The Annals of Probability
1992, Vol. 20, No. 1, 538–548

REPRESENTATION OF MEASURES BY BALAYAGE FROM A REGULAR RECURRENT POINT

BY J. BERTOIN AND Y. LE JAN

Université Pierre et Marie Curie

Let X be a Hunt process starting from a regular recurrent point 0 and ν a smooth probability measure on the state space. We show that $T = \inf\{s: A_s > L_s\}$, where A is the continuous additive functional associated to ν and L the local time at 0, solves the Skorokhod problem for ν, that is, X_T has law ν. We construct another solution which minimizes $\mathbb{E}_0(B_S)$ among all the solutions S of the Skorokhod problem, where B is any positive continuous additive functional. The special case where X is a symmetric Lévy process is discussed.

0. Introduction. Let X be a Markov process with state space E. The problem of representing a probability measure on E by the distribution of X taken at a suitable stopping time (often called the Skorokhod problem), has been considered by many authors including Skorokhod (1965), Dubins (1968) and Root (1969) when X is a one-dimensional Brownian motion and Rost (1971) for general Markov processes; see Dellacherie and Meyer (1983) for further references. Usually, the stopping times that solve this problem are constructed by approximation [some explicit solutions are known when X is a one-dimensional Brownian motion, but they are specific to this case; see, for example, Azéma and Yor (1979)]. We show here that there are simple and direct solutions when the starting point $0 \in E$ is regular and recurrent. One of our solutions is optimal, in the sense that it minimizes the expected value of any positive continuous additive functional (p.c.a.f.) taken at this time among all the solutions of the Skorokhod problem. When we specialize our results to the case when 0 is a holding point, we partially recover the Skorokhod-type representation of Azéma and Meyer (1974) for transient Markov processes. The case of symmetric Lévy processes is of special interest and is discussed in the Appendix.

Our approach relies on excursion theory and on the correspondence between measures and additive functionals. Of course, things are simpler when E is discrete and the reader only concerned with continuous-time Markov chains will find the solutions in the examples below the statements.

1. Statements of results. Henceforth, we consider a Hunt process X with a reference measure on a locally compact space E with a countable base. We assume that $0 \in E$ is a regular recurrent point for X. We denote the local time at 0 by L [cf. Blumenthal and Getoor (1968)], its right-continuous inverse

Received June 1990; revised September 1990.
AMS 1980 *subject classifications*. Primary 60G40; secondary 60J55.
Key words and phrases. Skorokhod problem, additive functional, Revuz measure, excursions.

by L^{-1}, the characteristic measure of the corresponding excursion process by n [cf. Itô (1972)] and the first hitting time of 0 by ζ. If c is the delay coefficient at 0, then the measure μ

$$\int f\,d\mu = \int dn \int_0^{\zeta} f(X_s)\,ds + cf(0) \tag{1}$$

is invariant; see Getoor (1979). For every positive continuous additive functional B, we call the Revuz measure of B, the measure χ given by $\int f\,d\chi = (1/t)\mathbb{E}_{\mu}(\int_0^t f(X_s)\,dB_s)$; see Revuz (1970) for the correspondence between p.c.a.f. and measures. We denote by $V(f\chi)(x) = \mathbb{E}_x(\int_0^{\zeta} f(X_t)\,dB_t)$ and simply $V(\mathbf{1}\chi)$ by $V\chi$.

We consider a p.c.a.f. A with Revuz measure ν and we assume that ν gives no mass to $\{0\}$. Our claim is:

THEOREM. *Introduce the stopping time $T = \inf\{t\colon A_t > L_t\}$ and denote the $\mathbb{P}_0$-distribution of X_T by η, that is, $\mathbb{P}_0(X_T \in dx, T < \infty) = \eta(dx)$.*

(i) *If ν is a subprobability measure, then T solves the Skorokhod problem for ν, that is, $\nu = \eta$. Moreover, $\mathbb{E}_0(L_T) = \infty$.*

(ii) *If $\nu(E) > 1$, then $\mathbb{E}_0(L_T) = 1/a_0 < \infty$ and for every nonnegative Borel function f, we have*

$$\int f\,d\nu = \int (f + a_0 V(f\nu))\,d\eta.$$

REMARK. Monroe (1972) also considered passage times of additive functionals to solve the Skorokhod problem when X is a symmetric stable process in dimension 1. However, he has no explicit solution.

EXAMPLE. Assume that E is a discrete space. Then X is determined by its transition probabilities $P(x,y)$ and its holding time expectation $1/q(x)$. Assume that 0 is recurrent and that $\mathbb{P}_0(X \text{ hits } x) = 1$ for all $x \in E$. We denote by $P^{\dagger}$ the transition matrix of X killed at its first return to 0 and let $m(x) = \delta_0(x) + \sum_{k=1}^{\infty}(P^{\dagger})^k(0,x)$. This measure is uniquely determined by the two properties: $\sum_{y \in E} P(x,y)m(y) = m(x)$; and $m(0) = 1$. For every $x \in E$, we denote $(q(x)/m(x))\int_{[0,t]} \mathbf{1}_{\{X_s = x\}}\,ds$ by $L(x,t)$. Let ν be a subprobability measure on E that gives no mass to 0 and

$$T = \inf\Big\{t\colon \sum_{x \in E} L(x,t)\nu(x) > L(0,t)\Big\}.$$

Then $\mathbb{P}_0(X_T = x, T < \infty) = \nu(x)$ for all x.

When $\nu(E) \le 1$, the solution of the Skorokhod problem for ν that is given by the theorem is attractively simple. Nevertheless, T is often not optimal because $\mathbb{E}_0(L_T)$ is always infinite. Consider the hypothesis:

HYPOTHESIS 1. *There is a bounded Borel function* $\hat{V}\nu$ *such that for every Revuz measure* χ, $\int \hat{V}\nu \, d\chi = \int V\chi \, d\nu$. *We set* $\lambda_0 = \|\hat{V}\nu\|_\infty$.

We state:

COROLLARY. *Assume that* ν *is a probability measure and let* B *be a p.c.a.f. with Revuz measure* χ, $B \not\equiv 0$. *We have*:

(i) *If Hypothesis* 1 *is fulfilled, then for every* $\lambda \geq \lambda_0$,

$$T(\lambda) = \inf\left\{t \colon \lambda \int_0^t \left(\lambda - \hat{V}\nu(X_s)\right)^{-1} dA_s > L_t\right\}$$

solves the Skorokhod problem for ν *and* $\mathbb{E}_0(B_{T(\lambda)}) = \int (\lambda - \hat{V}\nu)\, d\chi$ [*in particular*, $\mathbb{E}_0(L_{T(\lambda)}) = \lambda$]. *Furthermore, for every solution* S *of the Skorokhod problem for* ν, $\mathbb{E}_0(B_S) \geq \int (\lambda_0 - \hat{V}\nu)\, d\chi$.

(ii) *If Hypothesis* 1 *is not fulfilled, then for every solution* S *of the Skorokhod problem for* ν, $\mathbb{E}_0(B_S) = \infty$.

REMARK. The probability measure ν is a cocapacitary measure, that is, $\hat{V}\nu \equiv \lambda_0, \nu$ a.s., if and only if $T(\lambda_0)$ is the first hitting time of the fine support of ν.

EXAMPLE. The hypotheses are the same as in the example following the theorem. Assume now that ν is a probability measure on E and introduce

$$\hat{V}\nu(x) = \sum_{y \in E} \nu(y) \mathbb{E}_y(L(x, \zeta)),$$

where $\zeta = \inf\{t\colon X_t = 0\}$. If $\lambda_0 = \sup\{\hat{V}\nu(x)\colon x \in E\} < \infty$, then

$$T(\lambda_0) = \inf\left\{t\colon \sum_{x \in E} \lambda_0 \left(\lambda_0 - \hat{V}\nu(x)\right)^{-1} L(x,t)\nu(x) > L(0,t)\right\}$$

solves the Skorokhod problem for ν and $\mathbb{E}_0[L(x, T(\lambda_0))] = \lambda_0 - \hat{V}\nu(x)$ for every $x \in E$. Moreover $\mathbb{E}_0(L(x,S)) \geq \mathbb{E}_0(L(x, T(\lambda_0)))$ for every solution S of the Skorokhod problem for ν.

2. Extension to transient Markov processes. It is interesting to note that this corollary enables us to recover the Azéma and Meyer (1974) theorem for transient Markov processes by the argument below. Let $X^\dagger$ be a standard Markov process on $E \setminus \{0\}$ with initial distribution κ and a.s. finite lifetime. Take for X the recurrent extension of $X^\dagger$ with excursion measure $n = \mathbb{P}_\kappa^\dagger$ and such that 0 is a holding point with parameter 1 for X. Denote by $g = \inf\{t\colon X_t \neq 0\}$ the first exit time from 0 and by $A^\dagger$ the additive functional of $X^\dagger$ given by $A^{\dagger}_t(\theta_g \omega) = A_{g+t}(\omega)$, $t < \zeta(\theta_g \omega)$. Applying Lemma 2 in Section 3, we find

$$\mathbb{E}^{\dagger}_{\kappa}\left(\int_0^{\zeta} f(X^{\dagger}_s)\, dA^{\dagger}_s\right) = \int f\, d\nu,$$

that is, $A^{\dagger}$ is the additive functional associated to $(\nu, \mathbb{P}^{\dagger}_{\kappa})$ by Azéma's theorem [see Azéma (1973), page 491]. Clearly, every solution of the Skorokhod problem for ν and X is larger than g. Recall that $g = L_g$ has an exponential distribution with parameter 1. Thus, when Hypothesis 1 is fulfilled, the stopping time $T(\lambda_0)$ occurs during the first excursion of X from 0 if and only if $\lambda_0 = 1$ [this is easily seen to be equivalent to Rost's (1971) balayage condition with regard to $X^{\dagger}$]. In this case, if e is an independent exponential r.v. with parameter 1 and

$$T^{\dagger} = \inf\left\{t \colon \int_0^t \left(1 - \hat{V}\nu\left(X^{\dagger}_s\right)\right)^{-1} dA^{\dagger}_s > e\right\},$$

then the $\mathbb{P}^{\dagger}_{\kappa}$-distribution of $X^{\dagger}$ at time $T^{\dagger}$ is ν (this is the Azéma and Meyer solution of the Skorokhod problem for $X^{\dagger}$ in an expanded filtration). By the corollary, if Hypothesis 1 is not fulfilled or if $\lambda_0 > 1$, then there is no solution of the Skorokhod problem for ν and $X^{\dagger}$. Of course, this method immediately extends to subprobability measures, and, by change of time, to transient Markov processes with infinite lifetimes.

Another interesting consequence of the corollary is that it yields the solution of the Skorokhod problem in the *natural filtration* when the starting point is regular but not recurrent. Denote by $X^{\dagger}$ a standard transient Markov process on E, starting from a regular but not recurrent point 0. For simplicity, assume that $X^{\dagger}$ has lifetime $\xi < \infty$ a.s. Let $L^{\dagger}$ be the local time at 0 and $A^{\dagger}$ the p.c.a.f. associated to the smooth probability measure ν $[\nu(\{0\}) = 0]$ by Azéma's theorem, that is,

$$\mathbb{E}^{\dagger}_0\left(\int_0^{\xi} f(X^{\dagger}_s)\, dL^{\dagger}_s\right) = f(0), \qquad \mathbb{E}^{\dagger}_0\left(\int_0^{\xi} f(X^{\dagger}_s)\, dA^{\dagger}_s\right) = \int f\, d\nu.$$

Let Δ be a cemetery point and consider the following recurrent extension X of $X^{\dagger}$: At the lifetime of $X^{\dagger}$, X is sent to Δ during an independent exponential time of parameter 1 and then it is resurrected at 0 (and so on). The local time L at 0 for X is the extension of $L^{\dagger}$ and one easily checks that the Revuz measure w.r.t. X of the extension A of $A^{\dagger}$ is still ν. Let L^{Δ} be the time spent by X at Δ. If Hypothesis 1 is not fulfilled, then $\mathbb{E}_0(L^{\Delta}_S) = \infty$ for every solution S of the Skorokhod problem for ν and X, and there is no solution of the Skorokhod problem for ν and $X^{\dagger}$. If Hypothesis 1 is fulfilled, then the stopping time $T(\lambda_0)$ occurs before X visits Δ if and only if $\mathbb{E}_0(L^{\Delta}_{T(\lambda_0)}) = 0$. By the corollary, this is equivalent to $\lambda_0 = \hat{V}\nu(\Delta)$, that is, $\int V\chi\, d\nu \le \hat{V}\nu(\Delta)$ for every smooth probability measure χ on E (of course, this is just a reformulation of Rost's balayage condition). In this case, a solution of the Skorokhod problem for ν and $X^{\dagger}$ is

$$T^{\dagger} = \inf\left\{t: \int_0^t \left(\lambda_0 - \hat{V}\nu(X_s^{\dagger})\right)^{-1} dA_s^{\dagger} > L_t^{\dagger}\right\}.$$

If Hypothesis 1 is fulfilled but $\lambda_0 > \hat{V}\nu(\Delta)$, then there is no solution of the Skorokhod problem for ν and $X^{\dagger}$.

3. Proofs. The key point is an elementary result on Lévy processes with bounded variation and no positive jumps. Consider a subordinator S with no drift and Lévy measure Π and set $Y_t = t - S_t$. Denote the law of Y by P and its characteristic exponent by Ψ: For every $a \geq 0$,

$$E_0(\exp aY_t) = \exp t\Psi(a), \qquad \Psi(a) = a + \int_{(0,\infty)} (e^{-ay} - 1)\Pi(dy).$$

Clearly, Ψ is a convex function on $[0, \infty)$ with $\lim_{\infty} \Psi = \infty$. Furthermore, if $\int y\Pi(dy) \leq 1$, then $\Psi'(0+) \geq 0$, Ψ is a bijection on $[0, \infty)$, and we denote its inverse function by Ψ^{-1}. If $\int y\Pi(dy) > 1$, then $\Psi'(0+) < 0$, there is a unique positive root a_0 of the equation $\Psi(a) = 0$, Ψ induces a bijection from $[a_0, \infty)$ into $[0, \infty)$ and we denote the inverse function by $\Psi^{-1}: [0, \infty) \to [a_0, \infty)$.

Set $\tau(y) = \inf\{t > 0: Y_t = y\} = \inf\{t > 0: Y_t \geq y\}$ $(y \geq 0)$. The optional sampling theorem applied to the martingale $\exp\{aY_t - t\Psi(a)\}$ yields

$$E_0(\exp\{-a\tau(y)\}) = \exp\{-y\Psi^{-1}(a)\}. \tag{2}$$

We will need the following:

Lemma 1. *Set $\sigma = \inf\{t: Y_t < 0\}$. For every bounded Borel function φ, we have*

$$E_0\left(\int_0^{\sigma} \varphi(Y_t)e^{-at}\,dt\right) = \int_0^{\infty} \varphi(y)\exp\{-y\Psi^{-1}(a)\}\,dy.$$

Moreover $1/E_0(\sigma) = \Psi^{-1}(0) = a_0$.

Proof. Set $\varphi \equiv 0$ on $(-\infty, 0)$. Since Y has no positive jumps, we have $E_0(\int_0^{\sigma}\varphi(Y_t)e^{-at}\,dt) = E_0(\int_0^{\tau(0)}\varphi(Y_t)e^{-at}\,dt)$. By the Markov property, we rewrite this last quantity as $E_0(1 - e^{-a\tau(0)})E_0(\int_0^{\infty}\varphi(Y_t)e^{-at}\,dt)$. First assume that Π has a finite total mass, so the graph of Y is a.s. a discontinuous broken line with right slope 1. Thus

$$\int_0^{\infty} \varphi(Y_t)e^{-at}\,dt = \int_0^{\infty} dy\,\varphi(y)\int_0^{\infty} e^{-at}\,dl_t^y,$$

where $l_t^y = \operatorname{card}\{0 \leq s < t: Y_s = y\}$. By the Markov property and (2), we have

$$E_0\left(\int_0^{\infty}\varphi(Y_t)e^{-at}\,dt\right) = \int_0^{\infty} dy\,\varphi(y)\,E_0(\exp\{-a\tau(y)\})\,E_0\left(\int_0^{\infty} e^{-at}\,dl_t^0\right)$$

$$= \int_0^{\infty} dy\,\varphi(y)\exp\{-y\Psi^{-1}(a)\}/E_0(1 - e^{-a\tau(0)}).$$

By approximation, the above equality also holds even if Π has an infinite mass [with $\int(1 \wedge y)\Pi(dy) < \infty$]. Taking $\varphi \equiv \mathbf{1}_{[0,\infty)}$, we get $E_0(e^{-a\sigma}) = 1 - a/\Psi^{-1}(a)$; so $E_0(\sigma) = 1/\Psi^{-1}(0)$. □

Now, recall the notation for excursions, additive functionals and so on. We rewrite ν in terms of the excursion measure n:

LEMMA 2. *For every nonnegative continuous function f, we have*

$$\int f\,d\nu = \int dn \int_0^{\zeta} f(X_s)\,dA_s.$$

REMARK. When 0 is a holding point, say of parameter 1, then n is the law of a transient Markov process. This lemma links the Revuz (1970) and Azéma (1973) results.

PROOF. Indeed, by (1), for every $a > 0$, we have

$$\begin{aligned}
\int f\,d\nu &= \mathbb{E}_{\mu}\left(\int_0^{\infty} ae^{-at} f(X_t)\,dA_t\right)\\
&= \int dn \int_0^{\zeta} \mathbb{E}_{X(s)}\left(\int_0^{\infty} ae^{-at} f(X_t)\,dA_t\right) ds + c\mathbb{E}_0\left(\int_0^{\infty} ae^{-at} f(X_t)\,dA_t\right)\\
&= \int dn \int_0^{\zeta} e^{as}\,ds \int_s^{\zeta} ae^{-at} f(X_t)\,dA_t\\
&\quad + \left(c + \int dn \int_0^{\zeta} ds\, e^{-a(\zeta - s)}\right)\mathbb{E}_0\left(\int_0^{\infty} ae^{-at} f(X_t)\,dA_t\right)
\end{aligned}$$

(by the Markov property for n and the strong Markov property for $\mathbb{P}$)

$$\begin{aligned}
&= \int dn \int_0^{\zeta} (1 - e^{-at})\, f(X_t)\,dA_t\\
&\quad + \left(c + \int a^{-1}(1 - e^{-a\zeta})\,dn\right)\mathbb{E}_0\left(\int_0^{\infty} ae^{-at} f(X_t)\,dA_t\right).
\end{aligned}$$

Since $\int(1 \wedge \zeta)\,dn < \infty$ [recall that $n(\zeta \in \cdot)$ is the Lévy measure of the subordinator L^{-1}], $\int a^{-1}(1 - e^{-a\zeta})\,dn$ converges to 0 as $a \uparrow \infty$. Letting a go to infinity in the above formula, we find that

$$\int f\,d\nu = \int dn \int_0^{\zeta} f(X_t)\,dA_t + c' f(0),$$

where $c' = \lim_{a\uparrow\infty} c\mathbb{E}_0(\int_0^{\infty} ae^{-at}\,dA_t)$. Since ν does not charge 0, $c' = 0$. □

We are now able to give:

PROOF OF THE THEOREM. Denote by A^{-1} the right-continuous inverse of A, and by G the set of left endpoints of excursion intervals. By the definition of T, $s \in G(\omega)$ is the left endpoint of the excursion interval straddling T if and only if $s \le T$ and $u < A_\zeta(\theta_s\omega)$, where $u = L_s(\omega) - A_s(\omega)$. In this case, $T = s + A_u^{-1}(\theta_s\omega)$. Introduce

$$\varphi(u) = \int \mathbf{1}_{\{u < A_\zeta\}} f\big(X(A_u^{-1})\big)\, dn.$$

Using Maisonneuve's exit system identity for the second equality below, for every $b > 0$,

$$\begin{aligned}
&\mathbb{E}_0(\exp\{-bL_T\} f(X_T)) \\
&\quad = \mathbb{E}_0\bigg(\sum_{s \in G} \mathbf{1}_{\{s \le T\}} \exp\{-bL_s\} f \circ X\big(s + A_u^{-1}(\theta_s\omega)\big)\mathbf{1}_{\{u < A_\zeta(\theta_s\omega)\}}\bigg) \\
&\quad = \mathbb{E}_0\bigg(\int_0^T dL_s \exp\{-bL_s\}\varphi(L_s - A_s)\bigg) \\
&\quad = \mathbb{E}_0\bigg(\int_0^\sigma dt \exp\{-bt\}\varphi(t - A \circ L^{-1}(t))\bigg),
\end{aligned}$$

where $\sigma = \inf\{t\colon t < A \circ L^{-1}(t)\}$. Recall Lemma 1 and the corresponding notation with $Y_t = t - A \circ L^{-1}(t)$. We rewrite the above quantity as

$$\int_0^\infty \varphi(u)\exp\{-\Psi^{-1}(b)u\}\, du = \int dn \int_0^{A_\zeta} \exp\{-\Psi^{-1}(b)u\} f\big(X(A^{-1}(u))\big)\, du.$$

Putting the pieces together, we obtain

$$\mathbb{E}_0(\exp\{-bL_T\} f(X_T)) = \int dn \int_0^\zeta \exp\{-\Psi^{-1}(b)A_t\} f(X_t)\, dA_t. \tag{3}$$

For every $a > 0$, introduce the measure η_a:

$$\int f\, d\eta_a = \int dn \int_0^\zeta \exp\{-aA_t\} f(X_t)\, dA_t.$$

Recall that $V(f\nu)(X_t) = n(\int_t^\zeta f(X_s)\, dA_s | \mathscr{F}_t)$ for $t < \zeta$, and rewrite $\exp\{-aA_t\}$ as $1 - a\int_0^t \exp\{-aA_s\}\, dA_s$. By Fubini, we get $\int f\, d\eta_a + a\int V(f\nu)\, d\eta_a = \int dn \int_0^\zeta f(X_s)\, dA_s$. Applying Lemma 2, we deduce that

$$\int (f + aV(f\nu))\, d\eta_a = \int f\, d\nu. \tag{4}$$

By (3), the $\mathbb{P}_0$-distribution of X_T is $\eta = \eta_{a_0}$, where $a_0 = \Psi^{-1}(0)$. According to Lemma 1, $1/a_0 = E_0(\sigma) = \mathbb{E}_0(L_T)$. Finally, the Lévy measure of the subordinator $A \circ L^{-1}$ is $\Pi(dy) = n(A_\zeta \in dy)$. By Lemma 2, $\int y\Pi(dy) = \nu(E)$ and $a_0 = 0$ if and only if $\nu(E) \le 1$. □

REMARK 1. An alternative proof can be given using the following easy fact: If Q_t is the semigroup of $X \circ A^{-1}$ and if f and g are such that $f(X_t) - \int_0^t g(X_s)\, dA_s$ is a local martingale, then $Q_{L_t - A_t} f(X_t) + \int_0^t Q_{L_s - A_s} g(X_s)\, dL_s$ is a local martingale up to time T.

REMARK 2. When $\nu(E) > 1$, $\Psi^{-1}(0) = a_0 > 0$ and (3) shows that η is the law of $X \circ A^{-1}(e)$, where e is an independent exponential time with parameter a_0.

REMARK 3. Assume that the α-resolvent kernel of $X \circ A^{-1}$ is absolutely continuous with respect to ν, with densities $u_\alpha(x, y)$. We easily deduce from (3) that

$$\mathbb{E}_0(\exp\{-aL_T\}|X_T = y) = (\Psi^{-1}(a) - a)u_{\Psi^{-1}(a)}(0, y).$$

Assume henceforth that ν is a probability measure. It is now very natural to consider the following problem:

> Find a positive Borel function ρ, $\rho \le 1$, such that $\inf\{t: A'_t > L_t\} = T'$ solves the Skorokhod problem for ν, where $A'_t = \int_0^t 1/\rho(X_s)\, dA_s$.

For every $a > 0$, introduce the measure η'_a:

$$\int f\, d\eta'_a = \int dn \int_0^\zeta \exp\{-aA'_t\} f(X_t)\, dA'_t.$$

We rewrite (4) as

$$\int \left(f + aV\left(\frac{f}{\rho}\nu\right)\right) d\eta'_a = \int \frac{f}{\rho}\, d\nu. \tag{4'}$$

By the theorem, the $\mathbb{P}_0$-distribution of X_T is $\eta' = \eta'_a$ for a certain $a = a'_0$. It follows from (4') that if $\eta'_a = \nu$, then $a \int V(g\nu)\, d\nu = \int (1 - \rho)\, dg\nu$ for every Borel nonnegative function g. Consequently, if Hypothesis 1 holds, then we should take $\rho = 1 - a\hat{V}\nu$.

PROOF OF THE COROLLARY. (i) Assume that Hypothesis 1 is fulfilled; take $\lambda > \lambda_0$ and $\rho = 1 - \lambda^{-1}\hat{V}\nu$. So $T' = T(\lambda)$. Writing $f = (f/\rho)(1 - \lambda^{-1}\hat{V}\nu)$, we get $\int (f/\rho)\, d\nu = \int (f + \lambda^{-1}V((f/\rho)\nu))\, d\nu$. Note that for $f(x) = \mathbb{E}_x(\int_0^\zeta \exp\{-\lambda^{-1}A'_t\} h(X_t)\, dA_t)$, $f + \lambda^{-1}V((f/\rho)\nu) = V(h\nu)$ (by the generalized resolvent equation). Comparing with (4'), we deduce that $\eta'_{1/\lambda} = \nu$. In particular, $\eta'_{1/\lambda}(\mathbf{1}) = 1$, that is, $\lambda^{-1} = \int dn(1 - \exp\{-\lambda^{-1}A'(\zeta)\})$. This means that $\lambda^{-1} = a'_0$, so $\eta' = \eta'_{1/\lambda} = \nu$, and by the theorem, $\mathbb{E}_0(L_{T(\lambda)}) = \lambda$.

Clearly, $T(\lambda)$ decreases to $T(\lambda_0)$ as $\lambda \downarrow \lambda_0$. Since X is right-continuous, $T(\lambda_0)$ solves the Skorokhod problem for ν and $\mathbb{E}_0(L_{T(\lambda_0)}) = \lambda_0$.

Take for B a p.c.a.f. with finite Revuz measure χ and such that $V\chi$ is bounded. One easily checks that $V\chi(X_t) + B_t - \chi(\mathbf{1})L_t$ is a local martingale. If S is a stopping time that solves the Skorokhod problem for ν, the optional sampling theorem and Fatou lemma imply that $\chi(\mathbf{1})\mathbb{E}_0(L_s) \ge \int V\chi\, d\nu = \int \hat{V}\nu\, d\chi$. We deduce that $\mathbb{E}_0(L_s) \ge \lambda_0$. Moreover, the same martingale arguments show that $\mathbb{E}_0(B_s) = \int (\mathbb{E}_0(L_s) - \hat{V}\nu(x))\, d\chi(x)$. This equality extends immediately by monotone convergence to any positive continuous additive functional B. Indeed, finite lifetime implies the existence of a positive function with bounded potential [cf. Blumenthal and Getoor (1968), Chapter 4]. Apply this fact to X killed at 0 and time-changed by B^{-1}. It follows that $\mathbb{E}_0(B_{T(\lambda_0)}) \le \mathbb{E}_0(B_S)$.

(ii) When Hypothesis 1 does not hold, the very same arguments as above show that $\mathbb{E}_0(L_S) = \infty$, and more generally that $\mathbb{E}_0(B_S) = \infty$ whenever $B \not\equiv 0$. □

APPENDIX

Symmetric Lévy processes. Let X be a symmetric real Lévy process, such that 0 is regular and recurrent. We denote its characteristic exponent by Φ, that is, $\exp - t\Phi(p) = \mathbb{E}_0(\exp\{ipX_t\})$. For simplicity, we assume that there is a measurable family $(L(a,t)\colon a \in \mathbb{R},\ t \ge 0)$ of local times such that $t \mapsto L(a,t)$ is continuous for all $a \in \mathbb{R}$. See Getoor and Kesten (1972) for (usually satisfied) sufficient conditions. In particular, $\int_1^\infty 1/\Phi(p)\,dp < \infty$.

In this situation, it is interesting to know when there is a solution of the Skorokhod problem with finite first moment.

PROPOSITION. *Consider ν, a probability on $\mathbb{R}\setminus\{0\}$, and introduce $\mathscr{F}\nu(p) = \int e^{ipx}\,d\nu(x)$, $u(x) = (\frac{1}{2})\mathbb{E}_x(L(x,\zeta))$. We have:*

(i) *Hypothesis 1 holds if and only if*

$$\int u\,d\nu = \int ((1 - \mathscr{F}\nu(p))/\Phi(p))\,dp < \infty.$$

(ii) *$\mathbb{E}_0(T(\lambda_0))$ is finite if and only if Hypothesis 1 holds and*

$$\mathscr{F}^{-1}\left(\frac{1-\mathscr{F}\nu(p)}{\Phi(p)}\right) = u * \nu - u$$

is nonnegative and integrable. Then $\mathbb{E}_0(T(\lambda_0)) = \int (u * \nu - u)\,dx$.

PROOF. (i) Let $v_\alpha(x,y) = \mathbb{E}_x(\int_0^\zeta e^{-\alpha t}\,dL(y,t))$ be the α-resolvent kernel for X killed at 0 and set $u_\alpha(x) = \mathbb{E}_0(\int_0^\infty e^{-\alpha t}(dL(0,t) - dL(x,t)))$. Then

$$\mathscr{F}u_\alpha(p) = \frac{-1}{\Phi(p)+\alpha} + \delta_0(p)\int \frac{1}{\Phi(q)+\alpha}\,dq$$

and

$$\int u_\alpha\,d\nu = \int \frac{1-\mathscr{F}\nu(p)}{\Phi(p)+\alpha}\,dp.$$

Moreover, one easily checks the identity $v_\alpha(x,y) = u_\alpha(x) + \mathbb{E}_x(e^{-\alpha\zeta})u_\alpha(y) - u_\alpha(x-y)$. In particular, $v_\alpha(x,x) = (1 + \mathbb{E}_x(e^{-\alpha\zeta}))u_\alpha(x)$. We deduce that $u_\alpha(x)$ increases to $u(x) = (\frac{1}{2})v(x,x)$ as $\alpha \downarrow 0$, and that the Green function of X killed at 0 is

$$v(x,y) = u(x) + u(y) - u(x-y). \tag{5}$$

Since $V\nu(x) = \int v(x,y)\,d\nu(y) \le \int v(y,y)\,d\nu(y) = 2\int u\,d\nu$, $V\nu$ is bounded whenever $\int u\,d\nu < \infty$.

Conversely, we first assume that ν is symmetric and we set $\mathbf{1}_{[-n,n]}\nu = \nu_n$. Note that $\int u\,d\nu_n < \infty$ and that

$$\begin{aligned}(u_\alpha - u_{\alpha*}\nu_n)(x) &= \left(\frac{1}{2}\right)\iint \frac{1}{\Phi(p)+\alpha}\left(e^{ip(x+a)} + e^{ip(x-a)} - 2e^{ipx}\right) d\nu_n(a)\,dp \\ &= -\iint \frac{1}{\Phi(p)+\alpha} e^{ipx} 2\sin^2\!\left(\frac{ap}{2}\right) d\nu_n(a)\,dp.\end{aligned}$$

By the Lévy–Khintchine formula, $\Phi(p) \ge (cp^2)$ $(p \cong 0)$ and by dominated convergence, we get

$$(u - u_*\nu_n)(x) = -2\iint \frac{1}{\Phi(p)} e^{ipx}\sin^2\!\left(\frac{ap}{2}\right) d\nu_n(a)\,dp,$$

which tends to 0 as x goes to infinity. By (5), $\lim_\infty V\nu_n = \int u\,d\nu_n$ and since $\nu_n \uparrow \nu$, $V\nu$ cannot be bounded unless $\int u\,d\nu < \infty$. Finally, we remove the symmetry hypothesis on ν by noting that $V\nu(x) + V\nu(-x) = 2V\tilde{\nu}(x)$, where $\tilde{\nu}$ is the symmetrization of ν.

(ii) By (i), Hypothesis 1 means that $\mathscr{F}(u_*\nu - u)$ is integrable. Since

$$\mathbb{E}_0(T(\lambda_0)) = \int_{\mathbb{R}} (\lambda_0 - V\nu)\,dx = \int_{\mathbb{R}} \left(\lambda_0 - \int u\,d\nu + (u_*\nu - u)\right) dx,$$

as soon as $\lim_\infty u_*\nu - u = 0$, $\mathbb{E}_0(T(\lambda_0))$ cannot be finite unless $\lambda_0 = \|V\nu\|_\infty = \int u\,d\nu$, or equivalently $u_*\nu - u$ is nonnegative and integrable. The converse is obvious. □

Finally, let us specify our results when X is a one-dimensional Brownian motion. The potential kernel for the Brownian motion killed at 0 is $v(x,y) = |x| + |y| - |x-y|$. We deduce that $\|V\nu\|_\infty = (2\int x^+\,d\nu) \vee (2\int x^-\,d\nu) = \lambda_0$. Hence Hypothesis 1 is equivalent to ν having a first moment. In this case, introduce

$$\rho(x) = \begin{cases} 2\int_{a>x}(a-x)\,d\nu(a), & \text{for } x \ge 0, \\ 2\int_{a<x}(x-a)\,d\nu(a), & \text{for } x < 0. \end{cases}$$

Then

$$T(\lambda_0) = \inf\left\{t\colon \lambda_0 \int L(x,t)\rho^{-1}(x)\,d\nu(x) > L(0,t)\right\}$$

solves the Skorokhod problem for ν and $\mathbb{E}_0(L(0,T(\lambda_0))) = \lambda_0$. Furthermore, $u_*\nu - \nu$ is positive if and only if ν is centered and then integrable if and only if ν has a second moment.

Note also that when ν is centered, $\lambda_0 = \int |x|\, d\nu(x) = \mathbb{E}_0(|X(T(\lambda_0))|)$. According to the Tanaka formula and Dellacherie and Meyer [(1975), Theorem 1.21], $|X|$ is then uniformly integrable on $[0, T(\lambda_0)]$. If furthermore ν has a finite second moment, then X is a square-integrable martingale on $[0, T(\lambda_0)]$ and $\mathbb{E}_0(T(\lambda_0)) = \int x^2\, d\nu$.

Acknowledgment. We would like to thank P. Fitzsimmons for pointing out to us the exit system formula that simplified our initial proof of the theorem.

REFERENCES

AZÉMA, J. (1973). Théorie générale des processus et retournement du temps. *Ann. Sci. École Norm. Sup.* (*4*) **6** 459–519.

AZÉMA, J. and MEYER, P. A. (1974). Une nouvelle représentation du type de Skorokhod. *Séminaire de Probabilités VIII. Lecture Notes in Math.* **381** 1–10. Springer, New York.

AZÉMA, J. and YOR, M. (1979). Une solution simple au problème de Skorokhod. *Séminaire de Probabilités XIII. Lecture Notes in Math.* **721** 90–115. Springer, New York.

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Process and Potential Theory*. Academic, New York.

DELLACHERIE, C. and MEYER, P. A. (1975). *Probabilités et Potentiel*, Chapitres 1–4. Hermann, Paris.

DELLACHERIE, C. and MEYER, P. A. (1983). *Probabilités et Potentiel, Théorie Discrète du Potentiel*, Chapitres 9–11. Hermann, Paris.

DUBINS, L. E. (1968). On a theorem of Skorokhod. *Ann. Math. Statist.* **39** 2094–2097.

GETOOR, R. K. (1979). Excursions of a Markov process. *Ann. Probab.* **7** 244–266.

GETOOR, R. K. and KESTEN, H. (1972). Continuity of local time for a Markov process. *Compositio Math.* **24** 277–303.

ITÔ, K. (1972). Poisson point process attached to Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **3** 225–239. Univ. California Press, Berkeley.

MONROE, I. (1972). Using additive functionals to embed preassigned distributions in symmetric stable processes. *Trans. Amer. Math. Soc.* **163** 131–146.

REVUZ, D. (1970). Mesures associées aux fonctionnelles additives. *Trans. Amer. Math. Soc.* **148** 501–531.

ROOT, D. H. (1969). On the existence of certain stopping times of Browian motion. *Ann. Math. Statist.* **40** 715–718.

ROST, H. (1971). Markoff-Ketten bei sich fuellenden Loechern in Zustandraum. *Ann. Inst. Fourier* (*Grenoble*) **21** 253–270.

SKOROKHOD, A. V. (1965). *Studies in the Theory of Random Process*. Addison-Wesley, Reading, Mass.

LABORATOIRE DE PROBABILITÉS (L. A. 224)
TOUR 56
UNIVERSITÉ PIERRE ET MARIE CURIE
4, PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE