The Annals of Probability
1992, Vol. 20, No. 1, 410–420

A NOTE ON TRANSLATION CONTINUITY OF PROBABILITY MEASURES

BY S. L. ZABELL

Northwestern University

A probability measure on the sphere is absolutely continuous with respect to the uniform measure on the sphere if and only if the probability of any open set varies continuously as the sphere is rotated. In general, if a topological group G acts transitively on a topological space S, and both are Hausdorff, locally compact and second countable, then a probability measure ν on the Borel sets of S is absolutely continuous with respect to the unique invariant measure class on S if and only if the ν-probability of an open set in S varies continuously under the action of the group G. If S is a Borel G-space, but the action is not assumed to be transitive, then $\nu(gE)$ is continuous in g for every Borel set E if and only if ν is absolutely continuous with respect to a quasi-invariant measure on S.

1. Introduction. Let ν be a probability measure on $\mathbb{R}$. It is well known that ν is continuous (i.e., has no point masses) if and only if

$$\lim_{t \to 0} \nu(t + U) = \nu(U) \tag{1.1}$$

for every open set U of the form (a, ∞). (That is, if and only if the cumulative distribution function of ν has no discontinuities.) It is easy to see that if (1.1) holds for such intervals, then it in fact holds more generally for an arbitrary finite disjoint union of open intervals. Since every open set in $\mathbb{R}$ is a countable union of disjoint open intervals [see, e.g., Royden (1988), page 42], one might conjecture that (1.1) holds for every open set U when ν is continuous, but a simple counterexample shows that this need not be the case.

EXAMPLE 1.1. Let ν be the Cantor singular measure on the unit interval [see, e.g., Feller (1971), pages 35–36], let

$$A =: \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \cup \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \cup \cdots$$

be the open set removed from the unit interval by excising "middle thirds," and let $C =: [0,1] - A$ be the resulting Cantor set. The set C is the set of numbers x in $[0,1]$ possessing a triadic expansion $x = .x_1 x_2 x_3 \cdots$ with every $x_i = 0$ or 2; for each $k \geq 1$, let

$$C_k =: \{x = .x_1 x_2 x_3 \cdots \in C : x_k = 2\}.$$

Then $\nu(C_k) = 1/2$ and $C_k - (1/3)^k \subseteq A$, hence $\nu(A) = 0$ but $\nu(A + (1/3)^k) \geq 1/2$.

Received June 1989; revised December 1990.
AMS 1980 *subject classifications*. Primary 60B15; secondary 28A70.
Key words and phrases. Absolute continuity, locally compact group, homogeneous space, invariant measure, quasi-invariant measure.

Thus (1.1) can fail for some open sets if ν is only assumed to be continuous. Let us say that ν is *translation continuous* if (1.1) in fact holds for all open sets U. It is a remarkable fact that this property characterizes absolute continuity with respect to Lebesgue measure.

THEOREM 1.1. *A probability measure ν on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure μ on $\mathbb{R}$ if and only if it is translation continuous.*

It is curious that this natural analog of the characterization of continuous probability measures does not appear in the textbook literature [see, however, Saks (1937), pages 90–93]. The result obviously depends on the algebraic and topological structure of the underlying sample space (since these appear in its statement), and thus is not purely measure-theoretic in nature. It does not depend, however, on any special properties of the real line, and is valid for measures on a locally compact topological group if Lebesgue measure is replaced by Haar measure [Rudin (1959)]; for a simple proof in the case of $\mathbb{R}^n$, see Malament and Zabell (1980). Brown, Graham and Moran (1977), pages 374–380, discuss several closely related characterizations of absolute continuity, and give a detailed account of their history.

In this note the extension of Theorem 1.1 to the more general setting of a group acting on a topological space is discussed; see Theorems 2.1 and 4.1 below. Although straightforward, the extension seems of interest because of several new examples that arise in this setting. Perhaps the simplest and most attractive of these is that a probability measure on a sphere is absolutely continuous with respect to the uniform measure on the sphere if and only if the probability of every open set varies continuously under rotations of the sphere (Example 2.1). Transitivity of the action, however, becomes a consideration: Although every absolutely continuous measure is translation continuous, translation continuity only suffices to establish absolute continuity when the action is transitive (Example 2.2); even ergodicity of the action does not suffice (Example 2.3). If the action does not possess an invariant measure [for example, when $SL(n, \mathbb{R})$ acts on projective space], translation continuity still forces absolutely continuity with respect to the invariant measure class of the action (Example 2.4), and one cannot in general avoid this phenomenon by passing to a subgroup for which an invariant measure does exist (Example 2.5).

In Section 2 the necessary definitions are given, Theorem 2.1 (covering the case of a transitive action) is stated, and the examples mentioned above illustrating its scope are discussed. A simple proof of Theorem 2.1 is then given in Section 3. In Section 4 the case of a nontransitive action is discussed. If the action of the group is not transitive, then more than one invariant measure class can exist on S, and a measure on S is translation continuous if and only if it is absolutely continuous with respect to a quasi-invariant measure on S (Theorem 4.1). Such a result was proven under restrictive topological conditions by Kleppner (1967) and Liu and van Rooij (1968), but is in fact true in considerable generality; Theorem 4.1 assumes only that S is a Borel G-space.

2. Translation continuity for group actions. Let G be a topological group with unit e. A *topological G-space* is a topological space S together with a continuous mapping $(g, s) \to gs$ of $G \times S$ into S such that $g_1(g_2 s) = (g_1 g_2)s$ and $es = s$ for all $g_1, g_2 \in G$ and $s \in S$. The basic definitions and facts about topological G-spaces needed for the statement and proof of Theorem 2.1 are summarized below; useful general references include Bourbaki (1963), Chapter 7, Nachbin (1965), Chapter 3, Gaal (1973), Chapter 3, Mackey (1978), Fell and Dorand (1988), Chapter 3, and Royden (1988), Chapter 14. For simplicity, *it is assumed throughout this section and the next that both G and S are Hausdorff, locally compact and second countable.*

A measure μ on the Borel sets of the G-space S is said to be *invariant* if $\mu(gE) = \mu(E)$ for every $g \in G$ and Borel set E, and *quasi-invariant* if $\mu(E) = 0$ implies $\mu(gE) = 0$. Two measures μ_1 and μ_2 on S having the same sets of measure 0 are said to belong to the same *measure class*; in this case we write $\mu_1 \sim \mu_2$ and let $[\mu]$ denote the measure class of μ. If μ_1 is quasi-invariant and $\mu_1 \sim \mu_2$, then μ_2 is quasi-invariant; thus either all members of a measure class are quasi-invariant or none are. If $\mu_g(E) =: \mu(gE)$, then a measure μ is quasi-invariant if and only if $[\mu_g] = [\mu]$ for every $g \in G$; in this case the measure class $[\mu]$ is said to be *invariant* under the action of the group.

A topological G-space is said to be *transitive* if for every $s_1, s_2 \in S$, there exists a $g \in G$ such that $s_2 = gs_1$. Although in general a G-space S need not have an invariant measure, if the action of the group is transitive, then S has a unique nontrivial invariant measure class [Mackey (1952), pages 68–69]. If γ is left Haar measure on G and $\pi_s(g) =: gs$, then $\gamma(\pi_s^{-1})$ is quasi-invariant for every $s \in S$, and every quasi-invariant measure on S is contained in the same measure class. If we denote this unique invariant measure class by $[\mu]$, then, employing an obvious notation, we have $[\gamma(\pi_s^{-1})] \ll [\mu]$ for every s, and

$$[\mu](E) = 0 \Rightarrow \gamma\big(\pi_s^{-1}(E)\big) = 0 \quad \text{for every } s \in S.$$

Finally we will say that a probability measure ν on S is *translation continuous* if $\nu(gU) \to \nu(U)$ as $g \to e$, for every open set U in S.

THEOREM 2.1. *Let S be a transitive topological G-space and* $[\mu]$ *its unique invariant measure class. If* ν *is a probability measure on the Borel sets of S, then* $\nu \ll [\mu]$ *if and only if*

$$\lim_{g \to e} \nu(gU) = \nu(U) \tag{2.1}$$

for every open set U in S.

Specifically, we will show that:

(i) *If S is a G-space,* μ *a measure on S such that* $\mu_g \ll \mu$ *for every* $g \in G$ *and* $(d\mu_g/d\mu)(s)$ *is continuous in g for every* $s \in S$ *(for example, if* μ *is invariant), and* $\nu \ll \mu$ *is a probability measure, then* (2.1) *holds for every measurable set U, uniformly in U.*

(ii) *Conversely, if S is a* transitive *G-space with $[\mu]$ the (necessarily unique) invariant measure class induced on S by left Haar measure γ on G, and if* (2.1) *holds for every* open *set U, then $\nu \ll [\mu]$.*

Note that in (i) the action is not assumed to be transitive, while in (ii) the translation continuity of $\nu(gU)$ is assumed only for U open. If the action of G is transitive, it can be shown that the condition on the measure μ in (i) is always satisfied for some quasi-invariant measure in the unique invariant measure class [Loomis (1960)]; thus for transitive actions translation continuity for open sets implies translation continuity *uniformly* over all Borel sets. Theorem 2.1 is proved in Section 3 below. We conclude this section with several examples illustrating the scope of the theorem.

EXAMPLE 2.1. Let G be the orthogonal group $O(n)$, $S = S^{n-1}$ (the unit sphere in $\mathbb{R}^n$) and μ the uniform measure on S^{n-1}. Then G acts transitively on S; thus Theorem 2.1 states that a probability measure on the sphere is absolutely continuous with respect to the uniform measure on that sphere if and only if the probability of every open subset of the sphere varies continuously under rotations.

EXAMPLE 2.2. Let $G = S^1$, $S = S^2$, and let S^1 act on S^2 in the obvious way by rotating it around an axis. If μ is taken to be the uniform probability measure on the equator of S^2 (relative to this axis), then any measure $\nu \ll \mu$ is translation continuous (this follows from Theorem 2.1 applied to the action $S^1 \times S^1 \to S^1$), but not every S^1-translation continuous measure on S^2 is absolutely continuous with respect to μ: Just take a point mass concentrated on either pole, or the uniform probability measure concentrated on any latitude other than the equator.

The group action in Example 2.2 is very far from transitive. The next example shows that even when the orbits are dense, the action ergodic and the measure translation continuous in any reasonable sense, no matter how strong, the measure need not be absolutely continuous with respect to the invariant measure.

EXAMPLE 2.3. Let G act ergodically but not transitively on (S, μ). Then, as is well known, there can exist probability measures ν on S which are invariant with respect to the action of G (and hence trivially translation continuous), but which are not absolutely continuous with respect to μ [see generally Sinai (1972)]. Thus transitivity is essential for the validity of Theorem 2.1.

Can one weaken the assumption of transitivity by strengthening the translation continuity condition in some way [for example, by assuming uniformity of convergence, as in part (i) of the theorem]? Since ν is an invariant measure (and hence would satisfy any reasonable continuity condition), the answer is clearly no.

An interesting application of Theorem 2.1 in statistical mechanics relates to the existence of measures which are not absolutely continuous when $G = \mathbb{R}$ represents time, S is a constant energy hypersurface, μ is Liouville measure and the action of G on S is the flow induced by the Hamiltonian of the system. In this case the theorem suggests that stationary measures which are not absolutely continuous with respect to μ should be disregarded as physically unrealistic, and in turn this uniquely identifies μ as the equilibrium measure; see Malament and Zabell (1980) for further details.

A transitive G-space can be identified in a natural way with a coset space of G: If s is any point in S, H the subgroup of G which leaves s fixed (the *isotropy subgroup* or *stabilizer* of s) and G/H is the quotient space of G and H endowed with the quotient topology, then H is closed (because S is Hausdorff) and the map of $G/H \to S$ which sends the coset gH to the element gs is a homeomorphism of the two spaces [because G and S are locally compact and G is second countable; see Helgason (1962), page 111, Theorem 3.2]. Conversely, given any closed subgroup H of G, the action of G on the coset space G/H gives rise to a transitive G-space.

This construction provides a simple way of generating transitive G-spaces without an invariant measure: If G is unimodular (in the sense that its left and right Haar measures coincide), but H is a closed subgroup of G which is not unimodular, then no measure exists on G/H invariant under the action of G [see, e.g., Gaal (1973), page 267, and Fell and Dorand (1988), page 245].

EXAMPLE 2.4. Thus, if $G = GL(n, \mathbb{R})$ and S is projective space (or more generally the Grassmann manifold $P_{n,m}$ of m-dimensional subspaces of $\mathbb{R}^n$), then G is unimodular, and S is the quotient of G by a nonunimodular closed subgroup, hence no measure exists on S that is invariant under the action of G [see, e.g., Nachbin (1965), page 143, and Fell and Dorand (1988), pages 181 and 245]. Nevertheless, if a probability measure ν on S varies continuously under the action of G, then ν is absolutely continuous with respect to the invariant measure class of the action.

In Example 2.4, no invariant measure exists on S because G is "too large": If we restrict our attention to the (maximal) compact subgroup $G_0 = O(n)$, then G_0 acts transitively on $P_{n,m}$ and, since every compact group is unimodular, there exists a measure μ on $P_{n,m}$ invariant under the action of G_0. If ν is translation continuous with respect to G, then it is translation continuous with respect to G_0, hence $\nu \ll \mu$. If it were in similar fashion always possible, in a transitive G-space S not having an invariant measure, to pass to a transitive subgroup G_0 for which an invariant measure did exist, it might be argued that the appearance of the invariant measure class in the statement of Theorem 2.1 was unnecessary. It is not hard to show, however, that there are G-spaces for which the absence of an invariant measure is "intrinsic."

EXAMPLE 2.5. Let $G = SL(n, \mathbb{R})$, $n \geq 4$, and let H be a closed two-dimensional subgroup of G that is not unimodular, so that G/H has no G-invariant measure. Suppose there exist proper subgroups H_0 and G_0 such that $G \supseteq G_0 \supseteq H_0$ and $G_0/H_0 \approx G/H$. Then the codimension of G_0 would have to be less than or equal to 2. On the other hand, it can be shown that any proper subgroup of $SL(n, \mathbb{R})$ must have codimension at least 3. (This observation is due to Robert Zimmer.)

3. Proof of Theorem 2.1. Let ν be a finite signed measure on S, $\|\nu\|_\infty =: \sup\{|\nu(E)|: E \text{ Borel}\}$ the sup norm of ν viewed as a function on the Borel sets of S, and

$$\|\nu\| = \sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)|: E_i \text{ disjoint}\right\}$$

the total variation norm of ν. If $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν into its positive and negative parts, then it follows from the Hahn decomposition theorem that $\|\nu\|_\infty = \max\{\nu^+(S), \nu^-(S)\}$ and $\|\nu\| = \nu^+(S) + \nu^-(S)$. In particular, if $\nu = \nu_1 - \nu_2$ is the difference of two positive finite measures such that $\nu_1(S) = \nu_2(S)$, and $A \cup B$ is a Hahn decomposition of S (with ν positive on A and negative on B), then $\nu^+(S) = \nu^-(S) = \nu_1(A) - \nu_2(A) = \nu_2(B) - \nu_1(B)$, and hence $\|\nu_1 - \nu_2\| = 2\|\nu_1 - \nu_2\|_\infty$. Thus the two norms are equivalent, and in order to show that $\nu_g \to \nu$ in the sup norm, it suffices to show convergence in total variation.

LEMMA 3.1. *If* $\nu \ll [\mu]$, *then* $\lim_{g \to e}\|\nu_g - \nu\| = 0$.

PROOF. There exists a quasi-invariant Radon measure μ in the unique invariant measure class on S such that $(d\mu_g/d\mu)(s)$ is jointly continuous in g and s [Loomis (1960), page 579, Theorem 3]. Suppose $\nu \ll [\mu]$. Because μ is quasi-invariant, $\nu_g \ll [\mu]$, and thus

$$\sup\{|\nu_g(E) - \nu(E)|: E \text{ Borel}\} = \frac{1}{2}\|\nu_g - \nu\| = \frac{1}{2}\left\|\frac{d\nu_g}{d\mu} - \frac{d\nu}{d\mu}\right\|_1;$$

the second equality follows because the map which identifies an element of $L^1(\mu)$ with a finite signed measure on S is an isometric embedding of Banach spaces [see, e.g., Rudin (1986), page 134, Theorem 6.13].

Let $f(s) = (d\nu/d\mu)(s)$. Then $(d\nu_g/d\mu_g)(s) = f(gs) =: f_g(s)$, hence

$$\frac{d\nu_g}{d\mu} = \frac{d\nu_g}{d\mu_g}\frac{d\mu_g}{d\mu} = f_g\frac{d\mu_g}{d\mu}$$

and it suffices to show that the right-hand side converges in L_1 to f as $g \to e$. If $f(s)$ is continuous, then this immediately follows from Scheffé's theorem [see, e.g., Billingsley (1979), page 184, Theorem 16.11]. If $f(s)$ is not continuous, then for every $\varepsilon > 0$ there exists a continuous probability density $f^*(s)$

such that $\|f - f^*\|_1 < \varepsilon$, and a neighborhood G_0 of e such that

$$\left\| f_g^* \frac{d\mu_g}{d\mu} - f^* \right\|_1 < \varepsilon$$

for every $g \in G_0$. Since

$$\left\| f_g \frac{d\mu_g}{d\mu} - f_g^* \frac{d\mu_g}{d\mu} \right\|_1 = \int |(f - f^*)_g| \, d\mu_g = \|f - f^*\|_1,$$

it follows that for $g \in G_0$,

$$\left\| f_g \frac{d\mu_g}{d\mu} - f \right\|_1 \le \left\| f_g \frac{d\mu_g}{d\mu} - f_g^* \frac{d\mu_g}{d\mu} \right\|_1 + \left\| f_g^* \frac{d\mu_g}{d\mu} - f^* \right\|_1 + \|f^* - f\|_1 < 3\varepsilon.$$

This completes the proof of Lemma 3.1.

For any compact set K, $\nu(gK)$ is upper semicontinuous, hence for every Borel set E, $\nu(gE)$ is measurable. In order to show that translation continuity implies that $\nu \ll [\mu]$, we need to show that the g-translates of μ-null sets are ν-null for γ-almost all g.

Lemma 3.2. *If E is a measurable subset of S such that $\mu(E) = 0$, and γ is left Haar measure, then $\gamma\{g: \nu(gE) > 0\} = 0$.*

Proof. Let I_E be the indicator function of E. Then

$$\int_G I_E(gs) \, d\gamma(g) = \gamma(\pi_s^{-1}(E)) = \mu(E) = 0,$$

hence by Tonelli's theorem,

$$\begin{aligned}\int_G \nu(g^{-1}E) \, d\gamma(g) &= \int_G \int_S I_E(gs) \, d\nu(s) \, d\gamma(g) \\ &= \int_S \left[\int_G I_E(gs) \, d\gamma(g) \right] d\nu(s) = 0,\end{aligned}$$

which implies that $\nu(gE) = 0$ γ-almost everywhere, proving Lemma 3.2. □

Now suppose that ν is not absolutely continuous with respect to μ. Then there exists a measurable set E such that $\mu(E) = 0$ but $\nu(E) > 0$. Because S is locally compact and second countable, ν is regular, hence there exists a closed set B contained in E such that $\nu(B) > 0$. Let $A = B^c$. Then $\nu(A) < 1$ but it follows from Lemma 3.2 (applied to B) that $\nu(gA) = 1$ for γ-almost all g. Since any neighborhood of e has positive γ-measure, it follows that $\nu(gA)$ cannot be continuous at $g = e$, and the theorem is proved. □

Thus $\nu \ll [\mu] \Leftrightarrow \lim_{g \to e} \|\nu_g - \nu\| = 0$. In fact, one can prove more. Let $\nu = \alpha + \sigma$ denote the Lebesgue decomposition of ν into measures α and σ such that $\alpha \ll [\mu]$ and $\sigma \perp [\mu]$.

COROLLARY 3.1. $\limsup_{g \to e} \|\nu_g - \nu\| = 2\|\sigma\|$.

PROOF. If $\nu_1 \perp \nu_2$, then $\|\nu_1 + \nu_2\| = \|\nu_1\| + \|\nu_2\|$. Since $\nu_g = \alpha_g + \sigma_g$, it follows that

$$\|\nu_g - \nu\| = \|(\alpha_g + \sigma_g) - (\alpha + \sigma)\| = \|\alpha_g - \alpha\| + \|\sigma_g - \sigma\|.$$

By Lemma 3.2, there exists a γ-null set A such that $\sigma_g \perp \sigma$ for $g \notin A$. By Theorem 2.1, $\lim_{g \to e}\|\alpha_g - \alpha\| = 0$, while $\|\sigma_g - \sigma\| \leq 2\|\sigma_g\|$ for all $g \in G$, and $\|\sigma_g - \sigma\| = 2\|\sigma_g\|$ for $g \notin A$. It follows that $\lim_{g \to e}\|\nu_g - \nu\| = 2\|\sigma\|$ for $g \notin A$, and $\limsup_{g \to e}\|\nu_g - \nu\| = 2\|\sigma\|$. □

In the special case $G = S = \mathbb{R}$, Corollary 3.1 is due to Wiener and Young (1935); see also Plessner (1929).

4. Nontransitive actions. In this section we consider the case of a nontransitive action on S. In such examples S can have more than one quasi-invariant measure, and the appropriate generalization of Theorem 2.1 is that a measure on S is translation continuous if and only if it is absolutely continuous with respect to *some* quasi-invariant measure on S.

It is in fact possible to prove such a result under conditions of considerable generality. Thus let G be a locally compact Hausdorff topological group and suppose only that S is a *Borel G-space*: That is, the set S is only assumed to be a *standard* Borel space (i.e., Borel isomorphic to a Borel subset of a complete separable metric space), and the action $G \times S \to S$ is only assumed to be measurable (rather than continuous). Since every second countable locally compact Hausdorff space X is homeomorphic to a complete separable metric space, every topological G-space is a Borel G-space. (The one-point compactification X^* of X is separable, metrizable and complete, and X is an open subset of X^*, hence a G_δ of X^*.) Basic information about standard Borel spaces and Borel G-spaces may be found in Arveson (1976), Chapter 3, Mackey (1978) and Zimmer (1984).

Rather than confine our attention to probability measures, ν will be permitted to be a finite positive measure on S; and because S is not endowed with a topology, ν will be said to be "translation continuous" if the map $g \to \nu(gE)$ is continuous in g for every Borel set $E \subset S$. [If S has in addition the structure of a topological space and ν is regular, then this condition may be replaced in Theorem 4.1 by the weaker condition that $\nu(gK)$ is continuous in g for every compact set K; see Remark 4.1 below.]

THEOREM 4.1. *Let G be σ-compact and S a Borel G-space. If ν is a finite positive Borel measure on S, then ν is translation continuous if and only if ν is absolutely continuous with respect to a σ-finite quasi-invariant measure on S.*

Theorem 4.1 will follow from Lemmas 4.1–4.3 and Remark 4.2.

LEMMA 4.1. *If μ is a σ-finite quasi-invariant measure on S and $\nu \ll \mu$, then* $\lim_{g \to e} \|\nu_g - \nu\| = 0$.

PROOF. Let $\mathbb{M}(S)$ denote the space of complex measures on S, and let $\Phi_\mu = \{\nu \in \mathbb{M}(S)\colon \nu \ll \mu\}$ denote the Banach space of complex measures absolutely continuous with respect to μ, endowed with the total variation norm. Because μ is quasi-invariant, $\nu \in \Phi_\mu \Rightarrow \nu_g \in \Phi_\mu$; and it is immediate that for each $g \in G$ the map $T(g)\colon \Phi_\mu \to \Phi_\mu$ given by $[T(g)](\nu) = \nu_g$ is a linear isometry of Φ_μ. Let $\mathbb{L}(\Phi_\mu)$ denote the topological group of linear isometries of Φ_μ, endowed with the operator topology; because Φ_μ is isomorphic to $L^1(S, \mu)$ and S is a standard Borel space, $\mathbb{L}(\Phi_\mu)$ is separable. Since the group homeomorphism $T\colon G \to \mathbb{L}(\Phi_\mu)$ is measurable, it follows from a theorem of Mackey [see, e.g., Zimmer (1984), page 198] that T is continuous; and thus that the map from G to Φ_μ given by $g \to \nu_g$ is continuous in g for every $\nu \in \Phi_\mu$. □

To show that a translation-continuous measure ν is absolutely continuous with respect to some quasi-invariant measure on S, we construct a quasi-invariant measure ν^* by "smoothing" ν, and then show that $\nu \ll \nu^*$. Thus let γ denote a quasi-invariant measure on G, and let

$$\nu^*(E) =: \int \nu(g^{-1}E)\, d\gamma(g);$$

the measurability of $\nu(g^{-1}E)$ follows from Tonelli's theorem. Clearly, $\nu^*(E) = 0 \Leftrightarrow \nu(gE) = 0$ γ-a.e.; thus the measure class of ν^* only depends on the (unique) measure class of γ.

LEMMA 4.2. *The measure ν^* is quasi-invariant.*

PROOF. Let $g_1 \in G$. If $\nu^*(E) = 0$, then $\nu(gE) = 0$ for γ-almost all g, hence $\nu(gg_1E) = 0$ for γ-almost all g, hence $\nu^*(g_1E) = 0$. □

LEMMA 4.3. *If ν is translation continuous, then $\nu \ll \nu^*$.*

PROOF. If $\nu^*(E) = 0$, then $\nu(gE) = 0$ γ-a.e. Since every neighborhood of G has positive γ-measure, $\nu(gE) = 0$ for every $g \in G$, hence $\nu(E) = 0$. □

REMARK 4.1. If S is a topological space, ν is regular and $\nu(gK)$ is continuous in g for every compact set K, then $\nu(gE)$ is continuous in g for every Borel set E, hence $\nu \ll \nu^*$. [Argue as immediately after Lemma 3.2; the analog of that lemma here is that $\nu^*(E) = 0 \Rightarrow \nu(gE) = 0$ γ-a.e.]

We note in passing two simple aspects of the smoothing ν^*: If ν is translation continuous, then ν^* is the minimal quasi-invariant measure such that $\nu \ll \nu^*$ (Lemma 4.4); and if ν is already quasi-invariant, then smoothing ν does not change its measure class (Lemma 4.5). (Note that in the statement of both lemmas ν is not assumed to be translation continuous.)

LEMMA 4.4. *If $\nu \ll \mu$ and μ is quasi-invariant, then $\nu^* \ll \mu$.*

PROOF. If $\mu(E) = 0$, then $\mu(gE) = 0$ for all $g \in G$, hence $\nu(gE) = 0$ for all $g \in G$, hence $\nu^*(E) = 0$. □

LEMMA 4.5. *The measure ν is equivalent to ν^* if and only if ν is quasi-invariant.*

PROOF. If ν is not quasi-invariant, then it is immediate that ν and ν^* are not equivalent; while if ν is quasi-invariant, then $\nu^* \ll \nu$ by Lemma 4.4. Thus it suffices to show that if ν is quasi-invariant, then $\nu \ll \nu^*$. But if $\nu^*(E) = 0$, then $\nu(gE) = 0$ γ-a.e., hence there exists a g_0 such that $\nu(g_0 E) = 0$; since ν is quasi-invariant, it follows that $\nu(E) = 0$. □

REMARK 4.2. If G is σ-compact, then γ may be chosen to be finite. Then ν^* is finite, and Theorem 4.1 follows. The assumption that G be σ-compact involves no essential restriction: Every locally compact group G contains an open, σ-compact subgroup $G_0 \subset G$, and a measure ν is G-translation continuous if and only if it is G_0-translation continuous.

Let ν^* be finite and let $\nu = \alpha + \sigma$ denote the Lebesgue decomposition of ν into measures α and σ such that $\alpha \ll [\nu^*]$ and $\sigma \perp [\nu^*]$. Arguing as in Corollary 3.1, we deduce the following.

COROLLARY 4.1. $\limsup_{g \to e} \|\nu_g - \nu\| = 2\|\sigma\|$.

The conditions on G and S in Theorem 4.1 would appear to be close to, if not the minimal ones necessary to ensure the validity of Theorem 4.1: G is assumed locally compact to guarantee the existence of a quasi-invariant measure giving open sets positive measure; S is assumed standard Borel to guarantee that $L^1(S, \mu)$ is separable. Theorem 4.1 was proved by Kleppner (1967) for a topological G-space S with G and S locally compact and second countable; Liu and van Rooij (1968) showed that the requirement that G and S be second countable in Kleppner's theorem could be removed if G were assumed to be σ-compact. Theorem 2.3 of Graham, Lau and Leinert (1988) is closely related to Mackey's theorem (employed in the proof of Lemma 4.1 above) and is a generalization of an earlier result due to Larsen (1968) and Tam (1969); see also Larsen (1969) and Liu, van Rooij and Wang (1970).

Acknowledgments. My thanks to Colin Graham, C. Ionescu-Tulcea and Robert Zimmer for helpful comments and references.

REFERENCES

ARVESON, W. (1976). *An Invitation to C *-Algebra*. Springer, New York.
BILLINGSLEY, P. (1979). *Probability and Measure*. Wiley, New York.
BOURBAKI, N. (1963). *Éléments de mathématique: Livre 6: Intégration,* Chapters 7 and 8. Hermann, Paris.

Brown, G., Graham, C. C. and Moran, W. (1977). Translation and symmetry in $M(G)$. *Symp. Math.* **22** 371–392.

Fell, J. M. G. and Dorand, R. S. (1988). *Representations of* $*$*-Algebras, Locally Compact Groups, and Banach* $*$*-Algebraic Bundles. Basic Representation Theory of Groups and Algebras* **1**. Academic, San Diego.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

Gaal, S. (1973). *Linear Analysis and Representation Theory*. Springer, New York.

Graham, C., Lau, A. and Leinert, M. (1988). Separable translation-invariant subspaces of $M(G)$ and other dual spaces on locally compact groups. *Colloq. Math.* **55** 131–145.

Helgason, S. (1962). *Differential Geometry and Symmetric Spaces*. Academic, New York.

Kleppner, A. (1967). Absolutely continuous measures on G-spaces. Unpublished manuscript.

Larsen, R. (1968). Measures with separable orbits. *Proc. Amer. Math. Soc.* **19** 569–572.

Larsen, R. (1969). Transformation groups and measures with separable orbits. *Monatsh. Math.* **73** 222–225.

Liu, T.-S. and van Rooij, A. (1968). Transformation groups and absolutely continuous measures. *Indag. Math.* **30** 225–231.

Liu, T.-S., van Rooij, A. and Wang, J.-K. (1970). Transformation groups and absolutely continuous measures. II. *Indag. Math.* **32** 57–61.

Loomis, L. H. (1960). Positive definite functions and induced representations. *Duke Math. J.* **27** 569–579.

Mackey, G. W. (1952). Induced representations of locally compact groups. I. *Ann. Math.* **55** 101–139.

Mackey, G. (1978). *Unitary Group Representations in Physics, Probability, and Number Theory*. Benjamin, Reading, Mass.

Malament, D. and Zabell, S. (1980). Why Gibbs phase averages work—the role of ergodic theory. *Philos. Sci.* **47** 339–349.

Nachbin, L. (1965). *The Haar Integral*. Van Nostrand, Princeton.

Plessner, A. (1929). Eine Kennzeichnung der totalstetigen Funktionen. *J. Reine Angew. Math.* **60** 26–32.

Royden, H. (1988). *Real Analysis*, 3rd ed. Macmillan, New York.

Rudin, W. (1959). Measure algebras on abelian groups. *Bull. Amer. Math. Soc.* **65** 227–247.

Rudin, W. (1986). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York.

Saks, S. (1937). *Theory of the Integral*, 2nd ed. Hafner, New York.

Sinai, Ya. G. (1972). Gibbs measures in ergodic theory. *Russian Math. Surveys* **166** 21–69.

Tam, K. W. (1969). On measures with separable orbit. *Proc. Amer. Math. Soc.* **23** 409–411.

Wiener, N. and Young, R. C. (1935). The total variation of $g(x + h) - g(x)$. *Trans. Amer. Math. Soc.* **33** 327–340.

Zimmer, R. J. (1984). *Ergodic Theory and Semisimple Groups*. Birkhaüser, Boston.

Department of Mathematics
Lunt Hall
Northwestern University
Evanston, Illinois 60208-2730