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BROWNIAN FLUCTUATIONS OF THE EDGE FOR CRITICAL REVERSIBLE NEAREST-PARTICLE SYSTEMS[1]

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We apply an invariance principle due to De Masi, Ferrari, Goldstein and Wick to the edge process for critical reversible nearest-particle systems. Their result also gives an upper bound for the diffusion constant that we compute explicitly. A comparison between the movement of the edge, when the other particles are frozen, and a random walk allows us to find a lower bound for the diffusion constant. This shows that the right renormalization for the edge to converge to a nondegenerate Brownian motion is the usual one. Note that analogous results for nearest-particle systems are only known for the contact process in the supercritical case.

1. Introduction and statement of results. The nearest-particle system on the integers Z is a Markov process introduced by Spitzer [7] which evolves on $\{0,1\}^Z$ in the following way. Let η be a configuration of the process: sites $x \in Z$ for which $\eta(x) = 1$ are considered to be occupied by a particle, and sites for which $\eta(x) = 0$ are vacant. Let

$$l_x(\eta) = x - \max\{y < x : \eta(y) = 1\},$$
$$r_x(\eta) = \min\{y > x : \eta(y) = 1\} - x,$$

which may be infinite. If $\eta(x) = 0$, then a particle appears at x at a rate $\lambda b(l_x(\eta), r_x(\eta))$, where λ is a positive real parameter. If $\eta(x) = 1$, then the particle at x disappears at rate 1. In this paper we will assume that there exists a strictly positive probability density b on the positive integers such that

$$\text{(R)} \qquad b(l,r) = \frac{b(l)b(r)}{b(l+r)},$$

(A) $\qquad b(l,r)$ is a decreasing function of l and r,

$$\text{(1)} \qquad b(l,\infty) = b(\infty,l) = b(l),$$

$$\text{(2)} \qquad \lim_{l\to\infty} \frac{b(l)}{b(l+1)} = 1.$$

Under the attractiveness assumption (A), it is known that the law of the process converges to some probability measure when time goes to ∞ and the initial distribution is the point mass on the configuration with all sites on Z

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occupied. Let λ_c be the infimum of the λ such that the limit distribution is not the point mass on the empty configuration. When the limit distribution is the point mass on the empty configuration, we will say that the system dies out.

Liggett [5] proved that under the assumptions (R), (A) and (2):

$$\lambda_c = 1.$$

He also proved that for $\lambda = 1$ the system does not die out if and only if

$$M = \sum_{l \geq 1} l b(l) < \infty. \tag{3}$$

Under the same assumptions, Spitzer [7] proved that for $\lambda > 1$ the stationary renewal measure with density $\gamma^l b(l)$ is reversible for the process with infinite particles on the left and on the right of the origin, where γ is the solution of the following equation:

$$\sum_{l \geq 1} \lambda \gamma^l b(l) = 1.$$

When $\lambda = 1$ and under (3), the preceding property remains true. A more detailed exposition of these results together with many other results known about reversible nearest-particle systems can be found in Chapter 7 of Liggett [5].

We will prove that the rightmost particle of a critical ($\lambda = 1$) reversible nearest-particle system has Brownian fluctuations. The only analogous result for nearest-particle systems was proved by Galves and Presutti [4] for the supercritical contact process. The problem is open for the critical contact process, but it is generally believed that the fluctuations are not Brownian in that case.

We need some more notation to state precisely our result. Let m be the renewal measure with density b on the set of configurations which have a particle at the origin and no particle on the right of the origin. Let r_t be the rightmost particle of the critical reversible nearest-particle system under the initial probability m. We will also assume that

$$\sum_{l \geq 1} l^2 b(l) < \infty. \tag{4}$$

We can now state the main result of this paper.

THEOREM 1. *In the critical case* ($\lambda = 1$) *and under the initial distribution* m,

$$a r_{a^{-2}t}$$

converges to a Brownian motion as $a \to 0$ *in the Skorohod space. Furthermore, if* D *is the corresponding diffusion constant, then we have*

$$2\left(\sum_{l \geq 1} l b(l)\right)^2 \leq D \leq 2 \sum_{l \geq 1} l^2 b(l).$$

The convergence to a Brownian motion as well as the upper bound for D are direct consequences of an invariance principle for reversible Markov processes proved by De Masi, Ferrari, Goldstein and Wick. Our task will be to check that their invariance principle applies to our case. This will be done in Section 2. The more interesting part of our work will be done in Section 3, where we will find a lower bound for D. To do so, we define a process where deaths can only occur at the rightmost particle and births can only occur on the right of the rightmost particle. These deaths and births occur at the same rates than for the reversible nearest-particle system. We show that the rightmost particle of this new process behaves at large times like a random walk. This allows us to compute explicitly the diffusion constant D' corresponding to the new process. We finally use an observation in [3] to show that $D \geq D'$ and this gives the lower bound in Theorem 1.

Finally, in Section 4 we discuss two open problems.

We are only able to treat the critical case because, to apply the results of [2] and [3], we need the existence of a reversible probability for the system seen from the edge. The probability m has this property in the critical case but we think that in the supercritical case there is no reversible probability.

2. The edge process converges to a Brownian motion. We are interested in semi-infinite reversible nearest-particle systems as seen from the edge. This means that births and deaths occur with the rates that we already described, but when the rightmost particle dies or when a particle appears on the right of the rightmost particle, then we translate the configuration to have always the rightmost particle at the origin. We will denote this process $\eta(t)$ which evolves on

$$X = \{\eta : \eta(0) = 1, \eta(x) = 0 \text{ for } x > 0\}.$$

Let $\tilde{X}$ be the subset of the configurations of X which have infinite particles. We will consider another process on X that we will call the frozen process. In this process particles can only appear on the right of the rightmost particle and the only particle which can disappear is the rightmost one, but these deaths and births occur at the same rates as for $\eta(t)$. We will denote this frozen process seen from the edge as $\eta'(t)$.

It is possible to use the techniques in Liggett [5] (Theorem 3.5) to construct the Markov processes $\eta(t)$ and $\eta'(t)$ on $\tilde{X}$.

We will prove that r_t and r'_t, the rightmost particles of the critical reversible nearest-particle system and the frozen process, under initial distribution m, converge to Brownian motions. To do so, we will use an invariance principle which holds for "antisymmetric" reversible processes (see [2] and [3]). We will now state this theorem in the particular case of an edge process.

THEOREM 2 (De Masi, Ferrari, Goldstein and Wick). *Let $\eta(t)$ be a semi-infinite Markov process on Z. Assume that $\eta(t)$ is reversible and ergodic with respect to a probability measure m. Let r_t be the edge process under the initial distribution m. Let F_t be the σ-algebra generated by $\{\eta(s);\ s \leq t\}$. If r_t is in*

$L^2(m)$ for all t and if

$$v = \lim_{h \to 0} \frac{1}{h} E_m(r_{t+h} - r_t | F_t)$$

exists as a limit in $L^1(m)$, then

$$a r_{a^{-2} t}$$

converges in distribution, when $a \to 0$, to a Brownian motion in the Skorohod space. If, in addition, v is in $L^2(m)$, then we have the following expression for the diffusion constant D:

$$D = C - 2\int_0^{+\infty} \langle v, S(t)v \rangle_m \, dt,$$

where $C = \lim_{h \to 0}(1/h)E_m(r_h^2)$, $S(t)$ is the Markov semigroup corresponding to $\eta(t)$ and $\langle f, g \rangle_m = \int fg \, dm$.

We will now check that the hypotheses of Theorem 2 are satisfied in our case. Let m be the renewal measure with density b on $\tilde{X}$. We begin with the following lemma.

LEMMA 1. *The probability measure m is reversible for $\eta(t)$ and $\eta'(t)$.*

PROOF. Let L and L' be the Markov generators corresponding respectively to the processes $\eta(t)$ and $\eta'(t)$. We have

$$L = L_0 + L',$$

where L_0 is also a Markov generator corresponding to a nearest-particle system which has a particle at the origin which cannot die and such that there are no births on the right of the origin. This form of writing L is just a way to distinguish the behavior of the rightmost particle from the behavior of the other particles.

Let f be a function on X that depends on finitely many coordinates. We have

$$L_0 f(\eta) = \sum_{x<0:\, \eta(x)=0} \lambda b(l_x(\eta), r_x(\eta))(f(\eta_x) - f(\eta)) + \sum_{x<0:\, \eta(x)=1} 1(f(\eta_x) - f(\eta)),$$

where $\eta_x(y) = \eta(y)$ for $x \neq y$ and $\eta_x(x) = 1 - \eta(x)$ and

$$L' f(\eta) = \sum_{r \geq 1} \lambda b(r, \infty)\left(f(\tau_r^1 \eta) - f(\eta)\right) + \sum_{l \geq 1} 1_{\{X_1 = l\}}\left(f(\tau_{-l}^2 \eta) - f(\eta)\right),$$

where $\tau_r^1\eta(0) = 1$, $\tau_r^1\eta(y) = \eta(y + r)$ for $y < 0$ and $\tau_r^1\eta(y) = 0$ for $y > 0$. Similarly, $\tau_{-l}^2\eta(y) = \eta(y - l)$ for $y \leq 0$ and $\tau_{-l}^2\eta(y) = 0$. X_1 is the distance between the rightmost particle and the next one.

To prove the lemma, it is sufficient to have that for any continuous bounded functions f and g:

$$\langle f, Lg\rangle_m = \langle Lf, g\rangle_m.$$

Using the reversibility of the renewal measure with density b for the process with infinite particles on the left and on the right of the origin, an easy computation shows that

$$\langle f, L_0 g\rangle_m = \langle L_0 f, g\rangle_m$$

so it is sufficient to prove the reversibility for L'.

Let $x_1 < x_2 < \cdots < x_n$ be positive integers and let us define A as the subset of the configurations of $\tilde{X}$ such that the sites $0, -x_1, -x_2, \ldots, -x_n$ are occupied and no other site is occupied between 0 and $-x_n$. By the definition of m we have

$$m(A) = b(x_1)b(x_2 - x_1)\cdots b(x_n - x_{n-1}).$$

Let B be a subset of the same form as A and define

$$f = 1_A \quad \text{and} \quad g = 1_B.$$

It is easy to verify, using the hypothesis (1), that

$$\langle f, L'g\rangle_m = \langle L'f, g\rangle_m$$

but the functions 1_A generate all the bounded continuous functions on $\tilde{X}$, so this proves that m is reversible for $\eta'(t)$ and this implies that m is reversible for $\eta(t)$. □

We will now prove the following lemma.

LEMMA 2. *The processes $\eta(t)$ and $\eta'(t)$ are ergodic with respect to the probability m.*

PROOF. Let D be the set of the functions on X which depend on finitely many sites. Let (P) be the following property:

(P) If $f \in D$ is such that $\langle f, Lf\rangle_m = 0$, then f is m-almost surely constant.

To prove ergodicity, we will use the following criterion: If (P) holds, then the process is ergodic with respect to m. This criterion is well known; see, for instance, Reed and Simon [6] for a proof. We are grateful to the referee for suggesting this reference.

Let f be a function in D such that

$$\langle f, Lf\rangle_m = 0.$$

We have

$$L = L_0 + L'.$$

Using that m is invariant for the processes $\eta(t)$ and $\eta'(t)$, we can write, for any function g in D:

$$\langle g, L_0 g\rangle_m = \frac{-1}{2}\sum_{x<0}\langle c(x,\eta)(g(\eta_x) - g(\eta)), g(\eta_x) - g(\eta)\rangle_m$$

and we have a similar formula for L' so that

$$\langle g, L_0 g\rangle_m \le 0 \quad \text{and} \quad \langle g, L'g\rangle_m \le 0.$$

This implies for f that

$$\langle f, L_0 f\rangle_m = \langle f, L'f\rangle_m = 0$$

and so for all $x < 0$:

$$f(\eta_x) = f(\eta) \quad m\text{-almost surely.}$$

Thus f is m-almost surely constant. So (P) holds for $\eta(t)$ and this process is ergodic with respect to m.

Using the same criterion, it is easy to see that $\eta'(t)$ is ergodic with respect to m. □

We continue checking the hypotheses of Theorem 2 by showing that the rightmost particles of $\eta(t)$ and $\eta'(t)$ are in $L^2(m)$.

Let $H(t)$ be the rightmost particle of a process where births occur with the usual rates but deaths do not occur. Let f_t be the number of births on the right of the rightmost particle. It is a Poisson process with rate 1 ($\lambda = 1$). We can write

$$H(t) = \sum_{i=0}^{f_t} W_i,$$

where $W_0 = 0$ and the W_i, for $i \ge 1$, are independent identically distributed random variables with probability density b. This is a consequence of the hypothesis $b(l,\infty) = b(l)$.

On the other hand, let $G(t)$ be the rightmost particle of a process where deaths occur with the usual rates but births do not occur. Let g_t be the number of particles which died by time t on the right of $G(t)$. The random variable g_t is geometric with parameter e^{-t}. Under the initial distribution m, we have

$$G(t) = -\sum_{i=0}^{g_t} Z_i,$$

where $Z_0 = 0$ and the Z_i, for $i \ge 1$, are independent identically distributed random variables with probability density b. This is a consequence of the definition of m.

We can construct the three processes jointly so that

$$G(t) \le r_t \le H(t).$$

Using that f_t and the W_i are independent, we have

$$\operatorname{Var}(H(t)) = E(f_t)\operatorname{Var}(W_1) + E(W_1)^2\operatorname{Var}(f_t).$$

This and a similar formula for $G(t)$ imply that under assumption (4), r_t is in $L^2(m)$ for all time t.

Observe that the same arguments show that r'_t is in $L^2(m)$.

Comparing r_t to $H(t)$ and $G(t)$, it is not difficult to see that

$$v = \lim_{h\to 0} \frac{1}{h} E(r_{t+h} - r_t | F_t) = \sum_{l\geq 1} lb(l) - \sum_{l\geq 1} l 1_{\{X_1 = l\}}$$

and that this limit occurs in $L^1(m)$, where X_1 is the distance between the first two particles. The same comparisons show that

$$C = \lim_{h\to 0} \frac{1}{h} E_m(X^2(h)) = 2\sum_{l\geq 1} l^2 b(l).$$

Observe that the corresponding terms C' and v' for the frozen process are respectively equal to C and v (this is a consequence of the nearest-particle interaction).

From (4) we see that v is in $L^2(m)$. From these remarks we can conclude that Theorem 2 can be applied to r_t and r'_t.

Using the reversibility of m, we have that, for any cylindrical function f:

$$\langle f, S(t) f \rangle_m \geq 0.$$

So the expression of the diffusion constant D given in Theorem 2 implies the following corollary.

COROLLARY 1. $D \leq 2\sum_{l\geq 1} l^2 b(l)$.

At this point we do not have a lower bound for D and the limit Brownian motion could be degenerate; in the next section we will prove that this is not the case.

3. The diffusion constant is strictly positive. We will use a strategy suggested in De Masi, Ferrari, Goldstein and Wick [2] to find a lower bound for the diffusion constant D. We will find the diffusion constant D' of the frozen process and then show that D is larger than D'.

To find D', we will compare the movement of the edge r'_t of the frozen process $\eta'(t)$ to a random walk that we will denote $P(t)$.

We define $P(0) = 0$. Every time we have a birth on the right of the rightmost particle of the frozen process $\eta'(t)$, $P(t)$ jumps to $P(t) + M$. Every time the rightmost particle of $\eta'(t)$ dies, $P(t)$ jumps to $P(t) - M$. Recall that

$$M = \sum_{l\geq 1} lb(l).$$

Note that the preceding rules imply that after an exponential time of parameter 2, $P(t)$ jumps either to $P(t) + M$ or $P(t) - M$ with probability $\frac{1}{2}$.

We have the following theorem.

THEOREM 3. $\lim_{t\to\infty}(1/\sqrt{t}) E|P(t) - r'_t| = 0$.

PROOF. Let Z_{-i} be the distance between the ith and the $(i+1)$th particle of $\eta'(t)$ at time 0, for $i \geq 1$. Using the definition of the initial distribution m, the Z_{-i} are independent of each other and have the same density b. Every time a particle appears on the right of the rightmost one, then r'_t jumps to $r'_t + W_i$, where W_i is a random distance. The W_i are independent of each other and of the Z_i. Using the assumption $b(k,\infty) = b(k)$, we see that the W_i also have density b.

Define

$$Q(t) = \frac{1}{M}P(t) \quad \text{and} \quad m(t) = \min_{s \leq t} Q(s).$$

We can describe r'_t only using $Q(t)$, $m(t)$ and the Z_i, W_i. We now give two examples. Assume that at time t, $m(t) = -2$ and $Q(t) = 1$. Then

$$r'_t = -Z_{-1} - Z_{-2} + W_{-2} + W_{-1} + W_1.$$

Assume that at time t, $m(t) = -3$ and $Q(t) = -2$. Then

$$r'_t = -Z_{-1} - Z_{-2} - Z_{-3} + W_{-3}.$$

More generally if $P(t) < 0$, then

$$r'_t = -\sum_{i=Q(t)}^{-1} Z_i + \sum_{i=m(t)}^{Q(t)-1} (W_i - Z_i);$$

if $P(t) \geq 0$, then

$$r'_t = \sum_{i=1}^{Q(t)} W_i + \sum_{i=m(t)}^{-1} (W_i - Z_i).$$

Let $a > 0$ be a real number which will tend to 0. We have

$$E\big(|r'_t - P(t)|1_{\{P(t)>0\}}\big) \leq E\left(\left|P(t) - \sum_{i=1}^{Q(t)} W_i\right|1_{\{P(t)>0\}}\right) + E\left(\left|\sum_{i=m(t)}^{-1}(W_i - Z_i)\right|\right).$$

Consider first the term

$$E\left(\left|P(t) - \sum_{i=1}^{Q(t)} W_i\right|1_{\{P(t)>0\}}\right).$$

Using that $P(t) = Q(t)M$, the preceding term is equal to

$$(5) \qquad E\left(\left|\sum_{i=1}^{Q(t)}(M - W_i)\right|1_{\{Q(t)<a\sqrt{t}\}}\right) + E\left(\left|\sum_{i=1}^{Q(t)}(M - W_i)\right|1_{\{Q(t)>a\sqrt{t}\}}\right).$$

Using the triangle inequality and the fact that the W_i all have the same distribution, we have

$$\frac{1}{\sqrt{t}} E\left(\left|\sum_{i=1}^{Q(t)} (M - W_i)\right| 1_{\{Q(t) < a\sqrt{t}\}}\right) \le aE|M - W_1|. \tag{6}$$

For the following term,

$$E\left(\left|\sum_{i=1}^{Q(t)} (M - W_i)\right| 1_{\{Q(t) > a\sqrt{t}\}}\right), \tag{7}$$

using the independence between $Q(t)$ and the W_i, we have that (7) is equal to

$$\sum_{l=a\sqrt{t}}^{\infty} P(Q(t) = l) E\left|\sum_{i=1}^{l} (M - W_i)\right|.$$

Since the W_i are an independent identically distributed sequence in $L^1(m)$, we have the following convergence:

$$\lim_{l\to\infty} E\left|\frac{1}{l} \sum_{i=1}^{l} (W_i - M)\right| = 0. \tag{8}$$

We use now the Cauchy–Schwarz inequality and

$$E\left(\frac{Q(t)}{\sqrt{t}} 1_{\{Q(t) > a\sqrt{t}\}}\right) \le E\left(\frac{Q^2(t)}{t}\right)^{1/2} P\left(Q(t) > a\sqrt{t}\right)^{1/2}. \tag{9}$$

But $E(Q^2(t)) = 2t$, for all $t \ge 0$. From (8) and (9) we have

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} E\left(\left|\sum_{i=1}^{Q(t)} (M - W_i)\right| 1_{\{Q(t) > a\sqrt{t}\}}\right) = 0.$$

And so we can conclude that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} E\left(\left|\sum_{i=1}^{Q(t)} W_i - P(t)\right| 1_{\{P(t) > 0\}}\right) = 0.$$

Consider now

$$E\left(\left|\sum_{i=m(t)}^{-1} (W_i - Z_i)\right| 1_{\{P(t) > 0\}}\right).$$

Setting $M(t) = -m(t)$, by symmetry we have that the preceding term is equal to

$$E\left(\left|\sum_{i=1}^{M(t)} (W_i - Z_i)\right| 1_{\{M(t) < a\sqrt{t}\}}\right) + E\left(\left|\sum_{i=1}^{M(t)} (W_i - Z_i)\right| 1_{\{M(t) > a\sqrt{t}\}}\right). \tag{10}$$

Using the triangle inequality and the fact that the $W_i - Z_i$ all have the same distribution, we have

$$E\left(\left|\sum_{i=1}^{M(t)} (W_i - Z_i)\right| 1_{\{M(t) < a\sqrt{t}\}}\right) \le a\sqrt{t}\, E|W_1 - Z_1|. \tag{11}$$

For the second term of (10), we use the independence between the $(W_i - Z_i)$ and $M(t)$ so that

$$E\left(\left|\sum_{i=1}^{M(t)} (W_i - Z_i)\right| 1_{\{M(t) > a\sqrt{t}\}}\right) = \sum_{l=a\sqrt{t}}^{\infty} P(M(t) = l) E\left|\sum_{i=1}^{l} (W_i - Z_i)\right|.$$

Since the $(W_i - Z_i)$ are an independent identically distributed sequence in $L^1(m)$, we have the following convergence:

$$\lim_{l\to\infty} \frac{1}{l} E\left|\sum_{i=1}^{l} (W_i - Z_i)\right| = 0. \tag{12}$$

Now we have to show that $(1/\sqrt{t})E(M(t)1_{\{M(t) > a\sqrt{t}\}})$ is bounded. To do so, we apply a lemma due to Skorohod (Lemma 3.21 in Breiman [1]):

$$P(|M(t)| \ge l) \le \frac{1}{1 - c(t,l)} P(|Q(t)| \ge l/2), \tag{13}$$

where

$$c(t,l) = \sup_{s\le t} P(|Q(s)| \ge l).$$

It is not difficult to see that

$$c(t,l) = P(|Q(t)| \ge l) \le P(|Q(t)| \ge a\sqrt{t}) = c(t),$$

where the inequality comes from $l > a\sqrt{t}$. We can use again the Cauchy–Schwarz inequality to obtain an upper bound for $E(2|Q(t)|1_{\{2Q(t) > a\sqrt{t}\}})$ and the central limit theorem to compute $\lim_{t\to\infty}(1 - c(t))$. These last two observations used in (13) prove that $(1/\sqrt{t})E(M(t)1_{\{M(t) > a\sqrt{t}\}})$ is bounded. This fact and (12) imply

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} E\left(\left|\sum_{i=1}^{M(t)} (W_i - Z_i)\right| 1_{\{M(t) > a\sqrt{t}\}}\right) = 0.$$

So we can conclude that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} E\left|\sum_{i=m(t)}^{-1} (W_i - Z_i)\right| = 0.$$

We now have

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} E\left(|r_t' - P(t)|1_{\{P(t) > 0\}}\right) = 0.$$

It is clear that the same proof will work for the term corresponding to $P(t) < 0$ and the proof of Theorem 3 is complete. □

An immediate consequence of Theorem 3 is the following corollary.

COROLLARY 2. $D' = 2(\sum_{l \geq 1} l b(l))^2$.

We are now able to find a lower bound for D.

COROLLARY 3. $D \geq 2(\sum_{l \geq 1} l b(l))^2$.

PROOF. The results in [2] and [3] give alternative expressions for D and D':

$$D = C + 2\langle v, L^{-1} v \rangle_m,$$
$$D' = C' + 2\langle v', L'^{-1} v' \rangle_m.$$

We know that $C = C'$ and $v = v'$. Using

$$L = L_0 + L'$$

and

$$\langle v, L_0 v \rangle_m \leq 0,$$

we have

(14) $\langle v, Lv \rangle_m$ is smaller than $\langle v, L'v \rangle_m$.

But L and L' are self-adjoint operators in $L^2(m)$ and it is shown in [3] (Lemma 3.1) that (14) implies

$$\langle v, L^{-1} v \rangle_m \text{ is larger than } \langle v, L'^{-1} v \rangle_m.$$

The proof of Corollary 3 is complete. □

4. Two open problems.

1. T. Liggett asked the following question: What happens if $\sum_{l \geq 1} l^2 b(l) = \infty$? Do we still have convergence to Brownian motion? The methods in [2] and [3] do not seem to apply if this second moment condition fails.

Related to this problem, we can make the following observations: For the proof of Theorem 3, we only need the first moment condition $\sum_{l \geq 1} l b(l) < \infty$. This shows that this last condition is enough to prove that the rightmost particle of the frozen process converges to a Brownian motion even if $\mathrm{Var}(r_t') = \infty$ [this is implied by $\sum_{l \geq 1} l^2 b(l) = \infty$].

2. Consider $\zeta(t)$, the critical reversible nearest-particle system (not seen from the edge) under the initial distribution m. If $\zeta(t)$ converges in law to a probability measure ν, using the convergence of the edge to a Brownian motion, it is easy to see that $\nu(\varnothing) \geq \frac{1}{2}$. In fact, our conjecture is that $\zeta(t)$ converges and

$$\nu = \tfrac{1}{2}\delta_{\varnothing} + \tfrac{1}{2}\mu,$$

where μ is the upper invariant measure which is a renewal measure with density b.

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REFERENCES

[1] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.

[2] DE MASI, A., FERRARI, P., GOLDSTEIN, S. and WICK, D. (1985). An invariance principle for reversible Markov processes. Application to diffusion in the percolation regime. In *Particle Systems, Random Media, and Large Deviations* (R. Durrett, ed.). *Contemp. Math.* **41** 71–85. Amer. Math. Soc., Providence, R.I.

[3] DE MASI, A., FERRARI, P., GOLDSTEIN, S. and WICK, D. (1989). An invariance principle for reversible Markov processes. Application to random motion in random environment. *J. Statist. Phys.* **55** 787–855.

[4] GALVES, A. and PRESUTTI, E. (1987). Edge fluctuations for the one dimensional supercritical contact process. *Ann. Probab.* **15** 1131–1145.

[5] LIGGETT, T. (1985). *Interacting Particles Systems*. Springer, New York.

[6] REED, M. and SIMON, B. (1972). *Methods of Modern Mathematical Physics*. Academic, New York.

[7] SPITZER, F. (1977). Stochastic time evolution of one dimensional infinite particle systems. *Bull. Amer. Math. Soc.* **83** 880–890.

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