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ASYMPTOTIC BEHAVIOR OF SELF-NORMALIZED TRIMMED SUMS: NONNORMAL LIMITS[1]

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Let $\{X_j\}$ be independent, identically distributed random variables with continuous nondegenerate distribution F which is symmetric about the origin. Let $\{X_n(1), X_n(2), \dots, X_n(n)\}$ denote the arrangement of $\{X_1, \dots, X_n\}$ in decreasing order of magnitude, so that with probability 1, $|X_n(1)| > |X_n(2)| > \cdots > |X_n(n)|$. For integers $r_n \to \infty$ such that $r_n/n \to 0$, define the self-normalized trimmed sum $T_n = \sum_{i=r_n}^n X_n(i)/\{\sum_{i=r_n}^n X_n^2(i)\}^{1/2}$. The asymptotic behavior of T_n is studied. Under a probabilistically meaningful analytic condition generalizing the asymptotic normality criterion for T_n, various interesting nonnormal limit laws for T_n are obtained and represented by means of infinite random series. In general, moreover, criteria for degenerate limits and stochastic compactness for $\{T_n\}$ are also obtained. Finally, more general results and technical difficulties are discussed.

1. Introduction. The purpose of this article is to investigate in general the asymptotic behavior of the studentized intermediate magnitude trimmed sum formed from a continuous symmetric distribution, and in particular to obtain its asymptotic distribution under an analytic condition generalizing the criterion for its asymptotic normality. The resulting nonnormal limit laws will be represented by means of random series whose form explains and is explained by the probabilistic interpretation of the analytic conditions used. The main results are contained in Proposition 2.1, Theorem 5.3 and Theorem 5.30. Discussion of more general results is found in Section 7 [see also Hahn and Weiner (1990)].

Let $X, X_1, X_2, \dots$ be independent, identically distributed (i.i.d.) random variables with common nondegenerate distribution F and partial sums $S_n = X_1 + \cdots + X_n$. Let $N(0,1)$ denote the standard normal law and let $\mathscr{L}(Y)$ denote the law of the random variable Y. When $EX^2 < \infty$, the central limit theorem asserts that

$$(1.1) \qquad \mathscr{L}\left(\frac{S_n - nEX}{\sigma\sqrt{n}}\right) \to N(0,1),$$

where σ^2 is the variance of X. Moreover, (1.1) remains valid when the constant normalizers $\sigma\sqrt{n}$ are replaced by their empirical, random analogues:

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Letting $\bar{X}_n = S_n/n$, we have

$$\mathscr{L}\left(\frac{S_n - nEX}{\left\{\sum_{j=1}^n \left(X_j - \bar{X}_n\right)^2\right\}^{1/2}}\right) \to N(0,1). \tag{1.2}$$

The left member of (1.2) is often called a *self-normalized* or *studentized* sum. Such quantities are of considerable importance in statistics.

When $EX^2 = \infty$, (1.1) must be modified. The analytic criterion for constant-normalized asymptotic normality for S_n [i.e., the existence of constants σ_n and b_n such that $\mathscr{L}((S_n - b_n)/\sigma_n) \to N(0,1)$] is due to Lévy (1937):

$$\lim_{t\to\infty} \frac{t^2 P(|X| > t)}{E(X^2 \wedge t^2)} = 0. \tag{1.3}$$

For symmetric F, (1.3) is also the precise criterion for (1.2) to hold without modification [as observed in Griffin and Mason (1990)]. This suggests that studentization can improve and simplify the behavior of sums vis á vis constant normalization.

But when (1.3) fails, the behavior of the left member of (1.2) is quite complicated, even in the case of variables X in the domain of attraction of a stable law of index $0 < \alpha < 2$, that is, X such that the limit in (1.3) is $(2-\alpha)/2$ [Csörgő and Horvath (1988), LePage, Woodroofe and Zinn (1981) and Logan, Mallows, Rice and Shepp (1973)]. The form of the subsequential limits for more general distributions and convergence criteria for them have not yet been completely obtained, even in the symmetric case.

To see how to increase the scope of asymptotic normality for the sums and also to simplify their general asymptotic behavior, recall that for symmetric F, (1.3) is equivalent [(Lévy (1937)] to

$$\max_{j\le n} \frac{|X_j|}{|S_n|} \to_p 0. \tag{1.4}$$

Thus asymptotic normality fails (in the symmetric case, for both constant- and self-normalizations) precisely when there are relatively nonnegligible summands in the sum. This raises the question of whether removing these extreme, relatively nonnegligible summands leads to a more robust theory of asymptotic normality in particular, and convergence in distribution in general, for both the sums and their self-normalized versions.

Henceforth assume that

(1.5) F is nondegenerate and symmetric about the origin

and

(1.6) F is continuous.

Continuity is not essential here, but it allows for a great deal of technical complication to be avoided, in order not to obscure the main techniques and results.

Arrange the random sample $\{X_1, \dots, X_n\}$ in decreasing order of magnitude: Since continuity prevents ties with probability 1, we can denote the results $\{X_n(1), \dots, X_n(n)\}$ with

$$(1.7) \qquad |X_n(1)| > |X_n(2)| > \cdots > |X_n(n)|.$$

Given an integer $1 \le r \le n$, define the magnitude r-trimmed sum $S_n(r)$ by

$$(1.8) \qquad S_n(r) = \sum_{i=r}^{n} X_n(i) = S_n - \sum_{i=1}^{r-1} X_n(i),$$

where $S_n = S_n(1) = X_1 + \cdots + X_n$. By convention, an empty sum is 0. Thus in $S_n(r)$, the $r - 1$ summands largest in modulus have been removed from the full sum, and the leading summand retained is $X_n(r)$, the rth largest in magnitude. [Here the notation differs slightly from that of previous work on magnitude trimmed sums, where r summands are deleted and $X_n(r + 1)$ is the leading term retained. This change is appropriate in that previous techniques conditioned on the last summand removed, but here conditioning takes place on the leading summand retained in the trimmed sum; see also Remark 3.24.] For fixed r, no improvement with respect to asymptotic normality for $S_n(r)$ results [Maller (1982) and Mori (1984)]. Thus we consider $r = r_n \to \infty$. But in order to obtain asymptotic results depending on and utilizing the entire distribution F, it is necessary to require $r_n/n \to 0$. When r_n are integers with $0 \le r_n \le n$ such that

$$(1.9) \qquad r_n \to \infty \quad \text{and} \quad \frac{r_n}{n} \to 0,$$

then $S_n(r_n)$ is called an *intermediate magnitude* trimmed sum. Restrict hereafter to this case.

Pruitt (1985) proved that $S_n(r_n)$ is (constant-normalized) asymptotically normal provided

$$(1.10) \qquad \limsup_{t \to \infty} \frac{t^2 P(|X| > t)}{E(X^2 \wedge t^2)} < 1,$$

that is, F belongs to the Feller class $\mathscr{F}$. [This is precisely the class of distributions generating full sums S_n which can be centered and scaled by constants so that the resulting affine-normed sums are *stochastically compact*, i.e., tight with only nondegenerate subsequential limit laws [Feller (1967)]. In particular, $\mathscr{F}$ contains every distribution with regularly (but not slowly) varying tail.] Thus intermediate trimming does, indeed, provide a more robust theory of asymptotic normality for sums, since (1.10) is considerably weaker than (1.3).

More generally, Griffin and Pruitt (1987) completely determine the asymptotic distribution of constant normalizations of $S_n(r_n)$ for symmetric F as well as criteria for convergence to each of the possible limits. In particular, a criterion is given for $S_n(r_n)$ to be constant-normalized asymptotically normal; this criterion is considerably weaker even than (1.10).

To give their criterion (which is important here), for $t > 0$, define

$$G(t) = P(|X| > t),$$
$$M(t) = EX^2 I(|X| \le t), \tag{1.11}$$
$$K(t) = \frac{M(t)}{t^2}.$$

For $0 \le t \le 1$, put

$$G^{-1}(t) = \inf\{s \ge 0: G(s) \le t\}. \tag{1.12}$$

For $\alpha \in \mathbb{R}$, define for $n \in \mathbb{N}$ and $j = \pm 0, 1, 2, \dots,$

$$b_{nj}(\alpha) = G^{-1}\left(\frac{r_n + j - \alpha\sqrt{n_n + j}}{n}\right), \tag{1.13}$$

and write $b_n = b_{n0}$ throughout. Note that $b_n(\cdot)$ is nondecreasing, and is everywhere eventually defined due to (1.9). Given α, continuity and (1.9) guarantee that for all sufficiently large n, $b_n(\alpha)$ satisfies

$$G(b_n(\beta)) = \frac{r_n - \beta\sqrt{r_n}}{n} \tag{1.14}$$

for all β with $|\beta| \le |\alpha|$. Finally, define functions f_n by

$$f_n(\alpha) = \begin{cases} nK(b_n(\alpha)) = nM(b_n(\alpha))/b_n^2(\alpha), & b_n(\alpha) > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.15}$$

and let $n_k \to \infty$ be a given sequence of integers.

Then one result of Griffin and Pruitt (1987) is that $S_{n_k}(r_{n_k})$ is (constant-normalized) asymptotically normal if and only if

$$\forall\, \alpha: \quad \frac{f_{n_k}(\alpha)}{\sqrt{r_{n_k}}} \to \infty. \tag{1.16}$$

This result raises the natural problem of completely determining the self-normalized asymptotic behavior of $S_n(r_n)$, in or out of the presence of constant-normalized asymptotic normality for $S_n(r_n)$ as delimited by (1.16).

Define the magnitude r-trimmed sum of squares $V_n(r)$ by

$$V_n(r) = \sum_{i=r}^{n} (X_n(i))^2. \tag{1.17}$$

By continuity, $P(V_n(r) = 0) \le P(X = 0) = 0$ when $r \le n$. Thus we can always define the studentized magnitude r-trimmed sum $t_n(r)$ by

$$t_n(r) = \frac{S_n(r)}{\sqrt{V_n(r)}}. \tag{1.18}$$

Finally, given integers $\{r_n\}$ satisfying (1.9), define

$$T_n = t_n(r_n). \tag{1.19}$$

T_n is called the *intermediate studentized magnitude trimmed sum*. Traditionally, the studentized form of a sum involves the sample variance, as opposed to the sum-of-squares being used here. The adjustment here to the technically easier form using $V_n(r_n)$ rather than

$$V_n'(r_n) \equiv V_n(r_n) - \frac{1}{n - r_n + 1}(S_n(r_n))^2$$

causes asymptotically no change in the studentized distribution. (This will follow from Remark 2.11 and Lemma 2.13 below.)

Under (1.5) and (1.6), Hahn, Kuelbs and Weiner (1990) proved that

$$T_n \to_D N(0,1), \tag{1.20}$$

provided (1.10) holds. Moreover, Hahn, Kuelbs and Weiner (1990) also showed that (1.20) still holds along a given subsequence $\{n_k\}$ provided only

$$f_{n_k}(\cdot) \to \infty \quad \text{uniformly on compact sets,} \tag{1.21}$$

which is considerably weaker than the criterion (1.16) for constant-normalized asymptotic normality for $S_{n_k}(r_{n_k})$. [Note that when (1.16) holds, it holds uniformly on compact sets, as is easily seen from the proof of Lemma 3.6 in Griffin and Pruitt (1987).] Thus, in contrast to the situation for full sums, self-normalization does actually enhance the possibility of asymptotic normality for trimmed sums as compared to constant normalization.

The precise criterion for (1.20) along a given subsequence $\{n_k\}$ is slightly weaker than (1.21), and was recently obtained by Griffin and Mason (1990). In our notation (cf. Remark 3.24 below), their result can be stated as follows: Given $\{n_k\}$, (1.20) holds along $\{n_k\}$ if and only if

$$f_{n_k} \to \infty \quad \text{in measure with respect to } N(0,1),$$

or equivalently,

$$f_{n_k}(Z) \to_p \infty, \tag{1.22}$$

where $Z \sim N(0,1)$. Clearly (1.22) implies (1.16).

The purpose of this paper is to elucidate further the asymptotic behavior of the self-normalized trimmed sum $T_n = S_n(r_n)/\sqrt{V_n(r_n)}$. Section 2 establishes that $\{T_n\}$ is *always* stochastically compact and makes some observations concerning when the "trivial most" possible limit, the Rademacher, arises. The main objective of the remainder of the paper is to determine the asymptotic distribution of the subsequence $\{T_{n_k}\}$ when

$$f_{n_k}(Z) \to_p c \quad \text{for some } c \in [0,\infty). \tag{1.23}$$

Toward this end, Section 3 establishes some useful representations, Section 4 develops the required analytic facts and Section 5 establishes the main theorems. In particular, Theorem 5.3 characterizes the Rademacher limit as occurring if and only if (1.23) holds with $c = 0$. When (1.23) holds with $0 < c < \infty$, Theorem 5.30 identifies the limit distributions that arise, representing them in a natural way via random series. Then Section 6 shows that each limit in Theorem 5.30 does actually arise by providing a "limit-generating law" which is attracted along different subsequences to each of the possible limit laws. Two essential new ingredients are (i) a refinement of a well-known representation of order statistics by means of an i.i.d. unit mean exponential sequence, and (ii) a special analytic relation between the inverse tail and normalized truncated second moment functions. Finally, Section 7 explores some results and technical difficulties in the study of the general case.

2. Stochastic compactness of $\{T_n\}$ and observations about the Rademacher limit

PROPOSITION 2.1. *$\{\mathscr{L}(T_n)\}$ is stochastically compact.*

The proof of this proposition depends on the following widely known fact whose proof is included both for completeness of exposition, and because it is a convenient mechanism by which to introduce certain facts and representations which are crucial later.

LEMMA 2.2. *For $t \geq 0$,*

$$\sup_n E \exp(tT_n) \leq e^{t^2/2}. \tag{2.3}$$

PROOF. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ be i.i.d., independent of $\{X_j\}$, with $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = 1/2$. Then

$$T_n \overset{D}{=} \frac{\sum_{j=r_n}^n |X_n(j)| \varepsilon_j}{\left\{\sum_{j=r_n}^n (X_n(j))^2\right\}^{1/2}} \equiv \sum_{j=r_n}^n c_{nj} \varepsilon_j, \tag{2.4}$$

where $\{c_{nj}: r_n \leq j \leq n\}$ is independent of $\{\varepsilon_j: r_n \leq j \leq n\}$,

$$1 \geq c_{n, r_n} \geq c_{n, r_n + 1} \geq \cdots \geq c_{nn} \geq 0 \tag{2.5}$$

and

$$\sum_{j=r_n}^n c_{nj}^2 = 1. \tag{2.6}$$

Let

$$\mathscr{F}_n = \sigma(X_1, \dots, X_n), \tag{2.7}$$

the σ-algebra generated by $X_1, \dots, X_n$. Then, using independence of $\mathscr{F}_n$

and $\{\varepsilon_j\}$,

$$
\begin{aligned}
E\exp(tT_n) &= E\left(E\left[\exp\left(t\sum_{j=r_n}^{n} c_{nj}\varepsilon_j\right)\middle|\mathscr{F}_n\right]\right)\\
&= E\prod_{j=r_n}^{n}\left(\frac{e^{tc_{nj}}+e^{-tc_{nj}}}{2}\right)\\
&\le E\prod_{j=r_n}^{n} e^{t^2c_{nj}^2/2} = \exp\left(\frac{t^2}{2}\sum_{j=r_n}^{n} c_{nj}^2\right) = e^{t^2/2},
\end{aligned}
\tag{2.8}
$$

where the last line uses $(e^x + e^{-x})/2 \le e^{x^2/2}$, which follows from a Taylor series expansion:

$$
(e^x + e^{-x})/2 = \sum_{m=0}^{\infty}\frac{x^{2m}}{(2m)!} \le \sum_{m=0}^{\infty}\frac{x^{2m}}{2^m m!} = e^{x^2/2},
$$

since $(2m)! \ge 2^m m!$. □

Proof of Proposition 2.1. Markov's inequality and (2.3) immediately yield tightness for $\{\mathscr{L}(T_n)\}$. It remains to establish that every subsequential limit is nondegenerate.

From (2.3) and standard arguments also follows uniform integrability of $\{T_n^p\colon n \ge 1\}$ for each integer $1 \le p < \infty$. Thus $\{T_n^2\}$ is uniformly integrable, yet $ET_n^2 = 1$ due to independence and

$$
\begin{aligned}
ET_n^2 &= E\left(E\left[\left(\sum_{j=r_n}^{n} c_{nj}\varepsilon_j\right)^2 \middle| \mathscr{F}_n\right]\right)\\
&= E\left(\sum_{j=r_n}^{n} c_{nj}^2\right) = E1 = 1,
\end{aligned}
\tag{2.9}
$$

recalling (2.6). Hence [cf. Chung (1974), page 97]

$$
\mathscr{L}(T_{n_k}) \to \mathscr{L}(Y) \Rightarrow 1 \equiv ET_{n_k}^2 \to EY^2. \tag{2.10}
$$

Moreover, $\mathscr{L}(Y)$ will obviously be symmetric about the origin. Hence $\{\mathscr{L}(T_n)\}$ possesses no degenerate subsequential limits. □

Remark 2.11. Ordinary studentization for $S_n(r_n)$ would take the form

$$
T_n' = S_n(r_n)\Bigg/\left\{V_n(r_n) - \frac{1}{n - r_n + 1}\big(S_n(r_n)\big)^2\right\}^{1/2}, \tag{2.12}
$$

as is easily checked. However, using Proposition 2.1, we obtain the following lemma.

Lemma 2.13. $T_n' - T_n \to_p 0$.

PROOF. Write $T_n' = T_n\{1 - J_n^2\}^{-1/2}$, where

$$J_n^2 = \frac{1}{n - r_n + 1} \frac{(S_n(r_n))^2}{V_n(r_n)} = \frac{1}{n - r_n + 1} T_n^2 \to_p 0, \tag{2.14}$$

due to Proposition 2.1. □

As is evidenced by the proof of Proposition 2.1, every subsequential limit law for $\{T_n\}$ is symmetric about the origin and has variance 1. Thus the Rademacher law $\Delta = \frac{1}{2}\{\delta_1 + \delta_{-1}\}$, is the most "trivial" limit. Naturally enough, the analogue opposite to "$T_{n_k} \to_D N(0,1) \Leftrightarrow f_{n_k}(Z) \to_p \infty$" holds, that is, "$T_{n_k} \to_D \Delta \Leftrightarrow f_{n_k}(Z) \to_p 0$." This will be fully established in Section 5, but some preliminary observations are useful here.

Griffin and Mason's (1990) proof of the asymptotic normality criterion (1.22) for T_{n_k} depended, in part, on establishing the equivalence of their criterion to the uniform asymptotic negligibility of the studentized sample, specifically, the condition $(X_{n_k}(r_{n_k}))^2/V_{n_k}(r_{n_k}) \to_p 0$. Analogously, characterizing the case of a Rademacher limit involves characterizing the dominance of the leading term $X_{n_k}^2(r_{n_k})$ relative to the trimmed sum-of-squares $V_{n_k}(r_{n_k})$.

LEMMA 2.15. *Given a subsequence $\{n_k\}$, $T_{n_k} \to_D \Delta$ if and only if*

$$\frac{(X_{n_k}(r_{n_k}))^2}{V_{n_k}(r_{n_k})} \to_p 1. \tag{2.16}$$

PROOF. Recall the notation and setup of the proofs of Proposition 2.1 and Lemma 2.2, and suppress the subscript k. Simple computations analogous to those in (2.9) lead [cf. Efron (1969)] to

$$ET_n^4 = 3 - 2 \sum_{j=r_n}^{n} Ec_{nj}^4. \tag{2.17}$$

Suppose $\mathscr{L}(T_n) \to \Delta$. Since $\{T_n^4\}$ is uniformly integrable, $ET_n^4 \to 1$, so that [recalling (2.4)–(2.6)]

$$\begin{aligned} 1 \leftarrow \sum_{j=r_n}^{n} Ec_{nj}^4 &\le E\left[c_{n,r_n}^2 \sum_{j=r_n}^{n} c_{nj}^2\right] \\ &= Ec_{n,r_n}^2 \le 1. \end{aligned} \tag{2.18}$$

Since $c_{n,r_n}^2 \le 1$, we obtain $E|1 - c_{n,r_n}^2| = 1 - Ec_{n,r_n}^2 \to 0$, whence $c_{n,r_n}^2 \to_p 1$. This is the same as (2.16).

Conversely, if (2.16) holds, then $\sum_{j=r_n+1}^{n} c_{nj}^2 \to_p 0$ due to (2.6), whence [as in (2.10)] $E(\sum_{j=r_n+1}^{n} c_{nj}\varepsilon_j)^2 \to 0$. Thus, since $c_{n,r_n} \to_p 1$ due to (2.16), it follows that

$$T_n \overset{D}{=} c_{n,r_n}\varepsilon_1 + \sum_{j=r_n+1}^{n} c_{nj}\varepsilon_j \to_p \varepsilon_1 \sim \Delta, \tag{2.19}$$

proving the lemma. □

Analytically, characterizing the dominance condition (2.16) is somewhat more involved; see Theorem 5.3 below.

3. Representations. Here we develop and adapt a well-known representation of order statistics involving exponential random variables. In addition, convergence of certain of these representations (in a strong sense vital to the subsequent analysis of the studentized quantity T_n) will be established. Some related facts needed later are also presented here. [Much of what is included here for convenience and completeness may already be known, but we were unable to find references. For some similar results, however, see Reiss (1989).]

Let $E_0, E_1, E_2, \dots$ be i.i.d. exponential random variables with unit mean. Put $\Gamma_n = E_1 + \cdots + E_n$, with $\Gamma_0 = 0$. Then [e.g., Breiman (1968)]

$$\begin{aligned}&(|X_n(1)|, |X_n(2)|, \dots, |X_n(n)|)\\ &\overset{D}{=} \big(G^{-1}(\Gamma_1/\Gamma_{n+1}), G^{-1}(\Gamma_2/\Gamma_{n+1}), \dots, G^{-1}(\Gamma_n/\Gamma_{n+1})\big),\end{aligned} \tag{3.1}$$

where we recall (1.12) for the definition of G^{-1}. Thus, given $0 \le N \le r_n$ such that $r_n + N \le n$, we have

$$\begin{aligned}(|X_n(r_n+j)|: -N \le j \le N) &\overset{D}{=} \left(G^{-1}\left(\frac{\Gamma_{r_n+j}}{\Gamma_{n+1}}\right): -N \le j \le N\right)\\ &\overset{D}{=} \left(b_n\left(\sqrt{r_n}\left(1 - \frac{n}{r_n}\frac{\Gamma_{r_n+j}}{\Gamma_{n+1}}\right)\right): -N \le j \le N\right)\\ &\equiv \big(b_n(Z_{nj}): -N \le j \le N\big),\end{aligned} \tag{3.2}$$

recalling (1.13).

In Hahn, Kuelbs and Weiner (1990), Lemma 5, it was shown that for j fixed, $Z_{nj} \to_D N(0,1)$. Here, however, much more detailed information is required.

For N as above, define

$$\begin{aligned}\xi_{nN} = \big(&Z_{n0}, \sqrt{r_n}(Z_{n0} - Z_{n1}), \sqrt{r_n}(Z_{n1} - Z_{n2}), \dots,\\ &\sqrt{r_n}(Z_{n,N-1} - Z_{nN})\big).\end{aligned} \tag{3.3}$$

Let $Z \sim N(0,1)$ be independent of E_0 and $\{E_j: j \ge 1\}$, and let

$$\xi_N = (Z, E_1, E_2, \dots, E_N). \tag{3.4}$$

PROPOSITION 3.5. *$\mathscr{L}(\xi_{nN})$ has a Lebesgue density which converges in $L^1(\mathbb{R}^{N+1})$ with respect to Lebesgue measure to that of $\mathscr{L}(\xi_N)$. In particular,*

$$\mathscr{L}(\xi_{nN}) \to_{\mathrm{TV}} \mathscr{L}(\xi_N), \tag{3.6}$$

where $\to_{\mathrm{TV}}$ denotes convergence in total variation for measures.

PROOF. Note that

$$\xi_{nN} = \left(\sqrt{r_n}\left(1 - \frac{n}{r_n}\frac{\Gamma_{r_n}}{\Gamma_{n+1}}\right), \frac{n}{\Gamma_{n+1}}E_{r_n+1}, \frac{n}{\Gamma_{n+1}}E_{r_n+2}, \dots, \frac{n}{\Gamma_{n+1}}E_{r_n+N}\right), \tag{3.7}$$

which certainly makes the lemma plausible, since $\Gamma_{n+1}/n \to 1$ a.s. by the strong law of large numbers, Γ_{r_n} is independent of the i.i.d. unit mean exponential collection $\{E_{r_n+1}, \dots, E_{r_n+N}\}$, and $(\Gamma_{r_n} - r_n)/\sqrt{r_n} \to_D \mathscr{N}(0,1)$ by the classical clt [since $\mathrm{Var}(E_0) = 1$].

Consider the random vector in $\mathbb{R}^{N+2}$ given by

$$\Xi_{nN} = \left(-\frac{\Gamma_{r_n} - r_n}{\sqrt{r_n}}, E_{r_n+1}, \dots, E_{r_n+N}, \frac{\Gamma_{n+1} - \Gamma_{r_n+N}(n+1-r_n-N)}{\sqrt{n+N-r_n}}\right). \tag{3.8}$$

It has independent components, the first and last of which have densities which converge uniformly to the standard normal density. [This follows easily from the remark following Theorem 15.5.2 in Feller (1971), applied to the unit exponential distribution.] Thus the density of $\mathscr{L}(\Xi_{nN})$ converges uniformly to that of $\mathscr{L}(\xi_N, Z')$, where $Z' \sim \mathscr{N}(0,1)$ is independent of ξ_N. Moreover, the density of $\mathscr{L}(\Xi_{nN})$ is continuous on $\mathbb{R} \times (0,\infty)^N \times \mathbb{R}$.

We can write

$$K_n \equiv \left(\xi_{nN}, \frac{\Gamma_{n+1} - \Gamma_{r_n+N}(n+1-r_n-N)}{\sqrt{n+1-r_n-N}}\right) = W_n(\Xi_{nN}), \tag{3.9}$$

where W_n: $\mathbb{R} \times (0,\infty)^N \times \mathbb{R} \to \mathbb{R} \times (0,\infty)^N \times \mathbb{R}$ can be explicitly (albeit tediously) identified, and is locally invertible and smooth. Moreover, it is easily checked (from the explicit form) that the Jacobian $\det(dW_n)$ converges uniformly on compact subsets of $\mathbb{R} \times (0,\infty)^N \times \mathbb{R}$ to unity. Clearly, $W_n \to I$ in the same manner on $\mathbb{R} \times (0,\infty)^N \times \mathbb{R}$, where I is the identity on $\mathbb{R} \times (0,\infty)^N \times \mathbb{R}$. The standard transformation/densities lemma [e.g., Billingsley (1986), page 229] now shows that the density of $\mathscr{L}(K_n)$ converges pointwise to that of $\mathscr{L}(Z', \xi_N)$. Thus, by the Scheffé lemma [e.g., Billingsley (1986), page 218], the convergence is actually in $L^1(\mathbb{R}^{N+2})$. By "integrating out the final coordinate" of these densities (using Fubini), it follows that the density of $\mathscr{L}(\xi_{nN})$ converges in $L^1(\mathbb{R}^{N+1})$, to that of $\mathscr{L}(\xi_N)$, proving the first assertion. The total variation assertion now follows immediately. □

Note that Proposition 3.5 immediately gives

$$(Z_{n0}, Z_{n1}, \dots, Z_{nN}) \to_D (Z, Z, \dots, Z). \tag{3.10}$$

Proposition 3.5 also guarantees total variation convergence of each marginal distribution. For example, for each $0 \le j \le N$,

$$\begin{gathered} \mathscr{L}(Z_{nj}) \to_{\mathrm{TV}} N(0,1), \\ \mathscr{L}\big(Z_{nj}, \sqrt{r_n}\,(Z_{nj} - Z_{n,j+1})\big) \to_{\mathrm{TV}} \mathscr{L}(Z, E_0). \end{gathered} \tag{3.11}$$

We will require a uniform version of the first convergence in (3.11).

LEMMA 3.12. *Given integers* $0 \le N_n \le n - r_n$ *such that*

$$\frac{N_n}{\sqrt{r_n}} \to 0, \tag{3.13}$$

we have

$$\mathscr{L}(Z_{nj}) \to_{\mathrm{TV}} N(0,1) \quad \textit{uniformly in } 0 \le j \le N_n. \tag{3.14}$$

PROOF. Letting ϕ_{nj}, ϕ denote (respectively) the density of $\mathscr{L}(Z_{nj})$, $\mathscr{N}(0,1)$, it will be enough to show that given any $\{j_n\}$ with $0 \le j_n \le N_n$,

$$\int |\phi_{nj_n} - \phi|\, d\lambda \to 0, \tag{3.15}$$

where λ denotes Lebesgue measure on the line. [That (3.15) suffices for (3.14) follows by a standard contradiction argument.]

So, given $0 \le j_n \le N_n$, consider the random vectors

$$A_n = \left(\frac{\Gamma_{r_n+j_n} - (r_n + j_n)}{\sqrt{r_n + j_n}}, \frac{\Gamma_{n+1} - \Gamma_{r_n+j_n} - (n+1-r_n-j_n)}{\sqrt{n+1-r_n-j_n}} \right). \tag{3.16}$$

The density α_n of $\mathscr{L}(A_n)$ converges uniformly to the uniformly continuous density $\phi_0(x,y) = \phi(x)\phi(y)$ [again quoting Feller (1971), note following 15.5.2, and using (3.13)]. As in the proof of Proposition 3.5, we can write

$$\beta_n \equiv \left(Z_{nj_n}, \frac{\Gamma_{n+1} - \Gamma_{r_n+j+n} - (n+1-r_n-j_n)}{\sqrt{n+1-r_n-j_n}} \right) = W_n(A_n), \tag{3.17}$$

where $W_n\colon \mathbb{R}^2 \to \mathbb{R}^2$, and where $W_n \to I$ in the norm of $C^1(\mathbb{R}^2)$. Thus the density of $\mathscr{L}(\beta_n)$ converges uniformly to ϕ_0 on $\mathbb{R}^2$. By the Scheffé lemma and then by Fubini, $\phi_{nj_n} \to \phi$ in $L^1(\mathbb{R})$, so that (3.15) holds. □

An immediate consequence of Lemma 3.12 is the tightness of the real triangular array $\{Z_{nj}\colon j \le N_n,\ n \ge 1\}$ for any $\{N_n\}$ satisfying (3.13).

Also needed is a related fact. Let Z, E_0 be as above (in particular, independent).

LEMMA 3.18. $\mathscr{L}(Z - E_0/\sqrt{r_n}) \to_{\mathrm{TV}} N(0,1)$.

PROOF. Let ϕ denote the standard normal density, and let ϕ_n denote the density of $\mathscr{L}(Z - E_0/\sqrt{r_n})$. By independence,

$$(3.19)\quad \phi_n(t) = \int \phi(t-x)\, d\mathscr{L}\big(E_0/\sqrt{r_n}\big)(x) \to \int \phi(t-x)\, d\delta_0(x) = \phi(t),$$

since ϕ is bounded and continuous, and $E_0/\sqrt{r_n} \to_p 0$. Now apply the Scheffé lemma. □

Finally, the following technical result will be required for use in the proof of Lemma 5.18. Recall notation (1.13) for b_{nj}.

LEMMA 3.20. *Given integers $N_n \to \infty$ such that $N_n/\sqrt{r_n} \to 0$, for $0 \le |j| \le N_n$, define*

$$(3.20)\qquad \tilde{Z}_{nj} = Z_{nj}\sqrt{\frac{r_n + j}{r_n}} - \frac{j}{\sqrt{r_n}}.$$

Then for each j,

$$(3.21)\qquad nK\big(b_{nj}(Z_{nj})\big) = f_n\big(\tilde{Z}_{nj}\big)$$

and

$$(3.22)\qquad \mathscr{L}\big(\tilde{Z}_{nj}\big) \to_{\mathrm{TV}} N(0,1) \quad \text{uniformly in } |j| \le N_n.$$

Finally, for each fixed $j \ge 0$,

$$(3.23)\qquad \mathscr{L}\big(Z_{n0}, \sqrt{r_n}\big(Z_{n0} - \tilde{Z}_{nj}\big)\big) \to_{\mathrm{TV}} \mathscr{L}(Z, \Gamma_j - j).$$

PROOF. The proof is a straightforward computation using the statements and patterned after the proofs of Proposition 3.5 and Lemma 3.12. In particular, note that the transformations $Z_{nj} \to \tilde{Z}_{nj}$ are linear, with slopes and intercepts converging uniformly, in $|j| \le N_n$, to 1 and 0, respectively, as $n \to \infty$. □

REMARK 3.24. Combining Lemma 3.20 with the result of Griffin and Mason (1990), we can easily see that their result may be stated in the form $\mathscr{L}(S_{n_k}(r_{n_k})/\sqrt{V_{n_k}(r_{n_k})}) \to N(0,1) \Leftrightarrow f_{n_k}(Z) \to_p \infty$. In our notation, they actually proved for each $\{r_n\}$ that

$$\mathscr{L}\Big(S_{n_k}(r_{n_k}+1)/\sqrt{V_{n_k}(r_{n_k}+1)}\Big) \to N(0,1) \Leftrightarrow f_{n_k}(Z) = n_k K\big(b_{n_k 0}(Z)\big) \to_p \infty.$$

But [cf. argument in (4.3) below], applying their result to $\{r_{n_k}-1\}$, we have

$$f_{n_k}(Z) \to_p \infty \Leftrightarrow f_{n_k}\big(\tilde{Z}_{n_k,-1}\big) \to_p \infty \Leftrightarrow n_k K\big(b_{n_k-1}(Z)\big) \to_p \infty$$
$$\Leftrightarrow \mathscr{L}\left(\frac{S_{n_k}(r_{n_k})}{\sqrt{V_{n_k}(r_{n_k})}}\right) \to N(0,1),$$

thereby establishing the claim.

4. Analytic facts. Fix a sequence of integers $n_k \to \infty$ and throughout this section assume

$$(4.1)\qquad Z \sim N(0,1), \qquad E_0 \sim \text{Exp}(1), \qquad Z \text{ is independent of } E_0.$$

Assume furthermore

$$(4.2)\qquad f_{n_k}(Z) \to_p c,$$

where $0 \le c < \infty$. (Often the case $c = 0$ must be separated from that of $0 < c < \infty$.)

By Lemma 3.18, $\mathscr{L}(Z - E_0/\sqrt{r_n}) \to_{\text{TV}} \mathscr{L}(Z)$, so that for any Borel set B,

$$(4.3)\qquad \begin{aligned} P\left(f_n\left(Z - \frac{E_0}{\sqrt{r_n}}\right) \in B\right) &= \mathscr{L}\left(Z - \frac{E_0}{\sqrt{r_n}}\right)\big(f_n^{-1}(B)\big) \\ &= \mathscr{L}(Z)\big(f_n^{-1}(B)\big) + o(1) \\ &= P\big(f_n(Z) \in B\big) + o(1). \end{aligned}$$

Choosing $B = [c-\varepsilon, c+\varepsilon]^c$, where $\varepsilon > 0$, we see that condition (4.2) forces also

$$(4.4)\qquad f_{n_k}\left(Z - \frac{E_0}{\sqrt{r_{n_k}}}\right) \to_p c.$$

The following convenient formula relating the inverse-tail function b_n to the normalized truncated second moment function f_n will be required.

LEMMA 4.5. *For $z \in \mathbb{R}$ and $a \ge 0$,*

$$(4.6)\qquad f_n(z) = f_n\left(z - \frac{a}{\sqrt{r_n}}\right)\frac{b_n^2\big(z - a/\sqrt{r_n}\big)}{b_n^2(z)} + \int_0^a \frac{b_n^2\big(z - s/\sqrt{r_n}\big)}{b_n^2(z)}\,ds.$$

PROOF. Since F is assumed continuous, we have [recalling (1.11)–(1.13)], for large enough n,

$$\begin{aligned} M(b_n(s)) &= EX^2 I(|X| \le b_n(s)) \\ &= EX^2 I(|X| < b_n(s)) \\ &= \int G^{-1}(u)^2 I(G^{-1}(u) < b_n(s))\,du \\ &= \int G^{-1}(u)^2 I\left(G^{-1}(u) < G^{-1}\left(\frac{r_n - s\sqrt{r_n}}{n}\right)\right) du \\ &= \int G^{-1}(u)^2 I\left(u > \frac{r_n - s\sqrt{r_n}}{n}\right) du = \int_{(r_n - s\sqrt{r_n})/n} G^{-1}(u)^2\,du, \end{aligned}$$

where continuity was required only in the second and fifth equalities. Changing variables,

$$M(b_n(s)) - M(b_n(t)) = \frac{\sqrt{r_n}}{n}\int_t^s b_n^2(y)\,dy.$$

Thus expanding and changing variables again,

$$\begin{aligned} f_n(z) &= n\frac{M(b_n(z))}{b_n^2(z)} \\ &= \frac{n}{b_n^2(z)}\left(M(b_n(z)) - M\left(b_n\left(z - \frac{a}{\sqrt{r_n}}\right)\right)\right) \\ &\quad + f_n\left(z - \frac{a}{\sqrt{r_n}}\right)\frac{b_n^2\left(z - a/\sqrt{r_n}\right)}{b_n^2(z)} \qquad (4.7) \\ &= \frac{\sqrt{r_n}}{b_n^2(z)}\int_{z-a/\sqrt{r_n}}^{z} b_n^2(y)\,dy + f_n\left(z - \frac{a}{\sqrt{r_n}}\right)\frac{b_n^2\left(z - a/\sqrt{r_n}\right)}{b_n^2(z)} \\ &= \int_0^a \frac{b_n^2\left(z - s/\sqrt{r_n}\right)}{b_n^2(z)}\,ds + f_n\left(z - \frac{a}{\sqrt{r_n}}\right)\frac{b_n^2\left(z - a/\sqrt{r_n}\right)}{b_n^2(z)}, \end{aligned}$$

proving the lemma. □

Define, for $z \in \mathbb{R}$ and $s \ge 0$,

$$g_n(z, s) = \frac{b_n^2\left(z - s/\sqrt{r_n}\right)}{b_n^2(z)}. \qquad (4.8)$$

Note that $0 \le g_n \le 1$, and for each z, $g_n(z, \cdot\,)$ is nonincreasing.

LEMMA 4.9. *If* (4.2) *holds with* $0 < c < \infty$, *then*

$$e^{E_0/c} g_{n_k}(Z, E_0) \to_p 1. \tag{4.10}$$

If (4.2) *holds with* $c = 0$, *then*

$$g_{n_k}(Z, E_0) \to_p 0. \tag{4.11}$$

PROOF. The proof makes repeated use of the fact that convergence in probability is equivalent to the property that inside every subsequence is a further subsequence whereon almost-sure convergence holds with the same limit independent of the selected subsequence.

Inside any subsequence of $\{n_k\}$, there is [due to (4.2) and (4.4)] another $\{n_k(1)\}$ along which both

$$f_{n_k(1)}(Z) \to c, \quad \text{a.s., and} \quad f_{n_k(1)}\left(Z - \frac{E_0}{\sqrt{r_{n_k(1)}}}\right) \to c, \quad \text{a.s.} \tag{4.12}$$

Hence, for almost every $(z, a) \in \mathbb{R} \times [0, \infty)$, applying (4.12) to (4.6) leads to

$$c = \int_0^a g_{n_k(1)}(z, s)\, ds + c g_{n_k(1)}(z, a) + o(1). \tag{4.13}$$

Fix z so that (4.13) holds for almost every $a \ge 0$. Given any further subsequence of $\{n_k(1)\}$, select an even further subsequence $\{n_k(2)\}$ and a monotone function g so that for *every* s,

$$g_{n_k(2)}(z, s) \to g(z, s). \tag{4.14}$$

[Here Helly selection is used in the form given in Taylor (1985), page 398.] Using this subsequence $\{n_k(2)\}$, (4.13) yields (using the bounded convergence theorem)

$$c = \int_0^a g(z, s)\, ds + c g(z, a) \tag{4.15}$$

for almost every $a \ge 0$. Now $g(z, \cdot\,)$ is monotone, so that (4.15) guarantees it is absolutely continuous when $c > 0$. Thus, noting $g(z, 0) = 1$, we obtain as the unique solution of (4.15) that if $c = 0$,

$$\forall\, a > 0: \quad g(z, a) = 0, \quad g(z, 0) = 1, \tag{4.16}$$

while if $0 < c < \infty$,

$$\forall\, a \ge 0: \quad g(z, a) = e^{-a/c}. \tag{4.17}$$

First suppose $0 < c < \infty$. Thus, for the subsequence $\{n_k(1)\}$ on which (4.13) holds, and for almost every z, inside any further subsequence is another subsequence $\{n_k(3)\}$ along which

$$e^{E_0/c} g_{n_k(3)}(z, E_0) \to 1, \quad \text{a.s.} \tag{4.18}$$

Thus, for the subsequence $\{n_k(1)\}$ on which (4.13) holds, and for almost every z,

$$e^{E_0/c} g_{n_k(1)}(z, E_0) \to_p 1. \tag{4.19}$$

Using independence, Fubini and bounded convergence, it follows from (4.19) that on this same subsequence $\{n_k(1)\}$ where (4.13) holds, given $\varepsilon > 0$,

$$\begin{aligned} &P\left(|e^{E_0/c} g_{n_k(1)}(Z, E_0) - 1| > \varepsilon\right) \\ &\quad = \int P\left(|e^{E_0/c} g_{n_k(1)}(z, E_0) - 1| > \varepsilon\right) d\mathscr{L}(Z)(z) \to 0, \end{aligned} \tag{4.20}$$

that is,

$$e^{E_0/c} g_{n_k(1)}(Z, E_0) \to_p 1. \tag{4.21}$$

We have shown, therefore, that given any subsequence of $\{n_k\}$, there is a further one $\{n_k(1)\}$ whereon (4.21) holds. It follows now that for $0 < c < \infty$, (4.10) holds on all of $\{n_k\}$.

When $c = 0$, a similar but easier argument leads to (4.11). □

5. Limit theorems. Before proving the main theorems, we recall a powerful formula for analyzing magnitude trimmed sums by conditioning on the magnitude of the last summand removed from the full sum, which is due in general to Mori (1984); it has been utilized elsewhere many times.

Recall the representation (3.2). For $0 \le j \le N$, let

$$Y_{nj}(\alpha), Y_{nj1}(\alpha), Y_{nj2}(\alpha), \dots, Y_{njn}(\alpha)$$

be i.i.d. with $\mathscr{L}(Y_{nj}(\alpha)) = \mathscr{L}(X^2 | \, |X| \le b_{nj}(\alpha))$, where [as in (1.13)]

$$b_{nj}(\alpha) = G^{-1}\left(\frac{r_n + j - \sqrt{r_n + j}}{n}\right). \tag{5.1}$$

By continuity of F, for any nonnegative or bounded Borel function ρ,

$$\begin{aligned} &E\rho\left(V_n(r_n + j + 1)/X_n^2(r_n + j)\right) \\ &\quad = \int E\rho\left(b_{nj}^{-2}(\alpha) \sum_{k=r_n+j+1}^{n} Y_{njk}(\alpha)\right) d\mathscr{L}(Z_{nj})(\alpha), \end{aligned} \tag{5.2}$$

where we understand the interval of integration on the right to be the region where b_{nj} is well defined and positive.

Fix integers $n_k \to \infty$ as in Section 4. First, the case of the Rademacher limit law for T_{n_k} will be completely resolved. [A similar result can be found in Griffin and Mason (1990).]

THEOREM 5.3. *$\mathscr{L}(T_{n_k}) \to \Delta$ if and only if*

$$f_{n_k}(Z) \to_p 0. \tag{5.4}$$

PROOF. In this proof we suppress the subscript k since no further subsequences are required. Due to Lemma 2.15, it suffices to show that (5.4) is equivalent to

$$\frac{X_n^2(r_n)}{V_n(r_n)} \to_p 1 \tag{5.5}$$

or, equivalently,

$$\frac{V_n(r_n+1)}{X_n^2(r_n)} \to_p 0. \tag{5.6}$$

Assume (5.4) holds. Let $\varepsilon > 0$, and choose $\rho(x) = I(x \geq \varepsilon)$ in (5.2). Then, by Markov's inequality (suppressing the index $j = 0$ in $Z_{nj}, b_{nj}, Y_{njk}, Y_{nj}$),

$$\begin{aligned} P\big(V_n(r_n+1)/X_n^2(r_n) \geq \varepsilon\big) &= \int P\left(b_n^{-2}(\alpha) \sum_{k=r_n+1}^{n} Y_{nk}(\alpha) \geq \varepsilon\right) d\mathscr{L}(Z_n)(\alpha) \\ &\leq \int \left(1 \wedge \frac{(n-r_n)EY_n(\alpha)}{\varepsilon b_n(\alpha)^2}\right) d\mathscr{L}(Z_n)(\alpha). \end{aligned} \tag{5.7}$$

Choose R so that $\sup_n P(|Z_n| \geq R) < \varepsilon$ [since $\mathscr{L}(Z_n) \to N(0,1)$]. On $[-R, R]$, it is easy to see [e.g., Hahn, Kuelbs and Weiner (1990), proof of Lemma 16] that

$$\begin{aligned} (n-r_n)\frac{EY_n(\alpha)}{b_n^2(\alpha)} &\sim n\frac{EX^2 I(|X| \leq b_n(\alpha))}{b_n^2(\alpha)} \\ &= nK(b_n(\alpha)) = f_n(\alpha) \end{aligned} \tag{5.8}$$

uniformly.

Thus, for all large enough n,

$$\begin{aligned} &P\big(V_n(r_n+1)/X_n^2(r_n) \geq \varepsilon\big) \\ &\quad \leq \varepsilon + (1+\varepsilon)\int_{-R}^{R}\left(\frac{f_n(\alpha)}{\varepsilon} \wedge 1\right) d\mathscr{L}(Z_n)(\alpha) \\ &\quad \leq \varepsilon + (1+\varepsilon)E\left\{\frac{f_n(Z_n)}{\varepsilon} \wedge 1\right\}. \end{aligned} \tag{5.9}$$

Now $\mathscr{L}(Z_n) \to N(0,1)$ in total variation distance by Proposition 3.5. Thus $f_n(Z) \to_p 0$ implies $f_n(Z_n) \to_p 0$ [as in (4.3)]. It follows from boundedness that $E\{f_n(Z_n)/\varepsilon \wedge 1\} \to 0$.

Thus (5.6) holds.

Conversely, suppose (5.6) holds. Then, given $0 < \varepsilon < 1$,

$$\begin{aligned} 0 &\leftarrow P\big(V_n(r_n+1)/X_n^2(r_n) \geq \varepsilon\big) \\ &= \int P\left(b_n^{-2}(\alpha) \sum_{k=r_n+1}^{n} Y_{nk}(\alpha) \geq \varepsilon\right) d\mathscr{L}(Z_n)(\alpha). \end{aligned} \tag{5.10}$$

Choose $R > 1$. Abusing notation slightly, write $\tilde{V}_n(\alpha) = b_n^{-2}(\alpha)\sum_{k=r_n+2}^{n} Y_{nk}(\alpha)$.

We have seen in (5.8) that $E\tilde{V}_n(\alpha) = f_n(\alpha)(1 + o(1))$ uniformly on compact sets; similarly,

$$\operatorname{Var} \tilde{V}_n(\alpha) \le n \frac{EX^4 I(|X| \le b_n(\alpha))}{b_n^4(\alpha)}(1 + o(1)) \le f_n(\alpha)(1 + o(1))$$

uniformly on compacts.

Put $F_n(R) = [f_n \ge R]$. For $\alpha \in F_n(R)$,

$$\begin{aligned} P(\tilde{V}_n(\alpha) < \varepsilon) &= P(\tilde{V}_n(\alpha) - E\tilde{V}_n(\alpha) < \varepsilon - E\tilde{V}_n(\alpha)) \\ &\le P(|\tilde{V}_n(\alpha) - E\tilde{V}_n(\alpha)| > E\tilde{V}_n(\alpha) - \varepsilon) \\ &\le \frac{\operatorname{Var} \tilde{V}_n(\alpha)}{(E\tilde{V}_n(\alpha) - \varepsilon)^2}, \end{aligned} \tag{5.11}$$

by Chebyshev's inequality. Let $T > \varepsilon$. Then from (5.10),

$$\begin{aligned} 0 \leftarrow &\int P(\tilde{V}_n(\alpha) \ge \varepsilon)\, d\mathscr{L}(Z_n)(\alpha) \\ &\ge \int_{[-T,T] \cap F_n(R)} P(\tilde{V}_n(\alpha) \ge \varepsilon)\, d\mathscr{L}(Z_n)(\alpha) \\ &\ge \int_{[-T,T] \cap F_n(R)} \left(1 - \frac{\operatorname{Var} \tilde{V}_n(\alpha)}{(E\tilde{V}_n(\alpha) - \varepsilon)^2}\right) d\mathscr{L}(Z_n)(\alpha) \\ &\ge P([Z_n \in F_n(R)] \cap [|Z_n| \le T]) \\ &\quad - (1 + o(1)) \int_{[-T,T] \cap F_n(R)} \frac{f_n(\alpha)}{(f_n(\alpha) - \varepsilon)^2}\, d\mathscr{L}(Z_n)(\alpha). \end{aligned} \tag{5.12}$$

The final line uses the fact that $x \to x/(x - \varepsilon)^2$ decreases for $x \ge R > 1 > \varepsilon$.

Now select R to obey $1 - R/(R - \varepsilon)^2 \ge 1/2$. Then for each T,

$$P(f_n(Z_n) \ge R) \le P(|Z_n| > T) + P([Z_n \in F_n(R)] \cap [|Z_n| \le T]).$$

Given $\delta > 0$, choose T so that $P(|Z_n| > T) < \delta$, since $\mathscr{L}(Z_n) \to N(0,1)$, and then—using (5.12)—choose n_0 so that $n \ge n_0$ implies $P([Z_n \in F_n(R)] \cap [|Z_n| \le T]) < \delta$. It follows that for sufficiently large R,

$$P(f_n(Z_n) \ge R) \to 0. \tag{5.13}$$

Recalling (4.1) and Proposition 3.5, it follows that

$$P\left(f_n\left(Z - E_0/\sqrt{r_n}\right) \ge R\right) \to 0. \tag{5.14}$$

Now $0 \le X_n^2(r_n+1) \le V_n(r_n+1)$, so (5.6) implies, in particular [recalling (3.2) and (3.8)]:

$$(5.15) \qquad 0 \leftarrow_p \left| \frac{X_n(r_n+1)}{X_n(r_n)} \right| = \frac{b_n(Z_{n1})}{b_n(Z_{n0})} = g_n\big(Z_{n0}, \sqrt{r_n}(Z_{n0}-Z_{n1})\big).$$

Now via (3.11), it follows that

$$(5.16) \qquad g_n(Z, E_0) \to_p 0.$$

Recalling Lemma 4.5, the conditions (5.14) and (5.16) together imply

$$(5.17) \qquad \begin{aligned} f_n(Z) &= f_n\big(Z - E_0/\sqrt{r_n}\big) g_n(Z, E_0) + \int_0^{E_0} g_n(Z, s)\, ds \\ &= o_p(1) + \int_0^{E_0} g_n(Z, s)\, ds. \end{aligned}$$

Finally, by passing to subsequences where (5.16) holds almost surely (as in the proof of Lemma 4.9), the bounded convergence theorem implies (since $0 \le g_n \le 1$) that the integral term in (5.17) tends to 0 in probability. Hence (5.4) holds. □

Note that "(5.4) implies (5.6)" means, in particular, that $(S_n(r_n)/|X_n(r_n)|, V_n(r_n)/X_n^2(r_n)) \to_D (\varepsilon_1, 1)$. (Compare Theorem 5.30 below.)

Turning to the sufficient conditions for nonnormal, non-Rademacher limits, we require first the following tightness lemma. [The case $\{N_n\}$ bounded is contained in Griffin and Mason (1990).]

LEMMA 5.18. *Assume*

$$(5.19) \qquad \{f_{n_k}(Z)\} \text{ is tight.}$$

Let $0 \le N_n \le n - r_n$ *be integers such that*

$$(5.20) \qquad \frac{N_n}{\sqrt{r_n}} \to 0.$$

Then

$$(5.21) \qquad \big\{V_{n_k}(r_{n_k}+j)/X_{n_k}^2(r_{n_k}+j) \colon 0 \le j \le N_{n_k}\big\} \text{ is tight.}$$

PROOF. Again subscripts are temporarily suppressed. Given $R_n \to \infty$, such that $R_n/\sqrt{r_n} \to 0$, put $\rho(x) = I(|x| \ge R_n)$ in (5.2). Note that $Eb_{nj}^{-2}(\alpha)Y_{nj}^2(\alpha) \sim K(b_{nj}(\alpha))$ uniformly on $[-R_n, R_n]$ (by direct computation, or as in the proof of Lemma 16 in Hahn, Kuelbs and Weiner (1990)]. Let $o(1)$ denote a quantity tending to 0 uniformly in $0 \le j \le N_n$. Using Markov's inequality in (5.2), we have, uniformly in $0 \le j \le N_n$,

$$\begin{aligned} &P\big(V_n(r_n + j + 1)/X_n^2(r_n + j) \ge R_n\big) \\ &\quad \le \int_{[-R_n, R_n]} \left(1 \wedge \frac{(n - r_n - j)EY_{nj}(\alpha)}{R_n b_{nj}^2(\alpha)}\right) d\mathscr{L}(Z_{nj})(\alpha) + o(1) \\ (5.22) \quad &\quad \le \int_{[-R_n, R_n]} \left(1 \wedge \left(\frac{nK(b_{nj}(\alpha))}{R_n}\right)\right) d\mathscr{L}(Z_{nj})(\alpha)(1 + o(1)) + o(1) \\ &\quad \le E\left(1 \wedge \left(\frac{nK(b_{nj}(Z_{nj}))}{R_n}\right)\right) + o(1) \\ &\quad = E\left(1 \wedge \left(\frac{f_n(\tilde{Z}_{nj})}{R_n}\right)\right) + o(1), \end{aligned}$$

utilizing (3.21). By (3.22), $\{f_{n_k}(Z)\}$ tight [by (5.19)] implies that the triangular array

$$\left\{ f_{n_k}\left(\tilde{Z}_{n_k j}\right): 0 \le j \le N_{n_k}, k \ge 1 \right\} \tag{5.23}$$

is tight. Thus

$$\max_{j \le N_{n_k}} P\left(f_{n_k}\left(\tilde{Z}_{n_k j}\right) \ge \sqrt{R_{n_k}} \right) \to_{k \to \infty} 0. \tag{5.24}$$

Hence, since $R_n \to \infty$, (5.22) leads to

$$\begin{aligned} &\max_{j \le N_k} P\left(V_{n_k}(r_{n_k} + j + 1)/X_{n_k}^2(r_{n_k} + j) \ge R_n\right) \\ (5.25) \quad &\quad \le \max_{j \le N_k} P\left(f_{n_k}\left(\tilde{Z}_{n_k, j}\right) \ge \sqrt{R_{n_k}} \right) + \frac{\sqrt{R_{n_k}}}{R_{n_k}} + o(1) \to_{k \to \infty} 0. \end{aligned}$$

Since $R_n \to \infty$ was arbitrary (subject to $R_n/\sqrt{r_n} \to 0$), (5.25) validates (5.21). □

Before stating the main theorem, we require some observations on certain random series which will appear there.

Recall that $\{E_j: j \ge 0\}$ is i.i.d. ~ Exp(1), independent of the Rademacher sequence $\{\varepsilon_j: j \ge 0\}$, and recall $\Gamma_n = E_1 + \cdots + E_n$, with $\Gamma_0 = 0$.

LEMMA 5.26. *For each* $0 < c < \infty$, *the random series*

$$S = S_c = \sum_{j=0}^{\infty} \varepsilon_j e^{-\Gamma_j/2c} \quad \text{and} \quad V = V_c = \sum_{j=0}^{\infty} e^{-\Gamma_j/c} \tag{5.27}$$

each converge absolutely almost surely to finite random variables. Consequently, $\mathscr{L}(S - \varepsilon_0)$ *and* $\mathscr{L}(V)$ *are each infinitely divisible, and the random* *variable*

$$T = T_c = \frac{S}{\sqrt{V}} \tag{5.28}$$

is well defined and symmetric with

$$ET = 0 \quad \textit{and} \quad \operatorname{Var} T = 1. \tag{5.29}$$

PROOF. By the strong law of large numbers, as $n \to \infty$, $\Gamma_n/n \to E(E_0) = 1$, almost surely. Thus almost surely, eventually, $e^{-\Gamma_j/c} \le e^{-j/2c}$, where $\sum e^{-j/2c}$ converges. So the V-series in (5.27) converges almost surely. Likewise, almost surely, eventually $|\varepsilon_j e^{-\Gamma_j/2c}| \le e^{-j/4c}$, where $\sum e^{-j/4c} < \infty$, so the S-series in (5.27) converges absolutely almost surely. Finally, notice that $V \ge 1 > 0$, so that (5.28) is indeed well defined. For infinite divisibility, see Remark 5.51 below. □

THEOREM 5.30. *Assume* $f_{n_k}(Z) \to_p c \in (0, \infty)$. *Then*

$$\mathscr{L}\left(\tilde{S}_{n_k}, \tilde{V}_{n_k}\right) \equiv \mathscr{L}\left(\frac{S_{n_k}(r_{n_k})}{|X_{n_k}(r_{n_k})|}, \frac{V_{n_k}(r_{n_k})}{X^2_{n_k}(r_{n_k})} \right) \to \mathscr{L}(S_c, V_c). \tag{5.31}$$

Consequently,

$$\mathscr{L}(T_{n_k}) \to \mathscr{L}(T_c) \equiv \mathscr{L}\left(\frac{S_c}{\sqrt{V_c}} \right). \tag{5.32}$$

PROOF. To see that (5.32) follows from (5.31), note $V \ge 1 > 0$.

Turning to (5.31), assume $f_{n_k}(Z) \to_p c \in (0, \infty)$. Given J, write [recalling (2.4)]

$$\tilde{S}_n = \frac{S_n(r_n)}{|X_n(r_n)|} \overset{D}{=} \sum_{j=0}^{J} \left| \frac{X_n(r_n + j)}{X_n(r_n)} \right| \varepsilon_j + \sum_{j=J+1}^{n-r_n} \left| \frac{X_n(r_n + j)}{X_n(r_n)} \right| \varepsilon_j \equiv \sigma_n'(J) + \sigma_n''(J). \tag{5.33}$$

Similarly, write

$$\tilde{V}_n \equiv \frac{V_n(r_n)}{X_n^2(r_n)} = \sum_{j=0}^{J} \frac{X_n^2(r_n + j)}{X_n^2(r_n)} + \sum_{j=J+1}^{n-r_n} \frac{X_n^2(r_n + j)}{X_n^2(r_n)} \equiv v_n'(J) + v_n''(J), \tag{5.34}$$

$$S_c = \sum_{j=0}^{J} \varepsilon_j e^{-\Gamma_j/2c} + \sum_{j=J+1}^{\infty} \varepsilon_j e^{-\Gamma_j/2c} \equiv \sigma'(J) + \sigma''(J) \tag{5.35}$$

and

$$V_c = \sum_{j=0}^{J} e^{-\Gamma_j/c} + \sum_{j=J+1}^{\infty} e^{-\Gamma_j/c} \equiv v'(J) + v''(J). \tag{5.36}$$

Here it may be advantageous to outline and motivate as well as explain the technical proof to follow. It can be seen, via Lemma 4.9 and the representations in Section 3, that under the assumption $f_{n_k}(Z) \to_p c \in (0, \infty)$, for each fixed J the key collection of successive ratios of intermediate magnitude order statistics $\{|X_{n_k}(r_{n_k} + j + 1)|/|X_{n_k}(r_{n_k} + j)|: 0 \le j \le J - 1\}$ behaves asymptotically like an i.i.d. sample from the distribution $\mathscr{L}(e^{-E_0/2c})$. Since $|X_n(r_n + j)/|X_n(r_n)|$ is a telescoping product of intermediate ratios, when combined with the independence of signs and magnitudes due to symmetry, this crucial probabilistic consequence of the analytic assumption $f_{n_k}(Z) \to_p c$ suggests that for each fixed J the pair $(\sigma'_{n_k}(J), v'_{n_k}(J))$ behaves asymptotically like $(\sigma'(J), v'(J))$. In order to be able to consider J large enough to render $(\sigma''(J), v''(J))$ negligible via Lemma 5.26, we also need to have $\{(\sigma''_{n_k}(J), v''_{n_k}(J)$: $J_0 \le J \le N_{n_k})\}$ uniformly negligible for some sufficiently large J_0 and some $N_{n_k} \to \infty$. This will be accomplished with the aid of Lemma 5.18, which applies since here $f_{n_k}(Z) \to_p c < \infty$. The success in implementing this outline relies on the "suitability" of $|X_{n_k}(r_{n_k})|$ as a normalizer for both the trimmed sum $S_{n_k}(r_{n_k})$ and the trimmed sum-of-squares $V_{n_k}(r_{n_k})^{1/2}$, which is a consequence of the assumption $f_{n_k}(Z) \to_p c$.

Hereinafter, suppress the subscripts k in this proof. Let $\varepsilon > 0$, and let $d(\cdot, \cdot)$ be any distance between probability measures, which metrizes weak convergence. Note that

$$v''_n(J) = \left(\frac{X_n(r_n + J)}{X_n(r_n)} \right)^2 \frac{V_n(r_n + J)}{X_n^2(r_n + J)}. \tag{5.37}$$

Fix integers $0 \le N_n \le n - r_n$ such that

$$N_n \to \infty, \qquad \frac{N_n}{\sqrt{r_n}} \to 0. \tag{5.38}$$

By Lemma 5.18, choose R so that uniformly in $0 \le j \le N_n$,

$$P\big(V_n(r_n + j)/X_n^2(r_n + j) \ge R\big) < \varepsilon/8. \tag{5.39}$$

By Lemma 5.26, choose J_1 so that $J \ge J_1$ implies

$$P\left(\sigma''(J) \ge \frac{\varepsilon}{16}\right) < \frac{\varepsilon}{16}, \qquad P\left(v''(J) \ge \frac{\varepsilon}{16}\right) < \frac{\varepsilon}{16}. \tag{5.40}$$

Choose $J_2 \geq J_1$ so that $J \geq J_2$ implies

$$P\left(e^{-\Gamma_J/c} \geq \frac{\varepsilon^3}{64R}\right) < \frac{\varepsilon}{32}. \tag{5.41}$$

Finally, choose $J_0 = J_2$. Consider [recalling representation (3.2) and also (4.8)]

$$\begin{aligned} M_n &= \mathscr{L}\left(\frac{X_n^2(r_n)}{X_n^2(r_n)}, \frac{X_n^2(r_n+1)}{X_n^2(r_n)}, \dots, \frac{X_n^2(r_n+J_0)}{X_n^2(r_n)}\right) \\ &= \mathscr{L}\Big(1, g_n\big(Z_{n0}, \sqrt{r_n}\,(Z_{n0} - Z_{n1})\big), \dots, \\ &\qquad g_n\big(Z_{n0}, \sqrt{r_n}/(Z_{n0} - Z_{nJ_0})\big)\Big). \end{aligned} \tag{5.42}$$

By Proposition 3.5, $M_n - M_n' \to_{\mathrm{TV}} 0$, where

$$\begin{aligned} M_n' &= \mathscr{L}\big(1, g_n(Z, E_1), \dots, g_n(Z, E_1 + \cdots + E_{J_0})\big) \\ &\to \mathscr{L}\big(1, e^{-\Gamma_1/c}, \dots, e^{-\Gamma_{J_0}/c}\big) \end{aligned} \tag{5.43}$$

by lemma 4.9. Utilizing independence of $\{\varepsilon_j\}, \{|X_j|\}$, it follows that

$$\mathscr{L}(\sigma_n'(J_0), v_n'(J_0)) \to \mathscr{L}(\sigma'(J_0), v'(J_0)). \tag{5.44}$$

It remains to consider the terms $\sigma_n''(J_0), v_n''(J_0)$. Noting (5.41)–(5.43) and the continuity of $\mathscr{L}(\Gamma_{J_0})$, choose n_0 so that $n \geq n_0$ implies

$$P\left(\left(\frac{X_n(r_n+J_0)}{X_n(r_n)}\right)^2 \geq \frac{\varepsilon^3}{64R}\right) < \frac{\varepsilon}{16}. \tag{5.45}$$

Now choose $n_1 \geq n_0$ so that $n \geq n_1$ implies $N_n \geq J_0$ [using (5.38)]. Then (5.37), (5.39) and (5.45) imply that for $n \geq n_1$,

$$P\left(v_n''(J_0) > \frac{\varepsilon^3}{64}\right) < \frac{\varepsilon}{8}. \tag{5.46}$$

Computing conditionally as in (2.9), note that (5.46) and Markov's inequality give, for $n \geq n_1$,

$$\begin{aligned} P\left(\sigma_n''(J_0) > \frac{\varepsilon}{4}\right) &\leq P\left(v_n''(J_0) > \frac{\varepsilon^3}{64}\right) + P\left(\sigma_n''(J_0)^2 I\left(v_n''(J_0) \leq \frac{\varepsilon^3}{64}\right) > \frac{\varepsilon}{64}\right) \\ &< \frac{\varepsilon}{8} + \frac{4^2}{\varepsilon^2} E\sigma_n''(J_0)^2 I\left(v_n''(J_0) \leq \frac{\varepsilon^3}{64}\right) \\ &= \frac{\varepsilon}{8} + \frac{16}{\varepsilon^2} E v_n''(J_0) I\left(v_n''(J_0) \leq \frac{\varepsilon^3}{64}\right) \\ &\leq \frac{\varepsilon}{8} + \frac{16}{\varepsilon^2}\frac{\varepsilon^3}{64} = \frac{3\varepsilon}{8}. \end{aligned} \tag{5.47}$$

To complete the argument, in (5.44), choose $n_2 \geq n_1$ so that $n \geq n_2$ implies

$$d(\mathscr{L}(\sigma_n'(J_0), v_n'(J_0)), \mathscr{L}(\sigma'(J_0), v'(J_0))) < \frac{\varepsilon}{8}. \tag{5.48}$$

Combining (5.48) with (5.40), $n \geq n_2$ implies

$$d(\mathscr{L}(\sigma_n'(J_0), v_n'(J_0)), \mathscr{L}(S, V)) < \frac{\varepsilon}{4}. \tag{5.49}$$

Assuming $0 < \varepsilon < 1$ is small enough that $\varepsilon^3/64 < \varepsilon/8$, (5.46) and (5.47) combine with (5.49) to yield, for $n \geq n_2$,

$$d\left(\mathscr{L}\left(\tilde{S}_n, \tilde{V}_n\right), \mathscr{L}(S, V)\right) < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} = \frac{6}{8}\varepsilon < \varepsilon. \tag{5.50}$$

Assertion (5.31) now follows. □

Consideration of a converse to Theorem 5.30 is complicated by the lack of availability at this time, of general sufficient conditions (but see Section 7 below).

REMARK 5.51. The laws $\mathscr{L}(S_c - \varepsilon_0)$ and $\mathscr{L}(V_c)$ in (5.27) are infinitely divisible. To see this, represent the series in (5.27) by stochastic integrals with respect to Poisson processes. Let $\{\tilde{N}(t): t \geq 0\}$ be a symmetrized unit intensity ordinary Poisson process with $\tilde{N}(0) = 0$ and let $\{N(t): t \geq 0\}$ be the total-variation process from $\{\tilde{N}(t): t \geq 0\}$ (so that $\{N(t): t \geq 0\}$ is a standard unit intensity Poisson process]. Let $\varepsilon \sim \Delta$ be independent of $\tilde{N}$. Then (viewing the integration sample-path-wise), we may write, in joint distribution,

$$S_c = \varepsilon + \int_0^\infty h(t)\, d\tilde{N}(t), \qquad V_c = 1 + \int_0^\infty h^2(t)\, dN(t), \tag{5.52}$$

where $h(t) = e^{-t/2c}$. Representation (5.52) is especially illuminating when compared to the series representations for stable laws and companion sums-of-squares laws consequent from the work of LePage, Woodroofe and Zinn (1981); in our notation, when F is in the domain of attraction of a stable law of index $0 < \alpha < 2$, there are constants $\{a_n\}$ such that

$$\mathscr{L}(S_n(1)/a_n, V_n(1)/a_n^2) \to \mathscr{L}\left(\int_0^\infty g(t)\, d\tilde{N}(t), \int_0^\infty g^2(t)\, dN(t)\right), \tag{5.53}$$

where $g(t) = t^{-1/\alpha}$ [see also Csörgő, Haeusler and Mason (1988)]. Here, the change from h to g (and from c to α) highlights the role played by the analytic condition $f_{n_k}(Z) \to_p c$ as analogous to the regular variation with exponent $-1/\alpha$ of G^{-1}; indeed, Lemma 4.9 shows that this analytic condition forces a certain "local" regular variation in G^{-1}.

6. Examples. It is convenient to note that one distribution can generate, along different subsequences, every sort of behavior (and only those) considered in Theorems 5.3 and 5.30 together with the asymptotic normality theorem [(1.20) along $\{n_k\} \Leftrightarrow$ (1.22)] of Griffin and Mason (1990). That is, a "limit-generating" law exists to exhibit these phenomena. The law selected here is a fairly standard one for considering pathologies for limit theorems for sums.

EXAMPLE 6.1. Let X be a symmetric random variable with continuous tail satisfying

$$G(t) = P(|X| > t) = \frac{1}{\log t}, \qquad t \geq e. \tag{6.2}$$

Then

$$K(t) \sim \frac{1}{2(\log t)^2} = \frac{1}{2}(G(t))^2, \qquad t \to \infty, \tag{6.3}$$

so that [recalling (1.14)], for each $\alpha \in \mathbb{R}$,

$$\begin{aligned} f_n(\alpha) &= nK(b_n(\alpha)) \sim \frac{1}{2}n(G(b_n(\alpha)))^2 \\ &= \frac{1}{2}n\left(\frac{r_n - \alpha\sqrt{r_n}}{n}\right)^2 \sim \frac{1}{2}\frac{r_n^2}{n} \end{aligned} \tag{6.4}$$

as $n \to \infty$.

Given any subsequence $\{r_{n_k}\}$ such that $r_{n_k}/\sqrt{n_k} \to c_0 \in [0, \infty]$, we have

$$f_{n_k}(Z) \to_p \frac{c_0^2}{2} \equiv c \tag{6.5}$$

(in fact, the convergence is almost sure). When $0 < c < \infty$,

$$\left(\frac{S_n(r_n)}{|X_n(r_n)|}, \frac{V_n(r_n)}{X_n^2(r_n)}\right) \to_D (S_c, V_c) \quad \text{and} \quad T_n \to_D T_c \tag{6.6}$$

by Theorem 5.18. Therefore, by Theorem 5.3,

$$\left(\frac{S_n(r_n)}{|X_n(r_n)|}, \frac{V_n(r_n)}{X_n^2(r_n)}\right) \to_D (\varepsilon, 1) \Leftrightarrow c = 0 \Leftrightarrow T_n \to_D \varepsilon_1 \tag{6.7}$$

and finally, by the theorem in Griffin and Mason (1990),

$$T_n \to_D N(0,1) \Leftrightarrow c = \infty. \tag{6.8}$$

The key feature of the preceding example, namely (6.3), can be somewhat generalized. For convenience, call a function W: $(0, \varepsilon) \to (0, \infty)$ *continuously vanishing* near 0 provided $\lim_{x \downarrow 0} W(x) = 0$, W is continuous, and near 0:

$$x_n \to 0 \text{ and } y_n \sim x_n \Rightarrow W(x_n) \sim W(y_n). \tag{6.9}$$

Also, recall the notation of (1.11).

PROPOSITION 6.10. *Suppose*

$$K(t) = W(G(t)), \tag{6.11}$$

where W *is continuously vanishing near* 0. *Then, given* $0 \le c \le \infty$, *there exists integers* $r_n \to \infty$ *such that* $r_n/n \to 0$ *and*

$$nW\left(\frac{r_n}{n}\right) \to c. \tag{6.12}$$

Then (6.6) *holds if* $0 < c < \infty$, *and* (6.7) *and* (6.8) *each hold.*

PROOF. Certainly $x_n \to 0$ exists such that $nW(x_n) \to c$, using the properties of W. Thus arrange integers r_n with $r_n/n \sim x_n$, and then (6.12) holds via (6.9).

Then, for each $\alpha \in \mathbb{R}$, by (1.14) and (6.9),

$$\begin{aligned} f_n(\alpha) &= nK(b_n(\alpha)) = nW(G(b_n(\alpha))) = nW\left(\frac{r_n - \alpha\sqrt{r_n}}{n}\right) \\ &\sim nW\left(\frac{r_n}{n}\right) \to c. \end{aligned} \tag{6.13}$$

Thus

$$f_n(Z) \to_p c \in [0,\infty], \tag{6.14}$$

so that Theorems 5.3, 5.18 and the Griffin–Mason (1990) result each apply as in (6.6)–(6.8). □

Proposition 6.10 shows that in order for "$\exists\, c \in [0,\infty]$, $f_{n_k}(Z) \to_p c$" to fail, the tail G should be quite irregular. Section 7 discusses further results in the direction of constructing "limit-generating laws" for the most general kinds of behavior of $\{f_n(Z)\}$. [See also Hahn and Weiner (1990).]

REMARK 6.15. In order that in Example 6.1, there exist constants d_n such that for some probability measure $\mu \neq \delta_0$, $S_n(r_n)/d_n \to_D \mu$, it is necessary and sufficient that $r_n/n^{2/3} \to c$ for some $0 < c \le \infty$, with asymptotic normality if and only if $c = \infty$ [see Griffin and Pruitt (1987)]. In example 6.1, even when $0 < c < \infty$, such $\{r_n\}$ would lead directly to asymptotic normality for $T_n = S_n(r_n)/\sqrt{V_n(r_n)}$. In fact, for $r_n \sim \sqrt{n}$, we have $S_n(r_n)/\sqrt{V_n(r_n)}$ converging in distribution, but for no $\{d_n\}$ will $\{\mathscr{L}(S_n(r_n)/d_n)\}$ even be stochastically compact. Thus here, studentization partially compensates for the asymptotic-normality-damaging extreme values, even when constant normalization could not.

7. Progress toward the general case. In analogy with the case $f_{n_k}(Z) \to_p \infty$, Theorems 5.3 and 5.30 together cover the cases of constant convergence $f_{n_k}(Z) \to_p c \in [0,\infty)$. In general, of course, $\{T_n\}$ is stochastically compact (by Proposition 2.1) and $\{\mathscr{L}(f_n(Z))\}$ is relatively compact in the topology of vague convergence. Thus, at least when considering sufficient conditions for $\mathscr{L}(T_{n_k}) \to v$ and identification of possible limit laws v, we may assume $\mathscr{L}(f_{n_k}(Z)) \to \mu$ vaguely, where μ is a probability measure on $[0,\infty]$.

Now, the key to our present approach is the derivation of the form of the limit laws for intermediate successive ratios $|X_n(r_n+1)/X_n(r_n)|$ along the subsequence, that is, determining almost sure limits (along further subsequences) for $\{b_n(Z - E_0/\sqrt{r_n})/b_n(Z)\}$ explicitly represented as functions of (Z, E_0), in order to determine the asymptotic dependence (if any) among ratios $\{|X_n(r_n+j+1)|/|X_n(r_n+j)|\colon\ 0 \le j \le J\}$. Unfortunately, this approach depends on convergence in probability for $\{f_{n_k}(Z)\}$, rather than the vague convergence we are allowed to assume. (Of course, this holds when $\mu = \delta_c$, $0 \le c \le \infty$.)

Using methods similar to those developed here, with suitable refinements, but involving considerably more technical difficulties, it is possible to prove the following theorem, covering the case $f_{n_k}(Z) \to_p f(Z) \in [0,\infty]$, together with construction of "limit-generating" examples. The details appear in Hahn and Weiner (1990).

THEOREM. *Let F be continuous, symmetric and nondegenerate. Assume*

$$f_{n_k}(Z) \to_p f(Z), \tag{7.1}$$

where $f\colon \mathbb{R} \to [0,\infty]$ is Lebesgue measurable. Then

$$\begin{aligned} \mathscr{L}(T_{n_k}) &= \mathscr{L}\Big(S_{n_k}(r_{n_k})/\sqrt{V_{n_k}(r_{n_k})}\Big) \\ &\to \gamma_1 N(0,1) + \gamma_2\Delta + \gamma_3\mathscr{L}\Big(\Big(S_f/\sqrt{V_f}\Big)\big|0 < f(Z) < \infty\Big), \end{aligned} \tag{7.2}$$

where $\gamma_1 = P(f(Z) = \infty)$, $\gamma_2 = P(f(Z) = 0)$, $\gamma_3 = 1 - \gamma_1 - \gamma_2$ and

$$\begin{aligned} S_f &= \sum_{j=0}^{\infty} \varepsilon_j e^{-\Gamma_j/(2f(Z))}, \\ V_f &= \sum_{j=0}^{\infty} e^{-\Gamma_j/f(Z)}. \end{aligned} \tag{7.3}$$

Here Z is independent of $\{\varepsilon_j\}$ and $\{\Gamma_j\}$. Moreover, there exist $\{r_n\}$ and F such that $r_n \to \infty$, $r_n/n \to 0$, and such that given f, for some $\{n_k\}$, (7.1)–(7.3) all hold.

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REFERENCES

BILLINGSLEY, P. (1986). *Probability and Measure*, 2nd ed. Wiley, New York.
BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.
CHUNG, K. L. (1974). *A Course in Probability*, 2nd ed. Academic, New York.

CSÖRGŐ, S., HAEUSLER, E. and MASON, D. (1988). A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables. *Adv. in Appl. Math.* **9** 259–333.

CSÖRGŐ, M. and HORVATH, L. (1988). Asymptotic representations of self-normalized sums. *Probab. Math. Statist.* **9** 15–24.

EFRON (1969). Student's t-test under symmetry conditions. *J. Amer. Statist. Assoc.* **64** 1278–1302.

FELLER, W. (1967). On regular variation and local limit theorems. *Proc. Fifth Berkeley Symposium Math. Statist. Probab.* **2** 373–388 Univ. California Press, Berkeley.

GRIFFIN, P. S. and MASON, D. M. (1991). On the asymptotic normality of self-normalized sums. *Proc. Cambridge Phil. Soc.* **109** 597–610.

GRIFFIN, P. S. and PRUITT, W. E. (1987). The central limit problem for trimmed sums. *Math. Proc. Cambridge Philos. Soc.* **102** 329–349.

HAHN, M. G., KUELBS, J. and WEINER, D. C. (1990). The asymptotic distribution of magnitude-winsorized sums via self-normalization. *J. Theoret. Probab.* 137–168.

HAHN, M. G. and WEINER, D. C. (1990). Asymptotic behavior of self-normalized trimmed sums: Nonnormal limits, II. *J. Theor. Probab.* To appear.

LEPAGE, R., WOODROOFE, M. and ZINN, J. (1981). Convergence to a stable distribution via order statistics. *Ann. Probab.* **9** 624–632.

LÉVY, P. (1937). *Théorie de l'Addition des Variables Aléatoires*. Gauthier-Villars, Paris.

LOGAN, B., MALLOWS, C., RICE, S. and SHEPP, L. (1973). Limit distributions of self-normalized sums. *Ann. Probab.* **1** 788–809.

MALLER, R. (1982). Asymptotic normality of lightly trimmed means—a converse. *Math. Proc. Cambridge Philos. Soc.* **92** 535–545. *Z. Wahrsch. Verw. Gebiete* **41** 159–167.

MORI, T. (1984). On the limit distributions of lightly trimmed sums. *Math. Proc. Cambridge Philos. Soc.* **96** 507–516.

PRUITT, W. (1985). Sums of independent random variables with the extreme terms excluded. Preprint.

REISS, R. D. (1989). *Approximate Distributions of Order Statistics*. Springer, New York.

TAYLOR, A. (1985). *General Theory of Functions and Integration*. Dover, New York.

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