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# ON CHOQUET'S DICHOTOMY OF CAPACITY FOR MARKOV PROCESSES

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Following Choquet, the capacity associated with a Markov process is said to be dichotomous if each compact set $K$ contains two disjoint sets with the same capacity as $K$. In the context of right processes, we prove that the dichotomy of capacity is equivalent to Hunt's hypothesis that semipolar sets are polar. We also show that a weaker form of the dichotomy is valid for any Lévy process with absolutely continuous potential kernel.

**1. Introduction.** In two papers concerning the fine potential theory associated with a "regular" kernel $u(x, y)$, Choquet [3, 4] has remarked that if points are strongly polar in the sense that $u(x, x) = \infty$ for all $x$, then the capacity $C$ associated with $u(x, y)$ is *dichotomous*: For each compact $K$ and each $\varepsilon > 0$ there are disjoint compacts $K_1$ and $K_2$ contained in $K$ such that $C(K_i) \geq C(K) - \varepsilon$, $i = 1, 2$. Unfortunately, the proof of this assertion is only hinted at in [3]. A proof of the dichotomy for the Newtonian capacity was given by Feyel [7]. At about the same time Hansen [12] deduced the dichotomy property in the context of balayage spaces from a detailed study of semipolar sets. (Actually, both of these authors consider an $\varepsilon = 0$ form of the dichotomy.) Hansen showed that if points are polar, then the dichotomy property is equivalent to Hunt's hypothesis:

$(H)$ Semipolar sets are polar.

Subsequently Feyel [8, pages 50–51] extended the result of [7] to cover the case of certain capacities associated with Hunt potential kernels. Also see Bucur and Hansen [2] for a development similar to [12] in the context of standard $H$-cones.

Our object in this note is to give a new proof of Hansen's characterization of the dichotomy of capacity in a very general context. Namely, if $\Gamma$ is the Getoor–Steffens capacity associated with a transient Borel right Markov process and a given excessive measure $m$, then $\Gamma$ is dichotomous if and only if semipolar sets are $m$-polar. It should be noted that when $X$ is a standard process and $m$ is a reference measure relative to which $X$ has a standard dual process, then $\Gamma$ (restricted to compacts) agrees with Hunt's capacity as discussed in [1, Section 6.4].

The dichotomy property is closely related to the notion of *capacitance scissipare* [5], which plays an important role in the theory of semipolar sets.

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Indeed, our proof of the dichotomy relies heavily on a characterization of semipolar sets due to Mokobodzki [14] and developed in Dellacherie, Feyel and Mokobodzki [6].

We also present a related result concerning Lévy processes. Suppose that $X$ is a transient Lévy process in $\mathbb{R}^d$ whose potential kernel is absolutely continuous with respect to Lebesgue measure. Using a result of Zabczyk [15] we prove that if $G$ is a bounded open set then there is a Borel set $B \subset G$ with $\Gamma(B) = \Gamma(G \setminus B) = \Gamma(G)$. This variation on the dichotomy is valid even if $(H)$ fails. In view of the equivalence of $(H)$ to the dichotomy property, this last result lends moral support to Getoor's conjecture that $(H)$ holds for "most" Lévy processes.

**2. Main result.** Throughout the paper we shall work with a Borel right Markov process $X = (X_t, P^x)$. Thus $X$ is a strong Markov process with right continuous paths and Borel measurable transition semigroup $(P_t)$. The state space $E$ of $X$ is homeomorphic to a Borel subset of a compact metric space, and $\mathscr{E}$ denotes the Borel $\sigma$-field on $E$. The potential operator of $X$ is denoted $U = \int_0^\infty P_t\,dt$. We assume that $X$ is transient; this means that there is a strictly positive Borel function $f$ on $E$ such that $Uf$ is bounded. As a rule our notation is consistent with that found in [1].

A measure $m$ on $E$ is excessive provided it is $\sigma$-finite and $mP_t \le m$ for all $t > 0$. Since $X$ is transient, given an excessive measure $m$ there is a sequence of measures $(\mu_n)$ on $E$ such that $\mu_n U \uparrow m$ setwise. The capacity $\Gamma$ associated with $X$ and $m$ is defined by

$$\Gamma(B) = \uparrow \lim_n \mu_n P_B 1, \qquad B \in \mathscr{E}.$$

It is easy to see that $\Gamma$ does not depend on the particular approximating sequence $(\mu_n U)$. The set function $\Gamma\colon \mathscr{E} \to [0,\infty]$ is monotone increasing, strongly subadditive, countably subadditive and

$$(A_n) \subset \mathscr{E},\ A_n \uparrow A \quad \Rightarrow \quad \Gamma(A_n) \uparrow \Gamma(A), \tag{2.1}$$

$$A \in \mathscr{E} \quad \Rightarrow \quad \exists \text{ compacts } K_1, K_2, \ldots \text{ contained in } A \text{ with } \Gamma(K_n) \uparrow \Gamma(A). \tag{2.2}$$

For proofs of these facts see [9, Section 10] and [11]. If, for example, $X$ is Brownian motion in $\mathbb{R}^3$ and $m$ is Lebesgue measure, then $\Gamma$ is the familiar Newtonian capacity.

It should be noted that in the approximation $\mu_n U \uparrow m$ one can always arrange that $\mu_n \ll m$ for all $n$. It follows that

$$T_A = T_B \text{ a.e. } P^m \quad \Rightarrow \quad \Gamma(A) = \Gamma(B). \tag{2.3}$$

A set $B \in \mathscr{E}$ is $m$-polar provided $P^m(T_B < \infty) = 0$. Evidently $B$ is $m$-polar if and only if $\Gamma(B) = 0$. We say that $B \in \mathscr{E}$ is $m$-semipolar if $P^m$ $(X_t \in B$ for uncountably many $t$'s$) = 0$. It is known [10, (6.13)] that a Borel set is $m$-semipolar if and only if it can be written as the union of a Borel semipolar set and an $m$-polar set.

We now introduce the class

$$\mathscr{B} = \{B \in \mathscr{E}: B \text{ is finely closed and } \Gamma(B) < \infty\}.$$

In the "classical" context considered in [1, Section 6.4] every compact set lies in $\mathscr{B}$. Consider now the conditions:

$(H_m)$ Semipolar Borel sets are $m$-polar.

$(D_m)$ For each $B \in \mathscr{B}$ and each $\varepsilon > 0$ there are disjoint compacts $K_1$ and $K_2$ contained in $B$ such that $\Gamma(K_i) > \Gamma(B) - \varepsilon$, $i = 1, 2$.

$(D_m^{\sharp})$ For each $K \in \mathscr{B}$ there are disjoint Borel sets $A, B \subset K$ such that $\Gamma(A) = \Gamma(B) = \Gamma(K)$.

We call a point $x \in E$ *regular* provided $\{x\}^r = \{x\}$. (Recall that if $A \in \mathscr{E}$, then $A^r = \{x \in E: P^x(T_A = 0) = 1\}$ denotes the set of regular points for $A$.) Using (2.2) it is easy to see that $(D_m^{\sharp}) \Rightarrow (D_m)$. Thus our main result (Theorem 1) shows that the three conditions $(H_m)$, $(D_m^{\sharp})$ and $(D_m)$ are equivalent when the set of regular points is $m$-polar. Note that when $m$ is a reference measure, if points are polar then the set of regular points is necessarily empty.

THEOREM 1. (a) *If the set of regular points is m-polar, then* $(H_m)$ *implies* $(D_m^{\sharp})$.
(b) $(D_m)$ *implies* $(H_m)$ *and that singletons are m-polar.*

REMARKS. (a) Using a binary splitting argument (cf. Hansen [12, Section 4]) one can show that if the set of regular points is $m$-polar and $(H_m)$ holds, then the following holds:

$(D_m^{\sharp\sharp})$ For each $K \in \mathscr{B}$ there is a family $\{K_u,\ 0 \le u \le 1\}$ of disjoint Borel subsets of $K$ such that $\Gamma(K_u) = \Gamma(K)$ for all $u$.

The key point here is that the capacity $C$ introduced in the next section "descends" on compact sets (in the Ray topology). In fact, the sets $K_u$ in $(D_m^{\sharp\sharp})$ can be taken to be (Ray) $\mathscr{K}_\sigma$ sets. For a simpler version of $(D_m^{\sharp\sharp})$ see Proposition 2.

(b) To illustrate the gap between points (a) and (b) of Theorem 1, let $X$ be a compound Poisson process on $\mathbb{R}$ whose jump distribution is absolutely continuous. Let $m$ be Lebesgue measure, so that $m$ is excessive but not a reference measure. Evidently singletons are $m$-polar and since each point is regular, $(H_m)$ holds. However, it is easy to see that for any Borel set $B$, $\Gamma(B) > 0$ if and only if $m(B) > 0$. Therefore neither $(D_m^{\sharp})$ nor $(D_m)$ can hold.

(c) The gap in Theorem 1 also raises an interesting open question: Does $(D_m^{\sharp})$ imply that $\{x: x \text{ is regular}\}$ is $m$-polar?

(d) The $q$-subprocess of $X$ has transition semigroup $P_t^q = e^{-qt}P_t$, where $q > 0$ is a constant. Since $m$ is excessive for $X$, it is also excessive for $X^q$, and we have the associated $q$-capacity $\Gamma^q$. The process $X^q$ is always transient, so Theorem 1 applies to $X^q$ and $m$.

**3. Proof of Theorem 1.** Let $X$ be a transient Borel right process and $m$ an excessive measure as in the last section. For the proof of part (a) of Theorem 1 it will be convenient to work with a second capacity $C$ closely related to $\Gamma$. To this end we fix a probability measure $\nu$ equivalent to $m$, and a strictly positive Borel function $g$ such that the excessive function $h = Ug$ is bounded by 1. Such a function $g$ exists since $X$ is transient. We now define

$$C(B) = \nu P_B h = P^\nu\left(\int_{T_B}^\infty g(X_t)\,dt\right), \qquad B \in \mathscr{E}.$$

It is easy to see that $C$ has all the properties ascribed to $\Gamma$ in the last section. Moreover $C(E) = \nu(h) \le \nu(1) = 1$ and, because of (2.3),

$$C(B) = 0 \quad \Leftrightarrow \quad \Gamma(B) = 0, \qquad \forall\, B \in \mathscr{E}.$$

Recall now the *first entry* time $D_B = \inf\{t \ge 0\colon\ X_t \in B\}$ and the associated kernel $H_B(x, dy) = P^x(X_{D_B} \in dy;\ D_B < \infty)$. If $x \notin B \setminus B^r$, then $P^x(D_B = T_B) = 1$ and $H_B(x, \cdot\,) = P_B(x, \cdot\,)$. The excessive measure $m$ charges no semipolar set, so $\nu(B \setminus B^r) = 0$. Therefore,

$$C(B) = \nu H_B h = P^\nu\left(\int_{D_B}^\infty g(X_t)\,dt\right), \qquad B \in \mathscr{E}.$$

Finally, if $A \subset B$ and $C(A) = C(B)$, then clearly $T_A = T_B$ a.s. $P^\nu$. Invoking (2.3) we see that for $A, B \in \mathscr{E}$,

$$A \subset B, \quad C(A) = C(B) \quad \Rightarrow \quad \Gamma(A) = \Gamma(B). \tag{3.1}$$

Our proof of part (a) of Theorem 1 relies on the following special case of a theorem of Dellacherie, Feyel and Mokobodzki. This theorem was proved in [6] in case the state space of $X$ is compact. The extension to the case of a Lusin state space considered here is an easy exercise in the use of the Ray–Knight compactification.

LEMMA 1. *Assume that $\{x \in E\colon x$ is regular$\}$ is $m$-polar. Let $B \subset E$ be a Borel set and suppose there is a finite measure $\mu$ on $E$ such that $\mu(A) = 0$ implies $\Gamma(A) = 0$, for all compact sets $A \subset B$. Then $B$ is $m$-semipolar.*

The following result is the main step in the proof of Theorem 1. For the statement of the proposition we introduce the "capacitary measure" relevant to the capacity $C$:

$$\nu_B(A) = \nu P_B(1_A h) = \nu H_B(1_A h), \qquad A \in \mathscr{E}.$$

Note that $\nu_B$ is carried by the fine closure of $B$ and charges no $m$-polar set.

PROPOSITION 1. *Assume $(H_m)$ and that the set of regular points is $m$-polar. Given $K \in \mathscr{B}$, there is a Borel set $A \subset K$ with $C(A) = C(K)$ and $\nu_K(A) = 0$.*

PROOF. Let $\delta = \sup\{C(B)\colon B \in \mathscr{E},\ B \subset K,\ \nu_K(B) = 0\}$ and choose an increasing sequence of Borel sets $(A_n)$, each contained in $K$, such that $C(A_n) \uparrow \delta$

and $\nu_K(A_n) = 0$ for all $n$. Set $A = \bigcup_n A_n$, so that $C(A) = \delta$ and $\nu_K(A) = 0$. To see that $C(A) = C(K)$ put $B = K \setminus A$, $c = \nu_K(B \cap \{P_A h < h\})$ and compute

$$C(K) = \nu_K(K) = \nu_K(B) = \nu_K(B \cap \{P_A h = h\}) + c.$$

Since $h$ is excessive, so is $P_A h$; hence

$$\nu_K(B \cap \{P_A h = h\}) \le \nu P_K(1_{\{P_A h = h\}} h) \le \nu P_K P_A h \le \nu P_A h = C(A).$$

Thus $C(K) \le C(A) + c$, so we must show that $c = 0$. Since $\nu_K$ does not charge $m$-polars, it is enough to prove that $D = B \cap \{P_A h < h\}$ is $m$-polar. So let us assume that $D$ is not $m$-polar and try to reach a contradiction. Then $(H_m)$ implies that $D$ is not $m$-semipolar, so by (the contrapositive of) Lemma 1 there is a compact set $F \subset D$ with $\nu_K(F) = 0$ and $C(F) > 0$. Clearly $P_A h \le P_{A \cup F} h$, and if the finely open set $\{P_A h < P_{A\cup F} h\}$ were $\nu$-null ($= m$-null), then it would be $m$-polar. But $h = P_{A\cup F} h$ on $F^r$ and $F \subset \{P_A h < h\}$ by construction, so $F^r \subset \{P_A h < P_{A\cup F} h\}$. Also, $F \setminus F^r$ is semipolar, hence $m$-polar because of $(H_m)$; consequently $F^r$ is not $m$-polar. Therefore $\nu(P_A h < P_{A\cup F} h) > 0$, so

$$C(A) = \nu P_A h < \nu P_{A\cup F} h = C(A \cup F).$$

But this contradicts the construction of $A$ since $A \cup F \subset K$ and $\nu_K(A \cup F) = 0$. □

Proof of Theorem 1. (a) Assume $(H_m)$ and that the set of regular points is $m$-polar. Fix $K \in \mathscr{B}$. By Proposition 1, there is a Borel set $A \subset K$ such that $C(A) = C(K)$ and $\nu_K(A) = 0$. In particular, $\Gamma(A) = \Gamma(K)$ by (3.1). Moreover, setting $B = K \setminus A$ we have

$$\begin{aligned} C(B) \le C(K) &= \nu_K(B) = P^\nu\big(h(X_{D_K}); X_{D_K} \in B\big) \\ &\le P^\nu\big(h(X_{D_K}); D_K = D_B\big) = P^\nu\big(h(X_{D_B}); D_K = D_B\big) \\ &\le P^\nu\big(h(X_{D_B})\big) = C(B) \end{aligned}$$

and part (a) is proved.

(b) Assume $(D_m)$. It follows immediately that singletons are $m$-polar. Now suppose that $(H_m)$ fails. Then there is a non-$m$-polar, semipolar set $B$. In fact, since $X$ is transient, a result of Mertens [13] tells us that each semipolar set can be expressed as a countable union of strictly thin sets. Thus we can assume in addition that $B$ is strictly thin: There is a constant $0 < \delta < 1$ such that $P_B 1 \le \delta$ on $B$. Furthermore, replacing $B$ by $B \cap \{Uf \ge b\}$ (where $f > 0$ is $m$-integrable and $b > 0$ is sufficiently small) we can assume that $\Gamma(B) < \infty$. Now given $A \in \mathscr{E}$, define an excessive function $h_A$:

$$h_A(x) = P^x\Big(\sum_{t>0} 1_A(X_t)\Big).$$

Choosing potentials $\mu_n U \uparrow m$, we define a measure $\pi$ by

$$\pi(A) = \uparrow \lim_n \mu_n(h_A), \qquad A \in \mathscr{E}.$$

As in the definition of $\Gamma$, the R.H.S. is independent of the particular approximating sequence $(\mu_n U)$; see, for example, [9, (7.2)]. Since $B$ is strictly thin,

$$h_A = \sum_{n \geq 1} (P_A)^n 1 \leq \sum_{n \geq 0} \delta^n P_A 1 = (1-\delta)^{-1} P_A 1, \qquad \forall A \subset B,$$

so $\pi(A) \leq (1-\delta)^{-1}\Gamma(A) < \infty$ if $A \subset B$. Since $h_A \geq P_A 1$ we therefore have

$$\Gamma(A) \leq \pi(A) \leq (1-\delta)^{-1}\Gamma(A), \qquad \forall A \subset B. \tag{3.2}$$

Now fix an integer $n > (1-\delta)^{-1}$. By repeated application of $(D_m)$, given $\varepsilon > 0$, there are disjoint compacts $K_1, K_2, \dots, K_n$ contained in $B$ such that $\Gamma(K_i) > \Gamma(B) - \varepsilon$ for $i = 1, \dots, n$. Using (3.2) we have

$$\begin{aligned}(1-\delta)^{-1}&\Gamma(B)\\ &\geq \pi(B) \geq \pi\Big(\bigcup_i K_I\Big) = \sum_i \pi(K_i) \geq \sum_i \Gamma(K_i) \geq n(\Gamma(B) - \varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $(1-\delta)^{-1}\Gamma(B) \geq n\Gamma(B)$, which contradicts the choice of $n$ since $0 < \Gamma(B) < \infty$. □

Proposition 1 has other interesting consequences. For example, under the hypotheses of Proposition 1, given a Borel set $B$ and a measure $\mu$ on $E$, one can find a Borel set $A \subset B$ such that $\Gamma(A) = \Gamma(B)$ and $\mu(A) = 0$. This fact is an immediate consequence of the following result whose proof is adapted from [8, pages 51–52].

PROPOSITION 2. *Assume $(H_m)$ and that the set of regular points is $m$-polar. Given $B \in \mathscr{B}$, there is an uncountable collection $\{A_i,\ i \in I\}$ of disjoint Borel subsets of $B$ such that $\Gamma(A_i) = \Gamma(B)$ for all $i \in I$.*

PROOF. It suffices to prove the proposition with $\Gamma$ replaced by $C$. Consider the class of collections $\{A_i\}$ of disjoint Borel subsets of $B$ with $C(A_i) = C(B)$ and $\nu_B(A_i) = 0$ for all $i$. This class is nonempty because of Proposition 1; by Zorn's lemma it has a maximal element $\{A_i,\ i \in I\}$. Suppose that $I$ is countable. Then $A = \bigcup_{i \in I} A_i$ is a Borel set contained in $B$ with $\nu_B(A) = 0$. Let $L = B \setminus A$. Then as in the proof of Proposition 1 we have $C(L) = C(B)$. By Proposition 1 there is a Borel set $F$ contained in $L$ with $C(F) = C(L)$ and $\nu_L(F) = 0$. This forces $\nu_B(F) = 0$; indeed since $F \subset L \subset B$,

$$\begin{aligned}\nu_B(F) &= P^\nu\big(h(X_{D_B});\, X_{D_B} \in F\big) = P^\nu\big(h(X_{D_B});\, X_{D_B} \in F;\, D_B = D_L\big)\\ &\leq \nu_L(F) = 0.\end{aligned}$$

As $F$ is disjoint from each $A_i$, we have contradicted the maximality of $\{A_i,\ i \in I\}$. Thus $I$ is uncountable and the proposition is proved. □

**4. A dichotomy property of Lévy processes.** In this final section we prove a weakened form of the dichotomy property that is true for Lévy processes with absolutely continuous potential kernels. Hypothesis $(H)$ is not assumed. This result could be deduced from a proposition of Feyel [8, page 51], but the direct proof given below has its own interest.

The notation of previous sections is followed in this section, but $X$ is now a transient Lévy process in $\mathbb{R}^d$ and $m$ denotes Lebesgue measure. We assume that $m$ is a reference measure for $X$. Thus the potential kernel takes the form $U(x, dy) = u(y - x)m(dy)$ and the density $u$ is lower semicontinuous. Of course, since $m$ is a reference measure, "$m$-polar" is the same as "polar." Referring to [1, Section 6.4] we see that any bounded set $B$ has a cocapacitary measure $\pi_B$. That is, if $\mu_n U \uparrow m$, then $\mu_n P_B U \uparrow \pi_B U$. The measure $\pi_B$ is carried by $B \cup B^r$ and has total mass equal to $\Gamma(B)$. In what follows, if $A$ is a Borel set then $P_A^1 1(x) = P^x(e^{-T_A})$.

Theorem 2 is based on the following result of Zabczyk [15].

Proposition 3. *There is a non-semipolar $m$-null Borel set.*

Theorem 2. *If $G \subset \mathbb{R}^d$ is a bounded open set, then there is a Borel set $B \subset G$ such that $\Gamma(B) = \Gamma(G \setminus B) = \Gamma(G)$.*

Proof. By Proposition 3 we can choose a non-semipolar set $L$ with $m(L) = 0$. Then $L \cap L^r$ is also non-semipolar. Translating $L$ if necessary, we can assume that $L \cap L^r$ has a non-semipolar intersection with each neighborhood of the origin in $\mathbb{R}^d$. Let $\{x_i\}$ be a countable dense subset of the bounded open set $G$; set $B = \bigcup_i \{x_i + L\} \cap G$. Since $P_L^1 1$ is lower semicontinuous, given $\varepsilon > 0$, there is an open neighborhood $V$ of $L \cap L^r$ on which $P_L^1 1 > 1 - \varepsilon$. But $G = \bigcup_i \{x_i + V\} \cap G$, so $P_B^1 1 > 1 - \varepsilon$ on $G$ because $X$ is translation invariant. It follows that $P_B^1 1 = 1$ on $G$, hence $G \subset B^r$. Consequently the cocapacitary measure $\pi_G$, which is carried by $G \cup G^r$, does not charge $\mathbb{R}^d \setminus B^r$. Thus

$$\Gamma(G) = \pi_G(\mathbb{R}^d) = \pi_G(B^r) \leq \pi_G(P_B 1) \leq \Gamma(B),$$

where the second inequality follows from the inequality $P_G P_B 1 \leq P_B 1$. We have therefore shown that $\Gamma(B) = \Gamma(G)$. As for $G \setminus B$, note that $B \setminus (G \setminus B)^r$ is a finely open subset of $B$ and is therefore empty since $m(B) = 0$. Thus $B \subset (G \setminus B)^r$, so if $K$ is any compact subset of $B$ we have

$$\Gamma(K) = \pi_K(P_{G \setminus B} 1) \leq \Gamma(G \setminus B).$$

Now (2.2) allows us to conclude that $\Gamma(B) \leq \Gamma(G \setminus B)$. Since we have already shown that $\Gamma(B) = \Gamma(G)$, we obtain $\Gamma(B) = \Gamma(G \setminus B)$ as desired. □

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