The Annals of Probability
1992, Vol. 20, No. 1, 350–374

ON A CLASS OF STOCHASTIC RECURSIVE SEQUENCES ARISING IN QUEUEING THEORY

BY FRANÇOIS BACCELLI AND ZHEN LIU

INRIA

This paper is concerned with a class of stochastic recursive sequences that arise in various branches of queueing theory. First, we make use of Kingman's subadditive ergodic theorem to determine the stability region of this type of sequence, or equivalently, the condition under which they converge weakly to a finite limit. Under this stability condition, we also show that these sequences admit a unique finite stationary regime and that regardless of the initial condition, the transient sequence couples in finite time with this uniquely defined stationary regime. When this stability condition is not satisfied, we show that the sequence converges a.s. to ∞ and that certain increments of the process form another type of stochastic recursive sequence that always admit at least one stationary regime. Finally, we give sufficient conditions for this increment sequence to couple with this stationary regime.

1. Introduction. All of the random variables considered here are defined on a common probability space $(\Omega, \mathbb{F}, P, \theta)$, where θ is an ergodic shift that leaves P invariant. Let $K \geq 1 \in \mathbb{N}$ be the *dimension* of the equation.

The basic random data of the problem are:

1. The *initial condition*, which is an arbitrary nonnegative random vector $Y \in \mathbb{R}^{+K}$.
2. The *delay sequence*, which is a sequence of random matrices $\{l_n^{j,k}\}_{n=-\infty}^{\infty}$, where $l_n^{j,k} \in \mathbb{R}$, $1 \leq j, k \leq K$.
3. The *predecessor sets sequences*, which are K sequences of random sets $\{\pi_n^k\}_{n=-\infty}^{\infty}$, $k = 1, \dots, K$, where $\pi_n^k \in 2^{\{1,\dots,K\}}$, $1 \leq k \leq K$, where 2^S denotes the set of all subsets of set S.

The sequence of interest $\{W_n(Y)\}_{n=0}^{\infty}$ is given by the recursive equation

$$W_0(Y) = Y,$$

$$W_{n+1}^k(Y) = \max_{\{j \in \pi_n^k\}} \left(W_n^j(Y) + l_n^{j,k}\right)^+, \qquad 1 \leq k \leq K, \qquad n \geq 0, \tag{1.1}$$

where $a^+ = \max(a, 0)$.

Received January 1989; revised June 1990.
AMS 1980 *subject classifications*. 05C20, 60F20, 60G10, 60G17, 60G55, 60K25, 68Q75, 68Q90, 68R10, 93D05, 93E03, 93E15.
Key words and phrases. Stochastic recursive sequences, queueing theory, ergodic theory, stationary processes, coupling.

The aim of this paper is to analyze this type of equation under the following assumptions:

1. *Stationarity and ergodicity*: The sequences $\{l_n\}_{-\infty}^{\infty}$ and $\{\pi_n\}_{-\infty}^{\infty}$ are jointly ergodic and stationary. Within our abstract shift formalism, this translates into the assumption that the shift θ defined on $(\Omega, \mathbb{F})$ is compatible with the translation of sequences, namely $l_n = l \circ \theta^n$ and $\pi_n = \pi \circ \theta^n$ for all $n \in \mathbb{Z}$.
2. *Integrability*: The random variables $l^{j,k}$, $1 \le j, k \le K$, are integrable.
3. *Precedence*: For all $1 \le k \le K$, it is assumed that $k \in \pi^k$ a.s.

Under these assumptions we can rewrite the basic recursion as

$$W_0(Y) = Y,$$

$$W_{n+1}^k(Y) = \max_{\{j \in \pi^k \circ \theta^n\}} \left(W_n^j(Y) + l^{j,k} \circ \theta^n \right)^+, \qquad 1 \le k \le K, n \ge 0. \tag{1.2}$$

The aim of this paper is to analyze both the transient and the stationary solutions of (1.1), using ergodic theory arguments, and in particular Kingman's subadditive ergodic theory. For this, we introduce a pathwise increasing recursion, the nth term of which is equivalent in law to $W_n(0)$, and which generalizes the schema initially proposed by Loynes for $G/G/1$ queues [9]. In Section 2, this recursion is then used to determine the condition ensuring the existence of a stationary solution of Equation (1.1), which will be referred to as the stability condition.

The notion of irreducibility is also introduced, and it is shown that the stability condition of a reducible equation boils down to the intersection of the stability conditions for a set of equations of the same type, corresponding to certain communicating classes.

It is shown in Section 3, via some coupling argument, that when this stability condition holds, $W_n(Y)$ converges weakly to this stationary solution when n goes to ∞, regardless of the initial condition. Section 4 focuses on the case where the stability condition is not satisfied. Then $W_n(Y)$ converges a.s. to ∞, at least in the irreducible case. Nevertheless, it is shown that certain increments satisfy another related recursive equation that always admits at least one stationary solution. However, it is not always possible to reach this stationary regime from some adequate initial condition. A simple sufficient condition ensuring the reachability of this stationary regime is provided in Section 4.2. Finally, examples stemming from queueing theory are presented in Section 5.

The schema of Section 2.1 was first proposed in [4] for a specific computer system model, where the precedence structure is deterministic and the delay structure has the form $l^{j,k} = \sigma^{j,k} - \tau$, where $\sigma^{j,k}, \tau \in \mathbb{R}^+$. The schema for the increments of Section 4.1 was first considered in [1], for handling a class of Petri net models.

2. A Loynes schema. The basic idea for analyzing (1.1) consists of associating with this equation another recursive schema that generalizes in a sense the schema that was originally proposed by Loynes [9] for analyzing $G/G/1$ queues.

2.1. *Definition.* Consider the variables $\{M_n^k\}_{n=0}^{\infty}$, $1 \le k \le K$, defined by

$$M_0^k = 0,$$
$$M_{n+1}^k \circ \theta = \max_{j \in \pi^k} \left(M_n^j + l^{j,k} \right)^+. \tag{2.1}$$

LEMMA 2.1. *For every k, $1 \le l \le K$, the sequence M_n^k is increasing in n.*

PROOF. It is clear that for every k, $1 \le k \le K$, $M_1^k \ge M_0^k = 0$. Assume now that for some $n \ge 1$, $M_n^k \ge M_{n-1}^k$ holds for every k, $1 \le k \le K$. Then for any k, $1 \le k \le K$,

$$M_{n+1}^k \circ \theta = \max_{j \in \pi^k} \left(M_n^j + l^{j,k} \right)^+$$
$$\ge \max_{j \in \pi^k} \left(M_{n-1}^j + l^{j,k} \right)^+ = M_n^k \circ \theta. \qquad \square$$

Let M_∞^k be the limiting value of the increasing sequence M_n^k when n goes to ∞. A simple continuity argument yields

$$M_\infty^k \circ \theta = \max_{j \in \pi^k} \left(M_\infty^j + l^{j,k} \right)^+. \tag{2.2}$$

From this we get:

LEMMA 2.2. *For each k, $1 \le k \le K$, the event $\{M_\infty^k = \infty\}$ is of probability either 0 or 1.*

PROOF. For all $1 \le k \le K$, we have $k \in \pi^k$. Hence (2.2) entails

$$M_\infty^k \circ \theta \ge M_\infty^k + l^{k,k}.$$

Therefore, $M_\infty^k = \infty$ implies $M_\infty^k \circ \theta = \infty$. This immediately implies the result in view of the assumption that θ is P-ergodic. $\square$

The following expansion of the Loynes schema M_n^k will be used later on.

LEMMA 2.3. *For every n and k, $n \ge 1$, $1 \le k \le K$,*

$$M_n^k = \max\left(0, \max_{1 \le m \le n} H_m^k\right), \tag{2.3}$$

where

$$H_m^k = \max_{\{1 \le v_s \le K,\ s=0,\dots,m,\ v_0 = k,\ v_s \in \pi^{v_{s-1}} \circ \theta^{-s}\}} \sum_{s=1}^m l^{v_s, v_{s-1}} \circ \theta^{-s}. \tag{2.4}$$

PROOF. The proof proceeds by induction on n. For $n = 1$, (2.3) is simply a restatement of (2.1). Suppose it holds for some $n \geq 1$. Then, we get from equation (2.1) that

$$M_{n+1}^k = \max\Big(0, \max_{j \in \pi^k \circ \theta^{-1}} \big(M_n^j \circ \theta^{-1} + l^{j,k} \circ \theta^{-1}\big)\Big).$$

Using the inductive assumption, we obtain

$$\begin{aligned} M_{n+1}^k &= \max\Big(0, \max_{j \in \pi^k \circ \theta^{-1}} \max\Big(0, \max_{1 \leq m \leq n} H_m^j \circ \theta^{-1}\Big) + l^{j,k} \circ \theta^{-1}\Big) \\ &= \max\Big(0, \max_{1 \leq m \leq n} \max_{j \in \pi^k \circ \theta^{-1}} \max\big(H_m^j \circ \theta^{-1} + l^{j,k} \circ \theta^{-1}, l^{j,k} \circ \theta^{-1}\big)\Big) \\ &= \max\Big(0, \max\Big(\max_{1 \leq m \leq n} H_{m+1}^k, H_1^k\Big)\Big) \\ &= \max\Big(0, \max_{1 \leq m \leq n+1} H_m^k\Big). \end{aligned}$$

Therefore the equation holds for $n + 1$, which proves the lemma. □

2.2. *Decomposition of the equation.* In this subsection, we decompose equation (2.1) into a set of simpler equations of the same types which satisfy an *irreducibility* property.

Define the *communication graph* of the equation as the directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where

$$\begin{aligned} \mathscr{V} &= \{1, 2, \dots, K\}, \\ \mathscr{E} &= \big\{(j, k) | P[j \in \pi^k] > 0\big\}. \end{aligned}$$

Obviously, $\mathscr{G}$ can have cycles.

Decompose $\mathscr{G}$ into its *communicating classes*, namely the maximal strongly connected subgraphs of $\mathscr{G}$:

$$\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1), \dots, \mathscr{G}_g = (\mathscr{V}_{\mathscr{g}}, \mathscr{E}_g),$$

such that if there is a path from j to k in $\mathscr{G}_i$, there is also a path from k to j in $\mathscr{G}_i$. It is obvious that this decomposition satisfies the properties

$$\mathscr{V}_1 \cup \cdots \cup \mathscr{V}_g = \mathscr{V} \quad \text{and} \quad \mathscr{E}_1 \cup \cdots \cup \mathscr{E}_g \subseteq \mathscr{E},$$

and, for all $1 \leq i < j \leq g$,

$$\mathscr{V}_i \cap \mathscr{V}_j = \varnothing \quad \text{and} \quad \mathscr{E}_i \cap \mathscr{E}_j = \varnothing.$$

Furthermore, define the *reduced graph*, which is denoted by $\tilde{\mathscr{G}}$, to be the graph that describes the one-way relations that may exist between the communicating classes: $\tilde{\mathscr{G}} = (\tilde{\mathscr{V}}, \tilde{\mathscr{E}})$, where

$$\begin{aligned} \tilde{\mathscr{V}} &= \{1, 2, \dots, g\}, \\ \tilde{\mathscr{E}} &= \big\{(e, f) | e, f \in \{1, 2, \dots, g\}, e \neq f, \exists\, (j, k) \in \mathscr{E}, j \in \mathscr{V}_e, k \in \mathscr{V}_f\big\}. \end{aligned}$$

It follows from the very definition of strong connectedness that $\tilde{\mathscr{G}}$ is acyclic.

We associate now g equations to (2.1), one per communicating class. Equation i has for dimension $K_i = |\mathscr{V}_i|$, for state variables $M_n^{k,i}$ ($k \in \mathscr{V}_i$) and for evolution equation

$$M_0^{k,i} = 0,$$

$$M_{n+1}^{k,i} \circ \theta = \max_{j \in \pi^k \cap \mathscr{V}_i} \left(M_n^{j,i} + l^{j,k}\right)^+. \tag{2.5}$$

By analogy with the theory of Markov chains, we will say that the system (2.5) is *irreducible*.

LEMMA 2.4. *For every i, $1 \le i \le g$, either $M_\infty^k < \infty$ a.s. for all $k \in \mathscr{V}_i$ or $M_\infty^k = \infty$ a.s. for all $k \in \mathscr{V}_i$.*

PROOF. Owing to Lemma 2.2, either $M_\infty^k < \infty$ a.s. for all $k \in \mathscr{V}_i$, or there is some $k \in \mathscr{V}_i$ such that $M_\infty^k = \infty$ a.s. If we are in second case, then we argue that for all $h \in \mathscr{V}_i$ such that $P[k \in \pi^h] > 0$, the relation

$$M_\infty^h \circ \theta = \max_{j \in \pi^h} \left(M_\infty^j + l^{j,h}\right)^+ \ge \left(M_\infty^k + l^{k,h}\right)^+$$

holds with a positive probability, so that $M_\infty^h \circ \theta = \infty$ occurs with a positive probability, and hence, according to Lemma 2.2, is an almost sure event. Repeating this argument a finite number of times, we get that $M_\infty^h = \infty$ a.s. for all $h \in \mathscr{V}_i$. □

2.3. *Stability condition.* For every i, $1 \le i \le g$, we define

$$Q_n^i = \max_{\{v_1, \dots, v_{n+1} \in \mathscr{V}_i | v_{s+1} \in \pi^{v_s} \circ \theta^{-s}, s = 1, \dots, n\}} \sum_{s=1}^{n} l^{v_{s+1}, v_s} \circ \theta^{-s}, \tag{2.6}$$

$$n = 1, 2, \dots .$$

LEMMA 2.5. *For all i, $1 \le i \le g$, there exists a constant γ_i such that*

$$\lim_{n \to \infty} \frac{Q_n^i}{n} = \lim_{n \to \infty} \frac{E[Q_n^i]}{n} = \gamma_i \quad a.s. \tag{2.7}$$

PROOF. Observe first that the finiteness of $E[Q_n^i]$ follows from the integrability assumption on the delays [use the fact that

$$-|a| - |b| \le \max(a, b) \le |a| + |b|].$$

Now let $U_{m,m+n}^i = Q_n^i \circ \theta^{-m}$, $m \in \mathbb{Z}$, $n \ge 1$. Then for all $n \ge 1$ and all $p, q \ge 1$

such that $p + q = n$, we have

$$\begin{aligned} U^i_{m,m+n} &= \left(\max_{\{v_1,\dots,v_{n+1} \in \mathscr{V}_i | v_{s+1} \in \pi^{v_s} \circ \theta^{-s}, s=1,\dots,n\}} \sum_{s=1}^{n} l^{v_{s+1},v_s} \circ \theta^{-s} \right) \circ \theta^{-m} \\ &\le \left(\max_{\{v_1,\dots,v_{p+1} \in \mathscr{V}_i | v_{s+1} \in \pi^{v_s} \circ \theta^{-s}, s=1,\dots,p\}} \sum_{s=1}^{p} l^{v_{s+1},v_s} \circ \theta^{-s} \right) \circ \theta^{-m} \\ &\quad + \left(\max_{\{v_{p+1},\dots,v_{n+1} \in \mathscr{V}_i | v_{s+1} \in \pi^{v_s} \circ \theta^{-s}, s=p+1,\dots,n\}} \sum_{s=p+1}^{n} l^{v_{s+1},v_s} \circ \theta^{-s} \right) \circ \theta^{-m} \\ &= U^i_{m,m+p} + U^i_{m+p,m+p+q}. \end{aligned}$$

Therefore $U^i_{m,m+n}$ is a subadditive process. Applying Kingman's theorem on subadditive ergodic processes ([8]) readily yields

$$\lim_{n\to\infty} \frac{U^i_{0,n}}{n} = \lim_{n\to\infty} \frac{E\left[U^i_{0,n}\right]}{n} = \gamma_i \quad \text{a.s.},$$

which concludes the proof. □

THEOREM 2.6. *For all i, $1 \le i \le g$, if $\gamma_i < 0$, then $M^{k,i}_\infty < \infty$ a.s. for all $k \in \mathscr{V}_i$. If $\gamma_i > 0$, then $M^{k,i}_\infty = \infty$ a.s for all $k \in \mathscr{V}_i$.*

PROOF. The proof proceeds in two steps.

(i) It follows from Lemma 2.4 that the event $\{\forall\ k \in \mathscr{V}_i\colon M^{k,i}_\infty = \infty\}$ is of probability 0 or 1. Assume it is of probability 1. Then $\max_{k \in \mathscr{V}_i} M^{k,i}_\infty = \infty$ a.s. Let

$$H^{k,i}_n = \max_{\{v_s \in \mathscr{V}_i | s=0,\dots,n, v_0=k, v_s \in \pi^{v_{s-1}} \circ \theta^{-s}\}} \left(\sum_{s=1}^{m} l^{v_s,v_{s-1}} \circ \theta^{-s} \right). \tag{2.8}$$

In view of (2.3), $\max_{k \in \mathscr{V}_i} M^{k,i}_n \uparrow \infty$ a.s. is equivalent to

$$\limsup_{n\to\infty} \max_{k \in \mathscr{V}_i} H^{k,i}_n = \infty \quad \text{a.s.}$$

By using the identity $Q^i_n = \max_{k \in \mathscr{V}_i} H^{k,i}_n$ in the last relation, one gets

$$\limsup_{n\to\infty} \frac{Q^i_n}{n} \ge 0 \quad \text{a.s.}$$

Owing to the ergodic assumption and to Lemma 2.5, this entails

$$\gamma_i = \lim_{n\to\infty} \frac{Q^i_n}{n} \ge 0. \tag{2.9}$$

Therefore the fact that $\forall\ k \in \mathscr{V}_i\colon M^{k,i}_\infty = \infty$ a.s. implies that $\gamma_i \ge 0$. Taking the contrapositive of the above inference, we get that $\gamma_i < 0$ implies $M^{k,i}_\infty < \infty$ a.s. for all $k \in \mathscr{V}_i$. The first part of the theorem is thus proved.

(ii) Assume now that $\gamma_i > 0$. Then

$$\lim_{n\to\infty} \frac{Q_n^i}{n} = \gamma_i > 0 \quad \text{a.s.},$$

which implies

$$\lim_{n\to\infty} \max_{k\in\mathscr{V}_i} M_n^{k,i} = \lim_{n\to\infty} Q_n^i = \infty \quad \text{a.s.}$$

Owing to Lemma 2.4, the last fact implies that $M_\infty^{k,i} = \infty$ a.s. for all $k \in \mathscr{V}_i$. □

THEOREM 2.7. *For all* $1 \le h \le g$, *if* $\max_{1\le i\le h} \gamma_i < 0$, *then* $\max_{k\in\mathscr{V}_i,\, 1\le i\le h} M_\infty^k < \infty$ *a.s. If* $\max_{1\le i\le h}\gamma_i > 0$, *then* $\max_{k\in\mathscr{V}_i,\,1\le i\le h} M_\infty^k = \infty$ *a.s.*

PROOF. The second property follows immediately from the relation

$$M_n^j \ge M_n^{j,i}$$

that holds for all $j \in \mathscr{V}_i$ and $n \ge 0$.

We now prove the first property by induction on h. In view of Theorem 2.6, it is satisfied for $1 \le h \le g_0$, since the relation $M_n^j = M_n^{j,i}$, $n \ge 0$, holds for all i, $1 \le i \le g_0$ and $j \in \mathscr{V}_i$. Assume it is true up to rank $h-1$, where $g_0 < h < g$. From (2.1) and the definition of the communicating classes, we have

$$M_{n+1}^j \circ \theta = \max\left\{ \max_{k\in\pi^j\cap\mathscr{V}_h} \left(M_n^k + l^{k,j}\right), \max_{k\in\pi^j\cap\mathscr{V}_1\cdots\cap\mathscr{V}_{h-1}} \left(M_n^k + l^{k,j}\right)^+ \right\}, \qquad n \ge 0,$$

for all $j \in \mathscr{V}_h$. Let N_n^j, $j \in \mathscr{V}_h$, $n \ge 0$, be defined by $N_0^j = A^j$ and

$$N_{n+1}^j \circ \theta = \max\left\{ \max_{k\in\pi^j\cap\mathscr{V}_h} \left(N_n^k + l^{k,j}\right), A^j\circ\theta\right\}, \qquad n \ge 0,$$

where

$$A^j\circ\theta \quad \max_{k\in\pi^j\cap\mathscr{V}_1\cdots\mathscr{V}_{h-1}} \left(M_\infty^k + l^{k,j}\right)^+. \tag{2.10}$$

From the induction assumption, A^j is a.s. finite for all $j \in \mathscr{V}_h$. It is easily proved by induction that $M_n^j \le N_n^j$ for all $j \in \mathscr{V}_h$ and all $n \ge 0$. Define the vector $\tilde{M}_n^j = N_n^j - A^j$, $j \in \mathscr{V}_h$. It is immediate that $\tilde{M}_n$ satisfies the recursive equation

$$\tilde{M}_{n+1}^j \circ\theta = \max_{k\in\pi^j\cap\mathscr{V}_h} \left(\tilde{M}_n^k + \tilde{l}^{k,j}\right)^+, \qquad n \ge 0, \tag{2.11}$$

with $\tilde{M}_0^j = 0$ and $\tilde{l}^{k,j} = l^{k,j} + A^k - A^j\circ\theta$. If we apply Lemma 2.3 to the recursion (2.11), we get immediately $\tilde{M}_n^j = \max(0, \max_{1\le m\le n} \tilde{H}_m^j)$ with

$$\tilde{H}_m^j = \max_{\{v_s\in\mathscr{V}_h,\, s=0,\dots,m,\, v_0=j,\, v_s\in\pi^{v_{s-1}}\circ\theta^{-s}\}} A^{v_m}\circ\theta^{-m} - A^j + \sum_{s=1}^m l^{v_s, v_{s-1}}\circ\theta^{-s},$$

so that $\tilde{H}_m^j \le \max_{k\in\mathscr{V}_h} A^k\circ\theta^{-m} - A^j + H_m^j$. If $\gamma_h < 0$, we know from Lemmas 2.4 and 2.5 that H_m^j tends a.s. to $-\infty$ like $m\gamma_h$. On the other hand,

$$\lim_{m\to\infty} \frac{\max_{k\in\mathscr{V}_h} A^k \circ \theta^{-m}}{m} = 0 \quad \text{a.s.} \tag{2.12}$$

This property is immediately seen if A^k is integrable for all $k \in \mathscr{V}_h$, and it is proved under the weaker assumption that these random variables are finite and not necessarily integrable in the Appendix. Therefore, under the induction assumption, the hypothesis $\gamma_h < 0$ implies that $\lim_m \tilde{H}_m^j = -\infty$ a.s., which in turn implies that $\tilde{M}_\infty^j < \infty$ a.s., and this concludes the proof since the relation $M_n^j \le A^j + \tilde{M}_n^j$ implies $M_\infty^j \le A^j + \tilde{M}_\infty^j < \infty$ a.s. □

Now define

$$Q_n = \max_{\{v_0,\dots,v_n \in \mathscr{V} | v_s \in \pi^{v_{s-1}} \circ \theta^{-s},\, s=1,\dots,n\}} \sum_{s=1}^{n} l^{v_s, v_{s-1}} \circ \theta^{-s}, \tag{2.13}$$

$$n = 1, 2, \dots .$$

Using the same proof as in Lemma 2.5 allows one to establish the convergence

$$\lim_{n\to\infty} \frac{Q_n}{n} = \lim_{n\to\infty} \frac{E[Q_n]}{n} = \gamma \quad \text{a.s.,} \tag{2.14}$$

where γ is a constant.

COROLLARY 2.8. *We have* $\gamma = \max_{1\le i\le g} \gamma_i$.

PROOF. It is easily seen from (2.6) and (2.13) that for all $1 \le i \le g$, $\gamma \ge \gamma_i$, so that $\gamma \ge \max_{1\le i\le g} \gamma_i$.

Assume that $\gamma > \max_{1\le i\le g} \gamma_i$. Then $\delta = (\max_{1\le i\le g} \gamma_i - \gamma)/2 > 0$. Consider the variables $\{\overline{M}_n^k\}_{n=0}^\infty$, $1 \le k \le K$, defined by (2.1) with the delays $\bar{l}^{j,k}$ in place of $l^{j,k}$, where $\bar{l}^{j,k} = l^{j,k} - \gamma - \delta$, $1 \le j, k \le K$. Let $\bar{\gamma}$ and $\bar{\gamma}_i$, $1 \le i \le g$, be the associated constants as defined by (2.13) and (2.7), respectively. It is easily checked that

$$\max_{1\le i\le g} \bar{\gamma}_i = \max_{1\le i\le g} \gamma_i - \gamma - \delta = \delta > 0.$$

It then follows from Theorem 2.7 that $\max_{1\le k\le K} \overline{M}_n^k$ tends a.s. to ∞ when n goes to ∞. On the other hand, we also have $\bar{\gamma} = -\delta < 0$. Using the same type of arguments as in Theorem 2.6, one can show that $\bar{\gamma} < 0$ implies that $\max_{1\le k\le K} \overline{M}_n^k$ converges a.s. to a finite random variable when n goes to ∞, wherein lies the contradiction. Therefore, it is impossible that $\gamma > \max_{1\le i\le g} \gamma_i$, so that necessarily $\gamma = \max_{1\le i\le g} \gamma_i$. □

3. Existence and uniqueness of stationary solutions. We are now in a position to study the stationary solutions of equation (1.1). We first examine the conditions under which the solution of (1.1) converges weakly. Then we show that (1.1) has a unique finite stationary solution.

3.1. *Stability of the evolution equation.* As usual, we shall understand by stability of (1.1) the weak convergence of the state vector $W_n(Y)$ when n tends to infinity. Using the results of Section 2, it is easy to establish the stability condition of $W_n(0)$. Indeed, it can readily be checked that for all $n \geq 0$,

$$W_n^k(0) = M_n^k \circ \theta^n. \tag{3.1}$$

Consequently, the almost sure convergence of the schema M_n^k to a finite limit when n goes to ∞ translates into the weak convergence of the state variables $W_n^k(0)$.

THEOREM 3.1. *If $\gamma < 0$, then, for all k, $1 \leq k \leq K$, $W_n^k(0)$ converges weakly to a finite random $W_\infty^k(0)$ when n tends to ∞. If $\gamma > 0$, then there exists some k, $1 \leq k \leq K$, such that $W_n^k(0)$ converges a.s. to ∞ when n tends to ∞.*

The following lemma will be the basis for extending the preceding result to the case with arbitrary finite initial condition $Y \in \mathbb{R}^{+^K}$.

LEMMA 3.2. *Assume that $\gamma < 0$. Then for any $Y \in \mathbb{R}^{+^K}$, there exists an a.s. finite positive integer $N(Y)$ such that for all $n \geq N(Y)$, $W_n(Y) = W_n(0)$.*

PROOF. It can easily be checked by induction on n that for all $n \geq 0$, $W_n(Y) \geq W_n(0) \geq 0$. Assume that the statement of the theorem does not hold. Then $W_n(Y) > W_n(0)$ for all $n \geq 0$. For any fixed $n \geq 1$, let $k_n \in \{1, \dots, K\}$ be an index such that $W_n^{k_n}(Y) > W_n^{k_n}(0) \geq 0$. In view of (1.1), there exists an index k_{n-1} such that

$$\begin{aligned} W_n^{k_n}(Y) &= \max_{\{j \in \pi^{k_n} \circ \theta^{n-1}\}} \left(W_{n-1}^j(Y) + l^{j, k_n} \circ \theta^{n-1}\right)^+ \\ &= W_{n-1}^{k_{n-1}}(Y) + l^{k_{n-1}, k_n} \circ \theta^{n-1}. \end{aligned}$$

It is easy to see that necessarily $W_{n-1}^{k_{n-1}}(Y) > W_{n-1}^{k_{n-1}}(0) \geq 0$. If this were not true, we would then have

$$\begin{aligned} W_n^{k_n}(Y) &= W_{n-1}^{k_{n-1}}(Y) + l^{k_{n-1}, k_n} \circ \theta^{n-1} \\ &\leq W_{n-1}^{k_{n-1}}(0) + l^{k_{n-1}, k_n} \circ \theta^{n-1} \\ &\leq \max_{\{j \in \pi^{k_n} \circ \theta^{n-1}\}} \left(W_{n-1}^j(0) + l^{j, k_n} \circ \theta^{n-1}\right)^+ = W_n^{k_n}(0), \end{aligned}$$

and hence, $W_n^{k_n}(Y) \leq W_n^{k_n}(0)$, which would contradict the definition of k_n. Similarly, there exists an index k_{n-2} such that

$$\begin{aligned} W_{n-1}^{k_{n-1}}(Y) &= \max_{\{j \in \pi^{k_{n-1}} \circ \theta^{n-2}\}} \left(W_{n-2}^j(Y) + l^{j, k_{n-1}} \circ \theta^{n-2}\right)^+ \\ &= W_{n-2}^{k_{n-2}}(Y) + l^{k_{n-2}, k_{n-1}} \circ \theta^{n-2} \end{aligned}$$

and $W_{n-2}^{k_{n-2}}(Y) > W_{n-2}^{k_{n-2}}(0) \geq 0$. More generally, one can find a series of indices

k_{n-i}, $i = 1, 2, \ldots, n$, which satisfy the relations

$$W_{n-i+1}^{k_{n-i+1}}(Y) = W_{n-i}^{k_{n-i}}(Y) + l^{k_{n-i}, k_{n-i+1}} \circ \theta^{n-i}.$$

Therefore

$$W_n^{k_n}(Y) = Y^{k_0} + \sum_{i=1}^{n} l^{k_{i-1}, k_i} \circ \theta^{n-i}.$$

Obviously $\sum_{i=1}^{n} l^{k_{i-1}, k_i} \circ \theta^{n-i} \le Q_n \circ \theta^n$, where Q_n is defined by (2.13). Hence

$$W_n^{k_n}(Y) \le Y^{k_0} + Q_n. \tag{3.2}$$

Owing to Lemma 2.3, $(Q_n \circ \theta^n)/n \to \gamma$ when $n \to \infty$. Therefore, under the assumption $\gamma < 0$, (3.2) readily implies that $W_n^{k_n} \to -\infty$ when $n \to \infty$, whence comes the contradiction. □

The stability condition of (1.1) with arbitrarily initial condition Y is a direct consequence of Theorem 3.1 and Lemma 3.2.

THEOREM 3.3. *Let Y be an arbitrary nonnegative real vector in $\mathbb{R}^{+K}$. If $\gamma < 0$, then, for all k, $1 \le k \le K$, $W_n^k(Y)$ converges weakly to the finite random variable $W_\infty^k(0)$ when n tends to ∞. If $\gamma > 0$, then there exists some k, $1 \le k \le K$, such that $W_n^k(Y)$ converges a.s. to ∞ when n tends to ∞.*

3.2. *Existence and uniqueness of stationary solutions.* A sequence of finite nonnegative random variables V_n, $n \in \mathbb{Z}$, is said to be a stationary solution of (1.1) if $V_n = V_0 \circ \theta^n$ for all $n \in \mathbb{Z}$ and if $V = V_0$ satisfies the relation

$$V^k \circ \theta = \max_{\{j \in \pi^k\}} \left(V^j + l^{j,k}\right)^+. \tag{3.3}$$

Theorem 2.7 and Corollary 2.8 together with (2.2) show that the stochastic process $M_\infty \circ \theta^n$ is such a solution when $\gamma < 0$. This existence result is complemented by the following uniqueness property.

THEOREM 3.4. *Assume that $\gamma < 0$. Then M_∞ is the unique solution of* (1.1) *and for any initial condition Y, the sequence $W_n(Y)$ couples in finite time with the stationary sequence $M_\infty \circ \theta^n$.*

PROOF. Assume there is another solution V. From Lemma 3.2, there exists a finite integer $N(V) > 0$ such that for all $n \ge N(V)$,

$$V \circ \theta^n = W_n(V) = W_n(0) \quad \text{a.s.}$$

Using again Lemma 3.2 we obtain another finite integer $N(M_\infty) > 0$ such that for all $n \ge N(M_\infty)$,

$$M_\infty \circ \theta^n = W_n(M_\infty) = W_n(0) \quad \text{a.s.}$$

Hence for all $n \ge N = \max(N(V), N(M_\infty))$,

$$M_\infty \circ \theta^n = V \circ \theta^n \quad \text{a.s.},$$

which immediately implies that $V = M_\infty$ a.s. □

4. The unstable case. In the previous sections, it was established that if the constant γ associated with the equation [cf. Theorem 2.7 and Corollary 2.8] is negative, the equation is stable and has a unique stationary solution. It turns out that several examples of closed queueing network models satisfy an equation of the type (1.1), where the constant γ is strictly positive. An example of this type is provided in Section 5.2.2, where the matrices l_n have nonnegative entries. The aim of the present section is to show that this type of equation can be *stabilized* in the sense that certain increments of its solution can be made stationary. The discussion will be limited to the irreducible case.

4.1. *Stationary increment process.* The object of this section is the stochastic recursive sequence $W_n(Y)$ defined by

$$W_0(Y) = Y,$$
$$W_{n+1}^k(Y) = \max_{\{j \in \pi_n^k\}} \left(W_n^j(Y) + l_n^{j,k} \right), \qquad n \geq 0, 1 \leq k \leq K, \tag{4.1}$$

where the predecessor sets are assumed to be such that there is a single communicating class (see Section 2.2), and where the delays $l^{j,k}$ are assumed to be such that their associated constant γ (see Lemma 2.5) is positive.

This equation will now be transformed by adopting new state variables that satisfy an equation of the same nature, where the random sets π_n^k are all replaced by $\{1, \dots, K\}$. The reason for this transformation will become apparent later on.

Define

$$N = \inf\{n \geq 1 | \forall\, 1 \leq j, k \leq K, \exists\, v_0, v_1, \dots, v_n, v_{n+1} \in \{1, \dots, K\},$$
$$\text{with } v_0 = j, v_{n+1} = k \text{ and } v_i \in \pi_i^{v_i+1}, 0 \leq i \leq n\}. \tag{4.2}$$

LEMMA 4.1. *Under the irreducibility assumption, N is a.s. finite.*

PROOF. Owing to the irreducibility assumption, for all j, k, there exists a sequence $u_0, u_1, \dots, u_m, u_{m+1} \in \{1, \dots, K\}$ with $u_0 = j$, $u_{m+1} = k$, and such that for all $0 \leq i \leq m$, $P[u_i \in \pi^{u_{i+1}}] > 0$. Since $P[j \in \pi^{u_1}] > 0$, the event $\{j \in \pi_n^{u_1}\}$ occurs infinitely often, and there is hence a finite $n_1 \geq 1$ such that $\{j \in \pi_{n_1}^{v_1}\}$. This, together with the assumption $j \in \pi^j$ a.s., imply the existence of a finite sequence $v_0 = v_1 = \cdots = v_{n_1} = j$, $v_{n_1+1} = u_1$, such that $v_i \in \pi_i^{v_{i+1}}$, $0 \leq i \leq n_1$. Repeating this argument m times yields the finiteness of N. □

Let $\{N_n\}_0^\infty$ be the sequence defined by $N_0 = 0$ and

$$N_{n+1} = N_n + N \circ \theta^{N_n}. \tag{4.3}$$

Let Θ be the shift θ^N on $(\Omega, \mathbb{F}, P)$. Throughout Section 4, it will be assumed that Θ is P-invariant and ergodic. In the particular case where the sets π^k are deterministic, N is a constant and this assumption will for instance be satisfied whenever θ^k is P-ergodic for all $k \geq 1$.

For all $n \geq 0$, let L_n be the matrix $L \circ \Theta^n$, where

$$(4.4) \qquad L^{j,k} = \max_{\{v_0, v_1, \dots, v_N, v_{N+1} | v_0 = j, v_{N+1} = k, v_i \in \pi^{v_{i+1}}, 0 \leq i \leq N\}} \sum_{i=0}^{N} l_i^{v_i, v_{i+1}},$$

and let $V_n(Y)$ be defined by $V_n(Y) = W_{N_n}(Y)$. It is easily checked from (4.1) that the state variables V_n satisfy the relation

$$(4.5) \qquad \begin{aligned} &V_0(Y) = Y, \\ &V_{n+1}^k(Y) = \max_{\{1 \leq j \leq K\}} \left(V_n^j(Y) + L_n^{j,k}\right), \qquad n \geq 0, 1 \leq k \leq K. \end{aligned}$$

Owing to Lemma 2.5, $W_n(Y)$ diverges a.s. to ∞ like $n\gamma$ when n tends to ∞, so that the only variables that can be expected to become stationary in the long run are increments of the type $V_{n+1}(Y) - V_n(Y)$. Consider as new state variables the increments

$$(4.6) \qquad R_n^{j,k}(Y) = V_{n+1}^k(Y) - V_n^j(Y), \qquad n \geq 0, 1 \leq j, k \leq K.$$

LEMMA 4.2. *The state variables $R_n^{j,k}(Y)$ satisfy the recursion*

$$(4.7) \qquad R_0^{j,k}(Y) = Z^{j,k}(Y),$$

$$(4.8) \qquad R_{n+1}^{j,k}(Y) = \max_{1 \leq h \leq K} \min_{1 \leq i \leq K} \left(R_n^{i,h}(Y) + L_{n+1}^{h,k} - L_n^{i,j}\right), \qquad n \geq 0, 1 \leq j, k \leq K,$$

where

$$(4.9) \qquad Z^{j,k}(Y) = \max_{1 \leq h \leq K} \left(Y^h + L_0^{h,k} - Y^j\right), \qquad 1 \leq j, k \leq K.$$

PROOF. The proof of (4.7) follows immediately from the definition. For all $n \geq 1$, $1 \leq j, k \leq K$, write

$$\begin{aligned} R_{n+1}^{j,k}(Y) &= \max_{1 \leq h \leq K} \left(V_{n+1}^h(Y) + L_{n+1}^{h,k} - V_{n+1}^j(Y)\right) \\ &= \max_{1 \leq h \leq K} \left(V_{n+1}^k(Y) + L_{n+1}^{h,k} - \max_{1 \leq i \leq K} \left(V_n^i(Y) + L_n^{i,j}\right)\right) \\ &= \max_{1 \leq h \leq K} \min_{1 \leq i \leq K} \left(V_{n+1}^h(Y) - V_n^i(Y) + L_{n+1}^{h,k} - L_n^{i,j}\right). \qquad \square \end{aligned}$$

THEOREM 4.3. *Equation* (4.8) *has a stationary solution in the sense that there exist finite random variables $X^{j,k}$, $1 \leq j, k \leq K$, satisfying the relation*

$$(4.10) \qquad X^{j,k} \circ \Theta = \max_{1 \leq h \leq K} \min_{1 \leq i \leq K} \left(X^{i,h} + L^{h,k} \circ \Theta - L^{i,j}\right), \qquad 1 \leq j, k \leq K.$$

For proving this theorem, the following lemmas will be needed.

LEMMA 4.4. *For all $Y \in \mathbb{R}^{+^K}$, all $1 \leq j, k \leq K$, and all $n \geq 1$, $R_n^{j,k}(Y)$ satisfies the bounds*

$$(4.11) \qquad L_n^{j,k} \leq R_n^{j,k}(Y) \leq \|L_n\| + 2\|L_{n-1}\|,$$

where

$$\|L_n\| = \max_{1\le j,k\le K} |L_n^{j,k}|.$$

PROOF. Assume that for some $n \ge 0$, $R_n^{j,k}(Y) \ge L_n^{j,k}$ for all $1 \le j, k \le K$ [this is true for $n = 0$ in view of (4.9)]. Then

$$\begin{aligned} R_{n+1}^{j,k}(Y) &= \max_{1\le h\le K}\min_{1\le i\le K}\left(R_n^{i,h}(Y) + L_{n+1}^{h,k} - L_n^{i,j}\right) \\ &\ge \min_{1\le i\le K}\left(L_n^{i,j} + L_{n+1}^{j,k} - L_n^{i,j}\right) \qquad (4.12)\\ &= L_{n+1}^{j,k}, \qquad 1 \le j, k \le K, \end{aligned}$$

which completes the proof of the lower bound. As for the upper bound, we have

$$\begin{aligned} R_n^{j,k}(Y) &= V_{n+1}^k(Y) - V_n^j(Y) \\ &= \max_{1\le i\le K}\left(V_n^i(Y) + L_n^{i,k} - V_n^j(Y)\right) \\ &\le \max_{1\le i\le K} L_n^{i,k} + \max_{1\le i\le K}\left(V_n^i(Y) - V_n^j(Y)\right). \end{aligned}$$

Using the lower bound, we get

$$\begin{aligned} V_n^i(Y) - V_n^j(Y) &= \max_{1\le l\le K}\left(V_{n-1}^l(Y) - V_n^j(Y) + L_{n-1}^{l,i}\right) \\ &\le \max_{1\le l\le K}\left(L_{n-1}^{l,i} - L_{n-1}^{l,j}\right), \qquad (4.13) \end{aligned}$$

so that

$$R_n^{j,k}(Y) \le \max_{1\le i\le K} L_n^{i,k} + \max_{1\le i\le K}\max_{1\le l\le K}\left(L_{n-1}^{l,i} - L_{n-1}^{l,j}\right),$$

which concludes the proof of the upper bound. □

Consider now the Loynes schema defined by

$$\begin{aligned} S_0^{j,k} &= L^{j,k}, \qquad 1 \le j, k \le K, \\ S_{n+1}^{j,k}\circ\Theta &= \max_{1\le h\le K}\min_{1\le i\le K}\left(S_n^{i,h} + L^{h,k}\circ\Theta - L^{i,j}\right), \qquad (4.14)\\ & \qquad n \ge 0, 1 \le j, k \le K. \end{aligned}$$

LEMMA 4.5. *For all* $1 \le j, k \le K$, $S_n^{j,k}$ *is nondecreasing in n and*

$$L^{j,k} \le S_n^{j,k} \le \|L\| + 2\|L\circ\Theta^{-1}\|, \qquad n \ge 1. \qquad (4.15)$$

PROOF. The fact that $S_n^{j,k} \ge L^{j,k}$ is obtained by induction. For proving that $S_n^{j,k}$ is increasing, assume $S_n \ge S_{n-1}$ coordinatewise (this property is true for $n = 1$ in view of the preceding remark). Then for all $1 \le j, k \le K$,

$$\begin{aligned} S_{n+1}^{j,k} \circ \Theta &= \max_{1 \le h \le K} \min_{1 \le i \le K} \left(S_n^{i,h} + L^{h,k} \circ \Theta - L^{i,j} \right) \\ &\ge \max_{1 \le h \le K} \min_{1 \le i \le K} \left(S_{n-1}^{i,h} + L^{h,k} \circ \Theta - L^{i,j} \right) \\ &= S_n^{j,k} \circ \Theta. \end{aligned}$$

In order to prove the upper bound in (4.15), we first establish the property that whatever the value of Y,

$$S_n^{j,k} \le R_n^{j,k}(Y) \circ \Theta^{-n}, \qquad n \ge 0, 1 \le j, k \le K. \tag{4.16}$$

The proof is again by induction; the property clearly holds for $n = 0$, in view of (4.9) and of the relation $S_0^{j,k} = L^{j,k}$. Assuming it holds for some $n \ge 0$, we get then:

$$\begin{aligned} S_{n+1}^{j,k} \circ \Theta &= \max_{1 \le h \le K} \min_{1 \le i \le K} \left(S_n^{i,h} + L^{h,k} \circ \Theta - L^{i,j} \right) \\ &\le \max_{1 \le h \le K} \min_{1 \le i \le K} \left(R_n^{i,h}(Y) \circ \Theta^{-n} + L^{h,k} \circ \Theta - L^{i,j} \right) \\ &= R_{n+1}^{j,k}(Y) \circ \Theta^{-n}, \end{aligned}$$

which concludes the proof of (4.16). From (4.16) and (4.11), we get immediately that

$$S_n^{j,k} \le R_n^{j,k}(Y) \circ \Theta^{-n} \le \|L\| + 2\|L \circ \Theta^{-1}\|, \qquad n \ge 1, 1 \le j, k \le K,$$

which concludes the proof of the upper bound. □

PROOF OF THEOREM 4.3. From the preceding lemma, we get that the a.s. $\lim_{n \to \infty} S_n^{j,k} = S_\infty^{j,k}$ exists and is finite for all $1 \le j, k \le K$. Letting n go to ∞ in (4.14) yields

$$S_\infty^{j,k} \circ \Theta = \max_{1 \le h \le K} \min_{1 \le i \le K} \left(S_\infty^{i,h} + L^{h,k} \circ \Theta - L^{i,j} \right), \qquad 1 \le j, k \le K, \tag{4.17}$$

so that S_∞ is a solution of (4.10). □

In general, the solution of (4.10) is not unique. However, S_∞ satisfies the following extremal property.

COROLLARY 4.6. *$S_\infty \in \mathbb{R}^{+K \times K}$ is the smallest solution of* (4.10) *larger than L.*

PROOF. Let $S' \in \mathbb{R}^{+K \times K}$ be an arbitrary solution of (4.10) such that $S'^{j,k} \ge L^{j,k}$. One gets by induction that $S' \ge S_n$ for all $n \ge 0$. Thus, $S' \ge S_\infty$. □

4.2. *Reachability of the stationary regime and uniqueness.* We shall say that a stationary solution S of (4.10) is *reachable* if there exists an initial condition for which the increment process defined in (4.6) coincides with the stationary process defined by S, in the sense that

$$R_n^{j,k}(Y) = S^{j,k} \circ \Theta^n, \qquad n \geq 0, 1 \leq j, k \leq K.$$

Equivalently, S is reachable if and only if the system of equations

$$(4.18) \qquad S^{j,k} = \max_{1 \leq h \leq K} \left(Y^h + L_0^{h,k} - Y^j \right), \qquad 1 \leq j, k \leq K,$$

where the unknown is Y, has a finite solution. Indeed, if such a solution exists, the increment process (4.8) can then be made stationary by adopting Y as initial condition [see (4.9)]. It is the aim of this section to investigate the conditions under which the stationary solution of Theorem 4.3 satisfies these reachability and coupling properties. The proof for coupling is based on the notion of renovating events of Borovkov (see [6]).

For $n \geq 0$, let Φ_n denote the event

$$(4.19) \quad \exists\, A \in \mathbb{R}^K\colon S_n^{j,k} = \max_{1 \leq h \leq K} \left(A^h + L^{h,k} \right) - A^j, \qquad 1 \leq j, k \leq K,$$

with a similar definition for Φ_∞ with S_n replaced by S_∞.

Lemma 4.7. *The events Φ_n satisfy the following properties*:

(i) *For all $n \geq 0$, $\Phi_n \subseteq \Theta^{-1}\Phi_{n+1}$ and $\Phi_\infty \subseteq \Theta^{-1}\Phi_\infty$.*
(ii) *If $P[\Phi_n] > 0$ for some n, then $\limsup_{n \to \infty} \Phi_n = \Omega$ a.s.*
(iii) *If $P[\Phi_n] > 0$ for some n, then $P[\Phi_\infty] = 1$.*

Proof. (i) On Φ_n, there exists a random vector A such that

$$S_n^{j,k} = \max_{1 \leq h \leq K} \left(A^h + L^{h,k} \right) - A^j, \qquad 1 \leq j, k \leq K.$$

Hence, for all $1 \leq j, k \leq K$,

$$\begin{aligned}
S_{n+1}^{j,k} \circ \Theta &= \max_{1 \leq h \leq K} \min_{1 \leq i \leq K} \left(S_n^{i,h} + L^{h,k} \circ \Theta - L^{i,j} \right) \\
&= \max_{1 \leq h \leq K} \min_{1 \leq i \leq K} \left(\max_{1 \leq s \leq K} \left(\left(A^s + L^{s,h} \right) - A^i \right) + L^{h,k} \circ \Theta - L^{i,j} \right) \\
&= \max_{1 \leq h \leq K} \max_{1 \leq s \leq K} \left(A^s + L^{s,h} + L^{h,k} \circ \Theta - \left(\max_{1 \leq i \leq K} A^i + L^{i,j} \right) \right) \\
&= \max_{1 \leq h \leq K} \left(S_n^{1,h} + A^1 + L^{h,k} \circ \Theta - \left(S_n^{1,j} + A^1 \right) \right) \\
&= \max_{1 \leq h \leq K} \left(S_n^{1,h} + L^{h,k} \circ \Theta \right) - S_n^{1,j},
\end{aligned}$$

so that $\Theta^{-1}\Phi_{n+1}$ holds, with $A^j \circ \Theta = S_n^{1,j}$. The same argument shows that $\Phi_\infty \subseteq \Theta^{-1}\Phi_\infty$, so that the event Φ_∞ is of probability 0 or 1 due to the ergodic assumption.

(ii) If $P[\Phi_n] > 0$ for some $n \geq 0$, the ergodic assumption implies that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{m=1}^{k} 1_{\{\Theta^m \Phi_n\}} = P[\Phi_n] > 0 \quad \text{a.s.},$$

so that necessarily

$$\limsup_{m\to\infty} \Theta^m \Phi_n = \Omega \quad \text{a.s.}$$

From (i), for all $m \geq 0$, $\Phi_{m+n} \supseteq \Theta^m \Phi_n$. Thus, the last relation implies that

$$\limsup_{m\to\infty} \Phi_m = \Omega \quad \text{a.s.}$$

(iii) If $S \in \mathbb{R}^{K\times K}$ is such that the equation (4.18) has a solution A, then necessarily $A^h - A^j = S^{j,1} - S^{h,1}$ for all $1 \leq j \leq K$, so that

$$S^{j,k} = \max_{1\leq h\leq K}(S^{j,1} - S^{h,1} + L^{h,k}), \qquad 1 \leq j, k \leq K.$$

Conversely, if S satisfies the last relation, then $A^j = -S^{j,1}$ satisfies (4.18). Hence the set

$$\mathscr{D} = \left\{ S \in \mathbb{R}^{K\times K} \middle| \exists\, A \in \mathbb{R}^K \colon S^{j,k} = \max_{1\leq h\leq K}(A^h + L^{h,k}) - A^j,\ 1 \leq j, k \leq K \right\}$$

can be rewritten as

$$\mathscr{D} = \left\{ S \in \mathbb{R}^{K\times K} \middle| S^{j,k} = \max_{1\leq h\leq K}(S^{j,1} - S^{h,1} + L^{h,k}),\ 1 \leq j, k \leq K \right\}.$$

From this, it is immediate that $\mathscr{D}$ is a closed subset of $\mathbb{R}^{K\times K}$. From (ii), if $P[\Phi_n] > 0$ for some n, then for a.s. all $\omega \in \Omega$, there is a sequence of integers $\{n_k\}_k \uparrow \infty$ such that $\omega \in \Phi_{n_k}$, or equivalently, such that $S_{n_k} \in \mathscr{D}$, for all $k \geq 1$. Since $\mathscr{D}$ is closed, the a.s. limit S_∞ of S_{n_k} when k goes to ∞ is also in $\mathscr{D}$, so that $\omega \in \Phi_\infty$. □

Let Ψ be the event

$$\Psi = \{\exists\, 1 \leq h^* \leq K | L^{h^*,k} \circ \Theta \geq L^{h,k} \circ \Theta + L^{j,h} - L^{j,h^*}, \forall\, 1 \leq h, j, k \leq K\}. \tag{4.20}$$

THEOREM 4.8. *Under the condition* $P[\Psi] > 0$, *the stationary regime defined by* S_∞ *is reachable.*

PROOF. On Ψ, we have

$$\begin{aligned} L^{h^*,k} \circ \Theta &\geq L^{h,k} \circ \Theta + \max_{1\leq j\leq K}(L^{j,h} - L^{j,h^*}) \\ &= L^{h,k} \circ \Theta + \max_{1\leq j\leq K}(L^{j,i} - L^{j,h^*} + L^{j,h} - L^{j,i}) \\ &\geq L^{h,k} \circ \Theta + \max_{1\leq j\leq K}(L^{j,i} - L^{j,h^*}) + \min_{1\leq j\leq K}(L^{j,h} - L^{j,i}), \end{aligned}$$

namely

$$L^{h^*,k} \circ \Theta - \max_{1\leq j\leq K}(L^{j,i} - L^{j,h^*}) \geq L^{h,k} \circ \Theta - \max_{1\leq j\leq K}(L^{j,i} - L^{j,h}) \tag{4.21}$$

for all $1 \le h, i, j, k \le K$. Therefore, on the event Ψ, the relation

$$S_1^{j,k} \circ \Theta = \max_{1 \le h \le K} \Big(L^{h,k} \circ \Theta - \max_{1 \le i \le K} (L^{i,j} - L^{i,h}) \Big)$$
$$= L^{h^*,k} \circ \Theta - \max_{1 \le i \le K} (L^{i,j} - L^{i,h^*})$$

holds. Let $B^j = \max_{1 \le i \le K} (L^{i,j} - L^{i,h^*})$. We have

$$S_1^{j,k} \circ \Theta = L^{h^*,k} \circ \Theta - B^j$$
$$= \max_{1 \le h \le K} (B^h + L^{h,k} \circ \Theta) - B^j,$$

where we have used the definition of Ψ in order to get the last identity. Therefore, $\Psi \subseteq \Theta^{-1}\Phi_1$. If the event Ψ has a positive probability, we then get

$$P[\Phi_1] \ge P[\Psi] > 0,$$

which implies that $P[\Phi_\infty] = 1$, in view of Lemma 4.7. In other words, the condition $P[\Psi] > 0$ entails that the system of equations

$$S_\infty^{j,k} = \max_{1 \le h \le K} Y^h + L^{h,k} - Y^j, \qquad 1 \le j, k \le K, \tag{4.22}$$

has at least one solution. □

THEOREM 4.9. *Under the condition $P[\Psi] > 0$, the stationary sequence $S_\infty \circ \Theta^n$ is the unique stationary solution of* (4.8). *For any initial condition Y, the sequence $R_n(Y)$ couples in finite time with this stationary sequence.*

PROOF. We prove that under the condition $P[\Psi] > 0$, the sequence $R_n(Y)$ admits stationary renovating events of length 2. Let A_n, $n \ge 1$, be the event

$$\begin{aligned} A_n &= \Big\{ \exists\, 1 \le h^* \le K | \forall\, 1 \le h, j, k \le K,\ L_{n+1}^{h^*,k} - \max_{1 \le i \le K} \big(L_n^{i,j} - R_n^{i,h^*}(Y) \big) \\ &\qquad \ge L_{n+1}^{h,k} - \max_{1 \le i \le K} \big(L_n^{i,j} - R_n^{i,h}(Y) \big) \Big\} \\ &= \big\{ \exists\, 1 \le h^* \le K | \forall\, 1 \le h, j, k \le K,\ L_{n+1}^{h^*,k} \\ &\qquad \ge L_{n+1}^{h,k} + \big(V_{n+1}^h(Y) - V_{n+1}^{h^*}(Y) \big) \big\}. \end{aligned} \tag{4.23}$$

On the event A_n, the relation

$$R_{n+1}^{j,k}(Y) = L_{n+1}^{h^*,k} - \max_{1 \le i \le K} \big(L_n^{i,j} - R_n^{i,h^*}(Y) \big)$$

holds for all $1 \le j, k \le K$, so that

$$R_{n+1}^{j,k}(Y) - R_{n+1}^{j,k'}(Y) = L_{n+1}^{h^*,k} - L_{n+1}^{h^*,k'}.$$

Therefore, on A_n,

$$\begin{aligned} R_{n+2}^{j,k}(Y) &= \max_{1 \le i \le K} \big(V_{n+2}^i(Y) + L_{n+2}^{i,k} - V_{n+2}^j(Y) \big) \\ &= \max_{1 \le i \le K} \big(R_{n+1}^{1,i}(Y) - R_{n+1}^{1,j}(Y) + L_{n+2}^{i,k} \big) \\ &= \max_{1 \le i \le K} \big(L_{n+1}^{h^*,i} - L_{n+1}^{h^*,j} + L_{n+2}^{i,k} \big), \end{aligned}$$

which shows that the events A_n are renovating events of length 2. The inequality [cf. (4.13)]

$$V_{n+1}^{h}(Y) - V_{n+1}^{h^*}(Y) \le \max_{1 \le i \le K} \left(L_n^{i,h} - L_n^{i,h^*} \right)$$

entails $\Theta^{-n}\Psi \subseteq A_n$, which proves that these renovating events are included in a stationary sequence $\Theta^{-n}\Psi$ with $P(\Psi) > 0$. This proves that for any initial condition, $R_n(Y)$ couples in finite time with a uniquely defined stationary sequence, in view of [6] (Theorem 1, page 260). Since the process $S_\infty \circ \Theta^n$ can be reached by an appropriate choice of the initial condition (Theorem 4.8), it must coincide with this uniquely defined process. □

5. Queueing theory examples. In this section, we illustrate some simple queueing models where the state variables are described by the evolution equation (1.1).

5.1. *First come first serve queueing networks.* Consider a network of K single-server queues. Several variables are defined on the probability space $(\Omega, \mathbb{F}, P)$: $\tau_n \in \mathbb{R}^+$ is the nth interarrival variable; $m_n \ge 1$ is an integer-valued random variable representing the number of queues visited by customer n; $\{r_n^1, \dots, r^{m_n}\}$, where $r_n^i \in \{1, \dots, K\}$, $1 \le i \le m_n$, is the random route followed by customer n and σ_n^i is the service time (or request) of customer n in the ith queue of its route, $1 \le i \le m_n$. The queueing discipline is first come first serve (FCFS) in the sense that in each queue, the requests brought by customers $0, 1, 2, \dots, n$ must all be completed before any attention is given to those brought by customer $n+1$. This discipline has to be understood *locally* in the sense that it is possible for a specific server to start attending a request brought by customer $n+1$ even though some other servers have not yet completed all the requests brought by customers $0, 1, 2, \dots, n$.

Tandem queueing networks are particular cases of such FCFS networks, where the route is the sequence $(1, 2, \dots, K)$ for all customers. However, more complex systems can be contemplated where the length and the structure of the route may be random, with possible loops and so on. Such a model arises naturally when modeling computer systems that use two phase locking algorithms for keeping the consistency of their data (see [3] and [5]) and certain parallel processing systems ([4]).

For $1 \le k \le K$, let b_n^k (resp., e_n^k) be the index of the first visit (resp., last visit) of customer n to queue k:

$$b_n^k = \min_{1 \le i \le m_n, r_n^i = k} i,$$

$$e_n^k = \max_{1 \le i \le m_n, r_n^i = k} i,$$

with $b_n^k = 0$ and $e_n^k = 0$ if $k \notin \{r_n^1, \dots, r_n^{m_n}\}$, by convention. For k such that $e_n^k \ne 0$, let

$$\pi_n^k = \left\{ r_n^i, 1 \le i \le e_n^k \right\}$$

and for $j \in \pi_n^k$, let

$$l_n^{j,k} = \sum_{i=b_n^j}^{e_n^k} \sigma_n^i.$$

If $e_n^k = 0$, take $\pi_n^k = \{k\}$ and $l_n^{k,k} = 0$, by convention.

Let t_n, $n \geq 0$, be the nth arrival time to the network. The sequence $\{t_n\}$ is defined by the relations $t_0 = 0$ and $\tau_n = t_{n+1} - t_n$. Assume that the network has some initial workload to be cleared before any attention is given to the arriving customers and denote by $Y^k \geq 0$ the value of the initial workload in queue k. More generally, let T_n^k, $n \geq 0$, be the time when queue k has completed its initial workload and all the requests brought to it by customers $0, 1, \dots, n-1$.

LEMMA 5.1. *The state variables* $W_n^k(Y) = (T_n^k - t_n)^+$, $1 \leq k \leq K$, $n \geq 0$, *satisfy the recursion*

$$W_0(Y) = Y,$$
$$W_{n+1}^k(Y) = \max_{\{j \in \pi_n^k\}} \left(W_n^j(Y) + l_n^{j,k} - \tau_n\right)^+. \tag{5.1}$$

PROOF. There are two different cases.

(i) If $e_n^k = 0$, then $T_{n+1}^k = T_n^k$, from the very definition. Using this property together with the relation $(a-b)^+ = a \vee b - b$, one gets

$$\begin{aligned} W_{n+1}^k(Y) &= \left(T_n^k - t_{n+1}\right)^+ = T_n^k \vee t_{n+1} - t_{n+1} \\ &= T_n^k \vee t_n \vee t_{n+1} - t_{n+1} = \left(W_n^k(Y) + t_n\right) \vee t_{n+1} - t_{n+1} \\ &= \left(W_n^k(Y) - \tau_n\right)^+, \end{aligned}$$

which establishes (5.1) in this case.

(ii) If $e_n^k \neq 0$, then, for all $j \in \pi_n^k$, $T_{n+1}^k \geq T_n^j \vee t_n + l_n^{j,k}$. This relation follows from the queueing discipline which implies that the first request of customer n to queue j is attended at the earliest at time $T_n^j \vee t_n$. Since the additional delay due to the migration of customer n along the route $r_n^{b_n^j}, \dots, r_n^{e_n^k}$ cannot take place in less than $l_n^{j,k}$, queue k cannot have completed servicing the last request of customer n before that time. We get hence

$$T_{n+1}^k \geq \max_{\{j \in \pi_n^k\}} \left(T_n^j \vee t_n + l_n^{j,k}\right).$$

On the other hand, a request of customer n arriving in a queue can in no case find another request of customer n waiting in the queue. In other words, for all $1 \leq i \leq m$, either the ith request brought by customer n is attended immediately upon its arrival in queue r_n^i or its service is delayed by the completion of requests brought to this queue by customers $0, 1, \dots, n-1$ (including its initial workload). Let p_n^k be the largest j, $1 \leq j \leq e_n^k$, such that the service of customer n is delayed on queue r_n^j, with $p_n^k = 0$, if no such delay takes place on the route $\{r_n^1, \dots, r_n^{e_n^k}\}$.

If $p_n^k \neq 0$, then, obviously $T_n^{p_n^k} \geq t_n$ and

$$T_{n+1}^k = T_n^{p_n^k} + l_n^{p_n^k, k} = T_n^{p_n^k} \vee t_n + l_n^{p_n^k, k} \leq \max_{\{j \in \pi_n^k\}} \left(T_n^j \vee t_n + l_n^{j,k} \right).$$

If $p_n^k = 0$, then none of the e_n^k first requests brought by customer n has to wait, so that

$$T_{n+1}^k = t_n + l_n^{r_n^1, k} \leq \max_{\{j \in \pi_n^k\}} \left(T_n^j \vee t_n + l_n^{j,k} \right).$$

Therefore, whatever the value of p_n^k, we get from the preceding inequalities that

$$T_{n+1}^k = \max_{\{j \in \pi_n^k\}} \left(T_n^j \vee t_n + l_n^{j,k} \right),$$

so that the relation

$$\begin{aligned} W_{n+1}^k(Y) &= \left(T_{n+1}^k - t_{n+1} \right)^+ \\ &= \max_{\{j \in \pi_n^k\}} \left(T_n^j \vee t_n + l_n^{j,k} - t_n - \tau_n \right)^+ \\ &= \max_{\{j \in \pi_n^k\}} \left(W_n^j(Y) + l_n^{j,k} - \tau_n \right)^+ \end{aligned}$$

follows immediately. □

5.2. *Manufacturing blocking.*

5.2.1. *The open case.* Consider a network of K servers in tandem. The first server has an infinite buffer and is fed by an external arrival stream. There are no intermediate buffers between server k and $k+1$, $1 \leq k \leq K-1$, and a customer having completed its service in server k is blocked there as long as server $k+1$ is not empty (this is the so-called manufacturing blocking mechanism).

Several variables are defined on the probability space $(\Omega, \mathbb{F}, P)$: For $n \geq 0$, $\tau_n \in \mathbb{R}^+$ is the nth interarrival variable and σ_n^k is the service time of the nth customer to enter server k, $1 \leq k \leq K$. Let $Y^k \in \mathbb{R}$ be the time when server k gets free of all its initial workload. For $n \geq 0$ and $1 \leq j, k \leq K$, let

$$\lambda_n^{j,k} = \sum_{i=j}^{k} \sigma_n^i$$

and

$$l_n^j = \lambda_n^{1,j} - \lambda_{n+1}^{1,j-1},$$

with the convention that $\lambda_n^{j,k} = 0$ for all $j > k$. Denote by $t_n, n \geq 1$, the time of the nth external arrival to queue 1 ($t_1 = 0$ and $t_{n+1} = t_n + \tau_n$), by T_n^k the time when customer n leaves server k, and by $W_n^k(Y)$ the quantity $W_n^k(Y) = T_n^k - t_n - \lambda_n^{1,k}$.

LEMMA 5.2. *The state variables* $W_n^k(Y)$, $k = 1, \dots, K$, *satisfy the recursion*

$$W_{n+1}^k(Y) = \max_{\{1 \le j \le k+1\}} \left(W_n^j(Y) + l_n^j - \tau_n\right)^+, \qquad n \ge 0, \tag{5.2}$$

where

$$W_0^k(Y) = Y^k - \lambda_0^{1,k} + \tau_0,$$

and where $W_n^{K+1} = Y^{K+1} = -\infty$ *by convention.*

PROOF. Let $T_0 = Y$. Since server 1 gets free of customer n, $n \ge 1$ (resp., its initial workload), at $T_n^1 \ge 0$ (resp., T_0^1), customer $n+1$, $n \ge 0$ starts its service in server 1 at time $t_{n+1} \vee T_n^1$ and completes it at $t_{n+1} \vee T_n^1 + \sigma_{n+1}^1$. Since server 2 gets free of customer n, $n \ge 1$ (resp., its initial workload) at T_n^2 (resp., T_0^2), customer $n+1$, $n \ge 0$ will hence leave server 1 and start its service in 2 at

$$T_{n+1}^1 = \left(t_{n+1} + \sigma_{n+1}^1\right) \vee \left(T_n^1 + \sigma_{n+1}^1\right) \vee T_n^2.$$

More generally, if customer $n+1$ leaves server $k-1$, $1 < k < K$, at T_{n+1}^{k-1}, then it will complete its service in server k at $(T_{n+1}^{k-1} + \sigma_{n+1}^k)$ and leave server k at

$$T_{n+1}^k = \left(T_{n+1}^{k-1} + \sigma_{n+1}^k\right) \vee T_n^{k+1},$$

while for server K,

$$T_{n+1}^K = \left(T_n^{K-1} + \sigma_{n+1}^K\right).$$

Simple substitutions based on the last three relations then yield

$$T_{n+1}^k = \left(t_{n+1} + \lambda_{n+1}^{1,k}\right) \vee \max_{\{1 \le j \le k+1\}} \left(T_n^j + \lambda_{n+1}^{j,k}\right), \qquad n \ge 0, \tag{5.3}$$

with the convention $T_n^{K+1} = -\infty$. □

5.2.2. *The closed case.* Consider a closed network of K servers in tandem. There are no intermediate buffers between servers. The migration of customers is controlled by manufacturing blocking. When a customer finishes in server k, $0 \le k \le K-1$, it enters server $s(k) = k + 1 \bmod K$ if $s(k)$ is empty. Otherwise it is blocked in k until $s(k)$ is empty. In the sequel, all server indices are understood modulo K.

Let σ_n^k, $n \ge 1$, be the service time of the nth customer to be attended by server k after time 0. Let $M(k)$ denote the initial number of customers in server k at time 0 [$M(k) = 0$ or 1 and $0 < \sum_{k=0}^{K-1} M(k) < K$]. The initial condition is given under the form of the vector $Y \in \mathbb{R}^{+K}$, where Y^k represents the epoch when server k gets free for attending customers. If $M(k+1) = 1$, the epoch Y^k is also assumed to coincide with the time when the initial customer of server $k+1$ becomes available for service.

Let $T_n^k(Y)$, $n \geq 1$, be the time of the nth customer departure from server k and $T_0^k = Y^k$ by convention. For sake of simplicity, $T_n^k(Y)$ will be referred to as T_n^k.

Consider the first service in server k, $0 \leq k \leq K-1$. If $M(k) = 1$, then this service starts at time $T_0^{k-1} \vee T_0^k$ and finishes at time $(T_0^{k-1} + \sigma_1^k) \vee (T_0^k + \sigma_1^k)$. This customer may be blocked until time T_0^{k+1} if $M(k+1) = 0$ and T_1^{k+1} if $M(k+1) = 1$. Thus we obtain

$$T_1^k = \left(T_0^{k-1} + \sigma_1^k\right) \vee \left(T_0^k + \sigma_1^k\right) \vee T_{M(k+1)}^{k+1}, \quad \text{if } M(k) = 1.$$

Similarly, one can see that

$$T_1^k = \left(T_1^{k-1} + \sigma_1^k\right) \vee \left(T_0^k + \sigma_1^k\right) \vee T_{M(k+1)}^{k+1}, \quad \text{if } M(k) = 0.$$

Therefore, we get

$$T_1^k = \left(T_{1-M(k)}^{k-1} + \sigma_1^k\right) \vee \left(T_0^k + \sigma_1^k\right) \vee T_{M(k+1)}^{k+1}, \qquad 0 \leq k \leq K-1.$$

More generally, it can be checked that

$$T_n^k = \left(T_{n-M(k)}^{k-1} + \sigma_n^k\right) \vee \left(T_{n-1}^k + \sigma_n^k\right) \vee T_{n-1+M(k+1)}^{k+1}, \quad n \geq 1, \quad 0 \leq k \leq K-1. \tag{5.4}$$

Let $0 \leq d(k) \leq K-1$ denote the number of initially nonempty servers downstream of server k, excluding k:

$$d(k) = \min\{i | i \geq 0, M(k+i+1) = 0\},$$

that is, $M(k+1) = 1, \dots, M(k+d(k)) = 1$, $M(k+d(k)+1) = 0$.

Suppose that $d(k) > 0$ [i.e., $M(k+1) = 1$]; we get from (5.4) that

$$T_n^j = \left(T_{n-1}^{j-1} + \sigma_n^j\right) \vee \left(T_{n-1}^j + \sigma_n^j\right) \vee T_n^{j+1}, \qquad k < j < k + d(k),$$

$$T_n^{k+d(k)} = \left(T_{n-1}^{k+d(k)-1} + \sigma_n^{k+d(k)}\right) \vee \left(T_{n-1}^{k+d(k)} + \sigma_n^{k+d(k)}\right) \vee T_{n-1}^{k+d(k)+1}.$$

Hence we have the formula

$$T_n^k = \left(T_{n-M(k)}^{k-1} + \sigma_n^k\right) \vee \left(\bigvee_{j=k}^{k+d(k)-1} \left(T_{n-1}^j + \sigma_n^j \vee \sigma_n^{j+1}\right)\right) \vee \left(T_{n-1}^{k+d(k)} + \sigma_n^{k+d(k)}\right) \vee T_{n-1}^{k+d(k)+1}. \tag{5.5}$$

It is readily checked that the above equation also holds when $d(k) = 0$.

Now let $0 \leq u(k) \leq K-1$ be the number of initially empty servers upstream of server k, including k:

$$u(k) = \min\{i | i \geq 0, M(k-i) = 1\},$$

that is, $M(k) = 0, \dots, M(k-u(k)+1) = 0$, $M(k-u(k)) = 1$.

If $u(k) > 0$ [i.e., $M(k) = 0$], then we get from (5.4) that

$$T_n^j = \left(T_n^{j-1} + \sigma_n^j\right) \vee \left(T_n^j + \sigma_n^j\right) \vee T_{n-1}^{j+1}, \qquad k - u(k) < j < k,$$

$$T_n^{k-u(k)} = \left(T_{n-1}^{k-u(k)-1} + \sigma_n^{k-u(k)}\right) \vee \left(T_{n-1}^{k-u(k)} + \sigma_n^{k-u(k)}\right) \vee T_{n-1}^{k-u(k)+1}.$$

After some simple substitutions in the preceding relations, we get

$$(5.6)\quad T_n^{k-1} + \sigma_n^k = \left(T_{n-1}^{k-u(k)-1} + \lambda_n^{k-u(k),k}\right) \vee \left(\bigvee_{j=k-u(k)}^{k} \left(T_{n-1}^{j} + \lambda_n^{j,k}\right)\right),$$

where

$$\lambda_n^{j,k} = \sum_{i=j}^{j+[(k-j) \bmod K]} \sigma_n^i, \qquad 1 \le j,k \le K.$$

Therefore, if $M(k) = 0$, then the relation (5.5) can be rewritten as

$$(5.7)\quad \begin{aligned} T_n^k = {} & \left(T_{n-1}^{k-u(k)-1} + \lambda_n^{k-u(k),k}\right) \vee \left(\bigvee_{j=k-u(k)}^{k-1} \left(T_{n-1}^{j} + \lambda_n^{j,k}\right)\right) \\ & \vee \left(\bigvee_{j=k}^{k+d(k)-1} \left(T_{n-1}^{j} + \sigma_n^j \vee \sigma_n^{j+1}\right)\right) \\ & \vee \left(T_{n-1}^{k+d(k)} + \sigma_n^{k+d(k)}\right) \vee T_{n-1}^{k+d(k)+1}. \end{aligned}$$

This last relation trivially holds when $M(k) = 1$. Let

$$\pi^k = \{k - u(k) - 1, k - u(k), \dots, k, \dots, k + d(k), k + d(k) + 1\}$$

and

$$l_{n-1}^{j,k} = \begin{cases} \lambda_n^{k-u(k),k}, & j = k - u(k) - 1, \\ \lambda_n^{j,k}, & k - u(k) \le j < k, \\ \sigma_n^{k+1} I_{M(k+1)=1} \vee \sigma_n^k, & j = k, \\ \sigma_n^j \vee \sigma_n^{j+1}, & k < j < k + d(k), \\ \sigma_n^{k+d(k)}, & j = k + d(k), \\ 0, & j = k + d(k) + 1. \end{cases}$$

Owing to equation (5.7), we get immediately the final relation:

LEMMA 5.3. *The state variables $T_n^k(Y)$, $k = 1, \dots, K$, satisfy the recursion*

$$(5.8)\quad \begin{aligned} T_0^k(Y) &= Y^k, \\ T_{n+1}^k(Y) &= \max_{\{j \in \pi^k\}} \left(T_n^k(Y) + l_n^{j,k}\right), \qquad n \ge 0. \end{aligned}$$

APPENDIX

For any finite random variable B, we define its positive part $B^+ = \max(B, 0)$ and its negative part B^- by the relation $B = B^+ - B^-$.

LEMMA A.1. *Let C be a finite and nonnegative random variable. Define $B = C \circ \theta^{-1} - C$. If either $E[B^+] < \infty$ or $E[B^-] < \infty$, then $E[B] = 0$, $\lim_n C \circ \theta^n / n = 0$ a.s. and $\lim_n C \circ \theta^{-n}/n = 0$ a.s.*

PROOF. We first consider the case $E[B^+] < \infty$. For all $n > 0$, we have

$$\frac{C \circ \theta^{-n}}{n} = \frac{1}{n} \sum_{i=0}^{n-1} B \circ \theta^{-i} + \frac{C}{n}. \tag{A.1}$$

Since $E[B^+] < \infty$, we get from Birkhoff's theorem for quasi-integrable random variables that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} B \circ \theta^{-i} = E[B] \quad \text{a.s.},$$

where $E[B]$ is $-\infty$ if $E[B^-] = \infty$. We cannot have $E[B^-] = \infty$. If this were the case, by letting n go to ∞ in (A.1), we would get a left-hand side that is nonnegative by assumption, and a right-hand side that tends to $-\infty$ a.s. Therefore, necessarily $E[B^-] < \infty$, and B is hence in L^1. It then follows from [2], page 36, that $E[B] = 0$. Using this in (A.1), we get in turn from Birkhoff's theorem that

$$\lim_{n \to \infty} \frac{C \circ \theta^{-n}}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} B \circ \theta^{-i} + \lim_{n \to \infty} \frac{C}{n} = 0 \quad \text{a.s.}$$

If we have $E[B^-] < \infty$, we can write

$$\frac{C \circ \theta^n}{n} = \frac{1}{n} \sum_{i=1}^{n} (-B) \circ \theta^i + \frac{C}{n},$$

and the proof follows from the same arguments as above. □

PROOF OF (2.12). From (2.10), we get the bound

$$\max_{k \in \mathscr{V}_h} A^k \circ \theta \le \sum_{j \in \mathscr{V}_i, 1 \le i < h} M_\infty^j + \lambda,$$

where $\lambda = \max_{k,j} |l^{k,j}|$. Since λ is integrable, $\lambda \circ \theta^{-n}/n \to 0$ a.s., and it is enough to prove that $M_\infty^j \circ \theta^{-n}/n$ tends to zero a.s. for all $j \in \mathscr{V}_i$, $i < h$, to get the result. For this, we use the relation

$$M_\infty^j \circ \theta \ge M_\infty^j + l^{j,j},$$

which follows from the assumption that $j \in \pi^j$ a.s. Therefore

$$M_\infty^j \circ \theta^{-1} - M_\infty^j \le -l^{j,j} \circ \theta^{-1}.$$

Since $l^{j,j}$ is assumed to be integrable the proof is concluded from Lemma A.1. □

Acknowledgment. The authors would like to thank Professor Serguei Foss of Novosibirsk State University for his useful comments on the paper.

REFERENCES

[1] BACCELLI, F. (1989). Ergodic theory of stochastic decision free Petri nets. In *Proc. of the 28th IEEE COC* 1521–1527 Tampa.

[2] BACCELLI, F. and BRÉMAUD, P. (1987). *Palm Probabilities and Stationary Queues. Lecture Notes in Statist.* **41**. Springer, New York.

[3] BACCELLI, F., COURCOUBETIS, C. A. and REIMAN, M. I. (1987). Construction of the stationary regime of queues with locking. *Stochastic Process. Appl.* **26** 257–265.

[4] BACCELLI, F. and LIU, Z. (1990). On the executions of parallel programs on multiprocessor systems—a queueing theory approach. *J. Assoc. Comput. Mach.* **37**.

[5] BERNSTEIN, P. A. and GOODMAN, N. (1981). Concurrency control in distributed data base systems. *Comput. Surveys* **13** 185–221.

[6] BOROVKOV, A. A. (1984). *Asymptotic Methods in Queueing Theory*. Wiley, New York.

[7] FRANKEN, P., KÖNIG, D., ARNDT, U. and SCHMIDT, V. (1981). *Queues and Point Processes.* Akademie, Berlin.

[8] KINGMAN, J. F. C. (1976). Subadditive processes. *Ecole d'Eté de Probabilités de Saint-Flour V. Lecture Notes in Math.* **539** 165–223. Springer, New York.

[9] LOYNES, R. M. (1962). The stability of queues with non-independent inter-arrival and service times. *Proc. Cambridge Philos. Soc.* **58** 497–520.

INRIA
CENTRE DE SOPHIA ANTIPOLIS
2004, ROUTE DES LUCIOLES
06565 VALBONNE CEDEX
FRANCE