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FINITE REVERSIBLE NEAREST PARTICLE SYSTEMS IN INHOMOGENEOUS AND RANDOM ENVIRONMENTS[1]

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In this article we propose and study finite reversible nearest particle systems in inhomogeneous and random environments. Using the Dirichlet principle and the ergodic theorem we prove that a finite reversible nearest particle system in a random environment determined by an i.i.d. sequence λ_i survives if $E \log \lambda_i > 0$ and dies out if $E\lambda_i < 1$. Some discussion is provided to show that both survival and extinction may happen when $E \log \lambda_i < 0$ and $E\lambda_i > 1$.

1. Introduction. Reversible nearest particle systems are one-dimensional spin systems as described below. Associate with each site $x \in Z^1$ a stochastic process $\eta_t(x)$ with state space $\{0, 1\}$. We say that there is a particle at x when the number associated is 1 and we regard the site as vacant if the number associated is 0. The particle at x dies (i.e., $1 \to 0$) at rate 1, independently of the occupation of other sites. However, a particle is born to fill the vacancy at site x (i.e., $0 \to 1$) at rate

$$\lambda \frac{\beta(l_x)\beta(r_x)}{\beta(l_x + r_x)}, \tag{1}$$

where λ is a positive parameter, $\beta(\cdot)$ is a family of positive numbers with $\sum_{l=1}^{\infty}\beta(l) = 1$ and

$$\begin{aligned} l_x &= x - \max\{y < x, \eta(y) = 1\}, \\ r_x &= \min\{y > x, \eta(y) = 1\} - x. \end{aligned} \tag{2}$$

Let $l_x = \infty$ (respectively, $r_x = \infty$) if there is no y such that $\eta(y) = 1$ and $y < x$ (respectively, $y > x$), and use a convention that $\beta(\infty) = 0$, $\beta(\infty)/\beta(\infty) = 1$ and $\beta(\infty)\beta(\infty)/\beta(\infty) = 0$. Each individual $\eta_t(x)$ is not Markovian, but the collection $\eta_t = \{\eta_t(x), x \in Z^1\}$ is a Markov process on $\{0, 1\}^Z$. The process η_t is called an infinite nearest particle system if $\sum_x \eta_t(x) = \infty$ and is called a finite system if $\sum_x \eta_t(x) < \infty$ for all t. Infinite nearest particle systems were first introduced in [10] and were constructed in [2]. The finite version was introduced later in [3].

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Many articles have been written on this subject since then. It turned out to be one of the few interacting particle systems that exhibit rich properties and are fairly well understood. See Chapter VII of [4] for more detailed background.

Inspired by [9] and [6], we replace (1) by

$$\lambda_x \frac{\beta(l_x)\beta(r_x)}{\beta(l_x + r_x)} \tag{3}$$

and call the corresponding spin system a reversible *inhomogeneous* nearest particle system. The existence of such a spin system will be shown in Section 2 in the finite case. Instead of just one parameter λ, an inhomogeneous nearest particle system is parametrized by a sequence of $\{\lambda_x\}$. A special case is that λ_x is periodic in x, which will be examined in Section 3. By letting λ_x be i.i.d. random variables indexed by $x \in Z^1$, we obtain a nearest particle system in a random environment. Here a random environment is a realization of $\{\lambda_x\}$ and is then fixed throughout all time. This is the main subject of this article and is treated in Sections 5 and 6. When all λ_x are the same, (3) reduces to (1) and we call the corresponding spin system, which is exactly the one we described in the previous paragraph, a *homogeneous* reversible nearest particle system.

We briefly outline our goal and the way to achieve it before jumping into detailed studies. The homogeneous model will serve as a guide throughout this article. By identifying η_t with a finite set $A_t = \{x | \eta_t(x) = 1\}$, we can view a finite inhomogeneous nearest particle system as a set-valued process A_t. Similar to the homogeneous cases, the empty set $\varnothing$ is the unique absorbing state. We define the survival probability ρ^A as the probability that the process, starting from A, never hits the empty set; that is,

$$\rho^A = P^A(A_t \neq \varnothing, \text{for all } t > 0). \tag{4}$$

When $A = \{x\}$, we use ρ^x instead of $\rho^{\{x\}}$. A little thought reveals that $\rho^x > 0$ if and only if $\rho^y > 0$, for any $x, y \in Z^1$. So it is not ambiguous to say the system survives if $\rho^x > 0$ or dies out if $\rho^x = 0$, without mention of the initial state. In random environments, we say the system survives if $E\rho^x > 0$ and the system dies out if $E\rho^x = 0$. The Dirichlet principle is used in computing the survival probability of the periodic system, as we did in the homogeneous systems [5], but cannot be applied directly to general inhomogeneous systems. Our strategy is then to confine nearest particle systems first to a smaller sample space Ω_N. On Ω_N we construct two auxiliary models, named *Modified I* and *Modified II*, to which the Dirichlet principle is applied. Attractiveness is used in comparisons of various models. Sufficient conditions both for survival and for extinction are derived in Section 4 by passing from Ω_N to Ω, and are extended in Section 5 to nearest particle systems in random environments by using the ergodic theorem. Some discussion is provided in Section 6 concerning the lack of necessary and sufficient conditions for survival and for extinction. In particular, we prove the following result for attractive systems.

THEOREM. *Suppose* $\{\lambda_x,\ x \in Z^1\}$ *are i.i.d. and strictly positive and that* $\beta^2(n) \le \beta(n-1)\beta(n+1)$ *for* $n \ge 2$.

(i) *If* $E \log \lambda_x > 0$, *the nearest particle system in a random environment survives a.s.*

(ii) *If* $E\lambda_x < 1$, *the nearest particle system in a random environment dies out a.s.*

(iii) *If* $E \log \lambda_x < 0$, $E\lambda_x > 1$, *and* $\lambda_x \ge \varepsilon > 0$, *there is a probability distribution* $\{\beta(n)\}$ *so that* $E\rho^0 = 0$ *and there is another choice of* $\{\beta(n)\}$ *so that* $E\rho^0 > 0$.

Finally we would like to point out that analogues of the theorem are also valid for infinite inhomogeneous reversible nearest particle systems, although the proofs are entirely different [7].

2. General form of reversible measures. In this section we shall determine and construct all finite inhomogeneous reversible nearest particle systems. By the nature of nearest particle systems, the birth rate is determined completely by the location x of the newly born particle and distances l, r to its nearest particles to the left and right. Denote the birth rate by $b(x, l, r)$. Let the death rate be 1 in accordance with homogeneous nearest particle systems. We would like to know what kind of form the birth rate $b(x, l, r)$ must take if an inhomogeneous nearest particle system has a reversible measure in the sense of Definition 7.1.1 of [4].

Rephrase the question as follows. The state space Ω is the set of all finite subsets, including $\varnothing$, of Z^1. For each $A \in \Omega$, there is a positive number $\pi(A)$ called the reversible measure of A so that for any pair $A \ne \varnothing$, $x \notin A$,

$$\pi(A)q(A, A \cup \{x\}) = \pi(A \cup \{x\})q(A \cup \{x\}, A), \tag{5}$$

where $q(A, B)$ is the infinitesimal transition rate from A to B. Now the death rate $q(A \cup \{x\}, A) = 1$. So the birth rate $q(A, A \cup \{x\})$ is simply the ratio of the reversible measures of two relevant sets. In particular, say $x_1 < x_2 < \cdots < x_n$, we have

$$\frac{\pi(\{x_1, x_2, x_3\})}{\pi(\{x_1, x_2\})} = b(x_3, x_3 - x_2, \infty) = \frac{\pi(\{x_2, x_3\})}{\pi(\{x_2\})}.$$

So

$$\pi(\{x_1, x_2, x_3\}) = \frac{\pi(\{x_1, x_2\})\pi(\{x_2, x_3\})}{\pi(\{x_2\})}$$

and, in general,

$$(6)\quad \pi(\{x_1, x_2, x_3, \dots, x_n\}) = \frac{\pi(\{x_1, x_2\})\pi(\{x_2, x_3\}) \cdots \pi(\{x_{n-1}, x_n\})}{\pi(\{x_2\})\pi(\{x_3\}) \cdots \pi(\{x_{n-1}\})}.$$

Let $\pi(\{x\}) = \lambda_x$ and $b(x, y) = \pi(\{x, y\})/\pi(\{x\})\pi(\{y\})$, and rewrite (6) as

$$(7)\quad \begin{aligned}&\pi(\{x_1, x_2, x_3, \dots, x_n\}) \\ &\quad = \lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_n} b(x_1, x_2), b(x_2, x_3) \cdots b(x_{n-1}, x_n).\end{aligned}$$

This is the general form of the reversible measure of an inhomogeneous nearest particle system.

Now we may reverse the order and define a Markov process A_t on Ω as follows. Choose two families of positive numbers λ_x and $b(y, z)$ indexed by $x, y, z \in Z^1$ such that $b(y, z) = b(z, y)$. Assign to each finite subset of Z^1 a positive number according to (7). Define the infinitesimal rate for $A \neq \varnothing$ as

$$(8)\quad q(A, B) = \begin{cases} 1, & \text{if } B = A \setminus \{x\}, x \in A, \\ \pi(A \cup \{y\})/\pi(A), & \text{if } B = A \cup \{y\}, y \notin A, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, $q(\varnothing, A) = 0$ for all $A \in \Omega$. We call the corresponding process a reversible *inhomogeneous* nearest particle system. When $b(x, y) = \beta(|x - y|)$ for some probability measure $\beta(\cdot)$ on Z^+, $\beta(n) > 0$ for all n, then (7) and (8) can be rewritten, assuming $x_1 < x_2 < x_3 < \cdots < x_n$, as

$$(9)\quad \pi(\{x_1, x_2, x_3, \dots, x_n\}) = \lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_n} \prod_{i=2}^{n} \beta(x_i - x_{i-1}),$$

$$(10)\quad q(A, B) = \begin{cases} 1, & \text{if } B = A \setminus \{x\}, \\ \lambda_x \beta(l_x)\beta(r_x)/\beta(l_x + r_x), & \text{if } B = A \cup \{x\}, \\ 0, & \text{otherwise,} \end{cases}$$

where l_x and r_x are defined in (2).

To rule out possible explosions, we assume

$$(11)\quad \sup_x \sum_y b(x, y)\lambda_y < \infty.$$

Considering the motion of the two extreme points of A_t, we can apply Corollary 15.44 of [1] to conclude that the inhomogeneous system will not blow up in finite time. In the presence of (9) instead of (7), assumption (11) reduces to $\sup_x \lambda_x < \infty$. In the periodic case of Section 3 or the random environments of Section 5, assumption (11) either holds automatically or can be replaced. We shall return to this point at the beginning of Sections 3 and 5.

3. The periodic case. Take two positive numbers λ_1, λ_2 and a probability density $\beta(\cdot)$ on Z^+ such that $\beta(n) > 0$ for all n. Let $\lambda_{2n+1} = \lambda_1, \lambda_{2n} = \lambda_2$.

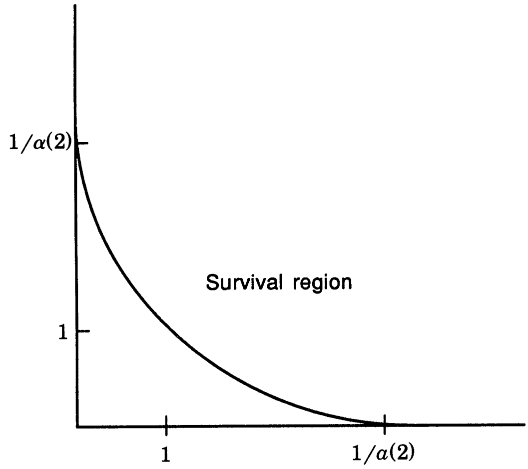

FIG. 1. *Survival region in the period* 2 *case.*

Then construct an inhomogeneous nearest particle system according to (9) and (10). Assumption (11) is always satisfied and A_t is well defined. The following cases are degenerate and can be viewed as *homogeneous* nearest particle systems: (a) $\lambda_1 = 0$, (b) $\lambda_2 = 0$, (c) $\lambda_1 = \lambda_2$. A phase transition occurs along each of these lines and the exact critical point is identified by Theorem 7.1.10 of [4] as $(0, 1/\Sigma\beta(2n))$, $(1/\Sigma\beta(2n), 0)$ and (1, 1) respectively. The next theorem, analogous to Theorem 7.1.10 of [4], completely distinguishes the survival region from the extinction region in the λ_1-λ_2 plane (see Figure 1). Let $\alpha_2 = \Sigma_{n=1}^{\infty}\beta(2n)$, $\alpha_1 = \Sigma_{n=1}^{\infty}\beta(2n-1) = 1 - \alpha_2$ and

$$\xi = \alpha_2(\lambda_1 + \lambda_2) + (1 - 2\alpha_2)\lambda_1\lambda_2.$$

Recall that ρ^1 and ρ^2 are the survival probabilities, defined in (4), for the systems with initial state {1} and {2}, respectively.

THEOREM 1. (i) *The system survives if and only if at least two of* $\xi, \lambda_1, \lambda_2$ *are larger than* 1.

(ii) *In the neighborhood of the curve* $\{(\lambda_1, \lambda_2)|\xi = 1, \min(\lambda_1, \lambda_2) \le 1\}$,

$$C_1(\xi - 1) \le \lambda_1\rho^1 + \lambda_2\rho^2 \le C_2|\log(\xi - 1)|^{-1}, \quad \textit{for } \xi > 1.$$

PROOF. Since both the reversible measure in (9) and the birth rate in (10) are invariant under translation by an even number, we define $A \sim B$ if A is a translation of B by $2k$ for some k. For later reference let $\tilde{\Omega}$ stand for the set of equivalence classes of Ω under this $\sim$. Consider the induced Markov process on the set $\tilde{\Omega}$ and manipulate it in a similar way as in proving Theorem 7.1.10 of [4]. Letting $\pi(\varnothing) = 1$ and $q(\varnothing, \{0\}) = \lambda_2$, $q(\varnothing, \{1\}) = \lambda_1$, we extend the validity of (5) to the empty set. Then apply the Dirichlet principle (Theorem 2.6.1 of [4]) to obtain

$$\lambda_1 \rho^1 + \lambda_2 \rho^2 = \inf_{h \in \mathscr{H}} \sum_{A \in \tilde{\Omega}} \pi(A) \sum_{x \in A} [h(A) - h(A \setminus \{x\})]^2, \tag{12}$$

where

$$\mathscr{H} = \Big\{ h | h : \tilde{\Omega} \to [0,1],\ h(\varnothing) = 0,\ h(A) = h(B)$$

$$\text{if } A \sim B,\ \lim_{n \to \infty} \inf_{[A]=n} h(A) = 1 \Big\},$$

and here $[A]$ stands for the number of even sites in A. Note that all terms on the right-hand side of (12) are increasing in λ_1 and in λ_2. One can compare it with the homogeneous nearest particle system with parameter $\min(\lambda_1, \lambda_2)$ or $\max(\lambda_1, \lambda_2)$ to conclude that the system survives if $\lambda_1 > 1$ and $\lambda_2 > 1$, and dies out if $\lambda_1 \le 1$ and $\lambda_2 \le 1$. So we may assume $\lambda_1 \le 1 < \lambda_2$ without loss of generality. Let

$$\pi_0 = \sum_{\substack{[A]=0 \\ A \in \tilde{\Omega}}} \pi(A) = \sum_{k=1}^{\infty} \alpha_2^{k-1} \lambda_1^k = \frac{\lambda_1}{1 - \alpha_2 \lambda_1},$$

$$\pi_1 = \sum_{\substack{[A]=1 \\ A \in \tilde{\Omega}}} \pi(A) = \lambda_2 [1 + \alpha_1 \lambda_1 + \alpha_1 \alpha_2 \lambda_1^2 + \alpha_1 \alpha_2^2 \lambda_1^3 + \cdots]^2$$

$$= \lambda_2 \left[\frac{1 + (1 - 2\alpha_2)\lambda_1}{1 - \alpha_2 \lambda_1} \right]^2$$

and

$$\pi_k = \sum_{\substack{[A]=k \\ A \in \tilde{\Omega}}} \pi(A) = \lambda_2^k \left[\frac{1 + (1 - 2\alpha_2)\lambda_1}{1 - \alpha_2 \lambda_1} \right]^2 \left(\frac{\alpha_2 + (1 - 2\alpha_2)\lambda_1}{1 - \alpha_2 \lambda_1} \right)^{k-1},$$

$$k = 2, 3, 4, \dots .$$

Choose

$$h(A) = \begin{cases} 0, & \text{if } [A] = 0, \\ \sum_{m=1}^{[A]} 1/m\pi_m \Big/ \sum_{m=1}^{N} 1/m\pi_m, & \text{if } 1 \le [A] < N, \\ 1, & \text{if } [A] \ge N. \end{cases}$$

Substitute it into (12) to get

$$\begin{aligned}\lambda_1\rho^1 + \lambda_2\rho^2 &\le \sum_{A\in\tilde{\Omega}} \pi(A) \sum_{x\in A} [h(A) - h(A\setminus\{x\})]^2 \\ &= \sum_{k=1}^{N} \sum_{[A]=k} k\pi(A)\left(\frac{1/k\pi_k}{\sum_{m=1}^{N} 1/m\pi_m}\right)^2 \\ &= \frac{1}{\sum_{k=1}^{N} 1/k\pi_k}.\end{aligned}$$

When $\xi = \alpha_2(\lambda_1 + \lambda_2) + (1 - 2\alpha_2)\lambda_1\lambda_2 \le 1$, $\lambda_2[\alpha_2 + (1 - 2\alpha_2)\lambda_1] \le 1 - \alpha_2\lambda_1$. Note that π_k is nonincreasing and $\sum_{k=1}^{N} 1/k\pi_k \to \infty$ as $N \to \infty$. So ρ^1, $\rho^2 = 0$.

When $\xi = \alpha_2(\lambda_1 + \lambda_2) + (1 - 2\alpha_2)\lambda_1\lambda_2 > 1$,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k\pi_k} &= \sum_{k=1}^{\infty} \frac{1}{k}\lambda_2^{-k}\left[\frac{1 + (1 - 2\alpha_2)\lambda_1}{1 - \alpha_2\lambda_1}\right]^{-2}\left(\frac{\alpha_2 + (1 - 2\alpha_2)\lambda_1}{1 - \alpha_2\lambda_1}\right)^{-k+1} \\ &= \frac{\alpha_2 - \alpha_1^2\lambda_1 - \alpha_2(1 - 2\alpha_1)\lambda_1^2}{[1 + (1 - 2\alpha_2)\lambda_1]^2}\left|\log\left(1 - \frac{1 - \alpha_2\lambda_1}{\alpha_2\lambda_2 + (1 - 2\alpha_2)\lambda_1\lambda_2}\right)\right|.\end{aligned}$$

Therefore,

$$\lambda_1\rho^1 + \lambda_2\rho^2 \le \frac{[1 + (1 - 2\alpha_2)\lambda_1]^2}{\alpha_2 - \alpha_1^2\lambda_1 - \alpha_2(1 - 2\alpha_1)\lambda_1^2}\left|\log\frac{\xi - 1}{\xi - \alpha_2\lambda_1}\right|^{-1}.$$

This proves the second inequality of the second part of the theorem. For the first inequality, consider

$$D_k = \{A | A \in \tilde{\Omega}, [A] = k, \text{ and the rightmost site of A is even}\}.$$

If x_n, y_l are even, $y_1, y_2, \dots, y_{l-1}$ are odd and $x_1 < x_2 < x_3 < \cdots < x_n < y_1 < y_2 < \cdots < y_l$, the sets

$$A_s = \{x_1, x_2, x_3, \dots, x_n, y_1, y_2, \dots, y_s\}, \qquad s = 1, 2, 3, \dots, l - 1,$$

form an interpolation sequence between $A_0 = \{x_1, x_2, x_3, \dots, x_n\} \in D_k$ and $A_l = \{x_1, x_2, x_3, \dots, x_n, y_1, y_2, \dots, y_l\} \in D_{k+1}$. For any $h \in \mathscr{H}$, let $I_0 = 0$,

$$\pi'_k = \sum_{A\in D_k} \pi(A) = \frac{1 - \alpha_2\lambda_1}{1 + (1 - 2\alpha_1)\lambda_1}\pi_k,$$

$$I_k = \frac{1}{\pi'_k}\sum_{A\in D_k} \pi(A)h(A), \qquad k = 1, 2, 3, \dots.$$

Then

$$\sum_{A\in\tilde{\Omega}} \pi(A) \sum_{x\in A} [h(A) - h(A \setminus \{x\})]^2$$

$$\geq \sum_{A\in\tilde{\Omega}} \pi(A)[h(A) - h(A \setminus \{x\})]^2 \text{ (where } x \text{ is the rightmost site of } A)$$

$$\geq C\sum_k \sum_{A_l\in D_k} \pi(A_l)\big([h(A_l) - h(A_{l-1})]^2 + [h(A_{l-1}) - h(A_{l-2})]^2 + \cdots + [h(A_1) - h(A_0)]^2\big)$$

$$\geq C\sum_k \sum_{A_l\in D_k} \frac{\pi(A_l)[h(A_l) - h(A_0)]^2}{l}$$

$$\geq C\sum_k \frac{\left[\sum_{A_l\in D_k}\pi(A_l)[h(A_l) - h(A_0)]\right]^2}{\sum_{A_l\in D_k}\pi(A_l)l}$$

$$\geq C\sum_k \frac{\pi_k^2}{\sum_{A\in D_k}\pi(A)l}[I_k - I_{k-l}]^2$$

$$= C_1\sum_k \pi_k[I_k - I_{k-l}]^2 \geq C_1\left[\sum_k \frac{1}{\pi_k}\right]^{-1}$$

$$= C_2\frac{\xi - 1}{\xi - \alpha_2\lambda_1} > C_3(\xi - 1).$$

Since h is arbitrary, we conclude that $\lambda_1\rho^1 + \lambda_2\rho^2 \geq C_3(\xi - 1) > 0$. This proves the second part of the theorem, which implies the first part. □

The periodic model with period 3 can be analyzed in the same way. Say $\lambda_{3n+1} = \lambda_1$, $\lambda_{3n+2} = \lambda_2$, $\lambda_{3n} = \lambda_0$ and $\alpha_0 = \sum\beta(3n)$, $\alpha_1 = \sum\beta(3n+1)$, $\alpha_2 = \sum\beta(3n+2)$. Now the analogue of ξ becomes

$$\left(\alpha_0^3 + \alpha_1^3 + \alpha_2^3 - 3\alpha_0\alpha_1\alpha_2\right)\lambda_1\lambda_2\lambda_0 + \left(\alpha_1\alpha_2 - \alpha_0^2\right)(\lambda_1\lambda_2 + \lambda_2\lambda_0 + \lambda_0\lambda_1) + \alpha_0(\lambda_0 + \lambda_1 + \lambda_2).$$

The corresponding expression is more complicated if the period ≥ 4. We do have, however, a simple expression for the following special case.

PROPOSITION 2. *If* $\alpha_k = \sum_{s=0}^{\infty}\beta(sN + k) = 1/N$ *for* $k = 0, 1, 2, \ldots, N - 1$, *then the periodic model of period* N *survives if and only if* $(\lambda_1 + \lambda_2 + \cdots + \lambda_N)/N > 1$.

The proof is similar to that of Theorem 1.

4. Comparisons of attractive models. In this section we consider attractive models defined by (9), (10) with (11) and

$$\frac{\beta(n+1)}{\beta(n)} \ge \frac{\beta(n)}{\beta(n-1)}. \tag{13}$$

The corresponding Markov process is denoted by A_t throughout this section. Because the state space Ω is "too big," we shall first consider the nearest particle system restricted to a "small" subset of Ω, to which we can apply the Dirichlet principle and then use these results to study A_t. The main theme of this section is to compare A_t with modified models.

Modified I. A Markov process A_{tN} is constructed on the state space

$$\Omega_N = \{A \in \Omega | A = \varnothing \text{ or } A \cap \{-N, -N+1, \dots, N-1, N\} \ne \varnothing\}$$

according to

$$B \to \begin{cases} B \cup \{x\}, & \text{at rate } \pi(B \cup \{x\})/\pi(B) \text{ if } x \notin B, \\ B \setminus \{x\}, & \text{at rate 1 if } x \in B \text{ and } B \setminus \{x\} \in \Omega_N, \\ B \setminus \{x\}, & \text{at rate 0 if } x \in B \text{ and } B \setminus \{x\} \notin \Omega_N. \end{cases}$$

This process is reversible with the same π defined in (9). Define

$$\tau = \inf\{t; A_t = \varnothing\}, \qquad \rho^x = P^{\{x\}}(\tau = \infty),$$

$$\tau' = \inf\{t; A_{tN} = \varnothing\}, \qquad \rho_N^x = P^{\{x\}}(\tau' = \infty).$$

By (13), both A_t and A_{tN} are attractive and can be coupled so that $A_t \subseteq A_{tN}$ for all t if initially $A_0 \subseteq A_{0N}$. We conclude that

$$\rho_N^x \ge \rho^x \quad \text{for all } x \in \{-N, \dots, N\}. \tag{14}$$

In particular, a crude estimate corresponding to the case $N = 0$ gives two practical criteria for extinction.

PROPOSITION 3. *If* $\sum_x (\lambda_x - 1)^+ < \infty$ *and* $\sum_k \beta(k) k^2 < \infty$, *then* $\rho^0 = 0$.

PROOF. The first assumption implies that $\prod_{x \in A} \lambda_x$ is uniformly bounded above for any $A \in \Omega_0$. Define the diameter of a finite set A to be

$$\|A\| = \max_{x \in A} x - \min_{x \in A} x.$$

Define the following for each $A \in \Omega_0$:

$$h(A) = \begin{cases} 0, & \text{if } \|A\| = 0, \\ \sum_{k=1}^{\|A\|} 1/k \Big/ \sum_{k=1}^{N} 1/k, & \text{if } 1 \le \|A\| \le N, \\ 1, & \text{if } \|A\| > N. \end{cases}$$

Note that $\sum_{\|A\|=n} \pi(A) \le Cn$. By the Dirichlet principle applied to the Modified

I process,

$$\rho^0 \le \rho_0^0 \le \sum_{n=0}^{N-1} \sum_{\|A\|=n} \sum_{x\notin A} \pi(A\cup\{x\})[h(A\cup\{x\}) - h(A)]^2$$

$$\le \sum_{n=0}^{N-1} \sum_{\|A\|=n} \sum_{l=1}^{\infty} C\pi(A)\beta(l)\left(\frac{l}{\|A\|+1}\Big/\sum_{k=1}^{N}\frac{1}{k}\right)^2$$

$$\le C_1 \sum_{n=0}^{N-1} \sum_{\|A\|=n} \frac{\pi(A)}{(1+\|A\|)^2} \sum \beta(l)l^2 \Big/ \left(\sum_{k=1}^{N}\frac{1}{k}\right)^2$$

$$\le C_2 \sum_{n=0}^{N-1} \frac{n+1}{(n+1)^2} \Big/ \left(\sum_{k=1}^{N}\frac{1}{k}\right)^2$$

$$\le \frac{C_2}{\sum_{k=1}^{N} 1/k} \to 0 \quad \text{is } N\to\infty.$$ □

Example 4. If $\lambda_x = 1 + C|x|^{-\alpha}$, $\alpha > 1$ and $\sum_n \beta(n)n^2 < \infty$, then the corresponding inhomogeneous system A_t dies out.

Proposition 5. *The survival probability* $\rho^0 = 0$ *if*

$$\sum_n \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} \lambda_{x_1}\lambda_{x_1+x_2} \cdots \lambda_{x_1+x_2+\cdots+x_n} \beta(x_1)\beta(x_2)\cdots\beta(x_n) < \infty,$$

and

$$\sum_m \sum_{y_1} \sum_{y_2} \cdots \sum_{y_m} \lambda_{-y_1}\lambda_{-y_1-y_2} \cdots \lambda_{-y_1-y_2-\cdots-y_m} \beta(y_1)\beta(y_2)\cdots\beta(y_m) < \infty.$$

Proof. Use the partition $D_n = \{A | A \in \Omega_0,\ |A| = n\}$, $\pi_n = \sum_{A\in D_n}\pi(A)$. Then the assumptions imply that $\sum_n \pi_n = \sum_{A\in\Omega_0}\pi(A) < \infty$. Therefore, $\sum_n 1/n\pi_n = \infty$. Define the following for each $A \in \Omega_0$:

$$h(A) = \begin{cases} 0, & \text{if } |A| = 0,\\ \sum_{k=1}^{|A|} 1/k\pi_k \Big/ \sum_{k=1}^{N} 1/k\pi_k, & \text{if } 1 \le |A| \le N,\\ 1, & \text{if } |A| > N. \end{cases}$$

A routine argument similar to that used in Section 3 yields $\rho_0^0 \le \lim_{N\to\infty}(\sum_1^N 1/k\pi_k)^{-1}$ and the conclusion follows from (1.4). □

On the other hand, it should not be surprising that as $N \to \infty$ the Modified I model behaves very much like the original A_t. In particular, the survival probabilities will coincide in the limit as shown in the next lemma, which will be used later.

LEMMA 6. *$\lim_{N\to\infty} \rho_N^x = \rho^x$ for any $x \in Z^1$.*

PROOF. Without loss of generality let us fix a point x and introduce

$$\tau_n = \inf\{t;\ |A_t| = n\}, \qquad \varrho^n = P^{\{x\}}(\tau_n < \infty),$$
$$\tau_n' = \inf\{t;\ |A_{tN}| = n\}, \qquad \varrho_N^n = P^{\{x\}}(\tau_n' < \infty).$$

Then $\varrho^n \searrow \rho^x$ and $\varrho_N^n \searrow \rho_N^x$ as $n \to \infty$. There is an n such that $\varrho^n < \rho^x + \varepsilon/2$. For that n, $\lim_{N\to\infty} \varrho_N^n = \varrho^n$. There is an N' such that when $N > N'$, $\varrho_N^n < \varrho^n + \varepsilon/2$. Therefore,

$$\rho_N^x < \varrho_N^n < \varrho^n + \frac{\varepsilon}{2} \le \rho^x + \varepsilon.$$

In view of (14), we are done. □

Modified II. Choose $\lambda_x, \beta(n)$ as before. In what follows, $x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots$. Define a Markov process on Ω according to

$$\{x_0, x_1, \dots, x_n\} \to \begin{cases} \{x_0, x_1, \dots, x_{n-1}\}, & \text{at rate } 1, \\ \{x_0, x_1, \dots, x_n, x_{n+1}\}, & \\ \qquad \text{at rate } \pi\{x_0, x_1, \dots, x_{n+1}\}/\pi\{x_0, x_1, \dots, x_n\}. & \end{cases}$$

This Markov process is called the Modified II process and is denoted by $A_{\mathrm{II}t}$. It may be interpreted as mountain climbing. A mountaineer climbs an infinitely high mountain, setting camps en route. From the highest camp he either abandons the site and goes down to the second highest camp or climbs up to set a higher camp. Then $\{x_0\}$ is (the height of) his base camp, $\{x_n\}$ is (the height of) his present camp and $\{x_0, x_1, \dots, x_n\}$ is (the heights of) camps he keeps. The survival probability ρ_{II}^x of the process $A_{\mathrm{II}t}$ starting at $\{x\}$ may be interpreted as the chance the mountaineer ever climbs up the infinitely high mountain.

For the sake of simplicity and convenience, let us agree to fix notation as follows throughout the rest of this article. Let $y_1, y_2, \dots, y_n, \dots$ be positive integers and $x_1 = y_1$, $x_2 = y_1 + y_2, \dots,$ $x_n = y_1 + y_2 + \cdots + y_n, \dots$.

PROPOSITION 7.

$$\lambda_0 \rho_{\mathrm{II}}^0 \ge \lim_{n\to\infty} \sum_{y_1} \sum_{y_2} \cdots \sum_{y_n} \frac{\beta(y_1)\beta(y_2)\cdots\beta(y_n)}{\sum_{k=1}^n 1/\lambda_0 \lambda_{x_1} \lambda_{x_2} \cdots \lambda_{x_k}}.$$

PROOF. Apply the Dirichlet principle again. For any $h \in \mathscr{H}$, take ε_1 as small as we wish. Then there is an N, so that when $|A| > N$, $h(A) > 1 - \varepsilon_1$. So,

$$\sum_{A \in \Omega_0} \pi(A)[h(A) - h(A \setminus \{x\})]^2$$

$$\geq \sum_{n=1}^{N} \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} \beta(y_1)\beta(y_2) \cdots \beta(y_n)\lambda_0\lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_n}$$

$$\times [h(\{x_1, \dots, x_n\}) - h(\{x_1, \dots, x_{n-1}\})]^2$$

$$\geq \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_N=1}^{\infty} \beta(y_1)\beta(y_2) \cdots \beta(y_N) \sum_{k=1}^{N} \lambda_0\lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_k}$$

$$\times [h(\{x_1, \dots, x_k\}) - h(\{x_1, \dots, x_{k-1}\})]^2$$

$$\geq \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_N=1}^{\infty} \beta(y_1)\beta(y_2) \cdots \beta(y_N) \frac{[h(\{x_1, \dots, x_N\}) - h(\{0\})]^2}{\sum_{k=1}^{N} 1/\lambda_0\lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_k}}$$

(by Cauchy's inequality)

$$\geq (1 - \varepsilon_1)^2 \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_N=1}^{\infty} \frac{\beta(y_1)\beta(y_2) \cdots \beta(y_N)}{\sum_{k=1}^{N} 1/\lambda_0\lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_k}}.$$

All numbers involved in the last line are positive. The sequence is monotone in N and is bounded below by its limit, say ε. We conclude that

$$\lambda_0\rho_{\mathrm{II}}^0 = \inf_{h \in \mathscr{H}} \sum_{A \in \Omega_0} \pi(A)[h(A) - h(A \setminus \{x\})]^2 \geq (1 - \varepsilon_1)^2 \varepsilon.$$

Letting $\varepsilon_1 \to 0$, $\lambda_0\rho_{\mathrm{II}}^0 \geq \varepsilon$. □

Note that all possible states a Modified II process will visit is a small subset of Ω if the initial state is fixed. For example, if initially $A_0 = \{0\}$, the possible states are $\{0, x_1, x_2, \dots, x_n\}$ for some positive integers $n, x_1, x_2, x_3, \dots, x_n$. If the initial state is $\{x\}$, $-N \leq x \leq N$, then we may replace Ω by Ω_N defined previously. This leads to comparison of two modified models, and by the Dirichlet principle we have the following.

PROPOSITION 8. $\sum_{x=-N}^{N} \lambda_x \rho_N^x \geq \sum_{x=-N}^{N} \lambda_x \rho_{\mathrm{II}}^x$.

PROOF. Some modifications are needed. With the change of $\pi(\varnothing) = 1$ and $q(\varnothing, \{x_i\}) = \lambda_i$, $i = -N, -N+1, \dots, N$, both the Modified I and Modified II processes are reversible in the sense of (5), with respect to the same $\pi(\cdot)$ given in (9). Now the Modified I process can start from $\varnothing$, the probability $\rho_N^{\varnothing}$ that it never returns to $\varnothing$ once it leaves $\varnothing$ is related to the quantity appearing in the statement and can be computed by applying the Dirichlet principle. Likewise, the counterpart for the Modified II is also computed and is compared with the terms of the Modified I. We have

$$\sum_{x=-N}^{N} \lambda_x \rho_N^x = \rho_N^{\varnothing} \sum_{x=-N}^{N} \lambda_x = \inf_{h \in \mathscr{H}} \sum_{A \in \Omega_N} \pi(A) \sum_{x \in A} [h(A) - h(A \setminus \{x\})]^2$$

$$\geq \inf_{h \in \mathscr{H}} \sum_{A \in \Omega'_N} \pi(A) [h(A) - h(A \setminus \{x\})]^2$$

$$= \rho_{\mathrm{II}}^{\varnothing} \sum_{x=-N}^{N} \lambda_x = \sum_{x=-N}^{N} \lambda_x \rho_{\mathrm{II}}^x,$$

where

$$\Omega'_N = \{A | A = \{x_0, x_1, x_2, \dots, x_n\},$$

$$x_0 < x_1 < x_2 < \cdots < x_n,\ x_0 \in [-N, N]\} \subset \Omega. \qquad \square$$

5. Random environments. From now on we consider nearest particle systems in random environments. Let λ_x be i.i.d. random variables with finite first moment, and let the birth rate be given by (10) with the reversible measure given by (9). In addition we assume (13), so that the nearest particle system is attractive. We first explain that (11) can be relaxed. We show that the nearest particle system without death will not explode by considering the motion of its edge. Conditioned on the λ'_x's, let x_n be the Markov chain on the positive integers that moves from m to $m+n$ with probability $\beta(n)\lambda_{m+n}/\mu_m$, where $\mu_m = \sum_{n=1}^{\infty} \beta(n)\lambda_{m+n}$. In order to show that a.s. explosions do not occur, it suffices to show

$$P\left(\frac{1}{\mu_{x_1}} + \cdots + \frac{1}{\mu_{x_k}} + \cdots < \infty\right) = 0 \quad \text{a.s.}$$

by Proposition 15.43 of [1]. This is equivalent to

$$\lim_{n \to \infty} P\left(\frac{1}{\mu_{x_1}} + \cdots + \frac{1}{\mu_{x_n}} < M\right) = 0, \quad \text{for any } M.$$

It is then enough to compute

$$E_\lambda \lim_{n\to\infty} P\left(\frac{1}{\mu_{x_1}} + \cdots + \frac{1}{\mu_{x_n}} < M\right)$$

$$= \lim_{n\to\infty} E_\lambda \sum_{y_i} \frac{\lambda_{x_1}}{\mu_0}\frac{\lambda_{x_2}}{\mu_{x_1}} \cdots \frac{\lambda_{x_n}}{\mu_{x_{n-1}}}$$

$$\times \beta(y_1)\beta(y_2)\cdots\beta(y_n)\mathbf{I}\left(\frac{1}{\mu_{x_1}} + \cdots + \frac{1}{\mu_{x_n}} < M\right)$$

$$\leq \lim_{n\to\infty} E_\lambda \sum_{y_i} \lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_n}\beta(y_1)\beta(y_2)\cdots\beta(y_n)\left(\frac{M}{n}\right)^n$$

$$= \lim_{n\to\infty} \sum_{y_i} \beta(y_1)\beta(y_2)\cdots\beta(y_n)\left(\frac{ME\lambda}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{ME\lambda}{n}\right)^n = 0.$$

In the previous argument $\mathbf{I}(\ )$ stands for the indicator function.

As functions of an i.i.d sequence $\{\lambda_x\}$, both sequences $\{\rho^x\}$ and $\{\lambda_x\rho^x\}$ are also ergodic and stationary [1, Proposition 6.31]. It follows from ergodicity that $E\rho^x > 0$ is equivalent to the fact that the nearest particle system in a random environment survives a.s. We say the system survives if $E\rho^x > 0$ and the system dies out if $E\rho^x = 0$.

LEMMA 9.

$$\lim_{N\to\infty} \frac{\sum_{x=-N}^{N}\lambda_x\rho^x}{\sum_{x=-N}^{N}\lambda_x} = \lim_{N\to\infty} \frac{\sum_{x=-N}^{N}\lambda_x\rho_N^x}{\sum_{x=-N}^{N}\lambda_x} \quad a.s.$$

and $E\rho^x > 0$ if and only if the limit is positive.

PROOF. As we did in constructing the Modified I model, we may confine the nearest particle system to

$$\Omega_{[N,M]} = \{A \in \Omega,\ A = \varnothing \text{ or } A \cap \{N, N+1,\ldots,M\} \neq \varnothing\}$$

and the corresponding survival probability is denoted by $\rho_{[N,M]}^x$. By attractiveness $\rho_{[A,B]}^x \leq \rho_{[C,D]}^x$ if $[A,B] \supseteq [C,D]$. Furthermore, $\rho_{[x-L,x+L]}^x$ is a stationary ergodic sequence indexed by x and by the ergodic theorem

$$\begin{aligned}
&\lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho_N^x}{\sum_{x=-N}^{N} \lambda_x} \\
&\quad \le \lim_{N\to\infty} \frac{\sum_{x=-N}^{-N+L-1} \lambda_x + \sum_{x=N-L+1}^{N} \lambda_x}{\sum_{x=-N}^{N} \lambda_x} + \lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho_{[x-L,x+L]}^x}{\sum_{x=-N}^{N} \lambda_x} \\
&\quad \le \lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho_{[x-L,x+L]}^x}{1+2N} \Bigg/ \frac{\sum_{x=-N}^{N} \lambda_x}{1+2N} \\
&\quad = \frac{E\lambda_0 \rho_{[-L,L]}^0}{E\lambda_0} \quad \text{a.s.}
\end{aligned}$$

By Lemma 6 and the dominated convergence theorem,

$$\lim_{L\to\infty} E\lambda_0 \rho_{[-L,L]}^0 = \lim_{L\to\infty} E\lambda_0 \rho_L^0 = E\lambda_0 \rho^0.$$

Together we have proved

$$\lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho_N^x}{\sum_{x=-N}^{N} \lambda_x} \le \frac{E\lambda_0 \rho^0}{E\lambda_0} = \lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho^x}{\sum_{x=-N}^{N} \lambda_x} \quad \text{a.s.}$$

Combining this with the opposite inequality from (14), we get the desired equality. □

THEOREM 10. *If* $E \log \lambda_x > 0$, *then* $E\rho^x > 0$.

PROOF.

$$\begin{aligned}
\frac{E\lambda_0 \rho^0}{E\lambda_0} &= \lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho^x}{\sum_{x=-N}^{N} \lambda_x} \quad \text{(by ergodicity)} \\
&= \lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho_N^x}{\sum_{x=-N}^{N} \lambda_x} \quad \text{(a.s. by Lemma 9)} \\
&\ge \lim_{N\to\infty} \frac{\sum_{x=-N}^{N} \lambda_x \rho_{\mathrm{II}}^x}{\sum_{x=-N}^{N} \lambda_x} \quad \text{(by Proposition 8)} \\
&= \frac{E\rho_{\mathrm{II}}^0 \lambda_0}{E\lambda_0} \quad \text{(by ergodicity)} \\
&\ge \frac{1}{E\lambda_0} E \lim_{N\to\infty} \sum_{y_1} \sum_{y_2} \cdots \sum_{y_N} \frac{\beta(y_1)\beta(y_2)\cdots\beta(y_N)}{\sum_{k=1}^{N} 1/\lambda_0 \lambda_{x_1} \lambda_{x_2} \cdots \lambda_{x_k}}
\end{aligned}$$

(by Proposition 7)

$$= \frac{1}{E\lambda_0} \lim_{N\to\infty} E \sum_{y_1} \sum_{y_2} \cdots \sum_{y_N} \frac{\beta(y_1)\beta(y_2)\cdots\beta(y_N)}{\sum_{k=1}^{N} 1/\lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_k}}$$

(by dominated convergence theorem)

$$= \frac{1}{E\lambda_0} \lim_{N\to\infty} E \frac{1}{\sum_{k=1}^{N} 1/\lambda_1\lambda_2\cdots\lambda_k}$$

$$= \frac{1}{E\lambda_0} E \frac{1}{\sum_{k=1}^{\infty} 1/\lambda_1\lambda_2\cdots\lambda_k}.$$

Since $E\log\lambda_x > 0$ implies $\sum_k 1/\lambda_1\lambda_2\cdots\lambda_k < \infty$ a.s., $E\lambda_0\rho^0 > 0$. Hence $E\rho^0 > 0$. □

THEOREM 11. *If* $E\lambda_x < 1$, *then* $E\rho^x = 0$.

PROOF.

$$E \sum_{n=1}^{\infty} \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} \lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_n}\beta(y_1)\beta(y_2)\cdots\beta(y_n)$$

$$= \sum_{n=1}^{\infty} \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} E\lambda_{x_1}E\lambda_{x_2}\cdots E\lambda_{x_n}\beta(y_1)\beta(y_2)\cdots\beta(y_n)$$

$$= \sum_{n=1}^{\infty} (E\lambda_1)^n \sum_{y_1=1}^{\infty} \sum_{y_2=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} \beta(y_1)\beta(y_2)\cdots\beta(y_n)$$

$$= \sum_{n=1}^{\infty} (E\lambda_1)^n < \infty.$$

So almost surely for a given sequence $\ldots, \lambda_{-3}, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots,$

$$\sum_n \sum_{y_1} \sum_{y_2} \cdots \sum_{y_n} \lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_n}\beta(y_1)\beta(y_2)\cdots\beta(y_n) < \infty.$$

Similarly,

$$\sum_m \sum_{z_1} \sum_{z_2} \cdots \sum_{z_m} \lambda_{-z_1}\lambda_{-z_1-z_2}\cdots\lambda_{-z_1-z_2-\cdots-z_m} \times \beta(z_1)\beta(z_2)\cdots\beta(z_m) < \infty \quad \text{a.s.}$$

By Proposition 5, $\rho^0 = 0$ a.s. Hence $E\rho^0 = 0$, and $E\rho^x = 0$ by stationarity. □

6. The intermediate case. The intermediate case $E\lambda > 1$ and $E\log\lambda < 0$ is more delicate than those treated in the previous section. In this case, we show that both extinction and survival can occur for appropriate choices of $\{\beta(n)\}$. It does not appear possible to find necessary and sufficient conditions for survival. Nevertheless, when combined with Theorems 10 and 11, the results below can be regarded as saying that $E\lambda = 1$ and $E\log\lambda = 0$ are both places where a change occurs in the answer to the question "Does survival occur for all, for some or for no choices of $\{\beta(n)\}$?"

THEOREM 12. *If $E\log\lambda < 0$, there is a probability distribution $\{\beta(n)\}$ so that $E\rho^0 = 0$.*

PROOF. Take a large positive number M such that

$$E\log\left(\lambda \vee \frac{1}{M}\right) = \log\gamma < 0.$$

By truncating, we may assume that $\lambda^{-1} \le M$. Choose $\varepsilon > 0$ so that $\gamma + \varepsilon < 1$. Then by the strong law of large numbers there is a random constant C so that $\lambda_1 \cdots \lambda_N \le C(\gamma+\varepsilon)^N$ for all N. Take a distribution $\beta(n)$ such that

$$\sum_n [(\gamma+\varepsilon)M]^n \beta(n) < M.$$

Regard $x_n = y_1 + y_2 + \cdots + y_n$ as a random walk on Z^1. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{y_1=1}^{\infty}\cdots\sum_{y_n=1}^{\infty}\lambda_{x_1}\cdots\lambda_{x_n}\beta(y_1)\cdots\beta(y_n)\\
&\quad\le \sum_{n=1}^{\infty}\sum_{N\ge n}\lambda_1\cdots\lambda_N M^{N-n}\sum_{x_n=N}\beta(y_1)\cdots\beta(y_n)\\
&\quad\le C\sum_{n=1}^{\infty}\sum_{N\ge n}(\gamma+\varepsilon)^N M^{N-n}P(x_n=N)\\
&\quad= C\sum_{n=1}^{\infty}M^{-n}\sum_{N=n}^{\infty}(\gamma+\varepsilon)^N M^N P(x_n=N)\\
&\quad= C\sum_{n=1}^{\infty}M^{-n}E[(\gamma+\varepsilon)M]^{x_n}\\
&\quad= C\sum_{n=1}^{\infty}\left[M^{-1}E[(\gamma+\varepsilon)M]^{x_1}\right]^n < \infty,
\end{aligned}$$

and therefore the process dies out a.s. by Proposition 5. □

The next result is motivated by Proposition 2.

THEOREM 13. *If $E\lambda > 1$, $E\lambda^2 < \infty$ and $\lambda \ge \varepsilon > 0$, there is a probability distribution $\{\beta(n)\}$ so that $E\rho^0 > 0$.*

It suffices to show $E\rho^0_{\mathrm{II}} > 0$, or equivalently, to show the Modified II is transient. By Royden's criterion [8] the Modified II is transient if there are real numbers $h_{A,B}$ with the following properties:

(1) $h_{A,B} = -h_{B,A}$.
(2) There is a set A' such that $\sum_B h_{A',B} \ne 0$ and $\sum_B h_{A,B} = 0$ for $A \ne A'$.
(3) $\sum_A\sum_B h^2_{A,B}/a_{A,B} < \infty$, with conventions $0/0 = 0$, $x/0 = \infty$ if $x \ne 0$, and $a_{A,B} = a_{B,A} = \lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_n}\beta(y_1)\beta(y_2)\cdots\beta(y_n)$ if $A = \{0, x_1, x_2, \dots, x_{n-1}\}$ and $B = \{0, x_1, x_2, \dots, x_n\}$ and $a_{A,B} = 0$ otherwise.

Let

$$\mu_n = \sum_{k=1}^{\infty} \beta(k)\lambda_{n+k} \quad \text{and} \quad \mu_{n,m} = \sum_{k=1}^{m-n} \beta(k)\lambda_{n+k} + \varepsilon\left(1 - \sum_{k=1}^{m-n} \beta(k)\right).$$

Set

$$h_{A,B} = -h_{B,A} = \frac{\lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_n}}{\mu_0\mu_{x_1}\cdots\mu_{x_{n-1}}}\beta(y_1)\beta(y_2)\cdots\beta(y_n)$$

if

$$A = \{0, x_1, x_2, \dots, x_{n-1}\} \quad \text{and} \quad B = \{0, x_1, x_2, \dots, x_n\}$$

and $h_{A,B} = 0$ otherwise. Then it is easy to verify properties 1 and 2 with $A' = \varnothing$. We will see that property 3 holds if for some N,

$$\text{(15)} \qquad \mathscr{A} = E \sum_{y_1,\dots,y_N} \frac{\lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_N}}{\mu^2_{0,x_N}\mu^2_{x_1,x_N}\cdots\mu^2_{x_{N-1},x_N}}\beta(y_1)\cdots\beta(y_N) < 1.$$

The proof of (15) consists of three lemmas.

LEMMA 14. *Suppose $U_1, U_2, \dots$ are independent random variables with the uniform distribution on* $[0,1]$. *Then*

$$\lim_{n\to\infty} E\frac{(E\lambda)^n}{\prod_{k=1}^n [(1 - U_1\cdots U_k)E\lambda + U_1\cdots U_k\varepsilon]^2} = 0.$$

Define two families of random variables $\{V_1, V_2, \dots, V_m\}, \{W_1, W_2, \dots, W_m\}$ by

$$P\left(V_1 = \sum_{k=1}^{x_1}\beta(k), V_2 = \sum_{k=1}^{x_2}\beta(k), \dots, V_m = \sum_{k=1}^{x_m}\beta(k)\right) = \beta(y_1)\cdots\beta(y_m),$$

$$W_m = 1 - U_1U_2\cdots U_m, \qquad m = 1, 2, 3, \dots.$$

LEMMA 15. *If $\beta(k) = qp^{k-1}$, then $(V_1, V_2, \dots, V_m)$ converges in distribution to $(W_1, W_2, \dots, W_m)$ as $q \to 0$.*

LEMMA 16. *Assume $\lambda \le M$ and $E\lambda^2 < \infty$. The difference between*

$$\sum_{y_1,\dots,y_N} \frac{\beta(y_1)\cdots\beta(y_N)}{\prod_{s=1}^N [(\sum_{k=1}^{x_s}\beta(k))E\lambda + (1 - \sum_{k=1}^{x_s}\beta(k))\varepsilon]^2}$$

and

$$\sum_{y_1,\dots,y_N} E\frac{\beta(y_1)\cdots\beta(y_N)}{\prod_{s=1}^N [\sum_{k=1}^{x_s}\beta(k)\lambda_k + (1 - \sum_{k=1}^{x_s}\beta(k))\varepsilon]^2}$$

is bounded by $C(N)\sqrt{\sum_k \beta^2(k)}$.

PROOF OF THEOREM 13. Without lost of generality we truncate λ so that $\lambda \le M$. By Lemma 14 we may choose N so that

$$E\frac{(E\lambda)^N}{\prod_{k=1}^N [(1 - U_1\cdots U_k)E\lambda + U_1\cdots U_k\varepsilon]^2} < \frac{1}{2}.$$

Now fix N. There is a family $\{\beta(k) = qp^{k-1}\}$ such that

$$\sum_{y_1,\dots,y_N} \frac{\beta(y_1)\cdots\beta(y_N)}{\prod_{s=1}^N[\sum_{k=1}^{x_s}\beta(k)E\lambda + (1-\sum_{k=1}^{x_s}\beta(k))\varepsilon]^2} < \frac{3}{4}$$

by Lemma 15. Then (15) is true by Lemma 16 if we take q small enough. Consequently,

$$(16)\qquad \begin{aligned} &E\sum_{l=1}^{\infty}\sum_{y_1,\dots,y_l}\frac{\lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_l}}{\mu_0^2\mu_{x_1}^2\cdots\mu_{x_{l-1}}^2}\beta(y_1)\cdots\beta(y_l)\\ &\quad\le\left[\sum_{l=1}^{N}E\sum_{y_1,\dots,y_l}\frac{\lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_l}}{\mu_{0,x_l}^2\mu_{x_1,x_l}^2\cdots\mu_{x_{l-1},x_l}^2}\beta(y_1)\cdots\beta(y_l)\right]\\ &\qquad\times[1+\mathscr{A}+\mathscr{A}^2+\cdots]\\ &\quad<\infty. \end{aligned}$$

So

$$(17)\qquad \sum_{l}\sum_{y_1,\dots,y_l}\frac{\lambda_{x_1}\lambda_{x_2}\cdots\lambda_{x_l}}{\mu_0^2\mu_{x_1}^2\cdots\mu_{x_{l-1}}^2}\beta(y_1)\cdots\beta(y_l)<\infty\quad\text{a.s.}$$

By Royden's criterion [8], $\rho_{\mathrm{II}} > 0$ a.s. □

PROOF OF LEMMA 14. Let $\tau = \min\{m;\ U_1U_2\cdots U_m \le 1 - 1/\sqrt{E\lambda}\}$. Take N to be the integer part of $E\lambda/\varepsilon^2$:

$$\begin{aligned} P(\tau\ge n) &= P\big(U_1U_2\cdots U_n \ge 1-1/\sqrt{E\lambda}\big)\\ &\le\left(\frac{\sqrt{E\lambda}}{\sqrt{E\lambda}-1}\right)^N(EU^N)^n=\left(\frac{\sqrt{E\lambda}}{\sqrt{E\lambda}-1}\right)^N\frac{1}{(N+1)^n}. \end{aligned}$$

Therefore,

$$E\left(\frac{E\lambda}{\varepsilon^2}\right)^\tau=\sum_n\left(\frac{E\lambda}{\varepsilon^2}\right)^nP(\tau=n)\le\left(\frac{\sqrt{E\lambda}}{\sqrt{E\lambda}-1}\right)^N\sum_n\left(\frac{E\lambda}{(N+1)\varepsilon^2}\right)^n$$

is finite by the choice of N. By the dominated convergence theorem,

$$\lim_{n\to\infty}E\frac{(E\lambda)^n}{\prod_{k=1}^n[(1-U_1\cdots U_k)E\lambda+U_1\cdots U_k\varepsilon]^2}=0. \qquad □$$

PROOF OF LEMMA 15. Let $G(n)=\sum_{k=1}^n\beta(k)$, $g(c)=\max\{n;\ G(n)\le c\}$,

$$\begin{aligned} F_n(c_1,c_2,\dots,c_n)&=P(W_1\le c_1,W_2\le c_2,\dots,W_n\le c_n),\\ \mathscr{F}_n(c_1,c_2,\dots,c_n)&=P(V_1\le c_1,V_2\le c_2,\dots,V_n\le c_n)\\ &=\sum_{x_1\le g(c_1)}\sum_{x_2\le g(c_2)}\cdots\sum_{x_n\le g(c_n)}\beta(y_1)\beta(y_2)\cdots\beta(y_n). \end{aligned}$$

We shall prove $\mathscr{F}_n(c_1, c_2, \dots, c_n) \to F_n(c_1, c_2, \dots, c_n)$ for any $\{c_1, c_2, \dots, c_n\}$ with $c_1 < c_2 < \cdots < c_n$ by induction on n. This is true when $n = 1$. For $n > 1$,

$$F_n(c_1, c_2, \dots, c_n) = \int_0^{c_1} \int_{u_1}^{c_2} \cdots \int_{u_{n-2}}^{c_{n-1}} \frac{c_n - u_{n-1}}{1 - u_{n-1}} \, dF_{n-1}(u_1, u_2, \dots, u_{n-1}),$$

$$\begin{aligned}
\mathscr{F}_n(c_1, c_2, \dots, c_n) &= \sum_{x_1 \le g(c_1)} \sum_{x_2 \le g(c_2)} \cdots \sum_{x_n \le g(c_n)} \beta(y_1)\beta(y_2) \cdots \beta(y_n) \\
&= \sum_{y_1=1}^{g(c_1)} \sum_{y_2=1}^{g(c_2)-x_1} \cdots \sum_{y_{n-1}=1}^{g(c_{n-1})-x_{n-2}} \sum_{y_n=1}^{g(c_n)-x_{n-1}} \\
&\quad \times \beta(y_n)\beta(y_1)\beta(y_2) \cdots \beta(y_{n-1}) \\
&= \sum_{y_1=1}^{g(c_1)} \sum_{y_2=1}^{g(c_2)-x_1} \cdots \sum_{y_{n-1}=1}^{g(c_{n-1})-x_{n-2}} \frac{1}{p^{x_{n-1}}} \sum_{y_n=x_{n-1}+1}^{g(c_n)} \\
&\quad \times \beta(y_n)\beta(y_1)\beta(y_2) \cdots \beta(y_{n-1}) \\
&= \sum_{y_1=1}^{g(c_1)} \sum_{y_2=1}^{g(c_2)-x_1} \cdots \sum_{y_{n-1}=1}^{g(c_{n-1})-x_{n-2}} \frac{c_n - G(x_{n-1})}{1 - G(x_{n-1})} \\
&\quad \times \beta(y_1)\beta(y_2) \cdots \beta(y_{n-1}) \\
&= \int_0^{c_1} \int_{u_1}^{c_2} \cdots \int_{u_{n-2}}^{c_{n-1}} \frac{c_n - u_{n-1}}{1 - u_{n-1}} \, d\mathscr{F}_{n-1}(u_1, u_2, \dots, u_{n-1}).
\end{aligned}$$

We conclude that $\mathscr{F}_n(c_1, c_2, \dots, c_n) \to F_n(c_1, c_2, \dots, c_n)$ from the induction hypothesis. □

PROOF OF LEMMA 16. Let $\mathscr{A}_k$ be

$$\sum_{y_1, \dots, y_N} E \frac{\beta(y_1) \cdots \beta(y_N)}{\prod_{s=1}^{k} \left[\left(\sum_{k=1}^{x_s} \beta(k) \right) E\lambda + \varepsilon \sum_{k=x_s+1}^{\infty} \beta(k) \right]^2}$$
$$\times \left\{ \prod_{s=k+1}^{N} \left[\sum_{k=1}^{x_s} \beta(k)\lambda_k + \varepsilon \sum_{k=x_s+1}^{\infty} \beta(k) \right]^2 \right\}^{-1}.$$

Then

$$\mathscr{A}_N = \sum_{y_1, \dots, y_N} \frac{\beta(y_1) \cdots \beta(y_N)}{\prod_{s=1}^{N} \left[\left(\sum_{k=1}^{x_s} \beta(k) \right) E\lambda + \varepsilon \left(1 - \sum_{k=1}^{x_s} \beta(k) \right) \right]^2},$$

$$\mathscr{A}_0 = \sum_{y_1, \dots, y_N} E \frac{\beta(y_1) \cdots \beta(y_N)}{\prod_{s=1}^{N} \left[\sum_{k=1}^{x_s} \beta(k)\lambda_k + \varepsilon \left(1 - \sum_{k=1}^{x_s} \beta(k) \right) \right]^2}$$

and

$$\mathscr{A}_l - \mathscr{A}_{l-1} = \sum_{y_1,\dots,y_N} E\left\{ \frac{\beta(y_1)\cdots\beta(y_N)\sum_{k=1}^{x_l}\beta(k)(\lambda_k - E\lambda)}{\prod_{s=1}^{l}\left[\left(\sum_{k=1}^{x_s}\beta(k)\right)E\lambda + \varepsilon\left(1 - \sum_{k=1}^{x_s}\beta(k)\right)\right]^2} \right.$$
$$\left. \times \frac{\sum_{k=1}^{x_l}\beta(k)(\lambda_k + E\lambda) + 2\varepsilon\left(1 - \sum_{k=1}^{x_s}\beta(k)\right)}{\prod_{s=l}^{N}\left[\sum_{k=1}^{x_s}\beta(k)\lambda_k + \varepsilon\left(1 - \sum_{k=1}^{x_s}\beta(k)\right)\right]^2} \right\},$$

$$(\mathscr{A}_l - \mathscr{A}_{l-1})^2 \le E \sum_{y_1,\dots,y_N} \beta(y_1)\cdots\beta(y_N)\left(\sum_{k=1}^{x_l}\beta(k)(\lambda_k - E\lambda)\right)^2 \frac{4M^2}{\varepsilon^{2N}}$$
$$= E(\lambda - E\lambda)^2 \sum \beta^2(n)\frac{4M^2}{\varepsilon^{2N}},$$

$$(\mathscr{A}_0 - \mathscr{A}_N)^2 \le N\sum_{l=1}^{N}(\mathscr{A}_l - \mathscr{A}_{l-1})^2 \le N^2\frac{4M^2}{\varepsilon^{2N}}\operatorname{var}(\lambda)\sum\beta^2(n). \qquad \square$$

Whereas Theorems 12 and 13 deal with the two extremes of the distribution $\beta(n)$, the next proposition provides an example of nonextreme $\beta(n)$. Once again, as in Lemma 15 as well as Theorem 13, the geometric distribution seems to be indispensable.

PROPOSITION 17. *If* $\beta(n) = pq^{n-1}$, $1 > p > 0$, $q = 1 - p$ *and* $E\log(p\lambda_x + q) < 0$, *then* $\rho^x = 0$ *a.s.*

PROOF. Let $k, y_1, \dots, y_k$ run over positive integers, $x_k = y_1 + y_2 + \cdots + y_k$ and

$$\pi_l^+ = \sum_k \sum_{x_k = l} \beta(y_1)\cdots\beta(y_k)\lambda_{x_1}\cdots\lambda_{x_k}.$$

Then

$$\begin{aligned}
\pi_{l+1}^+ &= \sum_k \sum_{x_k = l+1} \beta(y_1)\cdots\beta(y_k)\lambda_{x_1}\cdots\lambda_{x_k} \\
&= \lambda_{l+1}\sum_{s=1}^{l}\pi_s^+\beta(l+1-s) \\
&= \lambda_{l+1}\left[\pi_l^+\beta(1) + \sum_{s=1}^{l-1}\pi_s^+\beta(l-s)\frac{\beta(l+1-s)}{\beta(l-s)}\right] \\
&= \lambda_{l+1}p + \frac{q}{\lambda_l}\pi_l^+ \\
&= (p\lambda_l + q)\frac{\lambda_{l+1}}{\lambda_l}\pi_l^+ \\
&= \lambda_{l+1}\prod_{i=1}^{l}(p\lambda_i + q) \\
&\le \frac{1}{p}\prod_{i=1}^{l+1}(p\lambda_i + q).
\end{aligned}$$

When $E \log(p\lambda_x + q) < 0$, $\sum \pi_l^+ \leq \sum (1/p)\prod_{i=1}^{l+1}(p\lambda_i + q) < \infty$ a.s. This verifies the first half of the assumption of Proposition 5, and the same argument works for the other half. Then by Proposition 5, $\rho^x = 0$ a.s. □

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