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ON THE STABILITY OF A POPULATION GROWTH MODEL WITH SEXUAL REPRODUCTION ON $\mathbf{Z}^2$

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In this paper we study a growth model known as the "contact process with sexual reproduction" on $\mathbf{Z}^2$. We focus on the "symmetric" model in which a "child particle" can be produced at a vacant site whenever a pair of its neighboring sites is occupied by "parent particles." Two kinds of stability of the absorbing state ϕ (i.e., the state in which all the sites are vacant) are investigated in this paper. The first kind of stability concerns the behavior of the system when it starts close to the state ϕ. More explicitly, we consider the system starting with a random configuration in which the sites are occupied independently with occupation probability p, where p is a small positive parameter. The system is said to be stable if, for sufficiently small p, the probability that a site is occupied approaches 0 as time approaches infinity. The second kind of stability concerns the behavior of the system under the perturbation of adding a small quantity $\beta > 0$ to all the birth rates ("spontaneous birth at rate β"). In this case, stability means that there is an equilibrium state which is close to ϕ when β is small. It is proven in this paper that in the symmetric model the state ϕ is stable under the first kind of perturbation, but it is unstable under the second kind of perturbation.

0. Introduction. In this paper we investigate a growth model on $\mathbf{Z}^2$ which has sexual reproduction. We write ξ_t for the state of the system at time $t \ge 0$, which is the set of *sites* (points in $\mathbf{Z}^2$) that are *occupied* at time t. Sometimes we will also treat ξ_t as a function from $\mathbf{Z}^2$ to $\{0, 1\}$, with

$$\xi_t(x) = \begin{cases} 1, & \text{if } x \in \xi_t \ (x \text{ is occupied}), \\ 0, & \text{if } x \notin \xi_t \ (x \text{ is vacant}). \end{cases}$$

The system evolves according to the following rules:

(0.1) Occupied sites are vacated at a constant rate $\delta > 0$, that is, if $x \in \xi_t$, then

$$P(x \notin \xi_{t+s} | \xi_t) = \delta s + o(s), \quad \text{as } s \to 0.$$

(0.2) Vacant sites become occupied at rate $b_x(\xi)$, that is, if $x \notin \xi_t$, then

$$P(x \in \xi_{t+s} | \xi_t) = b_x(\xi_t)s + o(s), \quad \text{as } s \to 0.$$

We will call $b_x(\xi)$ the *birth rates* and δ the *death rate*. In particular, the death rate in the systems under our consideration is identically 1, that is, $\delta = 1$. To

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describe the birth rates, we first define for each $x \in \mathbf{Z}^2$, $\mathbf{N}_x = \{y: \|x - y\| = 1\}$ as the *neighbors* of x. In other words, if we let $\mathbf{e}_1 = (1,0)$, and $\mathbf{e}_2 = (0,1)$ be the standard basis vectors in $\mathbf{Z}^2$, then $\mathbf{N}_x = \{x \pm \mathbf{e}_1, x \pm \mathbf{e}_2\}$. For a particular site $x \in \mathbf{Z}^2$, we label its neighboring sites $\{x - \mathbf{e}_1, x - \mathbf{e}_2\}$ as *pair 1*, $\{x + \mathbf{e}_1, x - \mathbf{e}_2\}$ as *pair 2*, $\{x + \mathbf{e}_1, x + \mathbf{e}_2\}$ as *pair 3* and $\{x - \mathbf{e}_1, x + \mathbf{e}_2\}$ as *pair 4*.

We are mainly interested in the following five types of birth rates $b_x(\xi)$:

Type I:	$b_x(\xi) = \lambda$,	if pair 1 is occupied;
Type II(a):	$b_x(\xi) = \lambda$,	if pair 1 or pair 2 is occupied;
Type II(b):	$b_x(\xi) = \lambda$,	if pair 1 or pair 3 is occupied;
Type III:	$b_x(\xi) = \lambda$,	if any one of the pairs i, $i = 1, 2, 3$, is occupied;
Type IV:	$b_x(\xi) = \lambda$,	if any one of the pairs i, $i = 1, 2, 3, 4$, is occupied;

and for all of the preceding types,

$$b_x(\xi) = 0, \quad \text{otherwise}.$$

Notice that, in all these five types of birth rates, in order to produce a child particle at a vacant site x, at least one pair of neighboring sites needs to be occupied by parent particles. That is why these models are said to have sexual reproduction; they are also called the sexual contact processes on $\mathbf{Z}^2$. The type IV model of the system is often called the symmetric model.

Let us take a moment to have a look of the death and birth mechanisms in the contact process on $\mathbf{Z}^2$ with asexual reproduction, or the asexual contact process on $\mathbf{Z}^2$. In that system the death rate is also identically 1 but the birth rate $b_x(\xi)$ is defined as follows:

$$b_x(\xi) = \begin{cases} \lambda, & \text{if any site in } N_x \text{ is occupied;} \\ 0, & \text{otherwise.} \end{cases}$$

It is guaranteed by Liggett's theorem that rules (0.1) and (0.2) specify a unique Markov process [see Liggett (1985), Chapter 1]. Furthermore, all processes introduced previously with either sexual or asexual reproduction can be constructed explicitly by using a graphical representation that goes back to Harris (1978). A detailed construction which is well-suited for our purpose can be found in Durrett and Gray (1990). (We will give a brief description of this construction at the end of this section.) It is a consequence of that construction that there exists a single probability space (Ω, F, P) such that *all* the growth models under consideration in this paper can be defined jointly on (Ω, F, P). This fact enables us to make comparisons between processes with different rates and different initial states. For example, for any given set of rates described by statements (0.1) and (0.2), if we use ξ_t^A and ξ_t^B to denote the states of the system at time t when the initial states are A and B, respectively, we can define the processes ξ_t^A and ξ_t^B on (Ω, F, P) in such a way that if $A \subset B$, then $\xi_t^A \subset \xi_t^B$, for all $t \geq 0$. Also, if ξ_t is a process with birth rates $b_x(\xi)$ and death rate δ, and if ζ_t is another process with death rate $\delta^* \geq \delta$ and birth rate $b_x^*(\xi) \leq b_x(\xi)$, for all $x \in \mathbf{Z}^2$ and $\xi \in S$, then ξ_t and ζ_t can be defined in such a way that $\zeta_t \subset \xi_t$, for all $t \geq 0$, provided both processes have the same initial state. In this case we often simply say that ξ_t *dominates* ζ_t.

A system is called *attractive* if the birth and death rates b_x and d_x satisfy

$$b_x(\xi) \geq b_x(\eta) \quad \text{and} \quad d_x(\xi) \leq d_x(\eta), \quad \text{whenever } \eta \subset \xi \subset \mathbf{Z}^2.$$

In the systems with sexual or asexual reproduction described earlier, the death rates are identically 1 and the birth rates b_x are nondecreasing functions of the number of occupied sites in the set $\mathbf{N}_x$, so the preceding condition is satisfied. It was first shown by Holley (1972) that systems with attractive rates have certain useful monotonicity properties. Let ξ_t^0 and ξ_t^1 denote the state of the system at time t when the initial states are ϕ and $\mathbf{Z}^2$, respectively. Then, for all $A \subset \mathbf{Z}^2$ and $0 \leq s < t < \infty$,

$$P(\xi_t^0 \cap A \neq \phi) \geq P(\xi_s^0 \cap A \neq \phi) \quad \text{and} \quad P(\xi_t^1 \cap A \neq \phi) \leq P(\xi_s^1 \cap A \neq \phi).$$

Thus ξ_t^0 and ξ_t^1 converge weakly ($\Rightarrow$) as $t \to \infty$ to stationary distributions which we denote as ξ_∞^0 and ξ_∞^1, respectively. (We will use the notations ξ_∞^0 and ξ_∞^1 both for random variables which have these distributions as well as for the distributions themselves.) For the sexual contact processes types I–IV and the asexual contact process, we have $\xi_t^0 = \phi$ for all t, hence $\xi_\infty^0 = \delta_\phi$ (the point mass concentrated on the state ϕ). Thus, δ_ϕ is a trivial equilibrium. Let $\rho(\lambda) = \lim_{t\to\infty} P(0 \in \xi_t^1) = P(0 \in \xi_\infty^1)$. If $\rho(\lambda) = 0$, then $\xi_\infty^1 = \xi_\infty^0 = \delta_\phi$, and by attractiveness it follows that, for all initial configurations, $\xi(t) \Rightarrow \delta_\phi$ as $t \to \infty$. On the other hand, if $\rho(\lambda) > 0$, then $\xi_\infty^1 \neq \xi_\infty^0$. Let $\lambda_c = \inf\{\lambda : \rho(\lambda) > 0\}$. Then $\xi_\infty^1 = \xi_\infty^0 = \delta_\phi$ if $\lambda < \lambda_c$ and $\xi_\infty^1 \neq \xi_\infty^0$ if $\lambda > \lambda_c$. In both sexual and asexual contact processes on $\mathbf{Z}^2$, it was proven that $0 < \lambda_c < \infty$. [See, for example, Durrett and Gray (1990).] Therefore, we know that for both sexual and asexual contact processes on $\mathbf{Z}^2$, $\xi_\infty^0 = \delta_\phi$ is a trivial equilibrium for the systems regardless of the value of λ, whereas there exists a critical value $\lambda_c \in (0, \infty)$ such that ξ_∞^1 is nontrivial and distinct from ξ_∞^0 when $\lambda > \lambda_c$. It is natural to investigate the behavior of ξ_t as $t \to \infty$, when $\lambda > \lambda_c$ and ξ_t starts from simple initial distributions other than the ones concentrated at ϕ or $\mathbf{Z}^2$. In this study, we are particularly interested in the case that the initial state is a random configuration in which the sites are independently occupied with probability p, where p is a small positive number. More explicitly, we consider the process ξ_t^p whose initial distribution ξ_0^p satisfies the conditions that the events $\{x \in \xi_0^p\}$, $x \in \mathbf{Z}^2$, are independent and, for each $x \in \mathbf{Z}^2$, $P(x \in \xi_0^p) = p$. This initial distribution can be considered as a perturbation of the absorbing state ϕ. If $\xi_t^p \Rightarrow \delta_\phi$ as $t \to \infty$, we say δ_ϕ is *stable* under perturbation of the initial state; otherwise it is *unstable*. Another kind of perturbation that interests us is to add a small quantity $\beta > 0$ to all birth rates ("spontaneous births at rate β"). Namely, for each previously mentioned type of system, the birth rate for the corresponding new system is equal to $b_x(\xi) + \beta$. Let $\xi_t^{0,\beta}$ and $\xi_t^{1,\beta}$ denote the states at time t for the system with spontaneous births at rate β and initial states ϕ and $\mathbf{Z}^2$, respectively. It is clear that the new systems are still attractive. As we mentioned before, the monotonicity properties of systems with attractive rates imply that as $t \to \infty$, $\xi_t^{0,\beta}$ and $\xi_t^{1,\beta}$ converge to stationary distributions (denoted as) $\xi_\infty^{0,\beta}$ and $\xi_\infty^{1,\beta}$, respectively. The objective is to study the behavior of $\xi_\infty^{0,\beta}$ as $\beta \to 0$. If, as $\beta \to 0$, $\xi_\infty^{0,\beta} \Rightarrow \xi_\infty^0 = \delta_\phi$, then we say δ_ϕ is *stable* under perturbation of birth rate; otherwise it is *unstable*.

It is known that, for systems with asexual reproduction, δ_ϕ is *unstable* under either kind of perturbation [see, e.g., Durrett and Gray (1990) or Durrett (1985)]. For the systems with sexual reproduction, the results concerning the preceding two kinds of stability of δ_ϕ were only established for the type I system by Durrett and Gray (1990). They proved that the following hold for the type I system:

1. There exists a $p^* \in (0,1)$ which is independent of λ, such that if $p < p^*$, then $\xi_t^p \Rightarrow \delta_\phi$ as $t \to \infty$.
2. For any $\lambda > 0$, $\xi_\infty^{0,\beta} \Rightarrow \delta_\phi$ as $\beta \to 0$.

These results mean that in the type I system δ_ϕ is stable under either kind of perturbation.

The proof of statement 1 is a relatively simple application of the theory of oriented percolation, but it does not apply to any of the type II, III or IV systems. The proof of statement 2 uses what is commonly known as a contour argument, and the key to this argument is constructing a "dual process" for ξ_t. [See Durrett and Gray (1990) for details.] Because of the different features of the birth mechanism in the type II, III and IV systems, the dual processes become more complicated and the "contour argument" becomes accordingly more cumbersome (should it still work).

It was conjectured by Durrett (1985) that statements 1 and 2 are still true for the type II, III and IV systems, except that in statement 1 the value of p^* would then depend on λ. He also suggested that in order to solve these problems new ideas and methods are needed. This paper is devoted to studying the stability of δ_ϕ in the type IV system under the two kinds of perturbation described previously.

In Section 1 we study the stability of δ_ϕ in the type IV system under the perturbation of the initial states and obtain the following theorem.

THEOREM 1. *Let ξ_t^p denote the state of the type IV system at time t with the initial distribution ξ_0^p described earlier in this section, that is, the events $\{x \in \xi_0^p\}$, $x \in \mathbf{Z}^2$, are independent and, for each $x \in \mathbf{Z}^2$, $P(x \in \xi_0^p) = p$. For any given $\lambda \in (1, \infty)$, if $p > 0$ is sufficiently small (p may depend on λ), then, for large t,*

$$P(0 \in \xi_t^p) \le t^{-c \log_{2\lambda}(1/p)},$$

where c is a positive constant independent of λ and p.

In the proof of Theorem 1 a new method called successive rescaling or successive block renormalization is employed.

In Section 2 we study the stability of δ_ϕ in the type IV system under perturbation of birth rate and obtain the following theorem.

THEOREM 2. *Let $\xi_t^{0,\beta}$ denote the state of the type IV system at time t with spontaneous birth at rate $\beta > 0$ and initial state ϕ. Suppose that λ is sufficiently large. Then*

$$\lim_{\beta \to 0} P\big(0 \in \xi_\infty^{0,\beta}\big) > 0.$$

Theorems 1 and 2 show that, in the type IV system, δ_ϕ is stable under perturbation of the initial state, but is unstable under perturbation of birth rate. Since the type IV system dominates (in the sense that we described earlier in this section) all four other types of system, it follows that in all four other types of system the δ_ϕ are also stable under perturbation of the initial states. Thus Theorem 1 verifies the conjecture made by Durrett concerning the stability of δ_ϕ under perturbation of the initial states, whereas Theorem 2 disproves his conjecture concerning the stability of δ_ϕ in the type IV system under perturbation of birth rate.

There is a close relationship between the type IV system ξ_t^p and the so-called bootstrap percolation models. In fact, if we let the death rate $\delta = 0$ in ξ_t^p, then ξ_t^p turns out to be essentially the same as a special kind of bootstrap percolation model in $\mathbf{Z}^2$, the only difference being that in the bootstrap percolation model time is discrete. Namely, the bootstrap percolation model can be described as a discrete time process B_t, $t = 0, 1, 2, \dots,$ as follows:

1. Events $\{x \in B_0\}$, $x \in \mathbf{Z}^2$, are independent and $P(x \in B_0) = p$.
2. If $x \in B_{t_0}$ for some $t_0 \in \{0, 1, \dots\}$, then $x \in B_t$ for all $t > t_0$.
3. If $x \notin B_t$, then

$$x \in B_{t+1}, \quad \text{if any one of the pairs } i,\ i = 1,2,3,4, \text{ is occupied},$$

$$x \notin B_{t+1}, \quad \text{otherwise}.$$

Notice that in the system described previously, $B_t \subset B_{t+1}$, for all $t = 0, 1, \dots;$ thus when time goes to infinity there exists a limiting configuration which we will call the *final configuration*.

The general bootstrap percolation models in $\mathbf{Z}^d$, $d \geq 2$, have been studied by Aizenman and Lebowitz (1988). Their results indicate that if, for each $L = 1, 3, \dots,$ we restrict the dynamics to the cube $(-L/2, L/2]^d \cap \mathbf{Z}^d$ (i.e., at time 0 all sites in $\mathbf{Z}^d \setminus (-L/2, L/2]^d$ are vacant), then the density of the final configurations in the sequence of cubes $(-L/2, L/2]^d$ undergoes an abrupt transition, as L is increased, from being close to 0 to being close to 1.

This behavior is quite different from that of the type IV system ξ_t^p. Although the only significant difference between the dynamics of these two models is that there is a positive constant death rate (equal to 1) in the system ξ_t^p, we have shown in Theorem 1 that such a transition does not occur. Several numerical estimates established in Aizenman and Lebowitz (1988) have been found very useful in our proof of Theorem 1.

We now give a brief description of the graphical construction we will use throughout this paper. The details can be found in Durrett and Gray (1990).

For each $x \in \mathbf{Z}^2$, we let $S_n(x)$ and $L_n(x)$, $n \geq 1$, be independent Poisson processes with rates 1 and λ, respectively. Thus, if we let $S_0(x) = L_0(x) = 0$, then the increments $S_n(x) - S_{n-1}(x)$ and $L_n(x) - L_{n-1}(x)$, $n \geq 1$, are independent exponentially distributed random variables with means 1 and $1/\lambda$, respectively. We label certain points in the space–time graph $\mathbf{Z}^2 \times [0, \infty)$, using the following Poisson processes:

(0.3) Mark the points $D_x = \{(x, S_n(x)): n \geq 1\}$ with δ's (for death), and interpret the δ to vacate the site x at time $S_n(x)$ if x is occupied.

(0.4) Mark the points $B_x = \{(x, L_n(x)): n \geq 1\}$ with λ's (for life), and interpret the λ as a birth at the site x at time $L_n(x)$, provided the necessary conditions described in the definition of the birth mechanism for the particular process under construction are met. For example, if the type I process is under construction, then the conditions are

$$x \notin \xi_{L_n(x)^-} \quad \text{and both} \quad x - \mathbf{e}_1 \quad \text{and} \quad x - \mathbf{e}_2 \in \xi_{L_n(x)^-}.$$

Having marked points in the space–time graph, we can compute the evolution of the process according to the rules for interpreting the δ's and λ's given in (0.3) and (0.4). Note that the preceding construction takes care of all the processes without spontaneous birth under our consideration. For the processes with spontaneous births at rate β, we need to augment the construction to allow for the spontaneous births. For each $x \in \mathbf{Z}^2$, we let $U_n(x)$, $n \geq 1$, be a Poisson process with rate β, independent of the process $S_n(x)$ and $L_n(x)$. There is now a third rule in the description of the process, corresponding to spontaneous births at rate β:

(0.5) Mark the points $B_x^* = \{(x, U_n(x)): n \geq 1\}$ with β's (for birth), and interpret the β as a (spontaneous) birth at the site x at time $U_n(x)$ if x is vacant.

The preceding graphical construction also guarantees that the process constructed by using this method is unique [for details, see Durrett and Gray (1990)].

1. Proof of Theorem 1. In this section we will prove Theorem 1. Throughout the section we will assume that $p > 0$ is a sufficiently small real number. As we indicated in the statement of Theorem 1, how small p needs to be depends on the value of λ. We will clearly point out what value of p we should choose in terms of λ. To begin with, we need to introduce some related results concerning the bootstrap percolation model which will be useful later on.

1.1. *Some results concerning the bootstrap percolation model.* As we already indicated in Section 0, the general description of the bootstrap percolation model can be found in Aizenman and Lebowitz (1988). Let us first introduce the following definition employed by Aizenman and Lebowitz. (For our purpose, we will make some minor modifications.)

DEFINITION. Let $\Gamma \subset \mathbf{R}^2$ be a rectangular region. We say that Γ is *internally spanned* (by a configuration η), if $\Gamma \cap \mathbf{Z}^2$ is entirely covered by the final configuration when the initial state is $\eta \cap \Gamma$.

LEMMA 1. *Let Γ be a $w \times h$ rectangular region in the form of*

$$(m - \tfrac{1}{2}, m - \tfrac{1}{2} + w] \times (n - \tfrac{1}{2}, n - \tfrac{1}{2} + h],$$

where n, m, w, h are integers, $w, h \geq 1$. Then

$$P(\Gamma \text{ is internally spanned}) \leq (p(w \wedge h))^{w \vee h}, \tag{1.1}$$

where $w \vee h = \max\{w, h\}$ and $w \wedge h = \min\{w, h\}$.

PROOF. The idea of the proof can be found in the proof of Lemma 2 of Aizenman and Lebowitz (1988). Suppose first that $h \leq w$. We partition Γ into w disjoint strips of unit width parallel to the y axis. A necessary condition for Γ to be internally spanned is that each of those strips contains at least one occupied site. Hence

$$P(\Gamma \text{ is internally spanned}) \leq (ph)^w = (p(w \wedge h))^{w \vee h}.$$

The case $h > w$ can be proved similarly. □

Notice that in obtaining the estimate of $P(\Gamma$ is internally spanned) all we need is the following fact: If we partition Γ into disjoint strips of unit width, parallel to either the x axis or the y axis, each strip contains at least one occupied site. We would like to have a formal definition for this fact as follows.

DEFINITION. Let Γ be a $w \times h$ rectangular region as in Lemma 1. We say Γ is *weakly internally spanned* if both of the following conditions are met:

(a) If Γ is partitioned into h disjoint strips of unit width parallel to the x axis, then each of those strips contains at least one site occupied by ξ_0^p.

(b) If Γ is partitioned into w disjoint strips of unit width parallel to the y axis, then each of those strips contains at least one site occupied by ξ_0^p.

Based on this definition and Lemma 1 we may easily obtain the following corollary.

COROLLARY 1. *Suppose that Γ is a $w \times h$ rectangular region as in Lemma 1. Then*

$$P(\Gamma \text{ is weakly internally spanned}) \leq (p(w \wedge h))^{w \vee h}. \tag{1.2}$$

LEMMA 2. *Let Λ_L denote the square region $(-L/2, L/2]^2$ for $L \in \{1, 3, \dots\}$ as before. For all $k \in \{1, 2, \dots\}$ such that $L \geq k$, a necessary condition for Λ_L to be internally spanned is that it contains at least one internally spanned rectangular region whose longer side length is in the interval $[k, 2k+1]$.*

This is a special case of Lemma 1 of Aizenman and Lebowitz (1988). We will not reproduce its proof here.

1.2. *The procedure of block renormalization.* The first step of our argument is to partition the original lattice $\mathbf{Z}^2$ into a lattice of large blocks and to rescale the lattice in such a way that we regard each block as a site in the new lattice. The rescaled lattice will be called the level 1 lattice, denoted by $\mathbf{Z}^2[1]$, whereas the original lattice will be called the level 0 lattice. Our first major goal is to define "occupancy" and "vacancy" for each level 1 site according to the behavior of the process ξ_t^p, $t \geq 0$. We will define a set of conditions for the occupancy of each level 1 site. These conditions will depend on the initial configuration η of the original process ξ_t^p and on $\omega \in \Omega$. This construction gives us a random configuration on the level 1 lattice which will lead to the bootstrap percolation model on that level. We will then establish the relationship between the original process ξ_t^p and the bootstrap percolation model on level 1. Iterating the same procedure inductively, we will obtain, for all $n = 1, 2, \dots$, a level n bootstrap percolation model and relate it to the original process ξ_t^p. This approach allows us to study the asymptotic behavior of ξ_t^p, $t \geq 0$, by studying the corresponding bootstrap percolation model on the level n lattice instead of the original process itself.

To begin our procedure, we first consider the original lattice $\mathbf{Z}^2$ as a subset of $\mathbf{R}^2$ and partition $\mathbf{R}^2$ into a lattice of $L_1 \times L_1$ squares, where $L_1 = 2\lfloor 1/2p^{\varepsilon} \rfloor - 1$, $\varepsilon = \frac{1}{4}$ and $\lfloor x \rfloor$ = the greatest integer less than or equal to x. We situate these squares so that the origin is in the center of one of the squares. Let Γ be a rectangle of the form

$$((k - \tfrac{1}{2})L_1, (k + \tfrac{1}{2})L_1] \times (y_0 - \tfrac{1}{2}, y_0 - \tfrac{1}{2} + H],$$

where k, y_0, H are integers and $H \geq 1$.

For any given initial configuration η and $\omega \in \Omega$, we call Γ a *short vertical connector* if $H \leq L_1$ and in its partition into L_1 vertical strips of unit width, each strip contains at least one site which is occupied in the configuration η. We call Γ a *long vertical connector* if $H > L_1$ and Γ can be partitioned into j, $j \geq 2$, vertical strips such that each strip S_i is crossed horizontally by a rectangle

$$\Gamma_i = (m - \tfrac{1}{2}, m - \tfrac{1}{2} + w_i] \times (n - \tfrac{1}{2}, n - \tfrac{1}{2} + h_i] \subset \overline{\Gamma}$$

$$= ((k - \tfrac{3}{2})L_1, (k + \tfrac{3}{2})L_1] \times (y_0 - \tfrac{1}{2}, y_0 - \tfrac{1}{2} + H],$$

where m, n, w_i, h_i are integers, $w_i, h_i \geq 1$, such that Γ_i, $i = 1, 2, \dots, j$, are

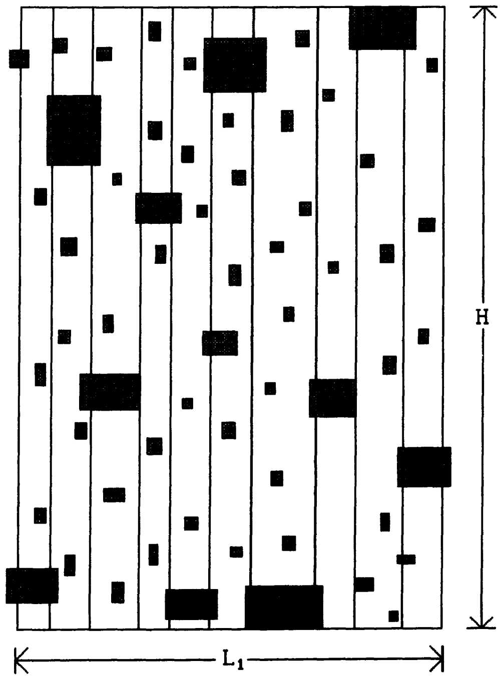

FIG. 1. *A typical long vertical connector.*

disjoint, internally spanned by η and satisfy at least one of the conditions (1.3), or (1.3′):

(1.3) For $i = 1, 2, \dots, j$, let y_i denote the y coordinate of the center of Γ_i, $z_i = y_i - y_{i-1}$, and $d_1 = |z_1| - h_1/2$, $d_i = |z_i| - (h_i/2 + h_{i-1}/2)$, $i = 2, \dots, j$. Let $\sigma_i = \inf\{t\colon \xi_t^{\Gamma_i \cap \mathbf{Z}^2} = \phi\}$ and, for $i = 1, 2, \dots, j-1$, let ζ_t^i denote the asexual contact process with death rate 1 and birth rate λ, confined to the right edge E_i of S_i, that is, E_i is the rightmost vertical $1 \times H$ strip in S_i. Let the initial state of ζ_t^i be $E_i \cap \Gamma_i \cap \mathbf{Z}^2$ and $\tau_i = \inf\{t\colon \zeta_t^i \cap \Gamma_{i+1} \neq \phi\}$. Then, for $i = 1, 2, \dots, j-1$, either $\sigma_{i+1} > d_{i+1}/4\lambda$ or $\tau_i < d_{i+1}/4\lambda$.

(1.3′) For each i let $S_i^T = S_{j-i+1}$, $\Gamma_i^T = \Gamma_{j-i+1}$, $\sigma_i^T = \sigma_{j-i+1}$, $h_i^T = h_{j-i+1}$, $w_i^T = w_{j-i+1}$ and $y_i^T = y_{j-i+1}$. Let $z_1^* = y_1^T - y_0$, $d_1^* = |z_1^*| - h_1^T/2$, $z_i^* = y_i^T - y_{i-1}^T$, $d_i^* = |z_i^*| - (h_i^T/2 + h_{i-1}^T/2)$, $i \geq 2$. For $i = 1, 2, \dots, j-1$, let $\tilde{\zeta}_t^i$ be the asexual process with death rate 1 and birth rate λ, confined to the left edge E_i^T of S_i^T. Let the initial state of $\tilde{\zeta}_t^i$ be $E_i^T \cap \Gamma_i^T \cap \mathbf{Z}^2$ and $\tau_i^* = \inf\{t\colon \tilde{\zeta}_t^i \cap \Gamma_{i+1}^T \neq \phi\}$. Then, for $i = 1, \dots, j-1$, either $\sigma_{i+1}^T > d_{i+1}^*/4\lambda$ or $\tau_i^* < d_{i+1}^*/4\lambda$.

We illustrate a typical vertical connector in Figure 1. The shaded regions represent internally spanned regions. Of course, more than one partition may be possible and condition (1.3) or (1.3′) is not reflected in the picture. Condition (1.3) or (1.3′) means intuitively that the rectangles Γ_i are close enough together so that a connection can be established between them through the spreading of individuals in the population. Condition (1.3) concerns possible spreading from left to right, whereas (1.3′) concerns possible spreading from right to left. We dominate the spreading of individuals from Γ_i to Γ_{i+1} by spreading in the asexual process ζ_t^i, so τ_i may be regarded as a lower bound on the time it takes for individuals to spread from Γ_i to Γ_{i+1} in the sexual process. The quantity d_i is a measure of distance between Γ_{i-1} and Γ_i.

Short and long *horizontal connectors* can be defined similarly by rotating the x-y axes 90°. In both cases we will use "length" L to denote the length of the side perpendicular to the partition strips and "height" H to denote the length of the side parallel to the partition strips. For convenience, in what follows we will discuss vertical connectors only; all conclusions apply to horizontal connectors as well.

For a particular vertical connector

$$\Gamma = ((k - \tfrac{1}{2})L_1, (k + \tfrac{1}{2})L_1] \times (y_0 - \tfrac{1}{2}, y_0 - \tfrac{1}{2} + H],$$

where k, y_0 and H are integers, $H \geq 1$, we call the line segments

$$((k - \tfrac{1}{2})L_1, (k + \tfrac{1}{2})L_1] \times \{y_0 - 1 + H\}$$

and

$$((k - \tfrac{1}{2})L_1, (k + \tfrac{1}{2})L_1] \times \{y_0\}$$

its *upper edge* and *lower edge*, respectively. The *left edge* and *right edge* of a horizontal connector can be defined similarly.

For our purposes, we are only interested in a subset of the set of all connectors. We use the following procedure to eliminate those connectors which do not play a crucial role in our block renormalization procedure.

First of all, we will only consider connectors which are *minimal*, namely, those connectors which do not contain any shorter connectors. Secondly, we want to eliminate overlapping between connectors. To do this, we give an inductive procedure to decide which connectors should be eliminated from a set of overlapping connectors. Let V_0 be the collection of all minimal vertical connectors. Suppose that, for $n \geq 0$, we have defined a set of minimal connectors V_n. Let V_{n+1}^* be the set of connectors in V_n whose lower edge does not intersect any other member of V_n. Let V_{n+1} be the union of V_{n+1}^* together with all those connectors in V_n which do not overlap any member of V_{n+1}^*. This procedure produces a sequence of sets $V_1^* \subset V_2^* \subset \cdots$ of minimal vertical connectors. Let $V^* = \bigcup_{n=1}^{\infty} V_n^*$. It is not difficult to check that the connectors in V_n^* are disjoint for all $n \geq 1$ and hence that the connectors in V^* are disjoint. Similarly (replacing lower edge by right edge), we may apply the same procedure to horizontal connectors and obtain our desired collection H^*.

REMARK (∗). For some initial configurations η and some ω, V^*, H^* could be empty, either because there are no vertical connectors at all or every vertical connector is part of an infinitely long chain of overlapping vertical connectors. For the kind of initial configurations that we will consider, namely, ξ_0^p for $p > 0$, this event has probability 0. The reason for this is that each rectangle with height $H = 1$ crossing a vertical strip $S = ((k - \frac{1}{2})L_1, (k + \frac{1}{2})L_1] \times \mathbf{Z}$ has positive probability of being a connector. By the Borel–Cantelli lemma, each infinite strip S will contain infinitely many such connectors almost surely, so V_0 is not empty, and infinitely long connected strips of overlapping connectors occur with probability 0.

Based on the preceding preliminaries we are now ready to define occupancy and vacancy of each level 1 site for any given initial configuration η and $\omega \in \Omega$. For each level 1 site $x[1] \in \mathbf{Z}^2[1]$, let $\Lambda(x[1])$ denote its corresponding square at level 0. We say $x[1]$ is *occupied* if at least one of the following conditions is met:

(1.4a) $\Lambda(x[1])$ contains the upper edge of a vertical connector in V^*;

(1.4b) $\Lambda(x[1])$ contains the left edge of a horizontal connector in H^*.

Intuitively, conditions (1.4a) and (1.4b) take care of two different situations. First, if in the initial configuration η the square $\Lambda(x[1])$ has a tendency to be filled even without the help of its neighboring squares [for instance, if $\Lambda(x[1])$ itself is internally spanned], then we will consider it as an occupied site in the level 1 lattice [because $\Lambda(x[1])$ is internally spanned implies that it is a short connector]. The second situation concerns those rectangles that may not be capable of filling up on their own, but which may maintain a "well-organized" structure for a relatively long time (for instance, they are long connectors) so that they may possibly be filled by spreading from neighboring squares.

The preceding intuitive statement can be formulated rigorously as follows.

PROPOSITION 1. *Let* $\Gamma = (m - \frac{1}{2}, m - \frac{1}{2} + L_1] \times (n - \frac{1}{2}, n - \frac{1}{2} + h]$ *be a level* 0 *rectangle where* m, n, h *are integers,* $h \geq 1$, *and* L_1 *is as before. Let* $\lambda \in (1, \infty)$. *Restrict the dynamics to the region* $(-\infty, \infty) \times (n - \frac{1}{2}, n - \frac{1}{2} + h]$. *If* Γ *does not contain any vertical connector, then there exists a vertical strip* S *contained in* Γ *such that* S *will be entirely vacated in the time interval* $[0, T]$, *where* $T = \max\{L_1, h\}$.

PROOF. Partition Γ into L_1 disjoint vertical strips with unit width, denoted by S_i, $i = 1, 2, \dots, L_1$. If, for some i, S_i is vacant initially, then by the nature of the process it will be vacant forever, and we are done. Suppose that each such strip S_i contains at least one occupied site initially. Then we can find (at least) a rectangle

$$\Gamma^* = (m - \tfrac{1}{2}, m - \tfrac{1}{2} + L_1] \times (n^* - \tfrac{1}{2}, n^* - \tfrac{1}{2} + h^*]$$

contained in Γ such that Γ^* can be partitioned into j, $j \geq 2$, disjoint vertical strips such that each strip is crossed horizontally by an internally spanned rectangle Γ_i, and Γ_i, $i = 1, 2,, \dots, j$, are disjoint. We call such a Γ^* a potential vertical connector. If $h^* \leq L_1$, then by the definition of a short connector, Γ^*

is actually a short connector and thus we obtain a contradiction. Therefore we may assume that all potential vertical connectors contained in Γ have vertical height greater than L_1.

Let $G^* = \{$all potential vertical connectors contained in $\Gamma\} = \{\Gamma_1^*, \dots, \Gamma_N^*\}$. For each Γ_k^* in G^*, we have a sequence of internally spanned rectangles corresponding to it. Let R be the collection consisting of all those internally spanned rectangles corresponding to $\Gamma_1^*, \dots, \Gamma_N^*$. For each $i = 1, 2, \dots, L_1$, let R_i be a subcollection of R such that R_i consists of all members in R which intersect the unit vertical strip S_i. We claim that there exists (at least) an $i_0 \in \{1, 2, \dots, L_1\}$ such that, for any member $\tilde{\Gamma}$ in R_{i_0}, $\tilde{\Gamma} \cap S_{i_0}$ will be entirely vacated by time T, thus completing the proof.

Proof of the claim. Suppose the opposite. Then there exist disjoint internally spanned rectangles $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_{\tilde{j}}$ in R, $2 \le \tilde{j} \le L_1$, corresponding to some potential vertical connector, such that each of them intersects at least one of the strips S_i, $i = 1, 2, \dots, L_1$, and none of those $\tilde{\Gamma}_i$, $i \in \{1, 2, \dots, \tilde{j}\}$, is vacated entirely by time T. We will prove that this statement implies that the potential vertical connector corresponding to $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_{\tilde{j}}$, $2 \le \tilde{j} \le L_1$, is actually a vertical connector and thus contradicts the hypothesis. To accomplish our proof, we recall the notations d_i, σ_i, τ_i and ζ_t^i defined in the definition of vertical connectors.

Since $T > d_i/4\lambda$, for all $i = 1, \dots, \tilde{j}$, the preceding statement implies that, for $i = 1, \dots, \tilde{j} - 1$, at least one of the following must be the case:

(a) Either $\sigma_{i+1} > d_{i+1}/4\lambda$ or the individuals spreading from $\tilde{\Gamma}_i$ reach $\tilde{\Gamma}_{i+1}$ before time $d_{i+1}/4\lambda$.

(b) Either $\sigma_i > d_{i+1}/4\lambda$ or the individuals spreading from $\tilde{\Gamma}_{i+1}$ reach $\tilde{\Gamma}_i$ before time $d_{i+1}/4\lambda$.

Since the speed of the spreading of the individuals from $\tilde{\Gamma}_i$ to $\tilde{\Gamma}_{i+1}$ is dominated by the asexual contact process ζ_t^i, (a) implies that either $\sigma_{i+1} > d_{i+1}/4\lambda$ or $\tau_i < d_{i+1}/4\lambda$ and hence implies (1.3) in the definition of vertical connectors. A similar argument verifies that (b) implies (1.3′) in the definition of vertical connectors.

The proof of Proposition 1 is complete. □

Remark. The analogous results hold for the level 0 rectangle

$$\Gamma = (n - \tfrac{1}{2}, n - \tfrac{1}{2} + h] \times (m - \tfrac{1}{2}, m - \tfrac{1}{2} + L_1],$$

if the dynamics is restricted to the region $(n - \frac{1}{2}, n - \frac{1}{2} + h] \times (-\infty, \infty)$ and Γ does not contain any horizontal connector.

Now, for any given initial configuration on level 0 and $\omega \in \Omega$ associated with the process ξ_t, we have determined the corresponding initial configuration on level 1, and thus we obtain a bootstrap percolation model defined on the level 1 lattice $\mathbf{Z}^2[1]$. We may consider the rescaled lattice $\mathbf{Z}^2[1]$ as a subset of $\mathbf{R}^2[1]$, which is the two-dimensional Euclidean space with the level 1 length unit. Notice that if for integers $m^{(1)}$, $n^{(1)}$, $w^{(1)}$ and $h^{(1)}$, where $w^{(1)}, h^{(1)} \geq 1$, $\Gamma[1]$ is the $w^{(1)} \times h^{(1)}$ rectangle

$$(m^{(1)} - \tfrac{1}{2}, m^{(1)} - \tfrac{1}{2} + w^{(1)}] \times (n^{(1)} - \tfrac{1}{2}, n^{(1)} - \tfrac{1}{2} + h^{(1)}]$$

at level 1, then it is the $w^{(1)}L_1 \times h^{(1)}L_1$ rectangle (denoted by)

$$\Gamma = \big((m^{(1)} - \tfrac{1}{2})L_1, (m^{(1)} - \tfrac{1}{2} + w^{(1)})L_1\big] \times \big((n^{(1)} - \tfrac{1}{2})L_1, (n^{(1)} - \tfrac{1}{2} + h^{(1)})L_1\big]$$

at level 0. We call Γ the *corresponding region* of $\Gamma[1]$ at level 0. Conversely, if Γ is the $w \times h$ rectangle $(m - \frac{1}{2}, m - \frac{1}{2} + w] \times (n - \frac{1}{2}, n - \frac{1}{2} + h]$ at level 0, then there is a unique minimal $w^{(1)} \times h^{(1)}$ rectangle $(m^{(1)} - \frac{1}{2}, m^{(1)} - \frac{1}{2} + w^{(1)}] \times (n^{(1)} - \frac{1}{2}, n^{(1)} - \frac{1}{2} + h^{(1)}]$ at level 1 denoted by $\Gamma[1]$, such that the corresponding region of $\Gamma[1]$ at level 0 contains Γ, that is, $(w^{(1)} - 1)L_1 < w \leq w^{(1)}L_1$ and $(h^{(1)} - 1)L_1 < h \leq h^{(1)}L_1$. We call $\Gamma[1]$ the *corresponding region* of Γ at level 1. For simplicity, in what follows when we use the phrase "a $w \times h$ rectangle" (at any level) we always mean a $w \times h$ rectangle of the form $(m - \frac{1}{2}, m - \frac{1}{2} + w] \times (n - \frac{1}{2}, n - \frac{1}{2} + h]$ (m, n are integers), as described previously. Although the rectangles described above are $\mathbf{R}^2$ regions and usually we use notation $\Gamma \cap \mathbf{Z}^2$ to describe a rectangle in $\mathbf{Z}^2$, for simplicity sometimes we also use Γ to denote a rectangle in $\mathbf{Z}^2$ in obvious situations.

Notice that at level 1 the occupancy of each site can be due to either condition (1.4a) or (1.4b) (or both). In the case that a level 1 site is occupied due to (1.4a) we say it is V-occupied (occupancy resulting from a vertical connector of level 0), and in the case that it is occupied due to (1.4b) we say it is H-occupied (occupancy resulting from a horizontal connector of level 0).

In terms of the definition of occupancy of sites in $\mathbf{Z}^2[1]$, for each initial configuration $\eta[1]$ we may define a level 1 rectangle $\Gamma[1]$ being internally spanned by $\eta[1]$ in the same manner as in the definition of a level 0 rectangle being internally spanned. We now introduce the definition of a level 1 rectangle being "weakly internally spanned" as follows.

DEFINITION. Let $\Gamma[1]$ be a $w^{(1)} \times h^{(1)}$ level 1 rectangle. We say $\Gamma[1]$ is *weakly internally spanned* at level 1, if both of the following two conditions are met:

(a) If $\Gamma[1]$ is partitioned into $h^{(1)}$ disjoint strips of unit width parallel to the x axis, then each of those strips contains at least one H-occupied site.

(b) If $\Gamma[1]$ is partioned into $w^{(1)}$ disjoint strips of unit width parallel to the y axis, then each of those strips contains at least one V-occupied site.

REMARK. At level 0, a rectangle Γ that is internally spanned is weakly internally spanned, but this is no longer the case at level 1.

DEFINITION. We say a region is *strongly internally spanned* if it is both internally spanned and weakly internally spanned.

We would like to build a link between the original process and the bootstrap percolation model on the level 1 lattice. We first introduce the following definition.

DEFINITION. Let Γ be a $W \times H$ level 0 rectangle. For a given initial configuration η we say Γ is *significantly spanned* by the process $\xi_\cdot^{\eta \cap \Gamma}$ if, for each $t \in [0, T]$, Γ is internally spanned by $\xi_t^{\eta \cap \Gamma}$, where $T = W \vee H$.

NOTE. Consider the bootstrap percolation model and restrict the dynamics to a rectangle Γ. In the final configuration, Γ either is entirely occupied or contains a collection of occupied rectangles strictly contained in Γ which are separated by a collection of vertical and/or horizontal vacant strips with width at least one unit. In particular, we can find a polygonal path of vacant strips connecting the top (left) edge and the bottom (right) edge of Γ (see Figure 2a). We call such a collection of strips a *separation* of Γ. Thus, the phrase "Γ is internally spanned by $\xi_t^{\eta \cap \Gamma}$ for each $t \in [0, T]$" means no separation of Γ occurs in $\xi_t^{\eta \cap \Gamma}$ for $t \in [0, T]$.

Let $\Gamma[1]$ be a level 1 rectangle which is *not* internally spanned. Suppose that in the final configuration $\Gamma[1]$ contains a collection of disjoint, entirely occupied rectangles $\Gamma_1[1], \dots, \Gamma_N[1]$. Let $S_{H(i)}[1]$, $i = 1, 2, \dots, m$, $S_{V(i)}[1]$, $i = 1, 2, \dots, n$, denote all vacant horizontal and vertical strips contained in $\Gamma[1]$, respectively. Let $\mathbf{S_H}[1] = \{S_{H(1)}[1], \dots, S_{H(m)}[1]\}$, $\mathbf{S_V}[1] = \{S_{V(1)}[1], \dots, S_{V(n)}[1]\}$. Suppose without loss of generality that the collection

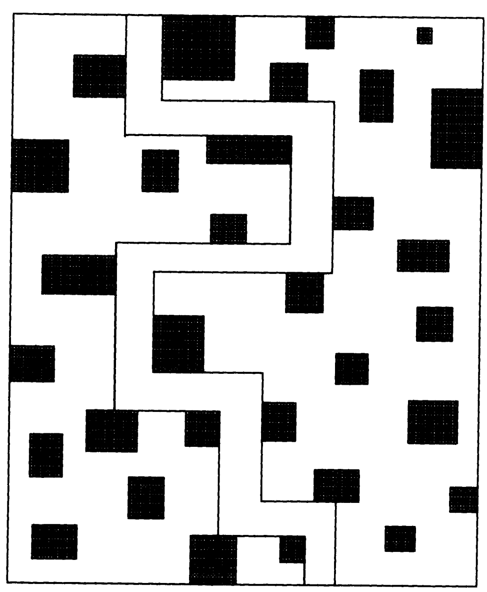

FIG. 2a.

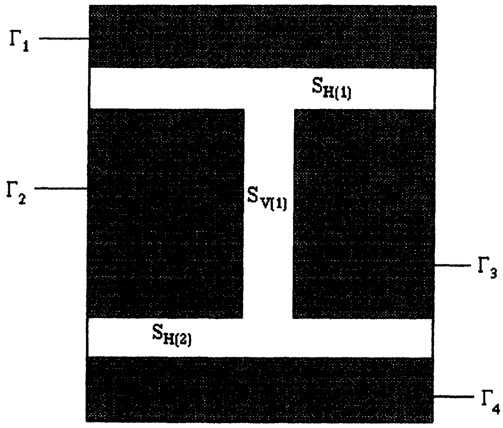

FIG. 2b.

$\{S_{H(1)}[1], \dots, S_{H(j-1)}[1], S_{V(1)}[1], \dots, S_{V(j)}[1]\}$, $1 \le j \le m \wedge n$, forms a polygonal path of vacant strips connecting the upper edge and bottom edge of $\Gamma[1]$. See Figure 2a for an illustration of this.

Let Γ, Γ_i be the corresponding regions of $\Gamma[1], \Gamma_i[1]$ at level 0. For $S_{H(i)}[1] \in \mathbf{S_H}[1]$ and $S_{V(i)}[1] \in \mathbf{S_V}[1]$, we use $S_{H(i)}$ and $S_{V(i)}$ to denote their corresponding strips at level 0, respectively. Let $\mathbf{S_H} = \{S_{H(1)}, \dots, S_{H(m)}\}$, $\mathbf{S_V} = \{S_{V(1)}, \dots, S_{V(n)}\}$. For $\Gamma[1]$ and Γ we claim the following proposition.

PROPOSITION 2. *Suppose the dynamics is restricted to Γ and $\lambda \in (1, \infty)$. Suppose, for $i = 1, \dots, m$, $S_{H(i)}$ does not contain any horizontal connector and, for $i = 1, \dots, n$, $S_{V(i)}$ does not contain any vertical connector. Then, for $i = 1, 2, \dots, j-1$, there is a horizontal substrip $\overline{S}_{H(i)}$ contained in $S_{H(i)}$ and, for $i = 1, 2, \dots, j$, there is a vertical substrip $\overline{S}_{V(i)}$ contained in $S_{V(i)}$ such that $\overline{S}_{H(1)}, \dots, \overline{S}_{H(j-1)}, \overline{S}_{V(1)}, \dots, \overline{S}_{V(j)}$ will be entirely vacated by time T and form a polygonal path of vacant strips which separates Γ at level 0 by time T. Thus Γ is not significantly spanned.*

PROOF. Since $\Gamma[1]$ is not internally spanned at level 1, in the final configuration each of the entirely occupied rectangles $\Gamma_1[1], \dots, \Gamma_N[1]$ is surrounded by vertical and horizontal vacant strips. By the hypotheses, at level 0 each of the rectangles $\Gamma_1, \dots, \Gamma_N$ is surrounded by horizontal strips in $\mathbf{S_H}$ which do not contain any horizontal connector and vertical strips in $\mathbf{S_V}$ which do not contain any vertical connector. Therefore, for each individual strip $S \in \mathbf{S_H} \vee \mathbf{S_V}$ we only need to consider that the dynamics is restricted to R, where R is the smallest rectangle that contains all Γ_i such that Γ_i adjoins the strip S. Without loss of generality we may consider the situation illustrated by Figure 2b, where the shaded regions are regarded as entirely occupied at level 0 when $t = 0$. We want to show that there are horizontal substrips $\overline{S}_{H(i)}$ contained in $S_{H(i)}$, $i = 1, 2$, and vertical substrip $\overline{S}_{V(1)}$ contained in $S_{V(1)}$, such that by time

T, $\bar{S}_{H(1)}$, $\bar{S}_{H(2)}$ and $\bar{S}_{V(1)}$ will all be vacated entirely and form a vacant polygonal path which separates Γ.

The proof is fairly straightforward. First we apply Proposition 1 and directly obtain that there are $\bar{S}_{H(i)}$ contained in $S_{H(i)}$, $i = 1, 2$, such that $\bar{S}_{H(1)}$, $\bar{S}_{H(2)}$ will be vacated entirely by time T, and they will remain vacant forever. That shows that we can ignore the influence from the regions Γ_1 and Γ_4 and restrict the dynamics to R_1, where R_1 is the smallest rectangle that contains the strips $S_{H(1)}$, $S_{H(2)}$ and $S_{V(1)}$. Applying Proposition 1 again we obtain the desired result. □

We are now ready to introduce the following proposition.

PROPOSITION 3. *Let* $\lambda \in (1, \infty)$. *Suppose* Γ *is a* $W \times H$ *rectangle at level* 0, $\Gamma[1]$ *is its corresponding region at level* 1 *with dimension* $W^{(1)} \times H^{(1)}$, $W^{(1)}, H^{(1)} > 2$. *Let* $\hat{\Gamma}[1]$ *be a* $(W^{(1)} - 2) \times (H^{(1)} - 2)$ *rectangle contained in* $\Gamma[1]$ *with the same center as* $\Gamma[1]$. *For almost all initial configurations* $\eta \in \xi_0^p$, *if* Γ *is significantly spanned at level* 0, *then at level* 1 *there exists an internally spanned rectangle* $\tilde{\Gamma}[1]$ *such that* $\hat{\Gamma}[1] \subset \tilde{\Gamma}[1] \subset \Gamma[1]$.

PROOF. Suppose we cannot find such a $\tilde{\Gamma}[1]$. Then there exists a separation at level 1 which separates both $\hat{\Gamma}[1]$ and $\Gamma[1]$. As in Proposition 2, we denote the separation by $\{S_{H(1)}[1], \dots, S_{H(j-1)}[1], S_{V(1)}[1], \dots, S_{V(j)}[1]\}$ and denote the collection of its corresponding strips at level 0 by $\{S_{H(1)}, \dots, S_{H(j-1)}, S_{V(1)}, \dots, S_{V(j)}\}$. We wish to show that, for each $i = 1, 2, \dots, j-1$, there is a horizontal substrip $\bar{S}_{H(i)}$ contained in $S_{H(i)}$ and, for each $i = 1, 2, \dots, j$, there is a vertical substrip $\bar{S}_{V(i)}$ contained in $S_{V(i)}$ such that $\bar{S}_{H(1)}, \dots, \bar{S}_{H(j-1)}$, $\bar{S}_{V(1)}, \dots, \bar{S}_{V(j)}$ will be entirely vacated by time T and form a polygonal path of vacant strips which separates Γ at level 0 by time T. It follows from Proposition 2 that we need only to show that, for each $i = 1, 2, \dots, j-1$, $S_{H(i)}[1]$ does not contain any horizontal connector and, for each $i = 1, 2, \dots, j$, $S_{V(j)}[1]$ does not contain any vertical connector. Since the argument is essentially the same whether the separation consists of one or more than one strip, we may assume without loss of generality that it consists of a single vertical strip of height $H^{(1)}$, denoted by $S[1]$. We may also assume that the width of $S[1]$ is one level 1 unit, and thus the corresponding strip of $S[1]$ at level 0 is an $L_1 \times H^{(1)}L_1$ strip, denoted by S.

It follows from the procedure of our rescaling scheme and the definition of occupancy of a level 1 site that, at level 0, S does not contain any *upper edge* of connectors in V^*. This means, of course, that S does not contain any connector in V^*. We claim that S does not contain any vertical connector at all. Suppose S contains a connector (say, G). Then the only possibility is that G is a member of a string of overlapping connectors whose bottom edge is beyond S. By Remark $(*)$, we may rule out the possibility that the string of connectors consists of infinitely many connectors whose bottom edge goes down infinitely far. Thus G is a member of a string of overlapping connectors whose bottom edge is within a finite distance of S. By the procedure in the definition of V^*, it is not hard to check that when n is sufficiently large either G or one of the connectors (say, G') whose upper edge overlaps with the lower edge of G will be in V_n^* and, thus, in V^*. But S contains *both* the upper and lower edges of G, so S contains the upper edges of both G and G'. Hence S contains the upper edge of some connector in V^*, a contradiction. □

For a level 0 rectangle Γ and its corresponding region $\Gamma[1]$ at level 1, if the corresponding region of $\Gamma[1]$ at level 0 happens to be equal to Γ, then we can improve the result of Proposition 3 and obtain the following corollary.

COROLLARY 2. *Let $\lambda \in (1, \infty)$. Suppose $\Gamma[1]$ is a $W^{(1)} \times H^{(1)}$ rectangle at level 1, Γ is its corresponding region at level 0 with dimension $W^{(1)}L_1 \times H^{(1)}L_1$. For almost all initial configurations $\eta \in \xi_0^p$, if Γ is significantly spanned at level 0, then $\Gamma[1]$ is strongly internally spanned at level 1.*

PROOF. We want to prove that if $\Gamma[1]$ is not strongly internally spanned at level 1, then Γ is not significantly spanned at level 0. If $\Gamma[1]$ is not strongly internally spanned at level 1, then at least one of the following is the case:

(a) There exists a separation $S[1]$ at level 1 which separates $\Gamma[1]$.

(b) If $\Gamma[1]$ is partitioned into disjoint vertical strips with unit width, there is at least one of them that contains no V-occupied site.

(b′) If $\Gamma[1]$ is partitioned into disjoint horizontal strips with unit width, there is at least one of them that contains no H-occupied site.

It is not hard to observe that the argument employed in the proof of Proposition 3 can be applied readily to prove that any of the cases (a), (b) or (b′) implies that Γ is not significantly spanned. □

Our next goal is to estimate the probability that a given level 1 site is occupied. We use A_1 to denote this event. To estimate $P(A_1)$, we need first to evaluate the probability that Γ is a connector, where Γ is a given rectangle which has the form described in the definition of connectors. In order to do so, we first prove the following lemma.

LEMMA 3. *Let $\lambda \in (1, \infty)$. Suppose that L is a sufficiently large integer, Λ is an $L \times L$ square region in $\mathbf{Z}^2$. Let ξ_t^Λ denote the type IV process with initial state Λ and $\sigma_L = \inf\{t: \xi_t^\Lambda = \phi\}$. Then*

$$P\Big(\sigma_L > (2\lambda)^L\Big) \le 3L \exp\left(-\frac{2^{L/2}}{L^2\theta}\right), \tag{1.5}$$

where θ is a positive constant independent of L and λ.

NOTE. For the system with sexual reproduction we always have $\xi_t^\Lambda \cap (\mathbf{Z}^2 \setminus \Lambda) = \phi$ for all $t \ge 0$; hence $\inf\{t: \xi_t^\Lambda = \phi\} = \inf\{t: \xi_t^\Lambda \cap \Lambda = \phi\}$.

PROOF OF LEMMA 3. We partition Λ into L disjoint horizontal unit strips and denote from the top to the bottom the ith strip as S_i, that is, $\Lambda = \bigcup_{i=1}^L S_i$.

By the nature of the birth mechanism, for each i, if S_i is vacated completely, it will never be occupied again. Hence, if we let

$$\sigma_L^{(1)} = \inf\{t: \xi_t^{\Lambda} \cap S_1 = \phi\} \quad \text{and} \quad \sigma_L^{(i)} = \inf\{t: \xi_t^{G_i} \cap S_i = \phi\}, \qquad \text{for } i = 2, \dots, L,$$

where $G_i = \bigcup_{j=i}^{L} S_j$, then by monotonicity and the Markov property it is clear that, for all $t \geq 0$,

$$P(\sigma_L > t) \leq P\left(\bigcup_{i=1}^{L} \left\{\sigma_L^{(i)} > \frac{t}{L}\right\}\right) \leq LP\left(\sigma_L^{(1)} > \frac{t}{L}\right).$$

Let ζ_t^L, $t \geq 0$, denote the one-dimensional asexual contact process defined on the finite set $\{1, 2, \dots, L\}$, with birth rate λ, death rate 1 and initial state $\zeta_0^L = \{1, 2, \dots, L\}$. Then the process $\xi_t^{\Lambda} \cap S_1$ is dominated by the process ζ_t^L. Let $\tau_L = \inf\{t: \zeta_t^L = \phi\}$. Then, for all $s > 0$, $P(\sigma_L^{(1)} > s) \leq P(\tau_L > s)$. It has been shown in the proof of Theorem 2 and Theorem 4 in Durrett and Liu (1988) that when $\lambda > 1$ and L is large, for any given $\gamma > 0$,

$$P(\tau_L > s) \leq [1 - \exp(-(1 + \gamma)L \log \lambda)]^{\lfloor s/L\theta \rfloor},$$

where $\theta > 0$ is a constant independent of L and λ and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Set

$$s = \frac{1}{L}\exp((1 + 2\gamma)L \log \lambda).$$

Then

$$\begin{aligned} &P\left(\tau_L > \frac{1}{L}\exp((1 + 2\gamma)L \log \lambda)\right) \\ &\qquad \leq [1 - \exp(-(1 + \gamma)L \log \lambda)]^{\lfloor (1/L^2\theta)\exp((1+2\gamma)L \log \lambda) \rfloor}. \end{aligned} \tag{1.6}$$

Since $1 - x \leq e^{-x}$, it follows from (1.6) that

$$\begin{aligned} &P\left(\tau_L > \frac{1}{L}\exp((1 + 2\gamma)L \log \lambda)\right) \\ &\quad \leq [\exp\{-\exp(-(1 + \gamma)L \log \lambda)\}]^{\lfloor (1/L^2\theta)\exp((1+2\gamma)L \log \lambda) \rfloor} \\ &\quad = \exp\left\{-\left\lfloor \frac{1}{L^2\theta} \exp((1 + 2\gamma)L \log \lambda) \right\rfloor \exp(-(1 + \gamma)L \log \lambda)\right\} \\ &\quad \leq \exp\left\{-\left(\frac{1}{L^2\theta} \exp((1 + 2\gamma)L \log \lambda) - 1\right)\exp(-(1 + \gamma)L \log \lambda)\right\} \\ &\quad = \exp\left\{-\frac{1}{L^2\theta} \exp(\gamma L \log \lambda) + \exp(-(1 + \gamma)L \log \lambda)\right\} \\ &\quad \leq \exp\left\{-\frac{1}{L^2\theta} \exp(\gamma L \log \lambda) + 1\right\} \\ &\quad \leq 3\exp\left\{-\frac{1}{L^2\theta} \exp(\gamma L \log \lambda)\right\}. \end{aligned}$$

Let $\gamma = (\log_\lambda 2)/2$. Then

$$s = \frac{1}{L}\exp\{(1+2\gamma)L\log\lambda\} = \frac{(2\lambda)^L}{L} \quad \text{and} \quad \gamma L\log\lambda = \frac{L\log 2}{2};$$

hence it follows that

$$P\left(\tau_L > \frac{(2\lambda)^L}{L}\right) \le 3\exp\left(-\frac{2^{L/2}}{L^2\theta}\right).$$

Hence

$$P\left(\sigma_L^{(1)} > \frac{(2\lambda)^L}{L}\right) \le P\left(\tau_L > \frac{(2\lambda)^L}{L}\right) \le 3\exp\left(-\frac{2^{L/2}}{L^2\theta}\right).$$

Therefore

$$P\left(\sigma_L > (2\lambda)^L\right) \le LP\left(\sigma_L^{(1)} > \frac{(2\lambda)^L}{L}\right) \le 3L\exp\left(-\frac{2^{L/2}}{L^2\theta}\right). \qquad \square$$

Now we are ready to prove the following proposition.

PROPOSITION 4. *Let λ be a given fixed number such that $\lambda \in (1,\infty)$. Suppose that Γ is a rectangle of the form*

$$((k-\tfrac{1}{2})L_1, (k+\tfrac{1}{2})L_1] \times (y_0 - \tfrac{1}{2}, y_0 - \tfrac{1}{2} + H],$$

where k, y_0, H are integers, $H \ge 1$, $L_1 = 2\lfloor 1/2p^\varepsilon \rfloor - 1$, $\varepsilon = \frac{1}{4}$, as described in the definition of connectors.

(a) *If $H \le L_1$, then*

$$P(\Gamma \text{ is a (short) vertical connector}) \le \exp\left\{-(1-\varepsilon)L_1\log\frac{1}{p}\right\}. \tag{1.7}$$

(b) *If $H > L_1$, then there exists a $p_0(\lambda) > 0$ such that*

$$\begin{aligned} &P(\Gamma \text{ is a (long) vertical connector}) \\ &\quad \le (1 + O(p^{1/4}))L_1^4 2^{L_1}\frac{1}{H^2}\exp\left(-(1-2\varepsilon)L_1\log\frac{1}{9p}\right), \end{aligned} \tag{1.8}$$

whenever $p < p_0(\lambda)$.

In what follows we will use C to denote a positive constant whose value may change from line to line.

PROOF OF PROPOSITION 4. (a) This case is relatively simple. Denote the probability in (1.7) as P_1. By the definition of short connector and the same arguments as we applied in the proof of Lemma 1, we have

$$P_1 \le (pH)^{L_1} \le (pL_1)^{L_1} = \left(p\left(2\left\lfloor \frac{1}{2p^\varepsilon}\right\rfloor - 1\right)\right)^{L_1}$$

$$\le (p^{1-\varepsilon})^{L_1} = \exp\left(-L_1 \log \frac{1}{p^{1-\varepsilon}}\right) \tag{1.9}$$

$$= \exp\left\{-(1-\varepsilon)L_1 \log \frac{1}{p}\right\}.$$

(b) We consider a specific partition π of Γ, $\pi = \{k_1, \dots, k_j\}$, $k_1 + \cdots + k_j = L_1$ and $j \ge 2$. Let S_i be the corresponding vertical strips. We will sum, over all possible rectangles Γ_i, the probability that the condition (1.3) or (1.3′) is met. We only need to sum over rectangles Γ_i whose right edge is contained in the right edge of S_i, except for the rightmost strip S_j. Throughout our estimate we will only deal with the case that the condition (1.3) is met; the argument applies to the case that the condition (1.3′) is met equally well.

For each $i \in \{1, 2, \dots, j\}$, recall that w_i, h_i denote the horizontal width and vertical height of Γ_i which crosses S_i. Let $\mu_i = h_i \wedge w_i$, $M_i = h_i \vee w_i$. Then $1 \le \mu_i \le 3L_1 - 1$, $k_i \le M_i \le H \vee 3L_1$. Let $\tilde{z} = (z_1, z_2, \dots, z_j)$, $\tilde{w} = (w_1, \dots, w_j)$ and $\tilde{h} = (h_1, \dots, h_j)$, and let x_j be the x coordinate of the center of Γ_j. Given y_0, $\tilde{z}$, $\tilde{w}$, $\tilde{h}$ and x_j, the rectangles $\Gamma_i = \Gamma_i(\tilde{z}, \tilde{w}, \tilde{h}, x_j)$ in a vertical connector are determined. Let

$$B\left(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j\right)$$
$$= \left\{\Gamma_i\left(\tilde{z}, \tilde{w}, \tilde{h}, x_j\right) \text{ is internally spanned and satisfies condition } (1.3)\right\}$$

and denote

$$P_2 = P(\Gamma \text{ is a (long) vertical connector}).$$

Then

$$P_2 \le \sum_{\pi} \sum_{(\tilde{z}, \tilde{w}, \tilde{h}, x_j) \in K} P\left(\bigcap_{i=1}^{j} B\left(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j\right)\right),$$

where π runs over all possible partitions of L_1 with constraint $k_i \ge 1$, and K is the set which consists of all possible choices of $(\tilde{z}, \tilde{w}, \tilde{h}, x_j)$ such that

$$\left(k + \frac{1}{2}\right)L_1 < x_j < \left(k + \frac{3}{2}\right)L_1, \quad 1 \le \mu_i \le 3L_1 - 1, \quad k_i \le M_i \le H \vee 3L_1,$$

$$\frac{h_1}{2} \le z_1 \le H - \frac{h_1}{2} \quad \text{and} \quad \left(\frac{h_i}{2} + \frac{h_{i-1}}{2}\right) \le |z_i| \le H - \left(\frac{h_i}{2} + \frac{h_{i-1}}{2}\right),$$
$$\text{for } i = 2, \dots, j,$$

and the rectangles Γ_i determined by $(\tilde{z}, \tilde{w}, \tilde{h}, x_j)$ are disjoint.

Since, for $(\tilde{z}, \tilde{w}, \tilde{h}, x_j) \in K$, the events $B(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j)$ are increasing and must happen disjointly, it follows from the van den Berg–Kesten inequality that, when $(\tilde{z}, \tilde{w}, \tilde{h}, x_j) \in K$,

$$P\left(\bigcap_{i=1}^{j} B\left(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j\right)\right) \le \prod_{i=1}^{j} P\left(B\left(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j\right)\right).$$

[See van den Berg and Kesten (1985) for meaning of the words "happen disjointly" and for the details of the van den Berg–Kesten inequality.] By translation invariance, for each i, $P(B(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j))$ depends only on d_i, w_i and h_i. Thus we may define $q(d_i, w_i, h_i) = P(B(i, \tilde{z}, \tilde{w}, \tilde{h}, x_j))$. Therefore,

$$\begin{aligned} P_2 &\le \sum_{\pi} \sum_{x_j} \sum_{(w_i, \dots, w_j)} \sum_{(d_1, \dots, d_j)} \sum_{(h_i, \dots, h_j)} \prod_{i=1}^{j} q(d_i, w_i, h_i) \\ &= \sum_{\pi} \sum_{x_j} \prod_{i=1}^{j} \left(\sum_{w_i} \left(\sum_{d_i} \left(\sum_{h_i} q(d_i, w_i, h_i) \right) \right) \right) \\ &= \sum_{\pi} \sum_{x_j} \prod_{i=1}^{j} \left(\sum_{M_i} \left(\sum_{d_i} \left(\sum_{\mu_i} q(d_i, \mu_i, M_i) \right) \right) \right) \\ &\quad + \sum_{\pi} \sum_{x_j} \prod_{i=1}^{j} \left(\sum_{M_i} \left(\sum_{d_i} \left(\sum_{\mu_i} q(d_i, M_i, \mu_i) \right) \right) \right) \\ &= \mathbf{I} + \mathbf{II}, \end{aligned}$$

where $k_i \le M_i \le H \vee 3L_1$ and $1 \le \mu_i \le (3L_1 - 1) \wedge M_i$, for $i = 1, 2, \dots, j$, and $0 \le d_1 \le H - h_1$ and $0 \le d_i \le H - (h_i + h_{i-1})$, for $i = 2, 3, \dots, j$.

We will only carry out the computation in finding an upper bound of **I**. All arguments apply to **II** as well.

Denote, for each i,

$$Q(d_i, M_i) = \sum_{\mu_i} q(d_i, \mu_i, M_i), \qquad N(M_i) = \sum_{d_i} Q(d_i, M_i)$$

and

$$V(i) = \sum_{M_i} N(M_i).$$

Then

$$\mathrm{I} = \sum_{\pi} \sum_{x_j} \prod_{i=1}^{j} V(i) \le L_1 \sum_{\pi} \prod_{i=1}^{j} V(i).$$

We first apply Lemma 1 to obtain, for each $i = 1, \dots, j$,

$$q(d_i, \mu_i, M_i) \le (p\mu_i)^{M_i}. \tag{1.10}$$

Although the conclusion of (1.10) holds for all d_i, $i = 1, \dots, j$, for our purposes we need to obtain a better upper bound of $q(d_i, \mu_i, M_i)$ in the case that

$d_i/4\lambda > (2\lambda)^{M_i \vee L_0}$, where L_0 is chosen so that when $L > L_0$ the conclusion of Lemma 3 holds. In this case we may improve (1.10) for $i \geq 2$ by using condition (1.3). As a matter of fact, when $d_i/4\lambda > (2\lambda)^{M_i \vee L_0}$, we let $l_i = \log_{2\lambda}(d_i/4\lambda)$ and σ_{l_i} be defined as in Lemma 3. Then it follows from Lemma 3 that

$$\begin{aligned} P\left(\sigma_{l_i} > \frac{d_i}{4\lambda}\right) &= P\left(\sigma_{l_i} > (2\lambda)^{l_i}\right) \leq 3l_i \exp\left(-\frac{2^{l_i/2}}{l_i^2\theta}\right) \\ &= 3\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)\exp\left\{-2^{(\log_{2\lambda}(d_i/4\lambda))/2}\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)^{-2}\theta^{-1}\right\} \\ &= 3\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)\exp\left\{-\left(\frac{d_i}{4\lambda}\right)^{1/[2(1+\log_2\lambda)]}\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)^{-2}\theta^{-1}\right\}. \end{aligned}$$

Since $l_i > M_i$, by monotonicity and translation invariance we obtain

$$\begin{aligned} P\left(\sigma_i > \frac{d_i}{4\lambda}\right) &\leq P\left(\sigma_{l_i} > \frac{d_i}{4\lambda}\right) \\ &\leq 3\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)\exp\left\{-\left(\frac{d_i}{4\lambda}\right)^{1/[2(1+\log_2\lambda)]}\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)^{-2}\theta^{-1}\right\}. \end{aligned}$$

On the other hand, as a consequence of Lemma 9 of Durrett [(1988), Chapter 1], there are constants $C, \gamma \in (0, \infty)$ such that

$$P\left(\tau_{i-1} < \frac{d_i}{4\lambda}\right) \leq C\exp(-\gamma d_i).$$

Therefore, for $i = 2, \ldots, j$, when $d_i/4\lambda > (2\lambda)^{M_i \vee L_0}$ we have

$$\begin{aligned} q(d_i, \mu_i, M_i) &\leq 3\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)\exp\left\{-\left(\frac{d_i}{4\lambda}\right)^{1/[2(1+\log_2\lambda)]}\left(\log_{2\lambda}\left(\frac{d_i}{4\lambda}\right)\right)^{-2}\theta^{-1}\right\} \\ &\quad + C\exp(-\gamma d_i) \\ &\leq C\exp(-d_i^a), \end{aligned} \tag{1.11}$$

where $a = a(\lambda) = 1/[4(1 + \log_2\lambda)]$.

Now we will obtain bounds for $Q(d_i, M_i)$ using (1.10) and (1.11). It follows from (1.10) that, for each $i = 1, 2, \ldots, j$,

$$Q(d_i, M_i) = \sum_{\mu_i} q(d_i, \mu_i, M_i) \leq \sum_{\mu_i=1}^{v_i} (p\mu_i)^{M_i},$$

where $v_i = (3L_1 - 1) \wedge M_i$.

Note that $L_1 = 2\lfloor 1/2p^{\varepsilon}\rfloor - 1$, $\varepsilon = \frac{1}{4}$ and $\mu_i \le v_i = (3L_1 - 1) \wedge M_i$. Therefore,

$$\begin{aligned} Q(d_i, M_i) &\le \sum_{\mu_i=1}^{v_i} (p\mu_i)^{M_i} = p^{M_i} \sum_{\mu_i=1}^{v_i} (\mu_i)^{M_i} \\ &\le p^{M_i} \int_0^{v_i+1} x^{M_i}\,dx = p^{M_i} \frac{(v_i+1)^{M_i+1}}{M_i+1} \le p^{M_i}(v_i+1)^{M_i} \\ &= (p(v_i+1))^{M_i} \le (3p^{1-\varepsilon})^{M_i} = \exp\left(-M_i \log \frac{1}{3p^{1-\varepsilon}}\right) \\ &\le \exp\left(-M_i \log \frac{1}{(9p)^{1-\varepsilon}}\right) \le \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right). \end{aligned} \tag{1.12}$$

Moreover, if in addition $d_i/4\lambda > (2\lambda)^{M_i \vee L_0}$, then by (1.11), for $i = 2, 3, \dots, j$,

$$Q(d_i, M_i) = \sum_{\mu_i} q(d_i, \mu_i, M_i) \le CM_i \exp(-d_i^a). \tag{1.13}$$

Furthermore, we are only considering the minimal connectors, so there exist $i_1, i_2 \in \{1, 2, \dots, j\}$, $i_1 + 1 \le i_2$, such that $|z_{i_1+1}| + \cdots + |z_{i_2}| \ge H - (h_{i_1}/2 + h_{i_2}/2)$. Thus there is an i_3, $i_1 + 1 \le i_3 \le i_2$, such that

$$H - \left(\frac{h_{i_1}}{2} + \frac{h_{i_2}}{2}\right) \le (i_2 - i_1)|z_{i_3}| \le (j-1)|z_{i_3}| \le (L_1 - 1)|z_{i_3}|,$$

or

$$\begin{aligned} H &\le (L_1 - 1)|z_{i_3}| + \frac{h_{i_1}}{2} + \frac{h_{i_2}}{2} \\ &= (L_1 - 1)\left(d_{i_3} + \frac{h_{i_3}}{2} + \frac{h_{i_3-1}}{2}\right) + \frac{h_{i_1}}{2} + \frac{h_{i_2}}{2} \\ &\le (L_1 - 1)\left(d_{i_3} + \frac{M_{i_3}}{2} + \frac{M_{i_3-1}}{2}\right) + \frac{M_{i_1}}{2} + \frac{M_{i_2}}{2}. \end{aligned}$$

Let i^* be such that $i_1 \le i^* \le i_2$ and $M_{i^*} = \max\{M_i: i_1 \le i \le i_2\}$. Then

$$H \le (L_1 - 1)(d_{i_3} + M_{i^*}) + M_{i^*} \le L_1(d_{i_3} + M_{i^*}).$$

Case 1. $d_{i_3} \vee M_{i^*} = d_{i_3}$. In this case, $H \le 2L_1 d_{i_3}$. When $d_{i_3}/4\lambda > (2\lambda)^{M_{i_3} \vee L_0}$, by (1.13) we get

$$\begin{aligned} Q(d_{i_3}, M_{i_3}) &\le CM_{i_3} \exp(-d_{i_3}^a) \le \frac{1}{H^3}(2L_1 d_{i_3})^3 CM_{i_3} \exp(-d_{i_3}^a) \\ &\le CL_1^3 \frac{1}{H^3} \exp\left(-d_{i_3}^a + \log 8 d_{i_3}^3 M_{i_3}\right). \end{aligned}$$

Note that $M_{i_3} < \log_{2\lambda}(d_{i_3}/4\lambda)$, hence we have

$$Q(d_{i_3}, M_{i_3}) \le CL_1^3 \frac{1}{H^3} \exp\left(-\frac{d_{i_3}^a}{2}\right). \tag{1.14}$$

When $d_{i_3}/4\lambda \le (2\lambda)^{M_{i_3} \vee L_0}$, then $H \le 2L_1 d_{i_3} \le 8\lambda L_1 (2\lambda)^{M_{i_3} \vee L_0}$, so by (1.12) we get

$$Q(d_{i_3}, M_{i_3}) \le (8\lambda)^3 L_1^3 \frac{1}{H^3} (2\lambda)^{3(M_{i_3} \vee L_0)} \exp\left(-(1-\varepsilon) M_{i_3} \log \frac{1}{9p}\right). \tag{1.15}$$

CASE 2. $d_{i_3} \vee M_{i^*} = M_{i^*}$. In this case, $H \le 2L_1 M_{i^*}$. When $d_{i^*}/4\lambda > (2\lambda)^{M_{i^*} \vee L_0}$, by (1.13) we get

$$\begin{aligned} Q(d_{i^*}, M_{i^*}) &\le CM_{i^*} \exp(-d_{i^*}^a) \le \frac{1}{H^3}(2L_1 M_{i^*})^3 CM_{i^*} \exp(-d_{i^*}^a) \\ &\le CL_1^3 \frac{1}{H^3} \exp(-d_{i^*}^a + \log 8M_{i^*}^4). \end{aligned}$$

Note that $M_{i^*} < \log_{2\lambda}(d_{i^*}^a/4\lambda)$, hence we have

$$Q(d_{i^*}, M_{i^*}) \le CL_1^3 \frac{1}{H^3} \exp\left(-\frac{d_{i^*}^a}{2}\right). \tag{1.16}$$

When $d_{i^*}/4\lambda \le (2\lambda)^{M_{i^*} \vee L_0}$, by (1.12) we get

$$\begin{aligned} Q(d_{i^*}, M_{i^*}) &\le \frac{1}{H^3}(2L_1 M_{i^*})^3 \exp\left(-(1-\varepsilon) M_{i^*} \log \frac{1}{9p}\right) \\ &= CL_1^3 \frac{1}{H^3} M_{i^*}^3 \exp\left(-(1-\varepsilon) M_{i^*} \log \frac{1}{9p}\right). \end{aligned} \tag{1.17}$$

From (1.12)–(1.17) we conclude that, for $i = 1$,

$$Q(d_1, M_1) \le \exp\left(-(1-\varepsilon) M_1 \log \frac{1}{9p}\right). \tag{1.18}$$

For $i = 2, 3, \dots, j$,

$$Q(d_i, M_i) \le \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right); \tag{1.19}$$

if, in addition $d_i/4\lambda > (2\lambda)^{M_i \vee L_0}$, then

$$Q(d_i, M_i) \le C \exp\left(-\frac{d_i^a}{2}\right), \quad \text{where } a = a(\lambda) = \frac{1}{4(1 + \log_2 \lambda)}.$$

We also conclude that there exists (at least) an $n \in \{1, 2, \dots, j\}$, such that when $d_n/4\lambda \le (2\lambda)^{M_n \vee L_0}$, then

$$Q(d_n, M_n) \le (8\lambda)^3 L_1^3 \frac{1}{H^3} (2\lambda)^{3(M_n \vee L_0)} \exp\left(-(1-\varepsilon) M_n \log \frac{1}{9p}\right), \tag{1.20}$$

and when $d_n/4\lambda > (2\lambda)^{M_n \vee L_0}$, then

$$Q(d_n, M_n) \leq CL_1^3 \frac{1}{H^3} \exp\left(-\frac{d_n^a}{2}\right). \tag{1.21}$$

We are now ready to find an upper bound for $N(M_i)$. For $i = 1$, we have

$$N(M_1) \leq H \exp\left(-(1-\varepsilon) M_1 \log \frac{1}{9p}\right). \tag{1.22}$$

For $i = 2, \dots, j$, let $\nu_i = \nu_i(\lambda) = 4\lambda(2\lambda)^{M_i \vee L_0} \vee (1/9p)^{\varepsilon/5}$. Then

$$\begin{aligned} N(M_i) &= \sum_{d_i} Q(d_i, M_i) \\ &= \sum_{d_i \leq \nu_i} Q(d_i, M_i) + \sum_{d_i > \nu_i} Q(d_i, M_i) \\ &\leq \sum_{d_i \leq \nu_i} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right) + \sum_{d_i > \nu_i} C \exp\left(-\frac{d_i^a}{2}\right) \\ &\leq \nu_i \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right) + C \exp(-\nu_i^{a'}), \quad \text{for some } a' \in (0, a). \end{aligned}$$

If $4\lambda(2\lambda)^{M_i \vee L_0} < (1/9p)^{\varepsilon/5}$, then $\nu_i = (1/9p)^{\varepsilon/5}$ and we have

$$N(M_i) \leq \left(\frac{1}{9p}\right)^{\varepsilon/5} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right) + C \exp\left(-\left(\frac{1}{9p}\right)^{\varepsilon a'/5}\right).$$

Note that $M_i < \log_{2\lambda}(1/9p)^{\varepsilon/5}$, hence

$$C \exp\left(-\left(\frac{1}{9p}\right)^{\varepsilon a'/5}\right) < \left(\frac{1}{9p}\right)^{\varepsilon/5} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right);$$

therefore,

$$\begin{aligned} N(M_i) &< 2\left(\frac{1}{9p}\right)^{\varepsilon/5} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right) \\ &\leq \exp\left(-(1-2\varepsilon) M_i \log \frac{1}{9p}\right). \end{aligned} \tag{1.23}$$

If $4\lambda(2\lambda)^{M_i \vee L_0} \geq (1/9p)^{\varepsilon/5}$, then $\nu_i = 4\lambda(2\lambda)^{M_i \vee L_0}$; it follows that

$$\begin{aligned} N(M_i) \leq\ & 4\lambda(2\lambda)^{M_i \vee L_0} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right) \\ & + C \exp\left(-\left(4\lambda(2\lambda)^{M_i \vee L_0}\right)^{a'}\right). \end{aligned}$$

For sufficiently small p,

$$C \exp\left(-\left(4\lambda(2\lambda)^{M_i \vee L_0}\right)^{a'}\right) < 4\lambda(2\lambda)^{M_i \vee L_0} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right).$$

Thus we obtain

$$N(M_i) \le 8\lambda(2\lambda)^{M_i \vee L_0} \exp\left(-(1-\varepsilon)M_i \log \frac{1}{9p}\right). \tag{1.24}$$

Moreover, for $i = n$, it follows from (1.20) and (1.21) that we have

$$\begin{aligned} N(M_n) &\le \sum_{d_n \le \nu_n} Q(d_n, M_n) + \sum_{d_n > \nu_n} Q(d_n, M_n) \\ &\le \nu_n (8\lambda)^3 L_1^3 \frac{1}{H^3} (2\lambda)^{3(M_n \vee L_0)} \exp\left(-(1-\varepsilon)M_n \log \frac{1}{9p}\right) \\ &\quad + C L_1^3 \frac{1}{H^3} \exp(-\nu_n^{a'}). \end{aligned}$$

If $4\lambda(2\lambda)^{M_n \vee L_0} < (1/9p)^{\varepsilon/5}$, then $\nu_n = (1/9p)^{\varepsilon/5}$. We have

$$\begin{aligned} N(M_n) &\le \left(\frac{1}{9p}\right)^{\varepsilon/5} (8\lambda)^3 L_1^3 \frac{1}{H^3} (4\lambda)^{-3} \left(\frac{1}{9p}\right)^{3\varepsilon/5} \exp\left(-(1-\varepsilon)M_n \log \frac{1}{9p}\right) \\ &\quad + C L_1^3 \frac{1}{H^3} \exp\left(-\left(\frac{1}{9p}\right)^{\varepsilon a'/5}\right) \\ &= 8 L_1^3 \frac{1}{H^3} \left(\frac{1}{9p}\right)^{4\varepsilon/5} \exp\left(-(1-\varepsilon)M_n \log \frac{1}{9p}\right) \\ &\quad + C L_1^3 \frac{1}{H^3} \exp\left(-\left(\frac{1}{9p}\right)^{\varepsilon a'/5}\right) \\ &\le 16 L_1^3 \frac{1}{H^3} \left(\frac{1}{9p}\right)^{4\varepsilon/5} \exp\left(-(1-\varepsilon)M_n \log \frac{1}{9p}\right) \\ &\le L_1^3 \frac{1}{H^3} \exp\left(-(1-2\varepsilon)M_n \log \frac{1}{9p}\right). \end{aligned} \tag{1.25}$$

If $4\lambda(2\lambda)^{M_n \vee L_0} \ge (1/9p)^{\varepsilon/5}$, then $\nu_n = 4\lambda(2\lambda)^{M_n \vee L_0}$ and

$$\begin{aligned} N(M_n) &\le 4\lambda(8\lambda)^3 L_1^3 \frac{1}{H^3} (2\lambda)^{4(M_n \vee L_0)} \exp\left(-(1-\varepsilon)M_n \log \frac{1}{9p}\right) \\ &\quad + C L_1^3 \frac{1}{H^3} \exp\left(-\left(4\lambda(2\lambda)^{M_n \vee L_0}\right)^{a'}\right) \\ &\le (8\lambda)^4 L_1^3 \frac{1}{H^3} (2\lambda)^{4(M_n \vee L_0)} \exp\left(-(1-\varepsilon)M_n \log \frac{1}{9p}\right). \end{aligned} \tag{1.26}$$

Choose $p_0 = p_0(\lambda)$ so that $(8\lambda)^4(2\lambda)^{4L_0} < 1/(9p_0)^{\varepsilon}$. Then, when $p < p_0(\lambda)$, we may rewrite the expression on the right-hand side of (1.24) and the last expression of (1.26) as follows:

$$8\lambda(2\lambda)^{M_i \vee L_0} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right)$$
$$= \left(8\lambda(2\lambda)^{M_i} \vee 8\lambda(2\lambda)^{L_0}\right) \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right)$$
$$\le \left(\left(\frac{1}{9p}\right)^{\varepsilon M_i} \vee \left(\frac{1}{9p}\right)^{\varepsilon}\right) \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right)$$
$$\le \left(\frac{1}{9p}\right)^{\varepsilon M_i} \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p}\right)$$
$$= \exp\left(-(1-\varepsilon) M_i \log \frac{1}{9p} + \varepsilon M_i \log \frac{1}{9p}\right)$$
$$= \exp\left(-(1-2\varepsilon) M_i \log \frac{1}{9p}\right)$$

and, similarly,

$$(8\lambda)^4 (2\lambda)^{4(M_n \vee L_0)} \exp\left(-(1-\varepsilon) M_n \log \frac{1}{9p}\right)$$
$$= (8\lambda)^4 \left((2\lambda)^{4M_n} \vee (2\lambda)^{4L_0}\right) \exp\left(-(1-\varepsilon) M_n \log \frac{1}{9p}\right)$$
$$\le \exp\left(-(1-\varepsilon) M_n \log \frac{1}{9p} + \varepsilon M_n \log \frac{1}{9p}\right)$$
$$= \exp\left(-(1-2\varepsilon) M_n \log \frac{1}{9p}\right).$$

It follows from (1.22)–(1.26) that, when $p < p_0(\lambda)$,

$$N(M_1) \le H \exp\left(-(1-\varepsilon) M_1 \log \frac{1}{9p}\right);$$

for $i = 2, \dots, j$,

$$N(M_i) \le \exp\left(-(1-2\varepsilon) M_i \log \frac{1}{9p}\right).$$

Moreover, for $i = n$,

$$N(M_n) \le L_1^3 \frac{1}{H^3} \exp\left(-(1-2\varepsilon) M_n \log \frac{1}{9p}\right).$$

Therefore, it follows that

$$\begin{aligned} V(1) &\le \sum_{M_1=k_1}^{I} N(M_1) \quad (\text{where } I = H \vee 3L_1) \\ &\le \sum_{M_1=k_1}^{I} H \exp\left(-(1-\varepsilon)M_1 \log \frac{1}{9p}\right) \\ &\le H\alpha(p)\exp\left(-(1-\varepsilon)k_1 \log \frac{1}{9p}\right) \end{aligned}$$

and, for $i = 2, \dots, j$,

$$\begin{aligned} V(i) &\le \sum_{M_i=k_i}^{I} N(M_i) \le \sum_{M_i=k_i}^{I} \exp\left(-(1-2\varepsilon)M_i \log \frac{1}{9p}\right) \\ &\le \alpha(p)\exp\left(-(1-2\varepsilon)k_i \log \frac{1}{9p}\right); \end{aligned}$$

moreover, there is an $n \in \{1, 2, \dots, j\}$ such that

$$\begin{aligned} V(n) &\le L_1^3 \frac{1}{H^3} \sum_{M_n=k_n}^{I} \exp\left(-(1-2\varepsilon)M_n \log \frac{1}{9p}\right) \\ &\le \alpha(p) L_1^3 \frac{1}{H^3} \exp\left(-(1-2\varepsilon)k_n \log \frac{1}{9p}\right), \end{aligned}$$

where

$$\alpha(p) \le \frac{1}{1-(9p)^{1-2\varepsilon}} = \frac{1}{1-3\sqrt{p}}.$$

We finally obtain that

$$\begin{aligned} \mathbf{I} &\le L_1 \sum_{\pi} \prod_{i=1}^{j} V(i) \\ &\le L_1^4 \frac{1}{H^2} \sum_{\pi} (\alpha(p))^j \exp\left(-(1-2\varepsilon)(k_1 + \cdots + k_j)\log \frac{1}{9p}\right) \\ &\le (\alpha(p))^{L_1} L_1^4 \frac{1}{H^2} \sum_{\pi} \exp\left(-(1-2\varepsilon)L_1 \log \frac{1}{9p}\right). \end{aligned}$$

Notice that $L_1 \le 1/p^{\varepsilon} = 1/p^{1/4}$, hence

$$(\alpha(p))^{L_1} \le \left(\frac{1}{1-3\sqrt{p}}\right)^{1/p^{1/4}}.$$

Since when $x \in (0,1)$, $1 - yx \le (1-x)^y$ for all $y > 1$, thus when p is sufficiently small,

$$\begin{aligned} (\alpha(p))^{L_1} &\le \left(\frac{1}{1-3\sqrt{p}}\right)^{1/p^{1/4}} \le \frac{1}{1-(1/p^{1/4})(3\sqrt{p})} = \frac{1}{1-3p^{1/4}} \\ &= 1 + O(p^{1/4}). \end{aligned}$$

Therefore

$$\mathbf{I} \le \left(1 + O(p^{1/4})\right) L_1^4 \frac{1}{H^2} \sum_{\pi} \exp\left(-(1-2\varepsilon) L_1 \log \frac{1}{9p} \right).$$

As we pointed out earlier, **II** is bounded above by the same quantity. It follows now that, when p is sufficiently small, $p < p_0(\lambda)$,

$$P_2 \le \mathbf{I} + \mathbf{II} \le 2\left(1 + O(p^{1/4})\right) L_1^4 \frac{1}{H^2} \sum_{\pi} \exp\left(-(1-2\varepsilon) L_1 \log \frac{1}{9p} \right). \tag{1.27}$$

For each $j = 2, \dots, L_1$, the number of possible different partitions $\{k_1, \dots, k_j\}$ of L_1 subject to the constraint $k_i \ge 1$ is equal to $(L_1 - 1)!/[(j-1)!(L_1 - 1)!]$. Therefore, the total number of all possible different partitions of L_1 subject to the constraint $k_i \ge 1$ is equal to $\sum_{j=2}^{L_1} (L_1 - 1)!/[(j-1)!(L_1 - 1)!]$, which is equal to $2^{L_1 - 1}$. Therefore, from (1.27), we have

$$\begin{aligned} P_2 &= P(\Gamma \text{ is a (long) vertical connector in } \xi_0^p) \\ &\le \left(1 + O(p^{1/4})\right) L_1^4 2^{L_1} \frac{1}{H^2} \exp\left(-(1-2\varepsilon) L_1 \log \frac{1}{9p} \right). \end{aligned}$$

The proof of Proposition 4 is now complete. □

It follows from Proposition 4 that, when p is sufficiently small, $p < p_0(\lambda)$,

$$\begin{aligned} P(A_1) &\le 2L_1 \sum_{H=1}^{L_1} \exp\left\{ -(1-\varepsilon) L_1 \log \frac{1}{p} \right\} \\ &\quad + 2L_1 \sum_{H=L_1+1}^{\infty} \left(1 + O(p^{1/4})\right) L_1^4 2^{L_1} \frac{1}{H^2} \exp\left\{ -(1-2\varepsilon) L_1 \log \frac{1}{9p} \right\} \\ &\le 2L_1^2 \exp\left\{ -(1-\varepsilon) L_1 \log \frac{1}{p} \right\} + \exp\left\{ -(1-2\varepsilon) L_1 \log \frac{1}{9p} + CL_1 \right\} \qquad (1.28) \\ &\le \exp\left\{ -(1-2\varepsilon)(1 - a(p)) L_1 \log \frac{1}{p} \right\} \\ &= \exp\left\{ -(1-2\varepsilon)(1 - a'(p)) \frac{1}{p^{\varepsilon}} \log \frac{1}{p} \right\}, \end{aligned}$$

where $a(p), a'(p) = O(1/\log(1/p))$.

We have now concluded that the probability that a level 1 site is occupied equals ρ_1, where

$$\begin{aligned} \rho_1 \le p_1 &\equiv \exp\left\{ -(1-2\varepsilon)(1 - a(p)) L_1 \log \frac{1}{p} \right\} \\ &= \exp\left\{ -(1-2\varepsilon)(1 - a'(p)) \frac{1}{p^{\varepsilon}} \log \frac{1}{p} \right\}. \end{aligned}$$

We will want to use this bound to estimate probabilities involving the bootstrap percolation model at level 1. However, at level 1 the events that different sites are occupied are not independent. To overcome this difficulty, we notice that in future estimates we will not need to use the accurate value ρ_1, but instead its upper bound p_1. We will show that, for any set of level 1 sites $x_1[1], \dots, x_k[1]$, the probability that they are all V-occupied is bounded above by p_1^k and the probability that they are all H-occupied is also bounded above by p_1^k, which will be good enough for our purposes. First we consider the case $k = 2$. Suppose that two level 1 sites $x[1]$ and $y[1]$ are V-occupied and suppose that the corresponding squares of $x[1]$ and $y[1]$ at level 0 contain the upper edges of vertical connectors $\Gamma_x(L_1, H)$ and $\Gamma_y(L_1, H')$ in V^*, respectively. $\Gamma_x(L_1, H)$ and $\Gamma_y(L_1, H')$ are either completely separate or they are adjoining. If they are completely separate, by the van den Berg–Kesten inequality we have $P(x[1] \text{ and } y[1] \text{ are occupied}) \leq p_1^2$, which is desired. The problem arises when $\Gamma_x(L_1, H)$ and $\Gamma_y(L_1, H')$ are adjoining and the two partition strips sharing their common border both are crossed by the same internally spanned region Γ_j which satisfies the conditions (1.3) or (1.3′) in the definition of connectors; see Figure 3.

We will show that in this case $P(x[1] \text{ and } y[1] \text{ are V-occupied}) \leq p_1^2$ still holds. Let $\pi_1 = \{k_1, \dots, k_j\}$ and $\pi_2 = \{k_{j+1}, \dots, k_m\}$ be two specific partitions for Γ_x and Γ_y, respectively, that is, $k_1 + \cdots + k_j = k_{j+1} + \cdots + k_m = L_1$ and $k_i \geq 1$, for $i = 1, 2, \dots, m$. Strips S_i, $i = 1, \dots, j-1, j+2, \dots, m$, are crossed by disjoint internally spanned rectangles Γ_i which satisfy the conditions (1.3) or (1.3′) in the definition of vertical connectors. Strips S_j and S_{j+1} share their common border and both are crossed by Γ_j (see Figure 3). This means the combined strip $S_j \cup S_{j+1}$ is crossed by Γ_j.

Consider the partition $\pi = \{k_1, \dots, k_{j-1}, k_j + k_{j+1}, k_{j+2}, \dots, k_m\}$ of the region $\Gamma_x \cup \Gamma_y$. Notice that now all strips are crossed by disjoint internally spanned rectangles. Hence the arguments we applied in the proof of Proposition 4 can be applied to $\Gamma_x \cup \Gamma_y$. In this case each of the two parts of the sum (**I** and **II**) in the proof of Proposition 4 is bounded above by $L_1 \Sigma_\pi \prod_{i=1}^{j} V(i)$, where

$$V(1) \leq H \exp\left(-(1-2\varepsilon) k_1 \log \frac{1}{9p}\right),$$

$$V(i) \leq \alpha(p) \exp\left(-(1-2\varepsilon) k_i \log \frac{1}{9p}\right), \qquad i = 2, 3, \dots, j-1,$$

$$V(i) \leq \alpha(p) \exp\left(-(1-2\varepsilon) k_{i+1} \log \frac{1}{9p}\right), \qquad i = j+1, j+2, \dots, m-1$$

and

$$V(j) \leq \alpha(p) \exp\left(-(1-2\varepsilon)(k_j + k_{j+1}) \log \frac{1}{9p}\right),$$

where $\alpha(p) \leq 1/(1 - 3\sqrt{p})$ is the same as in the proof of Proposition 4. Also,

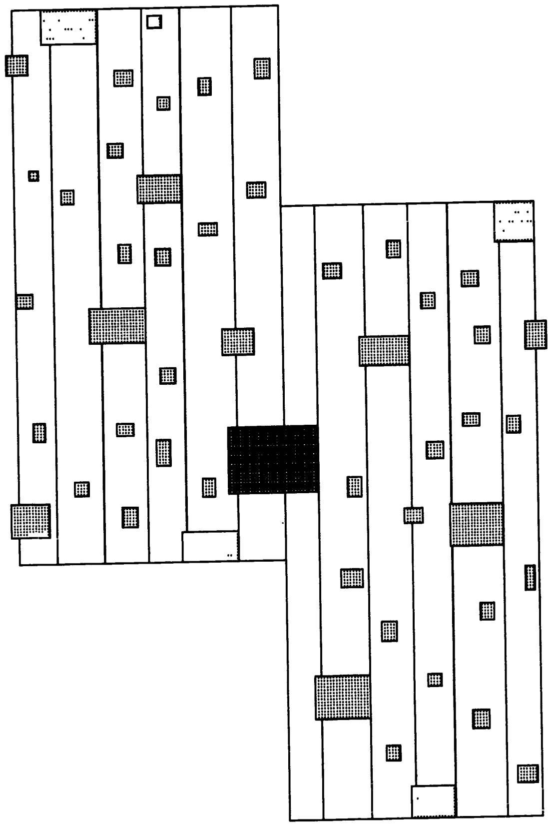

FIG. 3.

there exist at least $j_1 \in \{1, 2, \ldots, j\}$ and $j_2 \in \{j+1, j+2, \ldots, m\}$ such that

$$V(j_1) \le \alpha(p) L_1^3 \frac{1}{H^3} \exp\left(-(1-2\varepsilon) k_{j_1} \log \frac{1}{9p} \right)$$

and

$$V(j_2) \le \alpha(p) L_1^3 \frac{1}{H'^3} \exp\left(-(1-2\varepsilon) k_{j_2+1} \log \frac{1}{9p} \right).$$

Therefore,

$$\begin{aligned} \mathbf{I} &\le L_1^4 \frac{1}{H^2} L_1^3 \frac{1}{H'^3} \sum_{\pi} (\alpha(p))^m \exp\left\{ -(1-2\varepsilon)(k_1 + \cdots + k_m) \log \frac{1}{9p} \right\} \\ &\le (\alpha(p))^{2L_1} L_1^4 \frac{1}{H^2} L_1^3 \frac{1}{H'^3} \sum_{\pi} \exp\left\{ -(1-2\varepsilon)(k_1 + \cdots + k_m) \log \frac{1}{9p} \right\} \\ &\le \left(1 + O(p^{1/4})\right)^2 2^{2(L_1-1)} L_1^4 \frac{1}{H^2} L_1^3 \frac{1}{H'^3} \exp\left\{ -2(1-2\varepsilon) L_1 \log \frac{1}{9p} \right\}. \end{aligned}$$

Similarly, **II** is bounded above by the same quantity. This implies that the upper bound of $P(\Gamma_x$ and Γ_y are both long vertical connectors) is bounded above by the product of the upper bound of $P(\Gamma_x$ is a long vertical connector) and the upper bound of $P(\Gamma_y$ is a long vertical connector). It follows that

$$(1.29) \qquad P(x[1] \text{ and } y[1] \text{ are V-occupied}) \le p_1^2.$$

This argument can be extended easily by a routine (but tedious) inductive procedure to the case $k > 2$. Namely, if $x_1[1], x_2[1], \dots, x_k[1]$ are k different sites of level 1, then

$$(1.30) \qquad P(x_1[1], x_2[1], \dots, x_k[1] \text{ are V-occupied initially}) \le p_1^k.$$

The proof of $P(x_1[1], x_2[1], \dots, x_k[1]$ are H-occupied initially) $\le p_1^k$ is much the same. □

Because of these estimates and by the same argument we employed in the proof of Lemma 1, we obtain the following corollary.

COROLLARY 3. *Suppose that* $\Gamma^{(1)}$ *is a* $w^{(1)} \times h^{(1)}$ *level* 1 *rectangle. Let* $M = w^{(1)} \vee h^{(1)}$, $m = w^{(1)} \wedge h^{(1)}$. *Then*

$$P(\Gamma^{(1)} \textit{ is weakly internally spanned}) \le (p_1 m)^M,$$

and hence

$$P(\Gamma^{(1)} \textit{ is strongly internally spanned}) \le (p_1 m)^M;$$

moreover,

$$P(\Gamma^{(1)} \textit{ is internally spanned}) \le (4p_1 m)^{M/2}.$$

PROOF. Only the third assertion needs a little further explanation. For convenience we assume $w^{(1)} \ge h^{(1)}$, that is, $w^{(1)} = M$, $h^{(1)} = m$. As described in the proof of Lemma 1, if we partition $\Gamma^{(1)}$ into $w^{(1)}$ disjoint vertical strips, then each strip contains at least one occupied site (either H- or V-occupied). Denote

$N =$ the number of those strips containing at least one V-occupied site,

$K =$ the number of those strips containing at least one H-occupied site.

Then either $N \ge M/2$ or $K \ge M/2$. Assume without loss of generality that $N \ge M/2$. Then, it follows from (1.30) that

$$\begin{aligned} P(\Gamma^{(1)} \text{ is internally spanned}) &\le \sum_{N=M/2}^{M} C_M^N (p_1 m)^N \\ &\le (p_1 m)^{M/2} \sum_{N=M/2}^{M} C_M^N \\ &\le 2^M (p_1 m)^{M/2} = (4p_1 m)^{M/2}. \end{aligned}$$

□

Now we are going to continue our procedure inductively to define a bootstrap percolation model on the level n lattice. Suppose the following, for some $n \geq 2$:

1. We have defined the level $n-1$ lattice $\mathbf{Z}^2[n-1]$ such that each site $x[n-1]$ in $\mathbf{Z}^2[n-1]$ is a square region at level $n-2$ with edge size $2\lfloor 1/2p_{n-2}^{\varepsilon^{n-1}} \rfloor - 1$ level $n-2$ units, and the corresponding region of the level $n-1$ origin $0[n-1]$ at level $n-2$ is centered at $0[n-2]$.
2. We have defined the concepts of a level $n-2$ region being weakly internally spanned and strongly internally spanned.
3. We have defined the connectors at level $n-2$ and thus defined the meaning of occupancy and vacancy (also the meaning of V- and H-occupancy) for a level $n-1$ site and obtained that, for any set of k different level $n-1$ sites $x_1[n-1], \dots, x_k[n-1]$,

$$(1.31)\quad \begin{aligned} &P(x_1[n-1], \dots, x_k[n-1] \text{ are V-occupied}) \leq p_{n-1}^k \quad \text{and} \\ &P(x_1[n-1], \dots, x_k[n-1] \text{ are H-occupied}) \leq p_{n-1}^k, \end{aligned}$$

where

$$p_{n-1} = \exp\left\{-(1-2\varepsilon^{n-1})(1-a'(p_{n-2}))\frac{1}{p_{n-2}^{\varepsilon^{n-1}}}\log\frac{1}{p_{n-2}}\right\}$$

and

$$a'(p_{n-2}) = O\left(\frac{1}{\log(1/p_{n-2})}\right).$$

To define the level n lattice from level $n-1$, we repeat the same procedure.

Namely, each level n site $x[n]$ is a square region at level $n-1$ with edge length $2\lfloor 1/2p_{n-1}^{\varepsilon^n} \rfloor - 1$ (level $n-1$ units). Similar to the case $n = 1$, at level $n \geq 2$ if $\Gamma[n]$ is a $w^{(n)} \times h^{(n)}$ rectangle, then its corresponding region Γ at level 0 is a $w^{(n)}L_n \times h^{(n)}L_n$ rectangle. Conversely, if Γ is a $w \times h$ level 0 rectangle, then for each n there is a unique minimal $w^{(n)} \times h^{(n)}$ level n rectangle $\Gamma[n]$, such that the corresponding region of $\Gamma[n]$ at level 0 contains Γ. Furthermore, we may choose appropriate n so that $1 < h^{(n)} \vee w^{(n)} \leq 2\lfloor 1/2p_{n-1}^{\varepsilon^n} \rfloor - 1$.

Let L_n denote the edge size of the corresponding square region of a level n site at level 0. Then $L_n/L_{n-1} = 2\lfloor 1/2p_{n-1}^{\varepsilon^n} \rfloor - 1$.

We now define the sets $V^*[n-1]$ and $H^*[n-1]$ of connectors at level $n-1$ as follows (the coordinates and length unit are referred to the level $n-1$ lattice unless otherwise stated): Let

$$\Gamma[n-1] = \left(\left(k - \frac{1}{2}\right)\frac{L_n}{L_{n-1}}, \left(k + \frac{1}{2}\right)\frac{L_n}{L_{n-1}}\right] \times \left(y_0 - \frac{1}{2}, y_0 - \frac{1}{2} + H\right],$$

where k, y_0, H are integers, $H \geq 1$, be a level $n-1$ rectangle. For any given initial configuration $\eta[n-1]$ with respect to the level $n-1$ bootstrap percolation model, we call $\Gamma[n-1]$ a *short vertical connector at level* $n-1$ if $H \le L_n/L_{n-1}$ and, in its partition into L_n/L_{n-1} vertical strips of unit width, each strip contains at least one site which is V-occupied in $\eta[n-1]$. We call $\Gamma[n-1]$ a *long vertical connector at level* $n-1$ if $H > L_n/L_{n-1}$ and it can be partitioned into j, $j \ge 2$, vertical strips such that each strip $S_i[n-1]$ is crossed horizontally by a $w_i \times h_i$ rectangle,

$$\Gamma_i[n-1] \subset \bar{\Gamma}[n-1] = \left(\left(k-\frac{3}{2}\right)\frac{L_n}{L_{n-1}}, \left(k+\frac{3}{2}\right)\frac{L_n}{L_{n-1}}\right] \times \left(y_0-\frac{1}{2}, y_0-\frac{1}{2}+H\right],$$

such that the $\Gamma_i[n-1]$, $i = 1, 2, \dots, j$, are disjoint, strongly internally spanned by $\eta[n-1]$ and satisfy at least one of the conditions (1.3)[$n-1$] or (1.3′)[$n-1$]:

(1.3)[$n-1$] For $i = 1, 2, \dots, j$, let y_i denote the y coordinate of the centers of $\Gamma_i[n-1]$, $z_i = y_i - y_{i-1}$, and $d_1 = |z_1| - h_1/2$, $d_i = |z_i| - (h_i/2 + h_{i-1}/2)$, $i \ge 2$. Let Γ_i be the corresponding region of $\Gamma_i[n-1]$ at level 0, $\sigma_i = \inf\{t\colon \xi_t^{\Gamma_i \cap \mathbf{Z}^2} = \phi\}$ and, for $i = 1, \dots, j-1$, let ζ_t^i, $t \ge 0$, denote the asexual contact process with death rate 1 and birth rate λ, confined to the right edge E_i of the corresponding region of $S_i[n-1]$ at level 0, that is, E_i is the rightmost vertical $1 \times HL_{n-1}$ (level 0 length unit) strip in the corresponding region of $S_i[n-1]$ at level 0. Let the initial state of ζ_t^i be $E_i \cap \Gamma_i$, and $\tau_i = \inf\{t\colon \zeta_t^i \cap \Gamma_{i+1} \ne \phi\}$. Then, for $i = 1, 2, \dots, j-1$, either $\sigma_{i+1} > d_{i+1}/4\lambda$ or $\tau_i < d_{i+1}/4\lambda$.

(1.3′)[$n-1$] For each $i = 1, 2, \dots, j$, let $S_i^T[n-1] = S_{j-i+1}[n-1]$, $\Gamma_i^T[n-1] = \Gamma_{j-i+1}[n-1]$, $\Gamma_i^T = \Gamma_{j-i+1}$, $\sigma_i^T = \sigma_{j-i+1}$, $h_i^T = h_{j-i+1}$, $w_i^T = w_{j-i+1}$ and $y_i^T = y_{j-i+1}$. Let $z_1^* = y_1^T - y_0$, $d_1^* = |z_1^*| - h_1^T/2$, $z_i^* = y_i^T - y_{i-1}^T$ and $d_i^* = |z_i^*| - (h_i^T/2 + h_{i-1}^T/2)$, $i \ge 2$. For $i = 1, 2, \dots, j-1$, let $\tilde{\zeta}_t^i$ be the asexual contact process with death rate 1 and birth rate λ, confined to the left edge E_i^T of the corresponding region of $S_i^T[n-1]$ at level 0. Let the initial state of $\tilde{\zeta}_t^i$ be $E_i^T \cap \Gamma_i^T \cap \mathbf{Z}^2$, and $\tau_i^* = \inf\{t\colon \tilde{\zeta}_t^i \cap \Gamma_{i+1}^T \ne \phi\}$. Then, for $i = 1, \dots, j-1$, either $\sigma_{i+1}^T > d_{i+1}^*/4\lambda$ or $\tau_i^* < d_{i+1}^*/4\lambda$.

Note that conditions (1.3)[$n-1$] and (1.3′)[$n-1$] are not quite the level $n-1$ analogues of conditions (1.3) and (1.3′), since the process ζ_t^i and $\tilde{\zeta}_t^i$ used in (1.3)[$n-1$] and (1.3′)[$n-1$] are still level 0 processes, as is the process ξ_t.

The horizontal connectors are similarly defined by rotating the x-y axes 90°, and the collections $H^*[n-1], V^*[n-1]$ are defined in the same way as in the case $n = 1$.

Let $x[n]$ be a level n site and denote its corresponding square region at level $n-1$ as $\Lambda^{(n-1)}(x[n])$. We regard $x[n]$ as occupied if at least one of the following conditions is satisfied:

(1.32a) $\Lambda^{(n-1)}(x[n])$ contains the upper edge of a vertical connector in $V^*[n-1]$;

(1.32b) $\Lambda^{(n-1)}(x[n])$ contains the left edge of a horizontal connector in $H^*[n-1]$.

As in the case $n = 1$, if a level n site is occupied due to (1.32a), we say it is V-occupied, and if it is occupied due to (1.32b), we say it is H-occupied.

Now we state the following result, which is a generalization of Proposition 3.

PROPOSITION 5. *Let $\lambda \in (1, \infty)$. Suppose Γ is a $W \times H$ rectangle at level 0, $\Gamma[n]$ is its corresponding region at level n with dimension $W^{(n)} \times H^{(n)}$. Suppose $W^{(n)}, H^{(n)} > 2$. Let $\hat{\Gamma}[n]$ be a $(W^{(n)} - 2) \times (H^{(n)} - 2)$ rectangle properly contained in $\Gamma[n]$ with the same center. For almost all initial configurations $\eta \in \xi_0^p$, if Γ is significantly spanned at level 0, then at level n there exists an internally spanned rectangle $\tilde{\Gamma}[n]$ such that $\hat{\Gamma}[n] \subset \tilde{\Gamma}[n] \subset \Gamma[n]$.*

PROOF. Since the idea of the proof is essentially the same as in Proposition 3, we will only give a brief description. We prove it by induction. Proposition 3 proves that the conclusion is true when $n = 1$. Now we assume that $n \geq 2$ and the conclusion is true for $n - 1$. We want to prove that the conclusion is true for n as well. Suppose that we cannot find such a level n rectangle $\tilde{\Gamma}[n]$ as stated previously. Then there exists a separation $S[n]$ at level n which separates both $\hat{\Gamma}[n]$ and $\Gamma[n]$. The first half of the proof is the same as in Proposition 3. Without loss of generality, we may assume that $S[n]$ is a vertical separation with height $H^{(n)}$ and width 1 (at level n). Let $S[n-1]$ be the corresponding region of $S[n]$ at level $n-1$. We know that $S[n-1]$ has height $H^{(n)}L_n/L_{n-1}$ and width L_n/L_{n-1} level $n-1$ units. Then $S[n-1]$ does not contain any level $n-1$ connectors at all. (For details, see the proof of Proposition 3.)

By the definition of a level $n-1$ connector, the preceding facts mean that at level $n-1$ we can find a strip $\hat{S}[n-1]$ contained in $S[n-1]$ with height $H^{(n)}L_n/L_{n-1}$ such that it does not intersect any strongly internally spanned level $n-1$ rectangle satisfying conditions (1.3)[$n-1$] or (1.3′)[$n-1$]. By the induction hypothesis (namely, the level $n-1$ version of Corollary 2), the relationship between a level $n-1$ region being strongly internally spanned at level $n-1$ and its corresponding region at level 0 being significantly spanned is well-established. Therefore, the preceding fact means $\hat{S}[n-1]$ can only possibly contain occupied regions whose corresponding regions at level 0 will be vacated before any possible spreading from either side of the corresponding region of $\hat{S}[n-1]$ at level 0. Hence the corresponding region of $\hat{S}[n-1]$ at level 0 contains a region which will be vacated by time $T = \max\{W^{(n)}L_n, H^{(n)}L_n\}$ and separate both the corresponding regions of $\Gamma[n]$ and $\hat{\Gamma}[n]$, and thus separate Γ. Therefore Γ is not significantly spanned. □

Analogous to the case $n = 1$, for a level 0 rectangle Γ and its corresponding region $\Gamma[n]$ at level $n \geq 2$, if the corresponding region of $\Gamma[n]$ at level 0 happens to be equal to Γ, then we can improve the result of Proposition 5 as follows.

COROLLARY 4. *Let $\lambda \in (1, \infty)$. Suppose $\Gamma[n]$ is a $W^{(n)} \times H^{(n)}$ rectangle at level n, Γ is its corresponding region at level 0 with dimension $W^{(n)}L_n \times H^{(n)}L_n$. For almost all initial configurations $\eta \in \xi_0^p$, if Γ is significantly spanned at level 0, then $\Gamma[n]$ is strongly internally spanned at level n.*

Now we are going to estimate the probability of the occupancy of a level n site. We use A_n to denote the event that a given level n site $x[n]$ is initially occupied. As in the case $n = 1$, to evaluate $P(A_n)$, we need first evaluate the probability that a level $n - 1$ rectangular region $\Gamma[n-1]$ is a vertical connector. Applying the same method as in the Proof of Proposition 4 with ε, p and L_1 replaced by ε^n, p_{n-1} and L_n/L_{n-1}, respectively, we obtain

$$P(\Gamma^{(n-1)} \text{ is a short vertical connector at level } n-1) \leq \exp\left\{-(1-\varepsilon^n)\frac{L_n}{L_{n-1}}\log\frac{1}{p_{n-1}}\right\}, \tag{1.33}$$

$$P(\Gamma^{(n-1)} \text{ is a long vertical connector at level } n-1) \leq \left(1 + O(p_{n-1}^{1/4})\right)\left(\frac{L_n}{L_{n-1}}\right)^4 2^{L_n/L_{n-1}}\frac{1}{H^2} \times\exp\left\{-(1-2\varepsilon^n)\frac{L_n}{L_{n-1}}\log\frac{1}{9p_{n-1}}\right\}. \tag{1.34}$$

(Here we still use H to denote the vertical length of $\Gamma^{(n-1)}$, which is measured in level $n-1$ units.) Then, applying the same argument as used in evaluating $P(A_1)$ in (1.28) with the same replacement, we obtain

$$\begin{aligned} P(A_n) &\leq \exp\left\{-(1-2\varepsilon^n)(1-a(p_{n-1}))\frac{L_n}{L_{n-1}}\log\frac{1}{p_{n-1}}\right\} \\ &= \exp\left\{-(1-2\varepsilon^n)(1-a'(p_{n-1}))\frac{1}{p_{n-1}^{\varepsilon^n}}\log\frac{1}{p_{n-1}}\right\}, \end{aligned} \tag{1.35}$$

where $a(p_{n-1})$, $a'(p_{n-1}) = O(1/\log(1/p_{n-1}))$, $\varepsilon = \frac{1}{4}$. Hence,

$$\begin{aligned} \rho_n &= P(\text{a site of level } n \text{ is initially occupied}) = P(A_n) \\ &\leq p_n = \exp\left\{-(1-2\varepsilon^n)(1-a(p_{n-1}))\frac{L_n}{L_{n-1}}\log\frac{1}{p_{n-1}}\right\} \\ &= \exp\left\{-(1-2\varepsilon^n)(1-a'(p_{n-1}))\frac{1}{p_{n-1}^{\varepsilon^n}}\log\frac{1}{p_{n-1}}\right\}. \end{aligned}$$

The same argument as in the proof of (1.30) can be applied to the case $n \geq 2$, and we can obtain that, for any k, $k \geq 2$, level n sites $x_1[n], x_2[n], \dots, x_k[n]$,

$$P(x_1[n], x_2[n], \dots, x_k[n] \text{ are V-occupied}) \leq p_n^k$$

and

$$P(x_1[n], x_2[n], \dots, x_k[n] \text{ are H-occupied}) \leq p_n^k.$$

Also, the conclusions of Corollary 3 can be extended to the case $n \geq 2$ as follows.

COROLLARY 5. *Suppose that* $\Gamma[n]$ *is a* $w^{(n)} \times h^{(n)}$ *level* n *rectangle. Let* $M = w^{(n)} \vee h^{(n)}$, $m = w^{(n)} \wedge h^{(n)}$. *Then*

$$P(\Gamma[n] \textit{ is weakly internally spanned}) \leq (p_n m)^M$$

and hence

$$P(\Gamma[n] \textit{ is strongly internally spanned}) \leq (p_n m)^M;$$

moreover,

$$P(\Gamma[n] \textit{ is internally spanned}) \leq (4p_n m)^{M/2}.$$

This completes our block renormalization procedure.

1.3. Proof of Theorem 1. Let $\Lambda_L = (-L/2, L/2]^2$ as before, $\xi_t^{\Lambda_L, p}$ be the state at time t for the process with initial distribution $\xi_0^{\Lambda_L, p}$ described as follows: For all $x \in \Lambda_L \cap \mathbf{Z}^2$ events $\{x \in \xi_0^{\Lambda_L, p}\}$ are independent, $P(x \in \xi_0^{\Lambda_L, p}) = p$, and all sites in $\xi_0^{\Lambda_L, p} \cap \Lambda_L^c$ are vacant. We first prove the following lemma.

LEMMA 4. *Let* $\lambda \in (1, \infty)$, $t \in [0, \infty)$ *be fixed real numbers. Then there are constants* $\gamma, C \in (0, \infty)$ *such that*

$$P\big(\{0 \in \xi_t^p\} \setminus \{0 \in \xi_t^{\Lambda_{4\lambda t}, p}\}\big) < C \exp(-\gamma t). \tag{1.36}$$

PROOF. Let $\xi_s^{\theta_t, p}$ denote the state at time s for the system with initial distribution $\xi_0^{\theta_t, p}$ described as follows: For $x \in \Lambda_{4\lambda t} \cap \mathbf{Z}^2$, events $\{x \in \xi_0^{\theta_t, p}\}$ are independent, $P(x \in \xi_0^{\theta_t, p}) = p$ and all sites in $\xi_0^{\theta_t, p} \cap \Lambda_{4\lambda t}^c$ are occupied. It is obvious that $\{0 \in \xi_t^p\} \setminus \{0 \in \xi_t^{\Lambda_{4\lambda t}, p}\} \subset \{0 \in \xi_t^{\theta_t, p}\} \setminus \{0 \in \xi_t^{\Lambda_{4\lambda t}, p}\}$, hence we need only to show

$$P\big(\{0 \in \xi_t^{\theta_t, p}\} \setminus \{0 \in \xi_t^{\Lambda_{4\lambda t}, p}\}\big) \leq C \exp(-\gamma t). \tag{1.37}$$

To prove (1.37), we define $\chi_s = (\xi_s^{\theta_t, p}, \xi_s^{\Lambda_{4\lambda t}, p})$, $0 \leq s < \infty$. Since $\xi_0^{\theta_t, p} \supset \xi_0^{\Lambda_{4\lambda t}, p}$, we know by the discussion in Section 0 that $\xi_s^{\theta_t, p}(x) \geq \xi_s^{\Lambda_{4\lambda t}, p}(x)$, for all s and x. Thus $\chi_s(x)$ can only take three values: $(0,0)$, $(1,0)$ and $(1,1)$. We will regard in the system χ_s a site x as "occupied" at time s whenever $\chi_s(x) = (1,0)$ and as "vacant" at time s whenever $\chi_s(x) = (0,0)$ or $(1,1)$. Using this interpretation, it is clear that the initial state of the process χ_s is all sites in $\Lambda_{4\lambda t}^c \cap \mathbf{Z}^2$

are occupied and all sites in $\Lambda_{4\lambda t} \cap \mathbf{Z}^2$ are vacant. Furthermore,

$$\{0 \in \xi_t^{\theta_t, p}\} \setminus \{0 \in \xi_t^{\Lambda_{4\lambda t}, p}\} = \{\chi_t(0) = (1,0)\} = \{0 \text{ is occupied by } \chi_t\}.$$

A little thought reveals that the death rate of the system χ_s is at least 1, and the birth rate is dominated by that of the asexual contact process on $\mathbf{Z}^2$ with birth rate λ. Hence, if we denote $\zeta_s^{\Lambda_{4\lambda t}^c}$ as the state at time s for the asexual contact process with death rate identically 0 and birth rate λ, starting with the initial state that all sites in $\Lambda_{4\lambda t} \cap \mathbf{Z}^2$ are vacant but all sites in $\Lambda_{4\lambda t}^c \cap \mathbf{Z}^2$ are occupied, then the process χ_s is dominated by $\zeta_s^{\Lambda_{4\lambda t}^c}$, that is, $\chi_s \subset \zeta_s^{\Lambda_{4\lambda t}^c}$ for all $s \in [0, \infty)$. Therefore,

$$P(\{0 \in \xi_t^{\theta_t, p}\} \setminus \{0 \in \xi_t^{\Lambda_{4\lambda t}, p}\}) \le P(0 \in \zeta_t^{\Lambda_{4\lambda t}^c}). \tag{1.38}$$

An important property of the asexual contact process is that for all $s \in [0, \infty)$,

$$P(0 \in \zeta_s^{\Lambda_{4\lambda t}^c}) = P\Bigg(\bigcup_{x \in \Lambda_{4\lambda t}^c} \{0 \in \zeta_s^x\}\Bigg), \tag{1.39}$$

where ζ_s^x denote the same process with initial state $\{x\}$. [See Liggett (1985), Chapter VI]. Let $\Lambda_1 = \Lambda_{4\lambda t+1} \setminus \Lambda_{4\lambda t}$, $\Lambda_2 = \Lambda_{4\lambda t+2} \setminus \Lambda_{4\lambda t+1}, \dots, \Lambda_n = \Lambda_{4\lambda t+n} \setminus \Lambda_{4\lambda t+n-1}$. Then $\Lambda_{4\lambda t}^c = \bigcup_{n=1}^{\infty} \Lambda_n$. For any $x \in \Lambda_{4\lambda t}^c \cap \mathbf{Z}^2$, let $\tau(x) = \inf\{s\colon 0 \in \zeta_s^x\}$. Then for each $x \in \Lambda_n \cap \mathbf{Z}^2$, by Lemma 9 of Durrett (1988), there are constants $C, \gamma \in (0, \infty)$ such that

$$P(\tau(x) < t) \le C \exp(-\gamma(t+n)),$$

and hence

$$\begin{aligned} P(0 \in \zeta_t^x, \text{ for some } x \in \Lambda_{4\lambda t}^c) &= \sum_{n=1}^{\infty} P(0 \in \zeta_t^x, \text{ for some } x \in \Lambda_n) \\ &\le \sum_{n=1}^{\infty} 4Cn \exp(-\gamma(t+n)) = C \exp(-\gamma t). \end{aligned}$$

From (1.38) and (1.39) our assertion now follows. □

Conclusion of the proof of Theorem 1. As a consequence of Lemma 4,

$$P(0 \in \xi_t^p) < P(0 \in \xi_t^{\Lambda_{4\lambda t}, p}) + C \exp(-\gamma t).$$

Therefore, to prove Theorem 1, it suffices to prove that, for any fixed $\lambda > 1$, if $p > 0$ is sufficiently small (p may depend on λ), there exists a constant $c > 0$ such that, for all large t,

$$P(0 \in \xi_t^{\Lambda_{4\lambda t}, p}) \le \tfrac{1}{2} t^{-c \log_{2\lambda}(1/p)}. \tag{1.40}$$

To prove (1.40), we notice that, if we cannot find any rectangle inside the region $\Lambda_{4\lambda t}$ with edge length greater than $\log_{2\lambda} t$ which is significantly spanned and contains the origin, then, by Lemma 3, with very small probability the origin will be occupied at time t. To be more precise, we let

$E = \{\Lambda_{4\lambda t}$ contains some rectangular region containing the origin with length greater than $\log_{2\lambda} t$ which is significantly spanned$\}$.

Suppose Γ is a significantly spanned region contained in $\Lambda_{4\lambda t}$. If the longer edge length of Γ is less than $\log_{2\lambda} t$, then, by Lemma 3 and the monotonicity property, the probability that Γ is not completely vacated by time t is bounded by

$$3(\log_{2\lambda} t)\exp\{-2^{(\log_{2\lambda} t)/2}(\log_{2\lambda} t)^{-2}\theta^{-1}\}.$$

Hence,

$$\begin{aligned} P(0 \in \xi_t^{\Lambda_{4\lambda t}, p}, E^c) &\le 3(\log_{2\lambda} t)^2 \exp\{-2^{(\log_{2\lambda} t)/2}(\log_{2\lambda} t)^{-2}\theta^{-1}\} \\ &\ll \frac{1}{4}\exp\left\{-c(\log_{2\lambda} t)\log\frac{1}{p}\right\}. \end{aligned} \tag{1.41}$$

Thus, we need only to show that

$$P(E) \le \frac{1}{4}\exp\left\{-c(\log_{2\lambda} t)\log\frac{1}{p}\right\}.$$

To bound $P(E)$, we will apply the results established in Section 1.2. Our scheme is to rescale the original lattice up to level n, for some $n \ge 0$, so that at level n the corresponding region of the square region Λ_l ($l = \log_{2\lambda} t$) has "reasonable" size. We will obtain a good upper bound of the probability that the corresponding region of Λ_l at level n is internally spanned, and then Proposition 5 will give us the desired bound on $P(E)$.

To execute our scheme, we first need to verify that the lattice size of level n tends to infinity when $n \to \infty$. The verification is as follows: From the construction given in the block renormalization procedure, we notice that

$$\begin{aligned} \frac{L_2}{L_1} &= 2\left\lfloor \frac{1}{2p_1^{\varepsilon^2}} \right\rfloor - 1 \ge \frac{1}{p_1^{\varepsilon^2}} - 2 \\ &= \exp\left\{\varepsilon^2(1-2\varepsilon)(1-a'(p))\frac{1}{p^\varepsilon}\log\frac{1}{p}\right\} - 2 > \exp\left(\frac{1}{p^\varepsilon}\right). \end{aligned}$$

Inductively, suppose we have shown, for some $n \ge 3$, that $L_{n-1}/L_{n-2} \ge \exp^{(n-2)}(1/p^\varepsilon)$, where $\exp^{(n)}(\cdot)$ stands for iterating the exponential function n times, that is, $\exp^{(1)}(\cdot) = \exp(\cdot)$, $\exp^{(2)}(\cdot) = \exp\{\exp(\cdot)\}, \dots,$ and so forth. Then

$$\begin{aligned} \frac{L_n}{L_{n-1}} &\ge \frac{1}{p_{n-1}^{\varepsilon^n}} - 2 \\ &= \exp\left\{\varepsilon^n(1-2\varepsilon^{n-1})(1-a(p_{n-2}))\frac{L_{n-1}}{L_{n-2}}\log\frac{1}{p_{n-1}}\right\} - 2 \\ &= \exp\left\{\varepsilon(1-2\varepsilon^{n-1})(1-a(p_{n-2}))\frac{L_{n-1}}{L_{n-2}}\log\frac{1}{p_{n-2}^{\varepsilon^{n-1}}}\right\} - 2 \\ &= \exp\left\{\varepsilon(1-2\varepsilon^{n-1})(1-a(p_{n-2}))\frac{L_{n-1}}{L_{n-2}}\log\frac{L_{n-1}}{L_{n-2}}\right\} - 2 \\ &\ge \exp^{(n-1)}\left(\frac{1}{p^\varepsilon}\right). \end{aligned} \tag{1.42}$$

Hence $L_n/L_{n-1} \to \infty$, as $n \to \infty$.

Thus we may choose n such that $l^{(n)}L_n < \log_{2\lambda} t \le (l^{(n)} + 2)L_n$, $l^{(n)}$ is an odd integer and

$$1 \le l^{(n)} < 2\left\lfloor \frac{1}{2p_n^{\varepsilon^{n+1}}} \right\rfloor - 1.$$

Let $\Lambda^{(n)}(l^{(n)}) \equiv (-l^{(n)}/2, l^{(n)}/2]^2$. Note that the corresponding region of Λ_l at level n is $(-(l^{(n)}+2)/2, (l^{(n)}+2)/2]^2$. Let $\Lambda^{(n)}(M^{(n)}) \equiv (-M^{(n)}/2, M^{(n)}/2]^2$ denote the corresponding region of $\Lambda_{4\lambda t}$ at level n. Then

$$(1.43) \qquad M^{(n)} \le \frac{4\lambda(2\lambda)^{(l^{(n)}+2)L_n}}{L_n} = \left(\frac{4\lambda}{L_n}\right)\exp\{(l^{(n)}+2)L_n\log(2\lambda)\}.$$

Note that

$$p_n = \exp\left\{-(1-2\varepsilon^n)(1-a(p_{n-1}))\frac{L_n}{L_{n-1}}\log\frac{1}{p_{n-1}}\right\},$$

hence

$$\begin{aligned}\log\frac{1}{p_n} &= (1-2\varepsilon^n)(1-a(p_{n-1}))\frac{L_n}{L_{n-1}}\log\frac{1}{p_{n-1}}\\ &= \prod_{i=1}^{n}(1-2\varepsilon^i)(1-a(p_{i-1}))L_n\log\frac{1}{p},\end{aligned}$$

where

$$a(p_0) \equiv a(p) = O\left(\frac{1}{\log(1/p)}\right),$$

$$a(p_1) = O\left(\frac{1}{\log(1/p_1)}\right) \le p^{\varepsilon}$$

and

$$a(p_i) = O\left(\frac{1}{\log(1/p_i)}\right) \le \left(\frac{L_i}{L_{i-1}}\right)^{-1}, \quad \text{for all } i = 2,\dots,n.$$

By (1.42), $L_i/L_{i-1} > \exp^{(i-1)}(1/p^{\varepsilon})$, for all $i = 2,\dots,n$; thus

$$a(p_i) = O\left(\frac{1}{\log(1/p_i)}\right) \le \exp^{(i-1)}\left(-\frac{1}{p^{\varepsilon}}\right), \quad \text{for all } i = 2,\dots,n.$$

Hence $\prod_{i=1}^{n}(1-2\varepsilon^i)(1-a(p_{i-1})) > b > 0$. When p is sufficiently small, b can be chosen independent of p. Therefore,

$$(1.44) \qquad \log\frac{1}{p_n} \ge bL_n\log\frac{1}{p} \quad \text{or} \quad L_n \le \frac{\log(1/p_n)}{b\log(1/p)}.$$

It now follows from (1.43) that

$$M^{(n)} \le \left(\frac{4\lambda}{L_n}\right)\exp\left\{\frac{\log 2\lambda}{b\log(1/p)}(l^{(n)}+2)\log\frac{1}{p_n}\right\}. \tag{1.45}$$

Denote

$P_M = P(\Lambda^{(n)}(M^{(n)})$ contains some rectangular region containing the origin $0[n]$ with length greater than $l^{(n)}$ which is internally spanned).

By Lemma 2,

$P_M \le P(\Lambda^{(n)}(M^{(n)})$ contains some rectangular region with length between $[l^{(n)}, 2l^{(n)}+1]$ which is internally spanned).

Let $l^{*(n)} \in [l^{(n)}, 2l^{(n)}+1]$, and note that

$$1 < l^{(n)} \le 2\left\lfloor \frac{1}{2p_n^{\varepsilon^{n+1}}}\right\rfloor - 1 < \frac{1}{p_n^{\varepsilon^{n+1}}} - 1;$$

hence $l^{(n)} < l^{*(n)} \le 2/p_n^{\varepsilon^{n+1}}$. By Corollary 5, we have

$$\begin{aligned} &P\big(\Lambda^{(n)}(l^{*(n)}) \text{ is internally spanned at level } n\big)\\ &\le \big(4p_n l^{*(n)}\big)^{l^{*(n)}/2} \le \left(p_n\left(\frac{8}{p_n^{\varepsilon^{n+1}}}\right)\right)^{l^{*(n)}/2} = \exp\left\{\frac{1}{2}l^{*(n)}\log\big(8p_n^{1-\varepsilon^{n+1}}\big)\right\}\\ &= \exp\left\{-\frac{1}{2}l^{*(n)}\log\frac{1}{8p_n^{1-\varepsilon^{n+1}}}\right\} \le \exp\left\{-\frac{1}{2}(1-\varepsilon^{n+1})l^{*(n)}\log\frac{1}{64p_n}\right\}\\ &\le \exp\left\{-\frac{1}{2}(1-\varepsilon^{n+1})l^{(n)}\log\frac{1}{64p_n}\right\}. \end{aligned} \tag{1.46}$$

Note that

$$P_M \le \big(M^{(n)}(l^{(n)}+1)\big)^2 P\big(\Lambda^{(n)}(l^{*(n)}) \text{ is internally spanned at level } n\big);$$

it follows from (1.45) and (1.46) that

$$\begin{aligned} P_M &\le \big(M^{(n)}(l^{(n)}+1)\big)^2 \exp\left\{-\frac{1}{2}(1-\varepsilon^{n+1})l^{(n)}\log\frac{1}{64p_n}\right\}\\ &\le \left(\frac{4\lambda(l^{(n)}+1)}{L_n}\exp\left\{\frac{\log 2\lambda}{b\log(1/p)}(l^{(n)}+2)\log\frac{1}{p_n}\right\}\right)^2\\ &\quad\times\exp\left\{-\frac{1}{2}(1-\varepsilon^{n+1})l^{(n)}\log\frac{1}{64p_n}\right\}\\ &\le \frac{1}{4}\exp\left\{-\frac{1}{2}(1-\varepsilon^{n+1})(1-a^*(p))l^{(n)}\log\frac{1}{p_n}\right\}, \end{aligned} \tag{1.47}$$

where $a^*(p) = O\dfrac{1}{\log(1/p)}$.

Recall (1.44): $\log(1/p_n) \geq bL_n \log(1/p)$. It follows from (1.47) that

$$
\begin{aligned}
P_M &\leq \frac{1}{4}\exp\left\{-\frac{b}{2}(1-\varepsilon^{n+1})(1-a^*(p))l^{(n)}L_n \log\frac{1}{p}\right\} \\
&\leq \frac{1}{4}\exp\left\{-c(l^{(n)}+2)L_n \log\frac{1}{p}\right\} \qquad (1.48) \\
&\leq \frac{1}{4}\exp\left\{-c(\log_{2\lambda} t)\log\frac{1}{p}\right\},
\end{aligned}
$$

where c is a positive constant. By Proposition 5, (1.48) implies that

$$P(E) \leq \frac{1}{4}\exp\left\{-c(\log_{2\lambda} t)\log\frac{1}{p}\right\}.$$

Therefore,

$$P\left(0 \in \xi_t^{\Lambda_{4\lambda t}, p}, E\right) \leq P(E) \leq \frac{1}{4}\exp\left\{-c(\log_{2\lambda} t)\log\frac{1}{p}\right\}. \tag{1.49}$$

Combining (1.41) with (1.49) we finally obtain (1.40):

$$
\begin{aligned}
P\left(0 \in \xi_t^{\Lambda_{4\lambda t}, p}\right) &= P\left(0 \in \xi_t^{\Lambda_{4\lambda t}, p}, E\right) + P\left(0 \in \xi_t^{\Lambda_{4\lambda t}, p}, E^c\right) \\
&\leq \frac{1}{2}\exp\left\{-c(\log_{2\lambda} t)\log\frac{1}{p}\right\} = \frac{1}{2}t^{-c\log_{2\lambda}(1/p)}.
\end{aligned}
$$

The proof of Theorem 1 is now complete. □

2. Proof of Theorem 2. To prove Theorem 2, it suffices to show that, when λ is sufficiently large, there is a constant $c \in (0,1)$, such that, for any given $\beta > 0$ and $x \in \mathbf{Z}^2$, $P(x \in \xi_t^{0,\beta}) > c$ for large t. By the monotonicity property of the process, for each $x \in \mathbf{Z}^2$, $P(0 \in \xi_t^{0,\beta})$ increases when β increases; thus we may focus our attention only on the case that $\beta > 0$ is sufficiently small.

For any given $\beta > 0$, we let N be an odd integer such that $N > 1/\beta$. Recall that $\Lambda_N \equiv (-N/2, N/2]^2$. Let $\xi_t^{N,\beta}$, $t \geq 0$, denote the type IV system with spontaneous birth at rate β and initial state $\Lambda_N \cap \mathbf{Z}^2$.

As in Section 0, for each $x \in \mathbf{Z}^2$ we denote its neighboring sites $\{x - \mathbf{e}_1, x - \mathbf{e}_2\}$ as pair 1, $\{x + \mathbf{e}_1, x - \mathbf{e}_2\}$ as pair 2, $\{x + \mathbf{e}_1, x + \mathbf{e}_2\}$ as pair 3 and $\{x - \mathbf{e}_1, x + \mathbf{e}_2\}$ as pair 4. Let $\eta_t^{(i)}$, $t \geq 0$, $i = 1,2,3,4$, be the systems with death rate identically 1 and birth rate $b_x^{(i)}(\xi)$ defined as follows:

$$b_x^{(i)}(\xi) = \begin{cases} \lambda, & \text{if the pair } i \text{ is occupied,} \\ 0, & \text{otherwise,} \end{cases} \qquad i \in \{1,2,3,4\}.$$

The initial state $\eta_0^{(i)}$ for each $i = 1,2,3,4$ is $\mathbf{Z}^2$.

As we already discussed in Section 0, all of the preceding five systems $\xi_t^{N,\beta}$ and $\eta_t^{(i)}$, $i = 1,2,3,4$, can be constructed by the graphical representation in a common probability space (Ω, F, P). Define $\chi_t^{(i)} = (\xi_t^{N,\beta}, \eta_t^{(i)})$, for $t \geq 0$. We regard a site $x \in \mathbf{Z}^2$ as occupied by $\chi_t^{(i)}$ if $\chi_t^{(i)}(x) = (1,1), (1,0)$ or $(0,0)$, and a site $x \in \mathbf{Z}^2$ is vacant for $\chi_t^{(i)}$ if $\chi_t^{(i)}(x) = (0,1)$. Then, for each $i = 1,2,3,4$, the initial state of $\chi_t^{(i)}$ is that all sites in $\Lambda_N \cap \mathbf{Z}^2$ are occupied and all sites in $(\Lambda_N \cap \mathbf{Z}^2)^c$ are vacant. For convenience, in what follows we will use "$x \in \chi_t^{(i)}$" and "$x \notin \chi_t^{(i)}$" to denote "x is occupied" and "x is vacant" at time t in the system $\chi_t^{(i)}$ interpreted as before, and we use $(\chi_t^{(i)})^c$ to denote the set of sites in $\mathbf{Z}^2$ which are vacant at time t in the system $\chi_t^{(i)}$, although $\chi_t^{(i)}$ is not a set in $\mathbf{Z}^2$.

Let $Q(i)$, $i = 1,2,3,4$, denote the ith quadrant of $\mathbf{Z}^2$, respectively. We will first prove that, for $i = 1,2,3,4$, the probability that the set $\Lambda_N \cap Q(i)$ remains entirely occupied by $\chi_t^{(i)}$, for all $t \in [0, \exp(2N^{2/3})]$, is large. This will be used to show that, for $i = 1,2,3,4$, the probability that the set $\Lambda_{N+2} \cap Q(i)$ is entirely occupied by $\chi_t^{(i)}$, for all $t \in [T_{N+2}, \exp(2(N+2)^{2/3})]$, is large, where $T_{N+2} = \exp(N^{2/3}) + 1 + \frac{2}{3}N^2$. We will then conclude by induction that, for all $n = N, N+2, \dots$ and $i = 1,2,3,4$, the probability that the set $\Lambda_n \cap Q(i)$ is entirely occupied by $\chi_t^{(i)}$, for all $t \in [T_n, \exp(2n^{2/3})]$, is large, where $\{T_n\}$ is a time sequence such that

$$T_0 = 0 \quad \text{and} \quad T_n - T_{n-2} = \exp\bigl((n-2)^{2/3}\bigr) + 1 + \tfrac{2}{3}(n-2)^2,$$

$$\text{for } n = N+2, N+4, \dots .$$

That will imply that with large probability the process $\xi_t^{N,\beta}$ will dominate

$$\bigcup_{i=1}^{4} \bigl(\eta_t^{(i)} \cap Q(i)\bigr) \cap \Lambda_n$$

after time T_n, for all $n = N, N+2, \dots$. Since when λ is sufficiently large, for any given $x \in \mathbf{Z}^2$, $\lim_{t\to\infty} P(x \in \eta_t^{(i)})$ is strictly greater than 0, for all $i = 1,2,3,4$, it follows that for each given $\beta > 0$ and $x \in \mathbf{Z}^2$, $\lim_{t\to\infty} P(x \in \xi_t(^{N,\beta}) > c > 0$. Finally, we will prove that, with probability 1, there exists at least one $N \times N$ square region in $\mathbf{Z}^2$ which is entirely occupied by $\xi_1^{0,\beta}$; thus by translation invariance and the result obtained for $\xi_t^{N,\beta}$ we will obtain the desired result.

To achieve our goal we first introduce the following definitions.

Definition. A function π: $[s,t] \to \mathbf{Z}^2$ is called an $[s,t]$-path if π is a step function with jumps only to the nearest neighbors.

Definition. A space–time set $A \subset \mathbf{Z}^2 \times [0,\infty)$ is called *path-connected* if, for all $(x,s),(y,t) \in A$, there exists an $[s,t]$-path π such that $\pi(s) = x$, $\pi(t) = y$ and $(\pi(u), u) \in A$ for all $u \in [s,t]$.

We will focus on the coupled process $\chi_t^{(1)}$ only. Analogous results can be obtained for $\chi_t^{(i)}$, $i = 2,3,4$, in much the same manner. Fix $\omega \in \Omega$, for

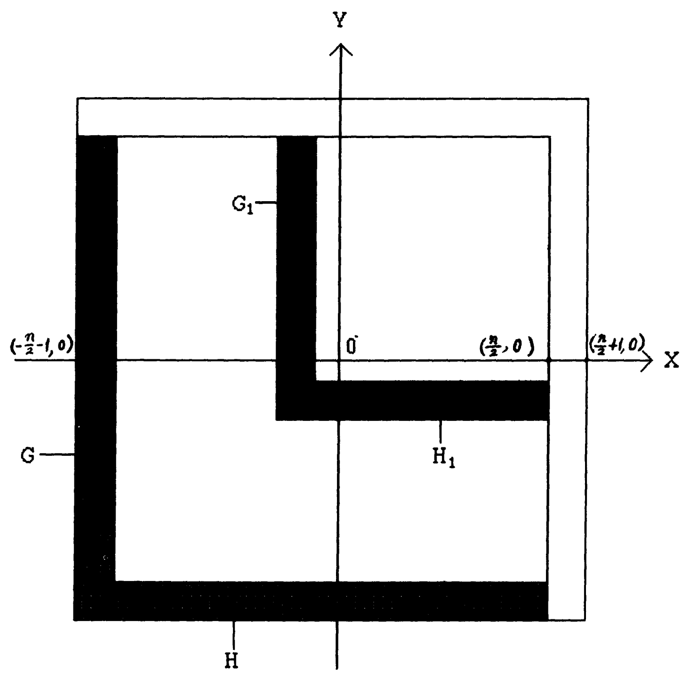

FIG. 4.

$n = N, N+2, \dots$ and $T > 0$. Let

$$G(n,T) = \left(-\frac{n}{2} - 1, -\frac{n}{2}\right) \times \left(-\frac{n}{2} - 1, \frac{n}{2}\right) \times [0,T],$$

$$H(n,T) = \left(-\frac{n}{2} - 1, \frac{n}{2}\right) \times \left(-\frac{n}{2} - 1, -\frac{n}{2}\right) \times [0,T],$$

$$G_1(n,T) = \left(-\frac{3}{2}, -\frac{1}{2}\right) \times \left(-\frac{3}{2}, \frac{n}{2}\right) \times [0,T], \quad \text{and}$$

$$H_1(n,T) = \left(-\frac{3}{2}, \frac{n}{2}\right) \times \left(-\frac{3}{2}, -\frac{1}{2}\right) \times [0,T].$$

We illustrate the cross sections of the preceding sets intersected by the plane $t = 0$ in Figure 4. We now claim the following.

LEMMA 5. *Fix* $\omega \in \Omega$, *let* $s \in [0,T]$, $V(T) = \bigcup_{t \in [0,T]}((\chi_t^{(1)})^c \times \{t\})$ *and* $V(N,T) = V(T) \cap ((-N/2 - 1, N/2)^2 \times [0,T])$. *Suppose there is a* $y_0 \in \Lambda_N \cap Q(1)$ *such that* $y_0 \notin \chi_s^{(1)}$. *Then there exists a path-connected set* $A = \bigcup_{t \in [u,s]}(A_t \times \{t\}) \subset V(N,T)$ *such that* $A_u \times \{u\} \subset G(N,T) \cup H(N,T) \cap \mathbf{Z}^2$ *and* $A_s \times \{s\} \ni (y_0, s)$.

PROOF. By the definition of $\chi_t^{(1)}$, for any $x \in \mathbf{Z}^2$, if $x \in \chi_{t-}^{(1)}$ but $x \notin \chi_t^{(1)}$, the only possibility is that $x \notin \xi_{t-}^{N,\beta}$, $x \notin \eta_{t-}^{(1)}$ and, at time t, x is replenished by $\eta_t^{(1)}$ but not by $\xi_t^{N,\beta}$. This means that $x - \mathbf{e}_1 \in \eta_{t-}^{(1)}$, $x - \mathbf{e}_2 \in \eta_{t-}^{(1)}$ but either $x - \mathbf{e}_1 \notin \xi_{t-}^{N,\beta}$ or $x - \mathbf{e}_2 \notin \xi_{t-}^{N,\beta}$, and thus either $x - \mathbf{e}_1 \notin \chi_{t-}^{(1)}$ or $x - \mathbf{e}_2 \notin \chi_{t-}^{(1)}$.

Note that the initial state $\chi_0^{(1)} = \Lambda_N \cap \mathbf{Z}^2$; it follows from the preceding argument that there exists $t_1 \in (0, s)$ such that $y_0 \in \chi_{t_1}^{(1)}$, $y_0 \notin \chi_t^{(1)}$, for all $t \in (t_1, s]$, and $y_1 \notin \chi_{t_1}^{(1)}$, for $y_1 = y_0 - \mathbf{e}_1$ or $y_0 - \mathbf{e}_2$. Apply the same argument to y_1: we find $t_2 \in (0, t_1)$ such that $y_1 \in \chi_{t_2}^{(1)}$, $y_1 \notin \chi_t^{(1)}$, for all $t \in (t_2, t_1]$, and $y_2 \notin \chi_{t_2}^{(1)}$, for $y_2 = y_1 - \mathbf{e}_1$ or $y_1 - \mathbf{e}_2$. By the fact that $\chi_0^{(1)} = \Lambda_N \cap \mathbf{Z}^2$ this procedure can be continued until we obtain a time sequence $0 \le t_k < \cdots < t_1 < s$ and a set of sites $\{y_0, \dots, y_k\}$ in $\mathbf{Z}^2$ such that, for $i = 1, \dots, k$, $y_{i-1} \in \chi_{t_i}^{(1)}$, $y_{i-1} \notin \chi_t^{(1)}$, for all $t \in (t_i, t_{i-1}]$, and $y_i \notin \chi_{t_i}^{(1)}$, for $y_i = y_{i-1} - \mathbf{e}_1$ or $y_{i-1} - \mathbf{e}_2$. Moreover, $(y_k, t_k) \in G(N,T) \cup H(N,T) \cap \mathbf{Z}^2$. It is easy to observe that the preceding procedure produces a path-connected set contained in $V(N,T)$ which contains (y_k, t_k) and (y_0, s). [A path-connected set produced as before will be simply called later "a $\chi^{(1)}$-vacant path starting from (y_k, t_k) and ending at (y_0, s)".] The proof is completed by letting $u = t_k$. □

COROLLARY 6. *Fix $\omega \in \Omega$. Suppose that for any $u, s \in [0,T]$, $u \le s$, there is no path-connected set $A = \bigcup_{t \in [u,s]} (A_t \times \{t\}) \subset V(N,T)$ such that*

$$(A_u \times \{u\}) \cap (G(N,T) \cup H(N,T)) \neq \phi$$

and

$$(A_s \times \{s\}) \cap \left(G_1(N,T) \cup H_1(N,T) \cap \mathbf{Z}^2\right) \neq \phi.$$

Then $\Lambda_N \cap Q(1)$ remains entirely occupied by $\chi_t^{(1)}$, for all $t \in [0,T]$.

Let $\Phi_1(N,T)$ denote the event described in the hypothesis of Corollary 6. Let $\Phi_i(N,T)$ denote the analogous events corresponding to the systems $\chi_t^{(i)}$, for $i = 2,3,4$. By Corollary 6 we know that if $\bigcap_{i=1}^4 \Phi_i(N,T)$ occurs, then $\Lambda_N \cap Q(i)$ remains entirely occupied by $\chi_t^{(i)}$ for all $t \in [0,T]$ and $i = 1,2,3,4$.

We would like first to prove that $P(\bigcap_{i=1}^4 \Phi_i(N, \exp(2N^{2/3})))$ is large. In order to do so we introduce the following lemma.

LEMMA 6. *Suppose λ is sufficiently large. For $n = N, N+2, \dots$ and $i = 1,2,3,4$, let*

$$K(i,n,T) = \bigcup_{t \in [0,T]} \left((\mathbf{Z}^2 \setminus \eta_t^{(i)}) \times \{t\}\right) \cap (\Lambda_n \times [0,T]).$$

Let $A \subset K(i,n,T)$ be a path-connected set. Then, for any real number $\gamma > 0$,

$$P(S(A) > \gamma) \le n^2 T \exp\{-a(\lambda)\gamma\},$$

where $a(\lambda)$ is an increasing function of λ independent of n and T and, for sufficiently large λ, $a(\lambda) > 0$; $S(A)$ denotes the surface area of A with respect *to* $\mu = \nu \times \lambda$ *on* $\mathbf{Z}^2 \times [0,\infty)$, *where* ν *is the counting measure on* $\mathbf{Z}^2$ *and* λ *is Lebesgue measure on* $[0,\infty)$.

This result is implied in the proof of Theorem 1 in Durrett and Gray (1990). The key to the proof is to define a "vacant dual process" for $\eta_t^{(i)}$ and then to apply the "contour method." We will take it as granted.

By the description given in the proof of Lemma 5, a $\chi^{(1)}$-vacant path contained in $V(N,T)$ can only start from the sites in the lower or the left boundary of Λ_{N+2} and it can only go up or right. Similar behaviors about $\chi^{(i)}$-vacant paths, $i = 2,3,4$, also can be obtained easily. Therefore, if a $\chi^{(1)}$-vacant path contained in $V(N,T)$ enters $\Lambda_N \cap Q(i)$, $i = 2,4$, it will stay there until it reaches $\Lambda_N \cap Q(1)$. Analogous consequences can be drawn for $\chi^{(i)}$-vacant paths, $i = 2,3,4$, as well.

Now we assume without loss of generality that the event

$$\left(\Phi_1(N, \exp(2N^{2/3}))\right)^c$$

occurs before any of the events $(\Phi_i(N,\exp(2N^{2/3})))^c$, $i = 2,3,4$, occurs. Then there is at least one $\chi^{(1)}$-vacant path π contained in $V(N,T)$ starting from the sites in $G(N,\exp(2N^{2/3})) \cup H(N,\exp(2N^{2/3}))$ and ending at the sites in $G_1(N,\exp(2N^{2/3})) \cup H_1(N,\exp(2N^{2/3}))$. It follows from the argument discussed in the preceding paragraph that there are two possibilities:

1. π entirely lies in one quadrant, namely, $\pi \subset \Lambda_{N+2} \cap Q(i)$, for some $i \in \{2,3,4\}$;
2. $\pi = \pi_1 \cup \pi_2$, where $\pi_1 = \pi \cap \Lambda_{N+2} \cap Q(3)$, $\pi_2 = \pi \cap \Lambda_{N+2} \cap Q(i)$, $i = 2$ or 4, and both π_1 and π_2 are connected paths with length at least 1.

Note that $S(\pi) \ge N/2$, hence in either case there is at least one $i \in \{2,3,4\}$ such that $S(\pi) \cap \Lambda_{N+2} \cap Q(i) \ge S(\pi)/2 \ge N/4$. Since none of the events $(\Phi_i(N,\exp(2N^{2/3})))^c, (i \in \{2,3,4\})$, has occurred, it follows that there is an $\eta^{(i)}$-vacant path $\pi^*(i)$ such that $\pi^*(i)$ coincides with $\pi \cap \Lambda_{N+2} \cap Q(i)$ and thus $S(\pi^*(i)) \ge N/4$. Applying Lemma 6 we now obtain the following corollary.

COROLLARY 7. *Suppose that* λ *is sufficiently large. Then*

$$P\left(\bigcup_{i=1}^{4} \left(\Phi_i(N, \exp(2N^{2/3}))\right)^c\right) \le CN^2 \exp(2N^{2/3}) \exp\{-a(\lambda)N\}, \tag{2.1}$$

and thus

$$P\left(\bigcap_{i=1}^{4} \Phi_i(N, \exp(2N^{2/3}))\right) \ge 1 - CN^2 \exp(2N^{2/3}) \exp\{-a(\lambda)N\}, \tag{2.2}$$

where C *is a positive constant and* $a(\lambda)$ *is the same as in Lemma* 6.

Combining Lemma 5 and (2.2) we obtain

$$P\big(\text{for } i = 1,2,3,4,\ \Lambda_N \cap Q(i) \text{ are entirely occupied by } \chi_t^{(i)},\ \text{for all } t \in [0, \exp(2N^{2/3})]\big) \tag{2.3}$$
$$\geq 1 - CN^2 \exp(2N^{2/3})\exp\{-a(\lambda)N\}.$$

Based on the preceding results we are now going to prove the following proposition.

PROPOSITION 6. *Suppose that* λ *is sufficiently large. Then*

$$P\Big(\textit{for } i = 1,2,3,4,\ \Lambda_{N+2} \cap Q(i) \textit{ are entirely occupied by } \chi_t^{(i)},\ \textit{for all } t \in \big[T_{N+2}, \exp\big(2(N+2)^{2/3}\big)\big]\Big)$$
$$\geq \big(1 - \exp\{-\tfrac{1}{2}a(\lambda)(N+2)\}\big)\big(1 - \exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}\big),$$

where $T_{N+2} = \exp(N^{2/3}) + 1 + \frac{2}{3}N^2$.

In order to prove Proposition 6, we need to introduce some preliminaries. We will still focus on the case $i = 1$.

For $n = N, N+2, \ldots,$ let $G(n,T)$ and $H(n,T)$ be as before. Let

$$G'(n,T) = \left(\frac{n}{2}, \frac{n}{2} + 1\right) \times \left(-\frac{n}{2} - 1, \frac{n}{2} + 1\right) \times [0,T],$$
$$H'(n,T) = \left(-\frac{n}{2} - 1, \frac{n}{2} + 1\right) \times \left(\frac{n}{2}, \frac{n}{2} + 1\right) \times [0,T].$$

Then

$$(\Lambda_{N+2} \setminus \Lambda_N) \times [0,T] = G(N,T) \cup H(N,T) \cup G'(N,T) \cup H'(N,T).$$

Let

$$I_1(n) = \left(\frac{n}{2}, \frac{n}{2} + 1\right) \times \left[-\sqrt{n}, -\frac{1}{2}\right),$$
$$I_2(n) = \left[-\sqrt{n}, -\frac{1}{2}\right) \times \left(\frac{n}{2}, \frac{n}{2} + 1\right).$$

See Figure 5 for illustration.

Recall the sets $D_x = \{(x, S_n(x)): n \geq 1\}$ and $B_x^* = \{(x, U_n(x)): n \geq 1\}$ introduced in Section 0. Define, for each $x \in \mathbf{Z}^2$ and $k \in \{0, 1, \ldots\}$,

$$J(x,k) = \{\text{there is an } s \in (k, k+1] \text{ such that } (x,s) \in B_x^* \text{ and } (x,t) \notin D_x \text{ for all } t \in (k, k+1]\},$$

$$E_1(N,k) = \bigcap_{x \in I_1(N)} J(x,k), \qquad E_2(N,k) = \bigcap_{x \in I_2(N)} J(x,k).$$

Let $E(N) = \bigcup_{k \in K}(E_1(N,k) \cap E_2(N,k))$, where $K = K(N) = \{0, 1, \ldots, \lfloor \exp(N^{2/3}) \rfloor\}$. We first claim the following lemma.

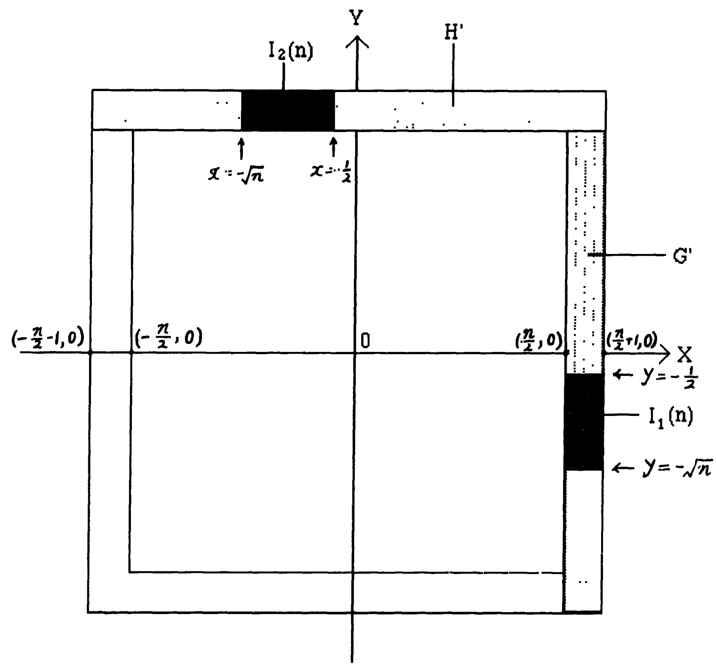

FIG. 5.

LEMMA 7. $P(E(N)) \geq 1 - \exp\{-\exp(N^{2/3}/2)\}$.

PROOF. Since, for each site $x \in \mathbf{Z}^2$ and $k \in \{0, 1, \dots\}$, the probability that there is an $s \in (k, k+1]$ such that $(x, s) \in B_x^*$ is equal to $1 - e^{-\beta}$, and that $(x, t) \notin D_x$ for all $t \in (k, k+1]$ is equal to e^{-1}, it follows that

$$P(J(x,k)) = e^{-1}(1 - e^{-\beta}).$$

For each $k \in \{0, 1, \dots\}$, the events $J(x, k), x \in \mathbf{Z}^2$ are independent. Hence, when β is sufficiently small, for each k,

$$\begin{aligned} P(E_1(N,k) \cap E_2(N,k)) &= 2^{-2\sqrt{N}}(1 - e^{-\beta})^{2\sqrt{N}} = \left(\frac{1}{e}\frac{e^{\beta} - 1}{e^{\beta}}\right)^{2\sqrt{N}} \\ &\geq \left(\frac{\beta}{e^{1+\beta}}\right)^{2\sqrt{N}} \geq \left(\frac{\beta}{3}\right)^{2\sqrt{N}} \geq \left(\frac{1}{3N}\right)^{2\sqrt{N}} \\ &= \exp\{-2\sqrt{N}\log 3N\}. \end{aligned}$$

Notice that the events $E_1(N,k) \cap E_2(N,k)$, $k = 0, 1, \dots,$ are independent.

Therefore,

$$P(E(N)) = P\Big(\bigcup_{k\in K}(E_1(N,k)\cap E_2(N,k))\Big)$$
$$= 1 - P\Big(\bigcap_{k\in K}(E_1(N,k)\cap E_2(N,k))^c\Big)$$
$$\geq 1 - \left(1 - \exp\{-2\sqrt{N}\log 3N\}\right)^{|K|},$$

where $|K|$ denotes the cardinality of K, which is equal to $\lfloor \exp(N^{2/3})\rfloor$. Since $1 - x \leq e^{-x}$, it follows that

$$P(E(N)) \geq 1 - \left(\exp\{-\exp\{-2\sqrt{N}\log 3N\}\}\right)^{|K|}$$
$$= 1 - \left(\exp\{-|K|\exp\{-2\sqrt{N}\log 3N\}\}\right)$$
$$= 1 - \exp\{-\lfloor \exp(N^{2/3})\rfloor \exp\{-2\sqrt{N}\log 3N\}\}$$
$$= 1 - \exp\{-\exp\{\lfloor N^{2/3}\rfloor - 2\sqrt{N}\log 3N\}\}$$
$$> 1 - \exp\{-\exp(N^{2/3}/2)\}. \qquad \square$$

REMARK. Notice that if $E_1(N,k)\cap E_2(N,k)$ occurs, then for all $x \in I_1(N)\cup I_2(N)$, either $\chi_t^{(1)}(x) = (1,1)$ or $\chi_t^{(1)}(x) = (1,0)$ at time $t = k+1$. Lemma 7 concludes that the probability that $E_1(N,k)\cap E_2(N,k)$ occurs for some $k \in K(N) = \{0,1,\dots,\lfloor \exp(N^{2/3})\rfloor\}$ is large.

Recall that

$$V(T) = \bigcup_{t\in[0,T]}\left((\chi_t^{(i)})^c \times \{t\}\right);$$

let

$$V^*(N,T) = V(T) = V(T)\cap(\Lambda_{N+2}\times[0,T]).$$

Let

$$I_1^*(N) = \left(\frac{N}{2}, \frac{N}{2}+1\right)\times\left(-\frac{N}{2}-1, -\sqrt{N}\right)$$

and

$$I_2^*(N) = \left(-\frac{N}{2}-1, -\sqrt{N}\right)\times\left(\frac{N}{2}, \frac{N}{2}+1\right).$$

Let $x_0 = (N/2 + \frac{1}{2}, 0)$, $z_0 = (0, N/2 + \frac{1}{2})$. We now claim the following lemma.

LEMMA 8. *Suppose that λ is sufficiently large. Let $\Psi(N,T)$, $\Theta(N)$ and $\Theta^*(N,k)$, $k \in K(N) = \{0,1,\dots,\lfloor \exp(N^{2/3})\rfloor\}$, be the events described as follows*:

$$\Psi(N,T) = \{\text{for } i = 1,2,3,4,\ \Lambda_N \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)},\ \text{for all } t \in [0,T]\};$$

$$\Theta(N) = \{\text{for any } u,s,\ 0 \le u < s < \lfloor \exp(N^{2/3})\rfloor + 1 + N^2, \text{ there is no path-connected set } \bigcup_{t\in[u,s]}(A_t \times \{t\}) \subset V^*(N, \exp(2N^{2/3})) \text{ such that } (A_u \times \{u\}) \cap (G(N,\exp(2N^{2/3})) \cup H(N,\exp(2N^{2/3})) \neq \phi \text{ and } A_s \times \{s\} \cap \{x_0 - \mathbf{e}_1, x_0 - \mathbf{e}_2, z_0 - \mathbf{e}_1, z_0 - \mathbf{e}_2\} \neq \phi\};$$

$$\Theta^*(N,k) = \{\text{for any } u,s \in (k+1, k+1+N^2],\ u<s, \text{ there is no path-connected set } \bigcup_{t\in[u,s]}(A_t \times \{t\}) \subset V^*(N,\exp(2N^{2/3})) \text{ such that either } (A_u \times \{u\}) \cap I_1^*(N) \times [k+1, K+1+N^2] \neq \phi \text{ and } (A_s \times \{s\}) \cap \{x_0 - \mathbf{e}_1, x_0 - \mathbf{e}_2\} \neq \phi \text{ or } (A_u \times \{u\}) \cap I_2^*(N) \times [k+1, k+1+N^2] \neq \phi \text{ and } (A_s \times \{s\}) \cap \{z_0 - \mathbf{e}_1, z_0 - \mathbf{e}_2\} \neq \phi\}.$$

Let $k^* = k^*(\omega) = \inf\{k\colon E_1(N,k) \cap E_2(N,k) \text{ occurs}\}$. *Then*

$$\{k^* \in K(N)\} = E(N) = \bigcup_{k\in K} (E_1(N,k) \cap E_2(N,k)).$$

Let $\Xi(N) = \Psi(N,\exp(2N^{2/3})) \cap \Theta(N) \cap \{k^* \in K(N)\} \cap \Theta^*(N,k^*)$. *Then*

$$P\big(\Lambda_{N+2} \cap Q(1) \text{ is entirely occupied by } \chi_t^{(1)} \text{ for all } t \in \big(k^* + 1 + \tfrac{2}{3}N^2, k^* + 1 + N^2\big) | \Xi(N)\big) > 1 - 2\exp(-bN),$$

where b is a positive constant.

PROOF. Since $\Psi(N,\exp(2N^{2/3}))$ occurs, it suffices to show that

$$P((N/2, N/2+1) \times (-\tfrac{1}{2}, N/2) \cup (-1/2, N/2+1) \times (N/2, N/2+1) \cap \mathbf{Z}^2 \text{ is entirely occupied by } \chi_t^{(1)}, \text{ for all } t \in (k^* + 1 + \tfrac{2}{3}N^2, k^* + 1 + N^2) | \Xi(N)) > 1 - 2\exp(-bN).$$

Under the condition $\Xi(N)$, if a spontaneous birth occurs at site x_0 at $t = \tau_0$, $k^* + 1 < \tau_0 < k^* + 1 + N^2$, x_0 will remain occupied by $\chi_t^{(1)}$ at least until $t = k^* + 1 + N^2$. After that happens, if a spontaneous birth occurs at site $x_0 + \mathbf{e}_2$ at $t = \tau_1$, $\tau_0 < \tau_1 < k^* + 1 + N^2$, then $x_0 + \mathbf{e}_2$ will remain occupied by $\chi_t^{(1)}$ at least until $t = k^* + 1 + N^2$. This procedure will continue until the last site $(N/2 + \frac{1}{2}, N/2 - \frac{1}{2})$ is hit by a spontaneous birth at time $t = \tau_{N/2-1/2}$. Let $\sigma_0 = \tau_0 - (k^* + 1)$, $\sigma_j = \tau_j - \tau_{j-1}$, $j = 1,\dots,N/2 - \frac{1}{2}$. Then $\sigma_0,\dots,\sigma_{N/2-1/2}$ are i.i.d. random variables with common distribution exponential with mean $1/\beta$. Therefore, by the large deviation principle, for

$\forall\ \varepsilon > 0$, there exists a $b'(\varepsilon) > 0$ such that

$$P\left(\sigma_0 + \sigma_1 + \cdots + \sigma_{N/2-1/2} > \frac{1}{2}N\left(\frac{1}{\beta} + \varepsilon\right)\right) < \exp\left(-\frac{1}{2}b'(\varepsilon)N\right);$$

in particular, we may choose $\varepsilon = 1$ and denote $b' = b'(1)$. Then

$$P\left(\sigma_0 + \sigma_1 + \cdots + \sigma_{N/2-1/2} > \frac{1}{2}N\left(\frac{1}{\beta} + 1\right)\right) < \exp\left(-\frac{1}{2}b'N\right).$$

Note that $N > 1/\beta$ and β is sufficiently small, hence

$$P\left(\sigma_0 + \sigma_1 + \cdots + \sigma_{N/2-1/2} \geq \tfrac{2}{3}N^2\right) < \exp\left(-\tfrac{1}{2}b'N\right).$$

Therefore

$$\begin{aligned}&P((N/2, N/2+1) \times (-\tfrac{1}{2}, N/2) \cap \mathbf{Z}^2 \text{ is entirely occupied by} \\ &\quad \chi_t^{(1)}, \text{ for all } t \in (k^* + 1 + \tfrac{2}{3}N^2, k^* + 1 + N^2) | \Xi(N)) \\ &\quad > 1 - \exp\left(-\tfrac{1}{2}b'N\right).\end{aligned}$$

Similarly, we may obtain that

$$\begin{aligned}&P((-\tfrac{1}{2}, N/2+1) \times (N/2, N/2+1) \cap \mathbf{Z}^2 \text{ is entirely occupied} \\ &\quad \text{by } \chi_t^{(1)} \text{ for all } t \in (k^* + 1 + \tfrac{2}{3}N^2, k^* + 1 + N^2) | \Xi(N)) \\ &\quad > 1 - \exp\left(-\tfrac{1}{2}b'N\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&P((N/2, N/2+1) \times (-\tfrac{1}{2}, N/2) \cup (-\tfrac{1}{2}, N/2+1) \times \\ &\qquad (N/2, N/2+1) \cap \mathbf{Z}^2 \text{ is entirely occupied by } \chi_t^{(1)}, \text{ for} \\ &\qquad \text{all } t \in (k^* + 1 + \tfrac{2}{3}N^2, k^* + 1 + N^2) | \Xi(N)) \\ &\qquad \geq 1 - 2\exp\left(-\tfrac{1}{2}b'N\right).\end{aligned}$$

The conclusion follows by letting $b = \frac{1}{2}b'$. □

When λ is sufficiently large, from (2.3) we know that

$$P\left(\Psi\left(N, \exp(2N^{2/3})\right)\right) > 1 - CN^2\exp(2N^{2/3})\exp\{-a(\lambda)N\}. \tag{2.4}$$

It follows from Corollary 7 that

$$P(\Theta(N)) \geq 1 - N^2\left(\exp(N^{2/3}) + 1 + N^2\right)^2\exp\{-a(\lambda)N\}. \tag{2.5}$$

By an argument similar to that employed in the proof of Corollary 7, combined with Lemma 7, we obtain

$$\begin{aligned}&P(\{k^* \in K(N)\} \cap \Theta^*(N, k^*)) \\ &\qquad > 1 - \exp\left\{-\exp\left(\frac{N^{2/3}}{2}\right)\right\} - \left(N^2\right)^2\exp\left\{-a(\lambda)\sqrt{N}\right\} \\ &\qquad = 1 - \exp\left\{-\exp\left(\frac{N^{2/3}}{2}\right)\right\} - N^4\exp\left\{-a(\lambda)\sqrt{N}\right\}.\end{aligned} \tag{2.6}$$

Therefore, it follows from (2.4), (2.5) and (2.6) that

$$
\begin{aligned}
P(\Xi(N)) &= P\big(\Psi(N, \exp(2N^{2/3})) \cap \Theta(N) \cap \{k^* \in K(N)\} \cap \Theta^*(N, k^*)\big) \\
&\geq 1 - CN^2 \exp(2N^{2/3}) \exp\{-a(\lambda)N\} \\
&\quad - N^2\big(\exp(N^{2/3}) + 1 + N^2\big)^2 \exp\{-a(\lambda)N\} \\
&\quad - \exp\left\{-\exp\left(\frac{N^{2/3}}{2}\right)\right\} - N^4 \exp\{-a(\lambda)\sqrt{N}\} \\
&\geq 1 - 4N^4 \exp\{-a(\lambda)\sqrt{N}\}.
\end{aligned}
\tag{2.7}
$$

Proof of Proposition 6. Combining Lemma 8 and (2.7), we obtain

$$
\begin{aligned}
&P(\exists\ k \in K(N) \text{ such that } \Lambda_{N+2} \cap Q(1) \text{ is entirely occupied} \\
&\qquad \text{by } \chi_t^{(1)}, \text{ for all } t \in (k + 1 + \tfrac{2}{3}N^2, k + 1 + N^2)) \\
&\qquad \geq (1 - 2\exp(-bN))\big(1 - 4N^4 \exp\{-a(\lambda)\sqrt{N}\}\big) \\
&\qquad \geq 1 - 2\exp(-bN) - 4N^4 \exp\{-a(\lambda)\sqrt{N}\} \\
&\qquad \geq 1 - 5N^4 \exp\{-a(\lambda)\sqrt{N}\}.
\end{aligned}
$$

Similarly, we may obtain the preceding conclusion for $i = 2, 3, 4$ as well. Therefore,

$$
\begin{aligned}
&P(\text{for } i = 1, 2, 3, 4,\ \exists\ k \in K(N) \text{ such that } \Lambda_{N+2} \cap Q(i) \text{ is} \\
&\qquad \text{entirely occupied by } \chi_t^{(i)}, \text{ for all } t \in (k^* + 1 + \tfrac{2}{3}N^2, \\
&\qquad k^* + 1 + N^2)) \\
&\qquad \geq 1 - 20N^4 \exp\{-a(\lambda)\sqrt{N}\}.
\end{aligned}
$$

By the strong Markov property, we may apply the same argument as in the proof of Corollary 7 and (2.3) to the process $\chi_t^{(i)}$, $i = 1, 2, 3, 4$, for $t \geq k^* + 1 + \frac{2}{3}N^2$ and obtain that

$$
\begin{aligned}
&P(\text{for all } i = 1, 2, 3, 4,\ \exists\ k \in K(N) \text{ such that } \Lambda_{N+2} \cap Q(i) \\
&\qquad \text{is entirely occupied by } \chi_t^{(i)}, \text{ for all } t \in (k^* + 1 + \\
&\qquad \tfrac{2}{3}N^2, \exp[2(N+2)^{2/3}])) \\
&\qquad \geq \big(1 - C(N+1)^2 \exp\big[2(N+2)^{2/3}\big] \exp\{-a(\lambda)(N+2)\}\big) \\
&\qquad\quad \times \big(1 - 20N^4 \exp\{-a(\lambda)\sqrt{N}\}\big).
\end{aligned}
$$

Note that $1 \leq k \leq \lfloor \exp(N^{2/3}) \rfloor$; it follows then that

$$
\begin{aligned}
&P\Big(\text{for all } i = 1, 2, 3, 4,\ \Lambda_{N+2} \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}, \\
&\qquad\qquad\qquad\qquad \text{for all } t \in \big(T_{N+2}, \exp\{2(N+2)^{2/3}\}\big)\Big) \\
&\qquad \geq \big(1 - C(N+1)^2 \exp\{2(N+2)^{2/3}\} \exp\{-a(\lambda)(N+2)\}\big) \\
&\qquad\quad \times \big(1 - 20N^4 \exp\{-a(\lambda)\sqrt{N}\}\big) \\
&\qquad \geq \big(1 - \exp\{-\tfrac{1}{2}a(\lambda)(N+2)\}\big)\big(1 - \exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}\big),
\end{aligned}
$$

where $T_{N+2} = \exp(N^{2/3}) + 1 + \frac{2}{3}N^2$. □

By the Markov property we may iterate the procedure employed in the Proof of Proposition 6 inductively and obtain, for all $n = N+4, N+6, \dots,$

$$\begin{aligned} &P\big(\text{for all } i = 1,2,3,4,\ \Lambda_n \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)},\\ &\qquad\qquad\qquad\qquad \text{for all } t \in \big(T_n, \exp\{2n^{2/3}\}\big)\big)\\ &\quad \geq \big(1 - \exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}\big)\big(1 - \exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N+2}\}\big)\cdots\\ &\qquad \times\big(1 - \exp\{-\tfrac{1}{2}a(\lambda)\sqrt{n-2}\}\big)\big(1 - \exp\{-\tfrac{1}{2}a(\lambda)n\}\big)\\ &\quad \geq 1 - 2\sqrt{N}\exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}, \end{aligned} \tag{2.8}$$

where T_n, $n = N+4, N+6, \dots,$ are so defined that $T_n - T_{n-2} = \exp[(n-2)^{2/3}] + 1 + \frac{2}{3}(n-2)^2$. Notice that, for each $n \in \{N, N+2, \dots\}$, $T_{n+2} \in (T_n, \exp[2n^{2/3}])$; hence (2.8) implies that

$$\begin{aligned} &P\big(\text{for all } i = 1,2,3,4,\ \Lambda_n \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)},\\ &\qquad\qquad\qquad\qquad \text{for all } t \in (T_n, \infty)\big)\\ &\quad \geq 1 - 2\sqrt{N}\exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}. \end{aligned} \tag{2.9}$$

For any given $x \in \mathbf{Z}^2$, there is an $N_0 \in \{N, N+2, \dots,\}$ such that $x \in \Lambda_{N_0} \cap Q(i)$ for some $i \in \{1,2,3,4\}$. Without loss of generality we may assume $i = 1$. It follows from (2.9) that

$$P\big(x \in \chi_t^{(1)}, \text{for all } t \in (T_{N_0}, \infty)\big) \geq 1 - 2\sqrt{N}\exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}.$$

That is,

$$\begin{aligned} &P\big((\xi_t^{N,\beta}(x), \eta_t^{(1)}(x)) = (1,1), (0,0) \text{ or } (1,0) \text{ for all } t \in (T_{N_0}, \infty)\big)\\ &\quad \geq 1 - 2\sqrt{N}\exp\{-\tfrac{1}{2}a(\lambda)\sqrt{N}\}. \end{aligned}$$

By Theorem 1 of Durrett and Gray (1990), when λ is sufficiently large, $\lim_{t\to\infty} P(x \in \eta_t^{(1)})$ is strictly greater than 0. Therefore, it follows that there exists $c \in (0,1)$ which is independent of β such that $\lim_{t\to\infty} P(x \in \xi_t^{N,\beta}) > c$.

Our final step of the proof of Theorem 2 is to claim that, with probability 1, there is at least one $N \times N$ square region in $\mathbf{Z}^2$ such that it is entirely occupied by $\xi_t^{0,\beta}$ at time $t = 1$. Then, by the Markov property and translation invariance and the preceding results, we finally conclude that for any given $\beta > 0$, $\lim_{t\to\infty} P(x \in \xi_t^{0,\beta}) > c$, provided λ is sufficiently large.

Proof of the claim. Partition $\mathbf{R}^2$ into $N \times N$ square regions and situate those squares so that one of them has the origin as its center (this one is actually Λ_N.) Remember that N is an odd integer; thus, if the square regions are so situated, their centers belong to $\mathbf{Z}^2$. Let $\mathbf{Y}$ be a subset of $\mathbf{Z}^2$ which consists of all centers of the previously mentioned square regions. Then each of those squares can be denoted by Λ_{N+y}, for some $y \in \mathbf{Y}$.

Recall the sets D_x and B_x^* defined in Section 0 and the definition of the event $J(x, 0)$:

$$J(x,0) = \{\text{there is an } s \in (0,1] \text{ such that } (x,s) \in B_x^* \text{ and } (x,t) \notin D_x \text{ for all } t \in (0,1]\}.$$

For $x \in \mathbf{Z}^2$ and $y \in \mathbf{Y}$, let

$$E_y = \bigcap_{x \in \Lambda_{N+y}} J(x,0).$$

In the proof of Lemma 7 we have already shown that

$$P(J(x,0)) = e^{-1}(1 - e^{-\beta}).$$

The events $J(x, 0)$, $x \in \mathbf{Z}^2$, are independent; hence, for each $y \in \mathbf{Y}$,

$$P(E_y) = e^{-N^2}(1 - e^{-\beta})^{N^2}$$
$$= \left(\frac{1}{e}\frac{e^{\beta}-1}{e^{\beta}}\right)^{N^2} \geq \left(\frac{\beta}{e^{1+\beta}}\right)^{N^2} \geq \left(\frac{\beta}{3}\right)^{N^2} \geq \left(\frac{1}{3N}\right)^{N^2} > 0.$$

Since the events E_y, $y \in \mathbf{Y}$, are independent, it follows from the Borel–Cantelli lemma that

$$P\left(\bigcup_{y \in \mathbf{Y}} E_y\right) = 1.$$

The proof of Theorem 2 is complete. □

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