition of Γ_u (Proposition 3.1) then leads to the desired result.

For any $i \in I$, the process Γ^i is measurable with respect to the σ-algebra generated by $\Gamma_{\alpha_i}(a)$ $[= \Gamma_{\beta_i}(a)]$, by the Brownian motion $(B^{e_{(i)}}(a + t) - B^{e_{(i)}}(a), t \geq 0)$ and by processes $(B^e, e \in E_i)$. Indeed let $u \in \mathscr{L}_f$ such that $u \in (\alpha_i, \beta_i)$. The associated excursion e satisfies

$$e_0 \ll e_1 \ll \cdots \ll e_p = e_{(i)} \ll \cdots \ll e_n = e$$

for some $e_{p+1}, \ldots, e_n \in E_i$; by Proposition 3.1, if $t \geq a$,

$$X^e(t) = X^e(a) + \sum_{j=p}^{n} \left(\int_a^t 1_{(x(e_j), x(e_{j+1})]}(s) \sigma(X^e(s))\, dB^{e_j}(s) + \int_a^t 1_{(x(e_j), x(e_{j+1})]}(s) b(X^e(s))\, ds \right).$$

We also have

$$X^e(a) = \Gamma_u(a) = \Gamma_{\alpha_i}(a).$$

The desired measurability property for Γ^i follows from these observations.

The fact that $\Gamma^{(a)}$ and the processes Γ^i, $i \in I$, are conditionally independent given Γ_{α_i}, $i \in I$, is now a consequence of the independence of the Brownian motions B^e, $e \in E_f$, together with the independence property of the increments of the processes B^{e_i}, $i \in I$.

We finally determine the conditional law of Γ^i. One may either proceed as for $\Gamma^{(a)}$ or argue as follows. The mapping $u \to \alpha_i + u$ is a bijection from $\mathscr{L}_{f_i} - \{0\}$ onto $(\alpha_i, \beta_i) \cap \mathscr{L}_f$. This mapping induces a one-to-one correspondence λ_i between E_{f_i} and the set $E_i \cup \{e_{(i)}\}$. Notice that this correspondence preserves the order structure on excursions. For every $e \in E_{f_i}$ and for any $t \geq 0$, let

$$B'^e(t) = B^{\lambda_i(e)}(a + t) - B^{\lambda_i(e)}(a).$$

The processes B'^e, $e \in E_{f_i}$, are independent Brownian motions, and are independent of the variable $\Gamma_{\alpha_i}(a)$ [which is measurable with respect to the σ-algebra generated by the processes B^e, $e \in E_{(a)}$, and $(B^{e_{(i)}}(t \wedge a),\ t \geq 0)$, $i \in I$]. It should now be clear that the process $(\Gamma_u^i,\ u \in \mathscr{L}_{f_i})$ is constructed for the function f_i, from $\Gamma_{\alpha_i}(a)$ and the Brownian motions B'^e, $e \in E_{f_i}$, by the same device that was used for the function f to construct $(\Gamma_u,\ u \in \mathscr{L}_f)$ from z and the Brownian motions B^e, $e \in E_f$. This completes the proof of Proposition 4.3. □

5. The construction of the measure-valued process. We now propose to apply the results of the previous sections to the construction of the superprocess associated with solutions of equation (A). As a matter of fact, we will first introduce the corresponding canonical measure, from which it is then easy to exhibit a construction of the superprocess itself. The function f, which was deterministic in the previous sections, will now be random, and more precisely it will be distributed according to the Itô measure of excursions of linear Brownian motion.

We shall work on the space

$$\Theta := C^* \times C(\mathbb{R}_+, C(\mathbb{R}_+, \mathbb{R}^d)).$$

An element θ of Θ is denoted $\theta = (f, \omega)$. For any $z \in \mathbb{R}^d$, we consider the σ-finite measure M_z on Θ defined by

$$M_z(df\,d\omega) = n(df)\mathbb{Q}_z^f(d\omega),$$

where $n(df)$ denotes the Itô measure of positive excursions of linear Brownian motion, normalized so that

$$n\Big(\sup_{s \geq 0} f(s) > \varepsilon\Big) = (2\varepsilon)^{-1}$$

for any $\varepsilon > 0$. Notice that the definition of M_z makes sense. On one hand, the well-known properties of Brownian motion show that $n(df)$ a.e., $f \in C^*$, so that n may be viewed as a measure on C^*. On the other hand, the mapping $f \to \mathbb{Q}_z^f$ is measurable by Proposition 4.2.

We will now introduce a measure-valued process defined on Θ. We need the notion of excursion local time. There exists a measurable mapping, $(a, t, f) \to l_t^a(f)$, defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times C^*$ and taking values in $\mathbb{R}_+$, such that the following three properties hold:

(i) for any $f \in C^*$, the map $(a,t) \to l_t^a(f)$ is continuous and increasing in the variable t;

(ii) for any $f \in C^*$, $a, t \in \mathbb{R}_+$, $l_t^a(f) = l_{t \wedge \tau(f)}^a(f)$;

(iii) $n(df)$ a.e., for any nonnegative measurable function φ and $t \geq 0$,

$$\int_0^{t \wedge \tau(f)} \varphi(f(s))\, ds = \int_0^\infty \varphi(a) l_t^a(f)\, da.$$

The existence of this mapping is easily derived from the well-known properties of Brownian local times [see, e.g., the last chapter of Rogers and Williams (1987)]. The uniqueness (up to a set of zero n-measure) follows from properties (i), (ii) and (iii). Interesting applications of the notion of excursion local time, somehow related to the present work, are developed in Pitman and Yor (1982).

The measure associated with the increasing function $t \to l_t^a(f)$ is denoted by $d_s l_s^a(f)$. This measure is supported on the set $\{s;\ f(s) = a\}$. It is well known that, $n(df)$ a.e.,

$$\sup_{s \geq 0} f(s) = \inf\{a > 0;\, l_\infty^a(f) = 0\}.$$

Finally, for any $\varepsilon > 0$, the law of $l_\infty^\varepsilon(f)$ under $n(df)$, conditionally on $\{\sup f(s) > \varepsilon\}$, is exponential with mean 2ε.

For any $(f, \omega) \in \Theta$ and $a \geq 0$, we define $Y_a(f, \omega) \in \mathscr{M}(\mathbb{R}^d)$ by setting

$$\langle Y_a(f, \omega), \varphi \rangle = \int_0^\infty d_s l_s^a(f) \varphi(\omega(s)(a))$$

for any measurable function $\varphi\colon \mathbb{R}^d \to \mathbb{R}_+$. Notice that for $a > 0$,

$$M_z(\langle Y_a(f, \omega), \varphi \rangle) = \int n(df) \int_0^\infty d_s l_s^a(f) \mathbb{Q}_z^f(\varphi(\omega(s)(a))) = P_a \varphi(z),$$

where (P_a) denotes the semigroup of solutions of equation (A). Indeed, we know by Proposition 4.1 that $\omega(s)$ is under $\mathbb{Q}_z^f$ distributed as a solution of (A) started from z and stopped at time $f(s)$. Moreover, it is plain that $n(l_\infty^a) = 1$ for any $a > 0$.

The continuity of $(a, t) \to l_t^a(f)$ implies that the mapping $a \to d_s l_s^a(f)$ is continuous [for the weak topology on $\mathscr{M}(\mathbb{R}_+)$], and then that for every $(f, \omega) \in \Theta$ the mapping $a \to Y_a(f, \omega)$ is continuous.

THEOREM 5.1. *If* $0 < a < b$,

$$M_z(\exp - \langle Y_b, \varphi \rangle | (Y_u, 0 \leq u \leq a)) = \exp - \langle Y_a, V_{b-a} \varphi \rangle,$$

where, for any $u > 0$, $y \in \mathbb{R}^d$,

$$V_u \varphi(y) = M_y(1 - \exp - \langle Y_u, \varphi \rangle).$$

REMARK. The measure M_z attributes a finite mass to the set $\{Y_a(f,\omega) \neq 0\} = \{l_\infty^a(f) > 0\}$, which makes it easy to define the conditional expectation involved in the statement of the theorem.

PROOF. We start with some notation. We will first introduce a filtration on Θ naturally associated with the process (Y_a). As in Section 4, we set, for $f \in C^*$ and $a > 0$,

$$\sigma_{(a)}^f(t) = \inf\left\{u;\ \int_0^u 1_{[0,a]}(f(v))\,dv > t\right\}.$$

We let $\mathscr{E}_a$, resp. $\mathscr{E}_a^0$, be the σ-field on Θ generated by the mapping $(f,\omega) \to (f \circ \sigma_{(a)}^f, \omega \circ \sigma_{(a)}^f)$, resp. the mapping $(f,\omega) \to f \circ \sigma_{(a)}^f$, and augmented by the sets that are M_z-negligible for every $z \in \mathbb{R}^d$. Observe that $(\mathscr{E}_a^0, a \geq 0)$ is the so-called excursion filtration, which in the case of Brownian motion has been used by various authors [see, e.g., Jeulin (1985) and McGill (1986)].

For every $a \geq 0$, Y_a is $\mathscr{E}_a$-measurable. Indeed, for $f \in C^*$ and $a \geq 0$, let $\lambda_f^a(ds)$ be the measure on $\mathbb{R}_+$ defined by

$$\lambda_f^a([0,t]) = l^a_{\sigma_{(a)}^f(t)}(f).$$

The mapping $(f,\omega) \to \lambda_f^a$ is then $\mathscr{E}_a^0$-measurable:

$$\lambda_f^a([0,t]) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{\varepsilon}\int_0^t 1_{[a-\varepsilon,a]}\big(f \circ \sigma_{(a)}^f(s)\big)\,ds$$

for every $t \geq 0$, $n(df)$ a.e. Furthermore,

$$\langle Y^a(f,\omega),\varphi\rangle = \int_0^\infty d_s l_s^a(f)\varphi(\omega(s)(a)) = \int_0^\infty \lambda_f^a(ds)\varphi\big(\omega \circ \sigma_{(a)}^f(s)(a)\big),$$

$M_z(df\,d\omega)$ a.e., and the described result follows. The same argument shows that $l_\infty^a(f) = \lambda_f^a(\mathbb{R}_+)$ is $\mathscr{E}_a^0$-measurable.

In view of the previous remarks, it is enough to show that

$$M_z\big(\exp - \langle Y_b,\varphi\rangle | \mathscr{E}_a\big) = \exp - \langle Y_a, V_{b-a}\varphi\rangle.$$

The proof proceeds in two steps. First, we will condition with respect to the σ-field $\mathscr{E}_a \vee \mathscr{E}_\infty^0$, and second, with respect to $\mathscr{E}_a$. The key ingredient of the first step is Proposition 4.3. For the second one, we shall need the next lemma.

LEMMA 5.2. *Let $a > 0$. For any $f \in C^*$, let $(a,(\alpha_i,\beta_i))$, $i \in I_f$, be the raw excursions of f above level a, and let f_i, $i \in I_f$, be defined as in Section 4. Let $N^f(dg\,dl)$ be the atomic measure on $C^* \times \mathbb{R}_+$ defined by*

$$N^f(dg\,dl) = \sum_{i \in I_f} \delta_{(f_i,l_i)}(dg\,dl),$$

where $l_i = l_{\alpha_i}^a(f)$.

The measure N^f is, under M_z, independent of $\mathscr{E}_a$ conditionally, given $l_\infty^a(f)$. Its conditional law is that of a Poisson measure with intensity

$$1_{(l \le l_\infty^a(f))} n(dg)\, dl.$$

PROOF. If $\mathscr{E}_a$ is replaced by $\mathscr{E}_a^0$, Lemma 5.2 becomes a standard result of excursion theory of linear Brownian motion, whose proof uses the independence of the excursions above and below a fixed level. In order to justify this replacement, it is enough to show that N^f and $\mathscr{E}_a$ are conditionally independent given $\mathscr{E}_a^0$. However, the first assertion of Proposition 4.3 shows that the conditional distribution under M_z of $\omega \circ \sigma_{(a)}^f$, given $\mathscr{E}_\infty^0$, is $\mathbb{Q}_z^{f \circ \sigma_{(a)}^f}$, hence depends only on $f \circ \sigma_{(a)}^f$. It follows that $\mathscr{E}_\infty^0$ and $\mathscr{E}_a$ are conditionally independent given $\mathscr{E}_a^0$, which yields the desired result. □

We now complete the proof of Theorem 5.1.

First step: For every $f \in C^*$ there is only a finite number $n = n(f)$ of excursions of f above level a which hit b. We denote by (α_j, β_j), $j \in \{1, \dots, n\}$, the corresponding intervals and we set

$$f_j(t) = f\big((\alpha_j + t) \wedge \beta_j\big) - a, \qquad \omega_j(t)(u) = \omega\big((\alpha_j + t) \wedge \beta_j\big)(a + u).$$

Then

$$\begin{aligned} &M_z\big(\exp - \langle Y_b, \varphi\rangle | \mathscr{E}_\infty^0 \vee \mathscr{E}_a\big) \\ &\quad = \mathbb{Q}_z^f\big(\exp - \langle Y_b, \varphi\rangle | \omega \circ \sigma_{(a)}^f\big) \\ &\quad = \mathbb{Q}_z^f\left(\exp - \sum_{j=1}^n \int_{\alpha_j}^{\beta_j} d_s l_s^b(f) \varphi(\omega(s)(b)) \middle| \omega \circ \sigma_{(a)}^f\right) \\ &\quad = \mathbb{Q}_z^f\left(\exp - \sum_{j=1}^n \int_0^\infty d_s l_s^{b-a}(f_j) \varphi\big(\omega_j(s)(b-a)\big) \middle| \omega \circ \sigma_{(a)}^f\right). \end{aligned}$$

The first equality is checked by multiplying by test functions of the form $F(f)G(\omega \circ \sigma_{(a)}^f)$. The other two are straightforward consequences of the definitions.

We then use Proposition 4.3 which implies that, under $\mathbb{Q}_z^f$, conditionally given $(\omega(\alpha_j)(a);\ j = 1, \dots, n)$, the processes ω_j are independent, and are independent of $\omega \circ \sigma_{(a)}^f$. Furthermore, the conditional law of ω_j is $\mathbb{Q}_{\omega(\alpha_j)(a)}^{f_j}$. Since the variables $\omega(\alpha_j)(a)$ are measurable functionals of $\omega \circ \sigma_{(a)}^f$ (when f is fixed), we obtain

$$M_z\big(\exp - \langle Y_b, \varphi\rangle | \mathscr{E}_\infty^0 \vee \mathscr{E}_a\big) = \prod_{j=1}^n H\big(f_j, \omega(\alpha_j)(a)\big),$$

where

$$H(f,y) = \mathbb{Q}_y^f\left(\exp - \int_0^\infty d_s l_s^{b-a}(f)\varphi(\omega(s)(b-a))\right)$$
$$= \mathbb{Q}_y^f(\exp - \langle Y_{b-a}(f,\cdot),\varphi\rangle).$$

Second step: We will now condition with respect to $\mathscr{E}_a$. We need the following notation:

$$\tau_a^f(u) = \inf\{t; l_t^a(f) > u\},$$

where by convention $\inf\varnothing = \tau(f)$. The (right-continuous) process $\omega\circ\tau_a^f$ is $\mathscr{E}_a$-measurable. Indeed,

$$\omega\circ\tau_a^f = \omega\circ\sigma_{(a)}^f\circ\gamma_{(a)}^f,$$

where $\gamma_a^f(u) = \inf\{t; \lambda_a^f([0,t]) > u\}$ is $\mathscr{E}_a^0$-measurable. Furthermore, for every $j\in\{1,\dots,n\}$,

$$\omega(\alpha_j) = \omega\circ\tau_a^f(l_j),$$

where $l_j = l_{\alpha_j}^a(f) = l_{\beta_j}^a(f)$.

Then

$$M_z(\exp - \langle Y_b,\varphi\rangle|\mathscr{E}_a) = M_z\left(\prod_{j=1}^n H(f_j,\omega(\alpha_j)(a))\middle|\mathscr{E}_a\right)$$
$$= M_z\left(\exp\int N^f(dg\,dl)\log(H(g,\omega\circ\tau_a^f(l)))\middle|\mathscr{E}_a\right)$$
$$= \exp - \int_0^\infty dl\int n(dg)(1-H(g,\omega\circ\tau_a^f(l)))$$
$$= \exp - \int_0^\infty d_s l_s^a(f)\int n(dg)(1-H(g,\omega(s)))$$
$$= \exp - \left\langle Y_a(f,\omega), \int n(dg)(1-H(g,\cdot))\right\rangle.$$

The only nontrivial equality is the third one. It follows from Lemma 5.2, the exponential formulas for Poisson measures, and the fact that $\omega\circ\tau_a^f$ is $\mathscr{E}_a$-measurable.

The proof of Theorem 5.1 is now easily completed. Indeed, from the definition of H, for any $y\in\mathbb{R}^d$,

$$\int n(dg)(1-H(g,y)) = \int n(dg)\mathbb{Q}_y^g(1-\exp - \langle Y_{b-a}(g,\cdot),\varphi\rangle)$$
$$= M_y(1-\exp - \langle Y_{b-a},\varphi\rangle). \qquad \square$$

We denote by $C_b(\mathbb{R}^d,\mathbb{R}_+)$ the set of all bounded continuous functions from $\mathbb{R}^d$ into $\mathbb{R}_+$, equipped with the topology of uniform convergence.

PROPOSITION 5.3. *Set $V_0\varphi = \varphi$. Then $(V_t)_{t\geq 0}$ is a semigroup of contractions of $C_b(\mathbb{R}^d, \mathbb{R}_+)$. For any $\varphi \in C_b(\mathbb{R}^d, \mathbb{R}_+)$, the mapping $(t, y) \to V_t\varphi(y)$ solves the integral equation*

$$V_t\varphi = P_t\varphi - 2\int_0^t P_{t-s}\big((V_s\varphi)^2\big)\,ds.$$

PROOF. It is easily seen from the definition that $V_t\varphi \in C_b(\mathbb{R}^d, \mathbb{R}_+)$ whenever $\varphi \in C_b(\mathbb{R}^d, \mathbb{R}_+)$. Also

$$V_t\varphi(y) = M_y(1 - \exp - \langle Y_t, \varphi\rangle) \leq M_y(\langle Y_t, \varphi\rangle) = P_t\varphi(y),$$

which shows that $V_t\varphi$ is a contraction of $C_b(\mathbb{R}^d, \mathbb{R}_+)$. To get the semigroup property, we fix $u, v > 0$, and, for any $a > 0$, we evaluate

$$E\big[\exp - \langle Y_{a+u+v}, \varphi\rangle | \mathscr{E}_a\big]$$

in two different ways, using Theorem 5.1. It follows that

$$\langle Y_a, V_v(V_u\varphi)\rangle = \langle Y_a, V_{u+v}\varphi\rangle,$$

M_y a.e. for any $y \in \mathbb{R}^d$. Integrating with respect to M_y and letting $a \to 0$, we get

$$V_v(V_u\varphi)(y) = V_{u+v}\varphi(y).$$

In order to get the integral equation satisfied by $V_t\varphi$, we first assume that φ is uniformly Lipschitz, with Lipschitz constant C_φ. Let $y, z \in \mathbb{R}^d$ and $T > 0$. Fix $f \in C^*$. We may perform the construction of Sections 3 and 4 simultaneously with the two starting points y, z, using the same Brownian motions B^e in Proposition 3.1. We get two processes (Γ_s^y) and (Γ_s^z) defined under a certain probability measure $\mathbb{P}^f$, such that (Γ_s^y), respectively (Γ_s^z), is distributed according to $\mathbb{Q}_y^f$, resp. $\mathbb{Q}_z^f$, and, for every $s \geq 0$,

$$\mathbb{E}^f\Big[\sup_{t\leq T}|\Gamma_s^y(t) - \Gamma_s^z(t)|\Big] \leq K_T|y - z|$$

where the constant K_T does not depend on f nor on s, y, z. The point is that for $s \in \mathscr{L}_f$, Γ_s^y, Γ_s^z are both solutions of (A) driven by the same Brownian motion, with respective starting points y, z, and we use our assumptions on the coefficients σ, b. Then, for $t \leq T$,

$$\begin{aligned}
|V_t\varphi(y) &- V_t\varphi(z)| \\
&= \bigg|\int n(df)\mathbb{E}^f\bigg[\exp - \bigg\{\int_0^\infty d_s l_s^t(f)\varphi(\Gamma_s^y(t))\bigg\} \\
&\qquad\qquad - \exp - \bigg\{\int_0^\infty d_s l_s^t(f)\varphi(\Gamma_s^z(t))\bigg\}\bigg]\bigg| \\
&\leq \int n(df)\int_0^\infty d_s l_s^t(f)\mathbb{E}^f[|\varphi(\Gamma_s^y(t)) - \varphi(\Gamma_s^z(t))|] \\
&\leq C_\varphi K_T|y - z|.
\end{aligned}$$

Next we study the behavior of $V_t\varphi$ as $t \to 0$. By Taylor's expansion,

$$V_t\varphi(y) = M_y(1 - \exp - \langle Y_t, \varphi\rangle) = M_y(\langle Y_t, \varphi\rangle) - \tfrac{1}{2}M_y\big(\langle Y_t, \varphi\rangle^2\big) + R_t^y\varphi,$$

where

$$0 \le R_t^y \varphi \le M_y\big(\langle Y_t, \varphi \rangle^3\big).$$

We have first

$$M_y(\langle Y_t, \varphi \rangle) = P_t \varphi(y).$$

Then

$$M_y\big(\langle Y_t, \varphi \rangle^2\big) = M_y\big(\langle Y_t, \varphi(y) \rangle^2\big) + M_y\big(\langle Y_t, \varphi - \varphi(y) \rangle(\langle Y_t, \varphi(y) \rangle + \langle Y_t, \varphi \rangle)\big).$$

Clearly,

$$M_y\big(\langle Y_t, \varphi(y) \rangle^2\big) = \varphi(y)^2 M_y\big(\langle Y_t, 1 \rangle^2\big) = 4t\varphi(y)^2,$$

whereas, by the Cauchy–Schwarz inequality,

$$\begin{aligned} &M_y\big(\langle Y_t, \varphi - \varphi(y) \rangle(\langle Y_t, \varphi(y) \rangle + \langle Y_t, \varphi \rangle)\big) \\ &\quad \le \big(16t\|\varphi\|_\infty^2\big)^{1/2}\big(M_y(\langle Y_t, \varphi - \varphi(y) \rangle)^2\big)^{1/2}. \end{aligned}$$

Then, if K is a Lipschitz constant for φ,

$$\begin{aligned} &M_y\big(\langle Y_t, \varphi - \varphi(y) \rangle^2\big) \\ &\quad \le K^2 \int n(df) \int_0^\infty d_s l_s^t(f) \int_0^\infty d_{s'} l_{s'}^t(f) \mathbb{Q}_y^f(|\omega(s)(t) - y|\, |\omega(s')(t) - y|) \\ &\quad \le CK^2 t^2 \end{aligned}$$

for some constant C. Therefore,

$$M_y\big(\langle Y_t, \varphi \rangle^2\big) = 4t\varphi(y)^2 + O(t^{3/2}),$$

where the estimate for $O(t^{3/2})$ holds uniformly in $y \in \mathbb{R}^d$ and depends only on a Lipschitz constant for φ. Similarly,

$$R_t^y \varphi \le M_y\big(\langle Y_t, \varphi \rangle^3\big) \le \|\varphi\|_\infty^3 M_y\big(\langle Y_t, 1 \rangle^3\big) = 24\|\varphi\|_\infty^3 t^2.$$

Finally, we get

$$V_t\varphi(y) = P_t\varphi(y) - 2t\varphi(y)^2 + O(t^{3/2}),$$

where the estimate $O(t^{3/2})$ holds uniformly in $y \in \mathbb{R}^d$ and depends only on $\|\varphi\|_\infty$ and on a Lipschitz constant for φ.

Next, since the functions $V_t\varphi$, $0 \le t \le T$, are uniformly bounded and satisfy a uniform Lipschitz condition, we have, for a fixed $t \in (0, T]$, for every $n \ge 1$,

$$\begin{aligned} V_t\varphi(y) &= V_{t/n}\big(\cdots(V_{t/n}\varphi)\big)(y) \\ &= P_{t/n}\big(V_{(n-1)t/n}\varphi\big)(y) - 2\frac{t}{n}\big(V_{(n-1)t/n}\varphi\big)^2(y) + O\big((T/n)^{3/2}\big) \\ &= P_t\varphi(y) - 2\frac{t}{n}\sum_{i=1}^n P_{(n-i)/n}\big(V_{(i-1)/n}\varphi\big)^2(y) + O(n^{-1/2}) \end{aligned}$$

by induction on n. The function $(s,u) \to P_s(V_u\varphi)^2(y)$ is clearly bounded and continuous [the continuity of $u \to V_u\varphi(y)$ derives from that of the map $u \to Y_u(f,\omega)$ when $u > 0$, and at $u = 0$ it follows from the previous asymptotics]. Therefore, passing to the limit as $n \to \infty$ leads to

$$V_t\varphi = P_t\varphi - 2\int_0^t P_{t-s}\big((V_s\varphi)^2\big)\,ds.$$

The general case when φ is only bounded and continuous can be easily treated using an approximation by suitable Lipschitz functions. We shall leave this extension to the reader. □

Theorem 5.1 and Proposition 5.3 imply that, for every $y \in \mathbb{R}^d$, the law of $(Y_a,\ a \geq 0)$ under M_y is the canonical measure of the superprocess associated to the solution of equation (A) starting from y. We will make this more precise by the following theorem, which shows how to construct the superprocess from its canonical measure.

THEOREM 5.4. *Let μ be a finite measure on $\mathbb{R}^d$ and let $\mathcal{N}(dz\,df\,d\omega)$ be a Poisson measure on $\mathbb{R}^d \times \Theta$ with intensity*

$$\mu(dz)M_z(df\,d\omega).$$

Let $Z = (Z_t,\ t \geq 0)$ be the measure-valued process defined by $Z_0 = \mu$ and, for every $t > 0$, let

$$\langle Z_t, \varphi\rangle = \int \mathcal{N}(dz\,df\,d\omega)\langle Y_t(f,\omega), \varphi\rangle.$$

Then Z is a superprocess associated with the solutions of equation (A).

PROOF. Let $0 < a_1 < \cdots < a_n = a < b$, and $\varphi_1,\ldots,\varphi_n,\ \psi \in C_b(\mathbb{R}^d,\mathbb{R}_+)$. We compute

$$\begin{aligned}
&E\left[\exp\left\{-\sum_{i=1}^n \langle Z_{a_i},\varphi_i\rangle - \langle Z_b,\psi\rangle\right\}\right]\\
&\quad= \exp - \int\mu(dz)\int M_z(df\,d\omega)\left(1 - \exp\left\{-\sum_{i=1}^n \langle Y_{a_i}(f,\omega),\varphi_i\rangle\right.\right.\\
&\qquad\qquad\left.\left.-\langle Y_b(f,\omega),\psi\rangle\right\}\right)\\
&\quad= \exp - \int\mu(dz)\int M_z(df\,d\omega)\left(1 - \exp\left\{-\sum_{i=1}^n \langle Y_{a_i}(f,\omega),\varphi_i\rangle\right.\right.\\
&\qquad\qquad\left.\left.-\langle Y_a(f,\omega),V_{b-a}\psi\rangle\right\}\right)\\
&\quad= E\left[\exp\left\{-\sum_{i=1}^n \langle Z_{a_i},\varphi_i\rangle - \langle Z_a, V_{b-a}\psi\rangle\right\}\right],
\end{aligned}$$

using Theorem 5.1. It follows that

$$E\big[\exp - \langle Z_b, \psi\rangle | (Z_u, 0 \le u \le a)\big] = \exp - \langle Z_a, V_{b-a}\psi\rangle.$$

On the other hand,

$$\begin{aligned} E[\exp - \langle Z_a, \varphi\rangle] &= E\left[\exp - \int \mathscr{N}(dz\,df\,d\omega)\langle Y_a(f,\omega), \varphi\rangle\right] \\ &= \exp - \int \mu(dz) M_z(df\,d\omega)\big(1 - e^{-\langle Y_a(f,\omega),\varphi\rangle}\big) \\ &= \exp - \langle \mu, V_a\varphi\rangle. \end{aligned}$$

This suffices to show that the finite-dimensional marginals of Z coincide with those of the superprocess started from μ.

Let us prove that Z is continuous. For $t > 0$, this is a consequence of the continuity of $t \to Y_t(f, \omega)$. Then a simple martingale argument (details are left to the reader) shows that $\lim_{t \to 0} Z_t$ exists almost surely, and hence is equal to μ. □

REMARK. Let us come back to the simple case described in the Introduction. In this case, the underlying diffusion process is d-dimensional Brownian motion, and $\mathbb{Q}_z^f(d\omega)$ may be described as the law of the Gaussian process with covariance:

$$\operatorname{cov}(\omega_s, \omega_t) = \inf_{[s,t]} f(u).$$

This description is special to the Gaussian case and does not extend to a more general situation. On the other hand, in the case when the superprocess starts from a Dirac measure, we may work with the law of reflected Brownian motion, instead of the Itô measure of excursions: This gives us directly the Poisson measure of excursions introduced in Theorem 5.4. These remarks explain the differences between the results of this section and the presentation given in the Introduction.

Let us briefly outline some consequences of our representation for the superprocess. The arguments of the proof of Proposition 4.1 imply that, $M_z(df\,d\omega)$ a.e., the function $(\omega(t), t \ge 0)$ satisfies a Hölder condition of order $1/4 - \varepsilon$ for any $\varepsilon > 0$. It is also known that $n(df)$ a.e. all the level sets of f have dimension at most $1/2$. It follows that, $M_z(df\,d\omega)$ a.e., for every $t > 0$, the set

$$\{\omega(s);\ f(s) = t\}$$

has Hausdorff dimension less than or equal to 2. But the latter set clearly contains the topological support of $Y_t(f, \omega)$ (see Section 9 for more information on this). We have thus proved that $M_z(df\,d\omega)$ a.e. the Hausdorff dimension of the support of $Y_t(f, \omega)$ is less than or equal to 2 for every $t > 0$. Obviously, the same property holds for the superprocess (Z_t) of Theorem 5.4

[see Perkins (1988, 1989a) for much more precise results in the case of Brownian motion].

Let us now consider the range of $(\omega(t),\ t \geq 0)$, whose dimension is, by the previous remarks, less than or equal to 4. It is not hard to show that this range coincides $M_z(df\,d\omega)$ a.e. with the range of Y, defined as the closure of the union of the topological supports of $Y_t(f, \omega)$ for $t \in (0, \infty)$. It follows that the range of Y has dimension less than or equal to 4. The same clearly holds for the range of Z, defined as in Dawson, Iscoe and Perkins (1989) (again, much more precise results may be found in the latter paper, at least in the case of Brownian motion).

Finally, the previous identification implies trivially that the range of Y is connected.

6. The historical process. We retain the notation and assumptions of Theorem 5.4: μ is a finite measure on $\mathbb{R}^d$, and $\mathscr{N}(dz\,df\,d\omega)$ denotes a Poisson measure on $\mathbb{R}^d \times \Theta$ with intensity $\mu(dz)M_z(df\,d\omega)$. The $\mathscr{M}(\mathbb{R}^d)$-valued process $(Z_t,\ t \geq 0)$ is defined by

$$\begin{aligned}\langle Z_t, \varphi\rangle &= \int \mathscr{N}(dz\,df\,d\omega)\langle Y_t(f, \omega), \varphi\rangle \\ &= \int \mathscr{N}(dz\,df\,d\omega)\int_0^\infty d_s l_s^t(f)\varphi(\omega(s)(t)).\end{aligned}$$

We now introduce a process $(\mathscr{Y}_t,\ t \geq 0)$ with values in $\mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$ by setting $\mathscr{Y}_0 = \mu$ (we identify a point z in $\mathbb{R}^d$ with the constant function equal to z), and, for $t > 0$, for any bounded measurable functional F on $\mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$,

$$\langle \mathscr{Y}_t, F\rangle = \int \mathscr{N}(dz\,df\,d\omega)\int_0^\infty d_s l_s^t(f)F(\omega(s)).$$

The obvious relation

$$\int Z_t(dx)\varphi(x) = \int \mathscr{Y}_t(d\gamma)\varphi(\gamma(t))$$

shows that, for every $t \geq 0$, Z_t is the image measure of $\mathscr{Y}_t$ through the map $\gamma \to \gamma(t)$. The same arguments as for Z imply that the process $\mathscr{Y}_t$ is continuous. It is also clear by construction that the measure $\mathscr{Y}_t$ is supported on the set of functions constant after time t. Heuristically, if we interpret Z_t as the empirical measure associated with a system of "particles," $\mathscr{Y}_t$ is the corresponding measure on the set of paths (stopped at time t) of these particles. To make this more precise, we will use the classical approximation of (Z_t) by systems of branching diffusions, for which the associated measure on the path space can be defined in a straightforward way.

We consider the standard binary branching diffusion process, which can be informally described as follows. One studies the evolution, in space and time, of a population of individuals (or particles). The individuals are located at time 0 according to a certain distribution. Then they move independently according to the law of the solutions of equation (A). Each individual dies at an exponential time of fixed parameter (the lifetimes of the different individuals are independent) and then gives rise to either zero or two new individuals, with the same probability $1/2$. The new individuals start from the final position of their "father," and then move independently, and so on.... .

We give a precise mathematical description of this model using the notion of a marked tree as formalized in Neveu (1986). This formalism will also be useful in Section 8. We set

$$K = \bigcup_{n=0}^{\infty} \{1,2\}^n,$$

where by convention $\{1,2\}^0 = \{\partial\}$. An element k of K will be written $k = k_1 \cdots k_n$, where $k_1 \cdots k_n \in \{1,2\}$. We set $|k| = n$ (by convention $|\partial| = 0$) and, if $|k| \geq 1$, we set $\bar{k} = k_1 \cdots k_{n-1}$ ($\bar{k} = \partial$ if $|k| = 1$). For $k = k_1 \cdots k_n$ and $h = h_1 \cdots h_m \in K$, we set $hk = h_1 \cdots h_m k_1 \cdots k_n$ (by convention $h\partial = \partial h = h$). Finally, we write $h < h'$ if there exists some $k \in K$ such that $h' = hk$.

A (finite binary) tree is a finite subset κ of K that satisfies the following conditions:

(i) $\partial \in \kappa$;

(ii) $\bar{k} \in \kappa$ whenever $k \in \kappa$ and $|k| \geq 1$;

(iii) if $k = k_1 \cdots k_n \in \kappa$ then either $k_1 \cdots k_n 1 \in \kappa$, $k_1 \cdots k_n 2 \in \kappa$ or $k_1 \cdots k_n 1 \notin \kappa$, $k_1 \cdots k_n 2 \notin \kappa$.

We denote by A the set of all (finite binary) trees. If $\kappa \in A$ and $k \in \kappa$, we set $v_k(\kappa) = 1_{\{k1 \in \kappa\}} = 1_{\{k2 \in \kappa\}}$.

We now introduce the notion of a marked tree. By definition, a marked tree is a pair (κ, ϑ), where κ is a tree and ϑ is a map from κ into a given space E (the space of marks). Here we take $E = \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d)$. The set of all marked trees is denoted by Ω, and an element of Ω will be written

$$w = \big(\kappa, (\tau_k, \psi_k)_{k \in \kappa}\big),$$

where $\kappa \in A$ and, for every $k \in \kappa$, $\tau_k \in \mathbb{R}_+$ and $\psi_k \in C(\mathbb{R}_+, \mathbb{R}^d)$.

Finally, we need the translation operators T_h. For any $h \in K$, the mapping T_h is defined on the subset $\{h \in \kappa\}$ of Ω by

$$T_h\big((\kappa, (\tau_k, \psi_k)_{k \in \kappa})\big) = \big(\kappa_h, (\tau_{hk}, \psi_{hk})_{k \in \kappa_h}\big),$$

where $\kappa_h = \{k \in K;\ hk \in \kappa\} \in A$.

We fix a parameter $\alpha > 0$.

PROPOSITION 6.1. *There exists a unique measurable collection* $(\Lambda_z^\alpha, z \in \mathbb{R}^d)$ *of probability measures on* Ω, *such that the following two properties hold*:

(i) *For every* $z \in \mathbb{R}^d$, *the random variables* $v_\partial, \tau_\partial$ *are independent under* Λ_z^α, *and* $\Lambda_z^\alpha[v_\partial = 1] = \Lambda_z^\alpha[v_\partial = 0] = 1/2$ *and* τ_∂ *is exponentially distributed with mean* α. *The conditional law of* ψ_∂, *given* $(v_\partial, \tau_\partial)$, *is that of a solution of* (A), *started from* z *and stopped at time* τ_∂.

(ii) *The conditional distribution under* $\Lambda_z^\alpha(dw)$ *of the pair* $(T_1 w, T_2 w)$, *given that* $\nu_\partial = 1$ *and given* $(\tau_\partial, \psi_\partial)$, *is* $\Lambda^\alpha_{\psi_\partial(\tau_\partial)} \otimes \Lambda^\alpha_{\psi_\partial(\tau_\partial)}$.

Proposition 6.1 can be proved using the general arguments outlined in Neveu (1986), Section 5. See also Chauvin (1988) or Sznitman (1988) for similar statements in slightly different situations.

We now use the measures Λ_z^α to construct the binary branching diffusion process that was informally described above, first in the situation when the initial population consists of only one individual located at z. We need some additional notation. For $w = (\kappa, (\tau_k, \psi_k)_{k\in\kappa})$, for any $h \in \kappa$, we set

$$x_h(w) = \sum_{\substack{k<h \\ k\neq h}} \tau_k, \qquad y_h(w) = x_h(w) + \tau_h$$

and we define a function $\xi_h\colon \mathbb{R}_+ \to \mathbb{R}^d$ by setting

$$\xi_h(t) = \begin{cases} \psi_k(t - x_k(w)), & \text{if } x_k(w) \le t < y_k(w) \text{ and } k < h, \\ \psi_h(\tau_h), & \text{if } t \ge y_h(w). \end{cases}$$

One easily checks that this definition is unambiguous, and that Λ_z^α a.s., for every $h \in \kappa$, the map ξ_h belongs to $C(\mathbb{R}_+, \mathbb{R}^d)$. For any $t \ge 0$, the variables $\xi_h(t)$, for all $h \in \kappa$ such that $x_h \le t < y_h$, represent the positions of the particles alive at time t, whereas the functions $(\xi_h(s), 0 \le s \le t)$ represent the paths followed by these particles (and before them by their ancestors) up to time t.

The associated "empirical measures" are defined by

$$\mathscr{U}_t(w) = \sum_{\substack{h\in\kappa \\ x_h \le t < y_h}} \delta_{\xi_h(\cdot\wedge t)}, \qquad U_t(w) = \sum_{\substack{h\in\kappa \\ x_h \le t < y_h}} \delta_{\xi_h(t)},$$

so that U_t takes values in $\mathscr{M}(\mathbb{R}^d)$ and, Λ_z^α a.s., $\mathscr{U}_t(w) \in \mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$ for every $t \ge 0$.

Let us now turn to the case when the initial distribution is random. Let η be a finite measure on $\mathbb{R}^d$ and let $N_\eta^\alpha(dz\,dw)$ be a Poisson measure on $\mathbb{R}^d \times \Omega$ with intensity

$$\eta(dz)\Lambda_z^\alpha(dw).$$

We then set

$$\mathscr{Y}_t^{\eta,\alpha} = \int N_\eta^\alpha(dz\,dw)\mathscr{U}_t(w), \qquad Z_t^{\eta,\alpha} = \int N_\eta^\alpha(dz\,dw)U_t(w).$$

($\mathscr{Y}_t^{\eta,\alpha}$ and $Z_t^{\eta,\alpha}$ both depend on the choice of N_η^α but their law is uniquely determined.)

THEOREM 6.2. *As $\alpha \to 0$, the laws of $(2\alpha\mathscr{Y}_t^{\mu/2\alpha,\alpha/2}, t \ge 0)$ converge, in the sense of weak convergence of the finite-dimensional marginals, toward the law of $(\mathscr{Y}_t, t \ge 0)$.*

REMARK. Theorem 6.2 implies that the laws of $(2\alpha Z_t^{\mu/2\alpha,\alpha/2}, t \ge 0)$ converge toward that of $(Z_t, t \ge 0)$. But the latter convergence is nothing but (a special case of) the classical approximation result for superprocesses. We will prove Theorem 6.2 independently of the results of Section 5. This provides an alternative proof of Theorem 5.4, together with a rigorous justification of the informal considerations of the beginning of this section.

PROOF. We will show that, on the same probability space where $\mathscr{Y}$ is defined, there exists a collection of processes $\mathscr{Y}^{(\alpha)}$, such that for every $\alpha > 0$,

$$\left(\mathscr{Y}_t^{(\alpha)}, t \geq 0\right) \overset{(d)}{=} \left(2\alpha \mathscr{Y}_t^{\mu/2\alpha, \alpha/2}, t \geq 0\right)$$

and such that for every $t \geq 0$,

$$\lim_{\alpha \to 0} \mathscr{Y}_t^{(\alpha)} = \mathscr{Y}_t, \quad \text{a.s.}$$

Until the last few lines of the proof we work with a fixed $\alpha > 0$. We will construct the process $\mathscr{Y}^{(\alpha)}$ and check that it has the desired distribution. For any $f \in C^*$, set

$$E_f^\alpha = \left\{e \in E_f;\, h(e) > \alpha\right\}.$$

Obviously, E_f^α is a finite set. For $(f, \omega) \in \Theta$ and $t \geq 0$, define

$$\mathscr{U}_t^{(\alpha)}(f, \omega) = \sum_{e \in E_f^\alpha} 1_{[x(e), y(e) - \alpha)}(t)\, \delta_{\omega(a(t, e, f))},$$

where $a(t, e, f) \in [0, \tau(f))$ is the initial time of the raw excursion above level t that represents e in E_f. If we fix $f \in C^*$, the distribution of $(\mathscr{U}_t^{(\alpha)}(f, \omega), t \geq 0)$ under $\mathbb{Q}_z^f(d\omega)$ coincides with that of

$$\left(\sum_{e \in E_f^\alpha} 1_{[x(e), y(e) - \alpha)}(t)\, \delta_{X^e(t)}, t \geq 0\right),$$

where the processes X^e are as in Proposition 3.1.

Set $n_\alpha(df) = n(df \,|\, \sup f(s) > \alpha)$ (n_α is a probability measure) and

$$M_z^\alpha(df\, d\omega) = n_\alpha(df)\, \mathbb{Q}_z^f(d\omega).$$

Our first goal is to check that the distribution of the process $(\mathscr{U}_t^{(\alpha)}(f, \omega))$ under $M_z^\alpha(df\, d\omega)$ coincides with that of $(\mathscr{U}_t(w))$ under $\Lambda_z^{\alpha/2}(dw)$. To this end, we shall need the so-called decomposition of the Brownian excursion at its deepest α-minimum [Neveu and Pitman (1989b)].

We first introduce some notation. For $f \in C^*$ such that $\sup f(s) > \alpha$, we let

$$\nu(f) = \begin{cases} 1, & \text{if } \operatorname{Card}(E_f^\alpha) > 1, \\ 0, & \text{otherwise}. \end{cases}$$

If $\nu(f) = 1$, we may define $\xi(f) \in (0, \infty)$ by imposing

$$\operatorname{Card}\left(E_f^\alpha \cap \{e; x(e) < \xi(f)\}\right) = 1,$$

$$\operatorname{Card}\left(E_f^\alpha \cap \{e; x(e) \leq \xi(f)\}\right) = 2.$$

Clearly, $\xi(f)$ is a local minimum of f. We denote by $\lambda(f)$ the corresponding time. Finally, if $\nu(f) = 0$, we set $\xi(f) = \sup f(s) - \alpha$, and we let $\lambda(f)$ be the time of the maximum of f.

THEOREM 6.3 [Neveu and Pitman (1989b)]. *The random variables* $\nu(f), \xi(f)$ *are independent under* $n_\alpha(df)$. *Furthermore,*

(i) $n_\alpha(\nu(f) = 1) = n_\alpha(\nu(f) = 0) = 1/2$;
(ii) $\xi(f)$ *is exponentially distributed with mean* $\alpha/2$;
(iii) *let*

$$T_\xi(f) = \sup\{t < \lambda(f);\ f(t) = \xi(f)\},$$
$$L_\xi(f) = \inf\{t > \lambda(f);\ f(t) = \xi(f)\};$$

then, conditionally given that $\nu(f) = 1$, *the functions*

$$f_1(t) = f\big((T_\xi(f) + t) \wedge \lambda(f)\big) - \xi(f),$$
$$f_2(t) = f\big((\lambda(f) + t) \wedge L_\xi(f)\big) - \xi(f),$$

are independent and independent of $\xi(f)$; *their conditional distribution is* n_α.

We now define a mapping $\mathscr{H}\colon \Theta \to \Omega$. For every $(f, \omega) \in \Theta$, $\mathscr{H}(f, \omega) = w$ satisfies the following properties. First,

$$\tau_\partial(w) = \xi(f),$$
$$\psi_\partial(w) = \omega(\lambda(f)),$$
$$\nu_\partial(w) = \nu(f).$$

Then, on $\{\nu(f) = 1\} = \{\nu_\partial(w) = 1\}$, the translated trees $T_1 w, T_2 w$ are such that

$$T_1(\mathscr{H}(f, \omega)) = \mathscr{H}(f_1, \omega_1),$$
$$T_2(\mathscr{H}(f, \omega)) = \mathscr{H}(f_2, \omega_2),$$

where f_1, f_2 are as in Lemma 6.3, and

$$\omega_1(t)(u) = \omega\big((T_\xi(f) + t) \wedge \lambda(f)\big)(\xi(f) + u),$$
$$\omega_2(t)(u) = \omega\big((\lambda(f) + t) \wedge L_\xi(f)\big)(\xi(f) + u)$$

with the notation of Lemma 6.3. Arguing by induction on $\mathrm{Card}(E_f^\alpha)$, one easily checks that the mapping $\mathscr{H}$ is well defined, measurable and uniquely characterized by the previous properties.

LEMMA 6.4. *For every* $z \in \mathbb{R}^d$, $\mathscr{H}(M_z^\alpha) = \Lambda_z^{\alpha/2}$.

PROOF. Except for the spatial motion along the branches of the tree, this is the Neveu–Pitman theorem on the branching tree in a Brownian excursion [see Neveu and Pitman (1989a, b)]. We check that $\mathscr{H}(M_z^\alpha)$ satisfies properties (i) and (ii) of Proposition 6.1. Property (i) follows from Lemma 6.3 and the fact that, conditionally given f, $\psi_\partial(\mathscr{H}(f,\omega)) = \omega(\lambda(f))$ is distributed as a solution of (A), stopped at time $\xi(f) = \tau_\partial(\mathscr{H}(f,\omega))$. To prove property (ii), we observe that, on $\{\nu_\partial(\mathscr{H}(f,\omega)) = 1\}$,

$$T_i(\mathscr{H}(f,\omega)) = \mathscr{H}(f_i, \omega_i),$$

by construction, and we use property (iii) of Lemma 6.3 together with Proposition 4.3. We get that, conditionally on $\nu(f) = 1$, conditionally given $\xi(f) = \tau_\partial(\mathscr{H}(f,\omega))$ and $\omega(\lambda(f)) = \Psi_\partial(\mathscr{H}(f,\omega))$, the pairs $(f_1,\omega_1),(f_2,\omega_2)$ are independent and distributed according to $M^\alpha_{\omega(\lambda(f))(\xi(f))}$. The desired result follows. □

We complete the proof of Theorem 6.2. We first observe that, for every $(f,\omega) \in \Theta$, such that $\sup f(s) > \alpha$, for every $t \ge 0$,

$$(*) \qquad \mathscr{U}_t^{(\alpha)}(f,\omega) = \mathscr{U}_t(\mathscr{H}(f,\omega)).$$

Indeed, we first notice that

$$\mathscr{U}_t^{(\alpha)}(f,\omega) = \delta_{\omega(\lambda(f))(\cdot\wedge t)} \quad \text{if } t < \xi(f),$$

which leads to the desired equality if $\mathrm{Card}\{E_f^\alpha) = 1$. In the general case, we argue by induction on $\mathrm{Card}(E_f^\alpha)$, observing that if $\nu(f) = 1$,

$$\mathscr{U}^{(\alpha)}_{\xi(f)+t}(f,\omega) = \mathscr{U}_t^{(\alpha)}(f_1,\omega_1) + \mathscr{U}_t^{(\alpha)}(f_2,\omega_2),$$

$$\mathscr{U}_{\xi(f)+t}(\mathscr{H}(f,\omega)) = \mathscr{U}_t(\mathscr{H}(f_1,\omega_1)) + \mathscr{U}_t(\mathscr{H}(f_2,\omega_2)).$$

We then set

$$\mathscr{Y}_t^{(\alpha)} = \int \mathscr{N}(dz\,df\,d\omega)\,\mathscr{U}_t^{(\alpha)}(f,\omega) = \int \mathscr{N}_\alpha(dz\,df\,d\omega)\,\mathscr{U}_t^{(\alpha)}(f,\omega),$$

where $\mathscr{N}_\alpha$ denotes the restriction of $\mathscr{N}$ to $\{\sup f(s) > \alpha\}$. Note that $\mathscr{N}_\alpha$ is a Poisson measure with intensity $(2\alpha)^{-1}\mu(dz)M_z^\alpha(df\,d\omega)$. It follows from Lemma 6.4 and the identity $(*)$ that $\mathscr{Y}^{(\alpha)}$ and $2\alpha\mathscr{Y}^{\mu/2\alpha,\alpha/2}$ are identically distributed.

It remains to prove that, for every $t \ge 0$, $\mathscr{Y}_t^{(\alpha)}$ converges almost surely to $\mathscr{Y}_t$ as $\alpha \to 0$. Let F be a bounded continuous functional on $C(\mathbb{R}_+, C(\mathbb{R}_+,\mathbb{R}^d))$. Then

$$\langle \mathscr{Y}_t^{(\alpha)}, F\rangle = 2\alpha\int \mathscr{N}(dz\,df\,d\omega)\langle \mathscr{U}_t^{(\alpha)}(f,\omega), F\rangle$$

$$= 2\alpha\int \mathscr{N}(dz\,df\,d\omega)\sum_{e\in E_f^\alpha} 1_{[x(e),y(e)-\alpha)}(t)F(\omega(a(t,e,f))).$$

However, from the well-known approximation of Brownian local time by upcrossing numbers, we know that for any $t > 0$,

$$\lim_{\alpha\to 0} 2\alpha \sum_{e\in E_f^\alpha} 1_{[x(e),y(e)-\alpha)}(t)\,\delta_{a(t,e,f)}(ds) = d_s l_s^t(f),$$

$n(df)$ a.e., in the sense of weak convergence of measures. It follows that for $t > 0$, a.s.,

$$\lim_{\alpha \to 0} \langle \mathscr{Z}_t^{(\alpha)}, F \rangle = \int \mathscr{N}(dz\, df\, d\omega) \int_0^\infty d_s l_s^t(f) F(\omega(s)) = \langle \mathscr{Z}_t, F \rangle.$$

The case $t = 0$ is simpler and will be left to the reader. □

REMARK. Using a suitable tightness criterion for measure-valued processes [see Roelly-Coppoletta (1986)], it is not hard to check that the convergence of Theorem 6.2 holds in the sense of weak convergence of probability measures on $\mathbb{D}([0,\infty), \mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d)))$.

7. Some properties of the historical process. Our goal in this section is to establish two important properties of the historical process and to stress the connection between these properties and analogous results of excursion theory. We will often use the "canonical process" associated with $\mathscr{Z}_t$, which is the $\mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$-valued process $(\mathscr{Y}_t, t > 0)$ defined on Θ by

$$\langle \mathscr{Y}_t(f, \omega), F \rangle = \int d_s l_s^t(f) F(\omega(s)).$$

We recall that for $a > 0$, $z \in \mathbb{R}^d$, $M_z^a(df\, d\omega) = n_a(df) \mathbb{Q}_z^f(d\omega)$, where $n_a(df) = n(df | \sup f(s) > a)$.

Our first result gives the (inhomogeneous) Markov property of the historical process, together with an explicit description of its transition kernel. We first need some notation. Let $\gamma \in C(\mathbb{R}_+, \mathbb{R}^d)$, $u > 0$ and $\zeta(d\gamma')$ a measure belonging to $\mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$ supported on $\{\gamma'; \gamma'(0) = \gamma(u)\}$. We denote by $\gamma \underset{u}{\circ} \zeta$ the image measure of $\zeta(d\gamma')$ by the mapping $\gamma' \to \gamma \underset{u}{\circ} \gamma'$, where

$$\gamma \underset{u}{\circ} \gamma' = \begin{cases} \gamma(t), & \text{if } t \le u, \\ \gamma'(t-u), & \text{if } t > u. \end{cases}$$

THEOREM 7.1. *Let* $0 \le a < b$. *Then*

$$\mathscr{Z}_b = \sum_{i=1}^{N_{a,b}} \gamma_i \underset{a}{\circ} \mathscr{Y}_{(i)},$$

where:

(i) *conditionally with respect to* $\sigma(\mathscr{Z}_s, 0 \le s \le a)$, *the measure*

$$\sum_{i=1}^{N_{a,b}} \delta_{\gamma_i}$$

is a Poisson measure on $C(\mathbb{R}_+, \mathbb{R}^d)$, *with intensity* $(1/2(b-a))\mathscr{Z}_a$;

(ii) *conditionally given* $N_{a,b}$ *and* $(\gamma_1, \dots, \gamma_{N_{a,b}})$, *the random measures* $\mathscr{Y}_{(1)}, \dots, \mathscr{Y}_{(N_{a,b})} \in \mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$ *are independent and independent of* $\sigma(\mathscr{Z}_s, 0 \le s \le a)$; *the conditional distribution of* $\mathscr{Y}_{(i)}$ *is the law of* $\mathscr{Y}_{b-a}$ *under* $M_{\gamma_i(a)}^{b-a}$.

PROOF. If $a = 0$, the result of Theorem 7.1 follows from our construction of $\mathscr{Z}$. Suppose $a > 0$; it is enough to prove the result of the theorem with $\mathscr{Z}_a, \mathscr{Z}_b$ replaced by $\mathscr{Y}_a, \mathscr{Y}_b$, the σ-field $\sigma(\mathscr{Z}_s, 0 \leq s \leq a)$ replaced by $\mathscr{E}_a$, the underlying probability measure being M_z^a (for some $z \in \mathbb{R}^d$). But then the desired result follows from the arguments developed in Section 5. For $(f, \omega) \in \Theta$, $N_{a,b} = N_{a,b}(f)$ is simply the number of excursions of f above level a that reach level b. If (α_i, β_i) are the corresponding time intervals, we may take $\gamma_i = \omega(\alpha_i) = \omega(\beta_i)$. Furthermore, in the notation of Lemma 5.2, we have

$$\mathscr{Y}_{(i)} = \mathscr{Y}_{b-a}(f_i, \omega_i),$$

where

$$\omega_i(t)(u) = \omega((\alpha_i + t) \wedge \beta_i)(a + u).$$

Details are left to the reader. □

REMARK. The intuitive meaning of Theorem 7.1 is as follows. There is only a finite number $N_{a,b}$ of individuals alive at time a which have descendants alive at time b. The paths of these individuals up to time a are the γ_i's. The measure $\mathscr{Z}_b$ is obtained by sticking together each path γ_i and the paths between a and b of the corresponding descendants (these paths are represented by $\mathscr{Y}_{(i)}$) and then summing over i.

Theorem 7.1 is related to the structure of excursions above a fixed level a. The proof of the next result involves considering the excursions of a Brownian motion above its minimum process. This theorem identifies the Palm measure associated with the historical process taken at a fixed time $a > 0$. This is a special case of a result proved in Dawson and Perkins (1989).

THEOREM 7.2. *Let F be a nonnegative measurable functional on $C(\mathbb{R}_+, \mathbb{R}^d) \times \mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$. Then, for every $a > 0$, $z \in \mathbb{R}^d$,*

$$M_z\Big(\int \mathscr{Y}_a(d\gamma) F(\gamma, \mathscr{Y}_a)\Big) = \int \mathscr{W}_z^a(d\gamma) E\Big[F\Big(\gamma, \int \mathscr{N}_\gamma(ds\, df\, d\omega)\, \gamma \underset{s}{\circ} \mathscr{Y}_{a-s}(f, \omega)\Big)\Big],$$

where:

(i) *$\mathscr{W}_z^a(d\gamma)$ denotes the law on $C(\mathbb{R}_+, \mathbb{R}^d)$ of a solution of* (A), *started from z and stopped at time a*;

(ii) *for every $\gamma \in C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathscr{N}_\gamma(ds\, df\, d\omega)$ is a Poisson measure on $[0, a) \times \Theta$, with intensity $4\, ds\, n(df) \mathbb{Q}^f_{\gamma(s)}(d\omega)$.*

PROOF. Notice that $\mathscr{Y}_{a-s}(f, \omega) = 0$ if $\sup f(u) < a - s$. Therefore, we may as well require that the intensity of $\mathscr{N}_\gamma(ds\, df\, d\omega)$ be

$$4 \cdot 1_{(\sup f(u) \geq a-s)}\, ds\, n(df) \mathbb{Q}^f_{\gamma(s)}(d\omega).$$

We will use the following simple fact about the measure $n(df)$. For every

$f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $s \ge 0$, define $\hat{f}_s, \check{f}_s \in C(\mathbb{R}_+, \mathbb{R}_+)$ by

$$\hat{f}_s(u) = f(s+u), \qquad \check{f}_s(u) = f((s-u)_+).$$

Then for any nonnegative measurable functional G on $C(\mathbb{R}_+, \mathbb{R}_+)^2$,

$$\int n(df) \int d_s l_s^a(f) G(\hat{f}_s, \check{f}_s) = \tilde{n}_a \otimes \tilde{n}_a(G),$$

where $\tilde{n}_a$ denotes the law of linear Brownian motion started at a and stopped at its first hitting time of 0. The easy proof is left to the reader.

By construction,

$$M_z\left(\int \mathscr{Y}_a(d\gamma) F(\gamma, \mathscr{Y}_a)\right) = \int n(df) \int \mathbb{Q}_z^f(d\omega) \int d_s l_s^a(f) F(\omega(s), \mathscr{Y}_a(f, \omega)).$$

To begin with, we fix $f \in C^*$ such that $\sup f(s) > a$, and $s > 0$ such that $f(s) = a$. We consider all intervals (α, β), such that:

(i) (α, β) is an excursion interval of f contained in $(0, s)$ or in $(s, +\infty)$;
(ii) $\sup_{(\alpha, \beta)} f(u) \ge a$;
(iii) if $\alpha > s$, $\inf_{(s, \alpha)} f(u) = f(\alpha)$, if $\alpha < s$, $\inf_{(\beta, s)} f(u) = f(\alpha)$.

It is clear that there are at most countably many such intervals and that they are disjoint. We denote them by $(\alpha_i, \beta_i)_{i \in I}$. Furthermore, $n(df)$ a.s. the measure $d_s l_s^a(f)$ coincides with

$$\sum 1_{(\alpha_i, \beta_i)}(s)\, d_s l_s^a(f).$$

Finally, for every $i \in I$ we set

$$f_i(u) = f((\alpha_i + u) \wedge \beta_i) - f(\alpha_i).$$

Proposition 4.3 and an induction argument imply that under $\mathbb{Q}_z^f(d\omega)$, the processes

$$(\omega((\alpha_i + u) \wedge \beta_i)(f(\alpha_i) + \cdot), u \ge 0)$$

are conditionally independent given $\omega(s)$. Furthermore, their conditional law is $\mathbb{Q}^{f_i}_{\omega(s)(\alpha_i)}$.

It follows that under $\mathbb{Q}_z^f(d\omega)$, conditionally given $\omega(s)$, the law of $\mathscr{Y}_a(f, \omega)$ coincides with that of

$$\sum_i \omega(s) \underset{f(\alpha_i)}{\circ} \mathscr{Y}_{a - f(\alpha_i)}(f_i, \omega_i)$$

under

$$\prod_i \mathbb{Q}^{f_i}_{\omega(s)(f(\alpha_i))}(d\omega_i).$$

Then, using the fact that the law of $\omega(s)$ under $\mathbb{Q}_z^f(d\omega)$ is $\mathscr{W}_z^a(d\gamma)$,

$$M_z\left(\int \mathscr{Y}_a(d\gamma)F(\gamma,\mathscr{Y}_a)\right)$$
$$= \int n(df)\int \mathbb{Q}_z^f(d\omega)\int d_s l_s^a(f)\int \prod_i \mathbb{Q}^{f_i}_{\omega(s)(f(\alpha_i))}(d\omega_i)$$
$$\times F\left(\omega(s), \sum_i \omega(s) \underset{f(\alpha_i)}{\circ} \mathscr{Y}_{a-f(\alpha_i)}(f_i,\omega_i)\right)$$
$$= \int \mathscr{W}_z^a(d\gamma)\int n(df)\int d_s l_s^a(f)\int \prod_i \mathbb{Q}^{f_i}_{\gamma(f(\alpha_i))}(d\omega_i)$$
$$\times F\left(\gamma, \sum_i \gamma \underset{f(\alpha_i)}{\circ} \mathscr{Y}_{a-f(\alpha_i)}(f_i,\omega_i)\right).$$

At this point we use the simple fact recalled at the beginning of the proof. We get

$$M_z\left(\int \mathscr{Y}_a(d\gamma)F(\gamma,\mathscr{Y}_a)\right)$$
$$= \int \mathscr{W}_z^a(d\gamma)\int \tilde{n}_a(dg)\tilde{n}_a(dh)\prod_i \mathbb{Q}^{g_i}_{\gamma(g(\alpha_i'))}(d\omega_i')\prod_j \mathbb{Q}^{h_j}_{\gamma(h(\alpha_j''))}(d\omega_j'')$$
$$\times F\left(\gamma, \sum_i \gamma \underset{g(\alpha_i')}{\circ} \mathscr{Y}_{a-g(\alpha_i')}(g_i,\omega_i') + \sum_j \gamma \underset{h(\alpha_j'')}{\circ} \mathscr{Y}_{a-h(\alpha_j'')}(h_j,\omega_j'')\right).$$

In the previous formula,

$$g_i(u) = g((\alpha_i' + u)\wedge \beta_i'), \qquad h_j(u) = h((\beta_j'' - u)\vee \alpha_j''),$$

the intervals (α_i', β_i') are all excursion intervals of g such that

$$\sup_{(\alpha_i',\beta_i')} g(u) \ge a, \qquad \inf_{[0,\alpha_i']} g(u) = g(\alpha_i')$$

(in other words, these are the excursion intervals of g, above its minimum function, corresponding to excursions reaching a) and the intervals (α_j'', β_j'') are defined similarly by replacing g with h.

To complete the proof, it suffices to note that, by the famous Lévy theorem on the supremum of linear Brownian motion and standard facts of excursion theory, the law under $\tilde{n}_a(dg)$ of the measure

$$\sum_i \delta_{(g(\alpha_i'),g_i)}$$

is that of a Poisson measure with intensity

$$2\cdot 1_{(\sup g(u) > a-s)}\,ds\,n(dg). \qquad \square$$

REMARK. Let us briefly explain Theorem 7.2. The path γ of a given "typical particle" alive at time a is distributed according to $\mathscr{W}_z^a$. Each term

$\gamma \circ_s \mathscr{Y}_{a-s}(f, \omega)$ represents the contribution of those particles that have the same ancestors as the given one, up to time s (the "cousins" at level s). See Dawson and Perkins (1989) for applications of this result.

8. A description of the measure at a fixed time. Our goal in this section is to give a description of the distribution of $\mathscr{Y}_a$, hence in particular of that of Y_a, for a fixed time $a > 0$. We will use a tree model similar to the one introduced in Section 6, except that we now restrict our attention to the infinite binary tree, represented by

$$K = \bigcup_{n=0}^{\infty} \{1,2\}^n.$$

As previously we will mark elements of K with elements of $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d)$. Our canonical space is therefore

$$\Omega' = \left(\mathbb{R}_+ \times C\left(\mathbb{R}_+, \mathbb{R}^d\right)\right)^K$$

and an element of Ω' is denoted by

$$w = (\tau_k, \psi_k)_{k \in K}.$$

The translation operators T_k, $k \in K$, are then defined in an obvious way, as in Section 6. The next result is similar to Proposition 6.1.

PROPOSITION 8.1. *There exists a unique measurable collection* $(\Delta_z^a,\ a > 0,\ z \in \mathbb{R}^d)$ *of probability measures on* Ω' *such that the following properties hold*:

(i) *Under* Δ_z^a, τ_∂ *is uniformly distributed over* $[0, a]$, *and*, *conditionally given* τ_∂, ψ_∂ *is distributed as a solution of* (A) *starting from* z, *stopped at time* τ_∂.

(ii) *Under* Δ_z^a, *conditionally given* $(\tau_\partial, \psi_\partial)$, *the translated trees* $T_1 w, T_2 w$ *are independent and follow the law* $\Delta_{\psi_\partial(\tau_\partial)}^{a-\tau_\partial}$.

The meaning of the probability measure Δ_z^a is as follows. Under Δ_z^a, the length τ_∂ of the first branch (the ancestor) is uniformly distributed over $[0, a]$. The lengths τ_1, τ_2 of its immediate descendants are, conditionally given τ_∂, independent and uniformly distributed over $[0, a - \tau_\partial]$, and so on. The marks ψ_∂ are obtained by running solutions of (A) along the branches of the tree, starting from z for the ancestor.

We now set

$$K_\infty = \{1,2\}^{\mathbb{N}^*},$$

where $\mathbb{N}^* = \mathbb{N} - \{0\}$. For $k_\infty = k_1 k_2 \cdots \in K_\infty$ and $n \in \mathbb{N}$, we set

$$[k_\infty]_n = k_1 k_2 \cdots k_n.$$

For $k \in K$, we write $k < k_\infty$ if $k = [k_\infty]_n$ for some $n \in \mathbb{N}$. Finally, for $w \in \Omega'$ and $k \in K$, we define $x_k(w)$, $y_k(w)$ and $\xi_k(w)$ as in Section 6.

THEOREM 8.2. (i) *With Δ_z^a probability* 1, *the limit*

$$\lim_{n\to\infty} \xi_{[k_\infty]_n} = \xi_{k_\infty}$$

exists for every $k_\infty \in K_\infty$, and the convergence is uniform.

(ii) *With Δ_z^a probability* 1, *there exists a unique measure ϑ on K_∞, such that for every $h \in K$,*

$$\vartheta(\{k_\infty; h < k_\infty\}) = \lim_{\varepsilon\to 0} 2\varepsilon \sum_{\substack{k\in K\\ h<k}} 1_{(x_k \le a-\varepsilon < y_k)}.$$

(iii) *The distribution under Δ_z^a of the random measure*

$$\int d\vartheta(k_\infty)\delta_{\xi_{k_\infty}}$$

coincides with that of $\mathscr{Y}_a$ under M_z^a.

Let us briefly discuss some consequences of Theorem 8.2. Starting from a marked tree which follows the law Δ_z^a, the compact set $\{\xi_{k_\infty}(a); k_\infty \in K_\infty\}$ is the limit set of $\{\psi_k(0); |k| \ge n\}$ as $n \to \infty$. This set is distributed as the support of Y_a under M_z^a. If we now want to get a set distributed as the support of the superprocess at time a, it suffices to take the union of a Poisson number of independent copies of this set. In order to recover the measure itself (not only its support), one constructs a measure on K_∞. This measure is not the natural product measure. It is constructed as the limit of the (suitably normalized) counting measures on the branches of the tree at level $a - \varepsilon$.

COROLLARY 8.3. *With the notation of Theorem* 8.2, *for any $\phi \in C_b(\mathbb{R}^d, \mathbb{R}_+)$,*

$$V_a\phi(z) = \frac{1}{2a}\Delta_z^a\Big(1 - \exp - \int d\vartheta(k_\infty)\phi\big(\xi_{k_\infty}(a)\big)\Big).$$

The corollary follows at once from the previous theorem, since we know that

$$V_a\phi(z) = M_z(1 - \exp - \langle Y_a, \phi\rangle) = \frac{1}{2a}M_z^a\Big(1 - \exp - \int \mathscr{Y}_a(d\gamma)\phi(\gamma(a))\Big).$$

The proof of Theorem 8.2 requires some notation and preliminary lemmas. To begin with, we fix $f \in C^*$ such that $\sup f(s) > a$. It is convenient to assume that f has no point of increase or point of decrease on $(0, \tau(f))$ and that a is not a local maximum of f. Notice that these assumptions are satisfied $n_a(df)$ a.s. We introduce a collection $((\alpha_k^a(f), \beta_k^a(f)),\ k \in K)$ of subintervals of $(0, \tau(f))$, such that

$$(\alpha_k^a(f), \beta_k^a(f)) \subset (\alpha_h^a(f), \beta_h^a(f)) \quad \text{if } h < k.$$

We start with $\alpha_\partial^a = 0$, $\beta_\partial^a = \tau(f)$. Then we set

$$T_a(f) = \inf\{u \geq 0;\ f(u) = a\},$$
$$L_a(f) = \sup\{u \geq 0;\ f(u) = a\},$$
$$m_a(f) = \inf_{[T_a(f),\, L_a(f)]} f(u).$$

Notice that $m_a(f) < a$ [otherwise $T_a(f)$ would be a point of increase of f]. We define

$$\alpha_1^a(f) = \sup\{u < T_a(f);\ f(u) = m_a(f)\},$$
$$\beta_2^a(f) = \inf\{u > L_a(f);\ f(u) = m_a(f)\},$$

and we let $\alpha_2^a(f) = \beta_1^a(f)$ be the unique $t \in (T_a(f), L_a(f))$ such that $f(t) = m_a(f)$. Observe that $(m_a(f), (\alpha_1^a(f), \beta_1^a(f)))$ and $(m_a(f), (\alpha_2^a(f), \beta_2^a(f)))$ are two raw excursions of f which hit level a.

We continue by induction as follows. We let

$$f_1(u) = f((\alpha_1^a(f) + u) \wedge \beta_1^a(f)) - m_a(f),$$
$$f_2(u) = f((\alpha_2^a(f) + u) \wedge \beta_2^a(f)) - m_a(f).$$

Note that f_1, f_2 satisfy the same assumptions as f, with a replaced by $a - m_a(f)$. In particular, $\sup f_i(s) > a - m_a(f)$ because a is not a local maximum of f. The collection $((\alpha_k^a(f), \beta_k^a(f)),\ k \in K)$ can then be uniquely defined by induction, for all $a > 0$ and all suitable functions f, in such a way that, for every $h \in K$, for $i = 1, 2$,

$$\alpha_{ih}^a(f) = \alpha_i^a(f) + \alpha_h^{a - m_a(f)}(f_i),$$
$$\beta_{ih}^a(f) = \alpha_i^a(f) + \beta_h^{a - m_a(f)}(f_i).$$

Moreover, for every $\varepsilon \in (0, a]$, the number

$$\sum_{h \in K} 1_{(f(\alpha_h^a(f)) \leq a - \varepsilon < f(\alpha_{h1}^a(f)))}$$

is exactly the number of excursions of f above level $a - \varepsilon$ that hit a (or equivalently the number of upcrossings of f from $a - \varepsilon$ to a).

LEMMA 8.4. *Under the probability measure $n_a(df)$,*

(i) *$m_a(f)$ is uniformly distributed over* $[0, a]$;

(ii) *conditionally given $m_a(f) = m$, f_1, f_2 are independent and follow the law n_{a-m}.*

PROOF. This is essentially a consequence of the Williams decomposition theorems for Brownian paths; see Le Gall (1989b) for a detailed proof. □

For every $k \in K$, we now set

$$\tau_k^a(f) = f(\alpha_{k1}^a(f)) - f(\alpha_k^a(f))$$

and, for $\omega \in C(\mathbb{R}_+, C(\mathbb{R}_+, \mathbb{R}^d))$, we define $\psi_k^a(f, \omega) \in C(\mathbb{R}_+, \mathbb{R}^d)$ by

$$\psi_k^a(f,\omega)(u) = \omega(\alpha_{k1}^a(f))(f(\alpha_k^a(f)) + u).$$

LEMMA 8.5. *Under the probability measure $M_z^a(df\,d\omega)$, the collection*

$$(\tau_k^a(f), \psi_k^a(f,\omega))_{k\in K}$$

is distributed according to Δ_z^a.

PROOF. We check properties (i) and (ii) of Proposition 8.1. Property (i) is clear using Lemma 8.4 and the fact that, conditionally given $\tau_\partial^a(f) = m_a(f)$, $\omega(\alpha_1^a(f))$ is distributed as a solution of (A) stopped at time $f(\alpha_1^a(f)) = \tau_\partial^a(f)$.

Next we investigate the joint distribution under $M_z^a(df\,d\omega)$ of

$$(\tau_{1k}^a(f), \psi_{1k}^a(f,\omega))_{k\in K}, \qquad (\tau_{2k}^a(f), \psi_{2k}^a(f,\omega))_{k\in K}.$$

The key observation is that, for $i = 1, 2$,

$$\tau_{ih}^a(f) = \tau_h^{a-m_a(f)}(f_i), \qquad \psi_{ih}^a(f,\omega) = \psi_h^{a-m_a(f)}(f_i, \omega_i),$$

where ω_1, ω_2 are defined by

$$\omega_i(t)(u) = \omega((\alpha_i^a(f) + t) \wedge \beta_i^a(f))(m_a(f) + u).$$

It follows from Proposition 4.3 and Lemma 8.4 that, conditionally given $(\tau_\partial^a(f), \psi_\partial^a(f,\omega)$, the pairs $(f_1, \omega_1), (f_2, \omega_2)$ are independent and identically distributed according to the law

$$M^{a-\tau_\partial(f)}_{\psi_\partial(f,\omega)(\tau_\partial(f))}.$$

This gives property (ii) of Theorem 8.1. □

PROOF OF THEOREM 8.2. By Lemma 8.5, it is enough to prove the statements of Theorem 8.2 for the marked tree

$$w = w(f,\omega) = (\tau_k^a(f), \psi_k^a(f,\omega))_{k\in K},$$

under the probability law $M_z^a(df\,d\omega)$. For simplicity, we now drop the superscript a in the notation. Notice that for every $k \in K$,

$$x_k(w) = f(\alpha_k(f)), \qquad y_k(w) = f(\alpha_{k1}(f)), \qquad \xi_k(w) = \omega(\alpha_{k1}(f)).$$

Then for any $h \in K$,

$$\sum_{h<k} 1_{(x_k \le a-\varepsilon < y_k)}$$

is exactly the number of upcrossings of f from $a - \varepsilon$ to a, during the time interval $[\alpha_h(f), \beta_h(f)]$. It follows that, $n_a(df)$ a.s.,

$$\lim_{\varepsilon\to 0} 2\varepsilon \sum_{h<k} 1_{(x_k \le a-\varepsilon < y_k)} = l_{\beta_h}^a(f) - l_{\alpha_h}^a(f).$$

We can define a measure ϑ on K_∞ by setting

$$\vartheta(\{k_\infty;\, h < k_\infty\}) = l^a_{\beta_h}(f) - l^a_{\alpha_h}(f).$$

This proves property (ii).

Next for every $k_\infty \in K_\infty$, the sequence $(\alpha_{[k_\infty]_n})$ is increasing. We denote by α_{k_∞} its limit. Note that $f(\alpha_{k_\infty}) = a$, and that by the continuity of ω,

$$\lim_{n\to\infty} \xi_{[k_\infty]_n} = \lim_{n\to\infty} \omega(\alpha_{[k_\infty]_{n_1}}) = \omega(\alpha_{k_\infty}).$$

The latter convergence holds simultaneously for all $k_\infty \in K_\infty$, thus proving (i) (the uniformity follows from the uniform continuity of ω).

To complete the proof, we need to show that for all nonnegative measurable functions Φ on $C(\mathbb{R}_+, C(\mathbb{R}_+, \mathbb{R}^d))$,

$$\int d_s l^a_s(f)\Phi(\omega(s)) = \int d\vartheta(k_\infty)\Phi(\omega(\alpha_{k_\infty})).$$

For every $s \in f^{-1}(a)$, let $k_\infty(s) \in K_\infty$ be defined by the conditions

$$\alpha_{[k_\infty(s)]_n} \le s \le \beta_{[k_\infty(s)]_n}$$

for every $n \in \mathbb{N}$. By construction, ϑ coincides with the measure

$$\int d_s l^a_s(f)\delta_{k_\infty(s)}$$

(both measures take the same value on cylinders). It follows that

$$\int d\vartheta(k_\infty)\Phi(\omega(\alpha_{k_\infty})) = \int d_s l^a_s(f)\Phi(\omega(\alpha_{k_\infty(s)})).$$

However, it is easy to check that $\omega(\alpha_{k_\infty(s)}) = \omega(s)$ for every s such that $f(s) = a$. Indeed, $\alpha_{k_\infty(s)} = s$ except if s is the right end of an excursion interval above a, in which case $\alpha_{k_\infty(s)}$ is the corresponding left end, and we still have the desired equality. This completes the proof. □

REMARKS. Theorem 8.2 (iii) and Theorem 7.2 imply that for any nonnegative measurable functional F on $C(\mathbb{R}_+, \mathbb{R}^d) \times \mathscr{M}(C(\mathbb{R}_+, \mathbb{R}^d))$,

$$\begin{aligned}\Delta^a_z\bigg(\int d\vartheta(k_\infty)F\Big(\xi_{k_\infty}, \int d\vartheta(k_\infty)\delta_{\xi_{k_\infty}}\Big)\bigg)\\ = \int \mathscr{W}^a_z(d\gamma)E\bigg[F\Big(\gamma, \int \mathscr{N}_\gamma(ds\,df\,d\omega)\gamma \underset{s}{\circ} \mathscr{Y}_{a-s}(f,\omega)\Big)\bigg],\end{aligned}$$

where we have used the notation of Theorem 7.2. In particular, for any $\gamma \in C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathscr{N}_\gamma(ds\,df\,d\omega)$ is a Poisson measure with intensity

$$4\,ds\,n(df)\mathbb{Q}^f_{\gamma(s)}(d\omega).$$

Now let θ be any deterministic probability measure on K_∞. It is not hard to check that

$$\Delta_z^a\left(\int d\theta(k_\infty) F\left(\xi_{k_\infty}, \int d\vartheta(k_\infty)\delta_{\xi_{k_\infty}}\right)\right)$$
$$= \int \mathscr{W}_z^a(d\gamma) E\left[F\left(\gamma, \int \mathscr{N}_\gamma'(ds\, df\, d\omega)\gamma \underset{s}{\circ} \mathscr{Y}_{a-s}(f, \omega)\right)\right],$$

where $\mathscr{N}_\gamma'(ds\, df\, d\omega)$ is now a Poisson measure with intensity

$$2\, ds\, n(df)\mathbb{Q}^f_{\gamma(s)}(d\omega).$$

The point is that if $x_1 > x_2 > \cdots > x_n > \cdots$ is a Poisson distribution of points on $(0, 1)$, with intensity ds/s, then x_1 is uniformly distributed over $(0, 1)$, x_2 is uniformly distributed over $(0, x_1)$ and so on. Notice the difference between the factors 4 and 2; such phenomena are common in the study of Palm measures.

9. An application. Let $Z = (Z_t, t \geq 0)$ be as in Theorem 5.4. Even if μ is not compactly supported, it readily follows from our construction that, a.s. for every $t > 0$, the (topological) support of Z_t is compact. We denote by $\mathscr{S}_t$ this support. The goal of this section is to investigate the continuity properties of the mapping $t \to \mathscr{S}_t$, which takes values in the set of compact subsets of $\mathbb{R}^d$, equipped with the Hausdorff metric.

Some of our results require the following assumptions.

(A1) Let $P(t, y, dz)$ be the transition kernel of solutions of equation (A). For every $t > 0$, $y \in \mathbb{R}^d$, the kernel $P(t, y, dz)$ is absolutely continuous with respect to Lebesgue measure.

(A2) For every $z \in \mathbb{R}^d$, $t > 0$, the Lebesgue measure of the support of $Y_t(f, \omega)$ is 0, $M_z(df\, d\omega)$ a.s.

As we have already observed in Section 5, assumption (A2) holds automatically if $d \geq 3$. It also holds for two-dimensional Brownian motion [see Perkins (1989)]. The following theorem is essentially due to Perkins (1990).

THEOREM 9.1. (i) *The mapping $t \to \mathscr{S}_t$ is a.s. right continuous with left limits on $(0, \infty)$. Let D denote the (countable) set of its discontinuities. Then, a.s. for every $t \in D$,*

$$\mathscr{S}_{t-} = \mathscr{S}_t \cup \{z_t\}$$

for some $z_t \in \mathbb{R}^d$.

(ii) *Under assumptions* (A1) *and* (A2) *the set $\{(t, z_t), t \in D\}$ is dense in*

$$\overline{\bigcup_{t>0} \{t\} \times \mathscr{S}_t}.$$

Here $\bar{A}$ denotes the closure of the set A.

REMARKS. (i) Discontinuities of the support process correspond to local extinctions of the "clusters" of the population. The first part of the theorem shows that, at every discontinuity point, exactly one cluster disappears. In our approach, this result will be seen to be equivalent to the fact that the local maxima of a linear Brownian motion are distinct.

(ii) When μ is compactly supported, it is not hard to check that the map $t \to \mathscr{S}_t$ is also right continuous at $t = 0$.

PROOF. With the notation of Theorem 5.4, we may write

$$\mathscr{N}(dz\,df\,d\omega) = \sum_{i\in I} \delta_{(z_i, f_i, \omega_i)}(dz\,df\,d\omega),$$

where I is a countable set of indices. With probability 1, the local maxima of the functions f_i are distinct. For every $t > 0$, $i \in I$, set

$$L_t^i = \{s \geq 0;\ f_i(s) = t\},$$

$$\hat{L}_t^i = \{s \geq 0;\ f_i(s) = t, \text{ and } s \text{ is not a time of local maximum of } f_i\},$$

$$\check{L}_t^i = \{s \geq 0;\ f_i(s) = t, \text{ and } s \text{ is not a time of local minimum of } f_i\}.$$

LEMMA 9.2. *A.s. for every $t > 0$,*

$$\mathscr{S}_t = \bigcup_{i\in I} \{\omega_i(s)(t);\ s \in \hat{L}_t^i\} = \bigcup_{i\in I} \omega_i(\hat{L}_t^i)(t).$$

PROOF. We first notice that for every $t > 0$ there is only a finite number of indices i such that $\hat{L}_t^i \neq \varnothing$. It follows that the sets

$$\bigcup_{i\in I} \omega_i(L_t^i)(t), \qquad \bigcup_{i\in I} \omega_i(\hat{L}_t^i)(t)$$

are closed. By construction, Z_t is supported on

$$\bigcup_{i\in I} \omega_i(L_t^i)(t).$$

However, since each of the measures $d_s l_s^t(f_i)$ attributes no mass to a single point, it is clear that Z_t is also supported on

$$\bigcup_{i\in I} \omega_i(\hat{L}_t^i)(t).$$

Next, if z belongs to the latter set, we may find $i \in I$ and s, such that $f_i(s) = t$, $\omega_i(s)(t) = z$ and s is not a time of local maximum of f_i. Assume first that s is not a time of local minimum of f_i. Then from well-known properties of Brownian local times, we have, for every $\varepsilon > 0$,

$$l_{s+\varepsilon}^t(f_i) - l_{s-\varepsilon}^t(f_i) > 0.$$

By the continuity of ω_i, this implies that $z \in \mathscr{S}_t$. If s is a local minimum of f_i, we simply replace s by $s' = \sup\{u < s;\ f_i(u) = f_i(s)\}$, observing that $\omega_i(s) = \omega_i(s')$.

A careful examination of the previous arguments shows that they hold for every $t > 0$ simultaneously. The point is that $n(df)$ a.e. for every $s \in (0, \tau(f))$, either s is a time of local extremum of f or s is a time of increase of $l^{f(s)}(f)$. This completes the proof of the lemma. □

We now complete the proof of part (i) of Theorem 9.1. Elementary arguments show that the map $t \to \hat{L}_t^i$ is right continuous with left limits, and that the left limit at t is $\check{L}_t^i$. By the continuity of the ω_i's, it follows that the map $t \to \mathscr{S}_t$ shares the same properties, and that its left limit at t is

$$\mathscr{S}_{t-} = \bigcup_{i \in I} \omega_i(\check{L}_t^i)(t) = \bigcup_{i \in I} \omega_i(L_t^i)(t),$$

where for the second equality we use the same argument as in the proof of the previous lemma.

For every $i \in I$, let D^i be the set of all local maxima of f_i. Set

$$D_0 = \bigcup_{i \in I} D^i.$$

Clearly, $\mathscr{S}_t = \mathscr{S}_{t-}$ if $t \notin D_0$. Next, if $t \in D_0$, we may find $i \in I$, and $s_t > 0$ such that t is a local maximum of f_i attained at s_t. Notice that with probability 1, i and s_t are unique for every $t \in D_0$. It then follows from the previous equalities that

$$\mathscr{S}_{t-} = \mathscr{S}_t \cup \{\omega_i(s_t)(t)\}.$$

This completes the proof of property (i).

We turn to property (ii). We first observe that

$$\overline{\bigcup_{t>0} \{t\} \times \mathscr{S}_t} = \overline{\bigcup_{i \in I} \{(f_i(s), \omega_i(s)(f_i(s))); s > 0\}}.$$

We will prove that, under assumptions (A1) and (A2), $D = D_0$, a.s. Since a.s. for every $i \in I$, the set $\{(s_t, t); t \in D^i\}$ is dense in the graph of f_i, this suffices to get the desired result.

We argue as follows. Let $t \in D^i$ and s_t as above. First, using assumptions (A1) and (A2) and the explicit knowledge of the conditional law of $\omega_i(s_t)$ given f_i, we get

$$\omega_i(s_t)(t) \notin \bigcup_{j \neq i} \omega_j(L_t^j)(t).$$

The point is that, conditionally given $(f_i, i \in I)$, the ω_j's are independent and the sets $\omega_j(L_t^j)(t)$ have Lebesgue measure 0 [by (A2)] whereas the law of $\omega_i(s_t)(t)$ is absolutely continuous with respect to Lebesgue measure [by (A1)].

Now choose $\delta > 0$ and let (α_k^i, β_k^i) be the excursion intervals of f_i above level $t - \delta$. Let k_0 be such that $s_t \in (\alpha_{k_0}^i, \beta_{k_0}^i)$. We argue as in the proof of Theorem 5.1, taking $a = t - \delta$ and $b = t$. If we condition with respect to $\mathscr{E}_{t-\delta}^i \vee \mathscr{E}_\infty^{0,i}$ (with an obvious notation) and use again assumptions (A1) and (A2), we obtain

$$\omega_i(s_t)(t) \notin \bigcup_{k \neq k_0} \{\omega_i(s)(t); s \in f^{-1}(t) \cap (\alpha_k^i, \beta_k^i)\}.$$

If δ is sufficiently small, this shows that

$$\omega_i(s_t)(t) \notin \omega_i(\hat{L}_t^i)(t).$$

Therefore, we conclude that t is a discontinuity point of $\mathscr{S}$. □

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