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BOOK REVIEW

XIANG-QUN YANG, *The Construction Theory of Denumerable Markov Processes*. Wiley Series in Probability and Mathematical Statistics, Wiley, New York, 1990, $95.00

REVIEW BY P. K. POLLETT

University of Queensland

The publication of the English edition of Yang's book is timely because it coincides with renewed interest in construction theory by researchers in the West. The book reports on the dramatic advances made by probabilists in the People's Republic of China over the last two decades. It is divided into six parts, the first of which provides an excellent introduction to modern chain theory. The rest is devoted to a detailed and exhaustive account of both the analytical and probabilistic approaches to construction theory and the relationship between them. The material is presented clearly and at a level accessible to those with a general background in probability. The book begins with a forward by David Kendall, to whom we owe a debt of gratitude for his enthusiasm in bringing to our attention the work of our Chinese colleagues.

What is construction theory? Let me begin from a purely practical standpoint. The procedure for modelling a stochastic system with discrete states using a Markov chain usually begins with writing down a collection (the q matrix) of transition rates, the form of which reflects the microscopic workings of the system in question. One may then look to the backward or, more usually, the forward equations to obtain the transition function (set of transition probabilities), a quantity which governs the behaviour of the process. Alternatively, if information is required about only the system's long-term behaviour, one may seek the solution of a collection of balance equations. But, even in this seemingly idyllic setting (I envisage, here, that the q matrix is stable and conservative) things can go wrong, and for reasons which provide a mandate for construction theory.

Because Yang's book gives particular attention to birth–death processes, let me illustrate this with a simple example of such a process, borrowed from Miller [3]. Let $r > 1$ and, in the usual notation, define $\lambda_n = r^{2n}$, $n \geq 0$, and $\mu_n = r^{2n-1}$, $n \geq 1$, to be the birth and the death rates, respectively. In this example, a set of detailed-balance equations can be solved. These provide an essentially unique invariant measure which, because $r > 1$, can be normalized to produce an "answer", $\pi_n = (1 - \rho)\rho^n$, $n \geq 0$, where $\rho = 1/r$, which might be regarded as an "equilibrium distribution". However, a careful examination of the rates reveals that, at a jump time, the likelihood of a birth is r times that of a death. The process must therefore be transient. So, what is going

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wrong? The problem is that the process, as it drifts toward infinity, "explodes" by performing an infinity of jumps in a finite time, and, the actual process for which π *is* the stationary distribution is not the one expected, but one of the infinity of processes, with the given set of birth and death rates, obtained by "restarting" after this and subsequent explosions. Using results in Part II of Yang's book we can identify the required process because these results provide a precise specification of the transition function, through its resolvent (Laplace transform) of *all* processes with the given set of rates. Additionally, using results contained in Part V, we can identify the structure of its sample paths. Those readers familiar with diffusion theory will probably have guessed that the process for which π is the stationary distribution is the one obtained by reflecting the process away from the boundary at infinity.

So, construction theory is one of identifying all those processes which satisfy a specified set of criteria; in the context of Yang's book, those which have a given set of transition rates. Although the book deals with both analytical and probabilistic methods, I shall sketch only the analytical approach because this leads naturally to the most recent advances in the field, which I shall also sketch briefly.

From an analytical point of view, one begins with a q matrix over a countable set S that is a collection $Q = (q_{ij},\ i, j \in S)$ of real numbers which satisfy

$$0 \leq q_{ij} < \infty, \qquad j \neq i,\ i, j \in S$$

and

$$\sum_{j \neq i} q_{ij} \leq q_i := -q_{ii} \leq \infty, \qquad i \in S.$$

Then, in its most general setting, the problem is to determine all Q processes; that is, to identify those real-valued functions $P(\cdot) = (p_{ij}(\cdot),\ i, j \in S)$, defined on $[0, \infty)$ which satisfy

$$p_{ij}(t) \geq 0, \qquad i, j \in S, t \geq 0, \tag{1}$$

$$\sum_{j \in S} p_{ij}(t) \leq 1, \qquad i \in S, t \geq 0, \tag{2}$$

$$p_{ij}(s + t) = \sum_{k \in S} p_{ik}(s) p_{kj}(t), \qquad i, j \in S,\ s, t \geq 0, \tag{3}$$

$$p_{ij}(0) = \delta_{ij} = \lim_{t \downarrow 0} p_{ij}(t), \qquad i, j \in S \tag{4}$$

(standard transition functions), and which, in addition, satisfy

$$\lim_{t \downarrow 0} \frac{p_{ij}(t) - \delta_{ij}}{t} = q_{ij}, \qquad i, j \in S.$$

Yang considers only the totally-stable case (that is, when $q_j < \infty$ for each $j \in S$), so that when Q is conservative, that is,

$$\sum_{j \neq i} q_{ij} = q_i, \qquad i \in S \tag{5}$$

(a very natural assumption, even when not all of the q_j's are finite), all Q processes satisfy the backward equations

$$p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t), \qquad t > 0, i, j \in S,$$

and so the construction problem can be viewed as one of determining all solutions to the backward equations which satisfy (1)–(4). But, even when Q is conservative, the solution appears to be a long way off. I would venture to label this problem "impossible" and hope to be proved wrong.

It is the thrust of Yang's book that one *can* make progress by imposing further structure. For example, if the space of bounded vectors x on S satisfying

$$\sum_{j \in S} q_{ij} x_j = \xi x_i, \qquad i \in S,$$

where $\xi > 0$, has *finite* dimension d (a quantity which does not depend on ξ), then the form of the resolvent of all Q processes can be written down explicitly. In a sense that can be made very precise, d represents the number of exit routes to infinity available to the process. On the strength of this assumption one can relax (5), stipulating only that there are finitely many states i for which

$$\sum_{j \neq i} q_{ij} < q_i,$$

and still obtain a complete picture. Examples of finite-exit processes are birth–death processes, these being single-exit, and birth–death processes taking values in $\mathbb{Z}$ (bilateral birth–death processes), these being double-exit processes. Thus a complete construction theory exists for these processes. It is one of the strengths of the book that birth–death processes are studied in some detail before a general theory is annunciated.

Since the book provides such a clear picture of classical construction theory, ideal for testing theoretical conjectures, it is not unreasonable to ask where we go from here. A direction which I find very appealing is to start with a general q matrix and specify a subinvariant measure for Q; that is, a collection $m = (m_j,\ j \in S)$ of strictly positive numbers which satisfy

$$\sum_{i \in S} m_i q_{ij} \le 0, \qquad j \in S. \tag{6}$$

Then seek to determine all Q processes P such that m is an invariant measure for P; that is,

$$\sum_{i \in S} m_i p_{ij}(t) = m_j, \qquad j \in S, t \ge 0. \tag{7}$$

This has particular appeal if $\sum_{i \in S} m_i < \infty$, for then one is essentially looking for those Q processes (of necessity, positive recurrent) whose stationary distribution has been specified in advance. The reason for an inequality in (6) is that an invariant measure for P is invariant for Q [that is, equality holds in

(6) for all $i \in S$] only if P satisfies the forward equations

$$p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}, \qquad t > 0, i, j \in S;$$

see [4]. In [9], Williams cites this problem as being challenging, but perhaps not impossible. Though, in view of the further achievements of the Chinese, reported in [9], on problems that were considered impossible, this challenge may already have been met! Some progress has been made with the stable, conservative case ([4] and [5]) following the work of Hou Chen-Ting and Chen Mufa [2], who studied the m-symmetrizable case, where (6) is specialized to

$$m_i q_{ij} = m_j q_{ji}, \qquad i, j \in S,$$

and (7) is specialized to

$$m_i p_{ij}(t) = m_j p_{ji}(t), \qquad j \in S, t \geq 0.$$

In what is perhaps the most significant recent advance in construction theory, it was shown that, even when some states are unstable, there exists an m-symmetrizable Q process if and only if

$$\{f \in l^2(m) : \mathscr{E}(f, f) < \infty\} \text{ is dense in } l^2(m),$$

where $\mathscr{E}(f, f)$ is the Dirichlet form

$$\mathscr{E}(f, f) := \tfrac{1}{2} \sum_{i, j, i \neq j} m_i q_{ij} (f_i - f_j)^2;$$

see [1], [6], [7] and [8]. However, it appears difficult to find a natural probabilistic interpretation of this condition.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF QUEENSLAND
QUEENSLAND 4072
AUSTRALIA