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A CENTRAL LIMIT THEOREM FOR THE RENORMALIZED SELF-INTERSECTION LOCAL TIME OF A STATIONARY VECTOR GAUSSIAN PROCESS[1]

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Let $\mathbf{X}(t)$ be a stationary vector Gaussian process in R^m whose components are independent copies of a real stationary Gaussian process with covariance function $r(t)$. Let $\phi(z)$ be the standard normal density and, for $t > 0$, $\varepsilon > 0$, consider the double integral

$$\int_0^t \int_0^t \varepsilon^{-m} \prod_{j=1}^{m} \phi\big(\varepsilon^{-1}(X_j(s) - X_j(s'))\big)\, ds\, ds',$$

which represents an approximate self-intersection local time of $\mathbf{X}(s)$, $0 \le s \le t$. Under the sole condition $r \in L_2$, the double integral has, upon suitable normalization, a limiting normal distribution under a class of limit operations in which $t \to \infty$ and $\varepsilon = \varepsilon(t)$ tends to 0 sufficiently slowly. The expected value and standard deviation of the double integral, which are the normalizing functions, are asymptotically equal to constant multiples of t^2 and $t^{3/2}$, respectively. These results are valid without any restrictions on the behavior of $r(t)$ for $t \to 0$ other than continuity.

1. Introduction and summary. Let $X(t)$, $t \ge 0$, be a real, measurable stationary Gaussian process. For simplicity, take $EX(t) = 0$ and $EX^2(t) = 1$ and let $r(t) = EX(0)X(t)$ be the covariance function, which is assumed to be continuous. For $m \ge 1$, let $X_1(t), \dots, X_m(t)$, $t \ge 0$, be independent copies of $X(t)$, and define the vector process $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$. Put

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), \tag{1.1}$$

and, for $\varepsilon > 0$ and $t > 0$, consider the random variable

$$\varepsilon^{-m} \int_0^t \int_0^t \prod_{j=1}^{m} \phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon}\right) ds\, ds'. \tag{1.2}$$

The following theorem is our main result.

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THEOREM 1.1. *Assume*

$$\int_{-\infty}^{\infty} r^2(s)\,ds < \infty. \tag{1.3}$$

Define

$$B(\varepsilon) = \int_0^1 \left(\frac{\varepsilon^2}{2} + 1 - r(s) \right)^{-m/2} ds, \tag{1.4}$$

and assume that $\varepsilon \to 0$ *and* $t \to \infty$ *in such a way that*

$$t^{-1/2} B(\varepsilon) \to 0. \tag{1.5}$$

Then the random variable (1.2), *with the normalization*

$$t^{-3/2} \left\{ \varepsilon^{-m} \int_0^t \int_0^t \prod_{j=1}^m \phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon} \right) ds\,ds' - t^2 2^{-m} \pi^{-m/2} \left(1 + \frac{\varepsilon^2}{2}\right)^{-m/2} \right\}, \tag{1.6}$$

has a limiting normal distribution with mean 0 *and variance*

$$8(2\pi)^{-m} \int_0^{\infty} \left\{ \left(4 - r^2(s)\right)^{-m/2} - 2^{-m} \right\} ds. \tag{1.7}$$

Limiting properties of the functional (1.2) for fixed t and $\varepsilon \to 0$ have been actively studied in the context of certain specific classes of processes under the heading "renormalized self-intersection local time." The first result of this type was that of Varadhan (1969) who showed, for Brownian motion in R^2, that the random variable (1.2), minus $(t/2\pi)\log(1/\varepsilon)$, converges in L_2 for $\varepsilon \to 0$. Such results have been extended within the context of Brownian motion by several authors, whose works are too numerous to list here. As an example of recent results, we mention the paper of Weinryb and Yor (1988). The results for Brownian motion were extended to other classes of processes by Rosen, who considered diffusion processes (1987a), fractional Brownian motion (1987b), and stable processes (1988).

Condition (1.5) does not hide any further restrictions on $r(t)$. In contrast to all previous work on the intersection local time for Gaussian processes [see Cuzick (1982), Rosen (1984) and Berman (1991)], the statement of the theorem is valid without restriction of the behavior of $E(X(t) - X(s))^2$, for $|s - t| \to 0$, other than continuity. This signifies that the theorem holds under the very wide variety of possible sample function behavior which is characteristic of Gaussian processes. In the stationary case $E(X(t) - X(s))^2 = 2(1 - r(s - t))$ and the local behavior of r enters the limit operation (1.5). If

$$\int_0^1 (1 - r(s))^{-m/2} ds < \infty, \tag{1.8}$$

then $B(0) < \infty$ and (1.5) is always satisfied.

In most previous work on Gaussian intersection local time, there is a hypothesis

$$E(X(t+s)-X(s))^2 \sim L_s(t)|t|^{\alpha}, \quad \text{for } t \to 0,$$

where $0 < \alpha < 2$ and $L_s(t)$ is slowly varying in t for each s. In the stationary case and for $L_s(t)$ a constant, this condition is equivalent to

$$1 - r(t) \sim c|t|^{\alpha}. \tag{1.9}$$

In this case, a direct calculation yields the following estimates of the function $B(\varepsilon)$, for $\varepsilon \to 0$:

$$B(\varepsilon)\begin{cases} \to B(0) < \infty, & \text{if } \alpha < 2/m, \\ \sim (\text{constant})\log(1/\varepsilon), & \text{if } \alpha = 2/m, \\ \sim (\text{constant})\varepsilon^{(2/\alpha)-m}, & \text{if } \alpha > 2/m. \end{cases} \tag{1.10}$$

Note that, in contrast to earlier work, our theorem includes the case $\alpha = 2$ in (1.9), where the sample function is mean-square differentiable or may even by differentiable of any order, almost surely. It also includes the case where $\alpha = 0$ and $1 - r(t)$ is slowly varying, when the sample functions may be unbounded everywhere, almost surely.

For $\varepsilon \to 0$, the primary contribution to the integral (1.2) is from points (s, s') where $X(s)$ is close to $\mathbf{X}(s')$. This set includes points near the diagonal as well as points bounded away from the diagonal. The former points are not necessarily self-intersection points, but may possibly contribute to (1.2) as a simple result of path continuity. We show in Theorem 6.1 that the contribution of such points is asymptotically negligible, so that (1.2) is a measure of genuine self-intersections.

The only assumption in the hypothesis of Theorem 1.1 is (1.3), which represents a mixing condition for the process. The basic idea of the proof of the main result is that the limiting distribution of the functional (1.2), for $t \to \infty$ and $\varepsilon \to 0$, is related to the limiting distribution of the functional

$$\int_0^t \exp\left(-\tfrac{1}{2}\sum_{j=1}^{m} X_j^2(s)\right) ds, \tag{1.11}$$

for $t \to \infty$. The distribution of the latter for $t \to \infty$, after suitable normalization, is obtained by applying a central limit theorem for integral functionals of stationary Gaussian processes,

$$\int_0^t h(\mathbf{X}(s))\, ds, \tag{1.12}$$

as $t \to \infty$, for a class of functions h. The condition (1.3) represents the mixing condition that is sufficient for the validity of this central limit theorem. In Sections 3 and 4, we prove a general theorem of this type for functionals (1.12), and then apply it to the specific functional (1.11).

Theorem 4.1, stating the general limit theorem for (1.12), is related to, yet distinct from, results of Breuer and Major (1983) and Chambers and Slud

(1989). Theorem 1 of the latter paper implies the conclusion of our Theorem 4.1 only in the uninteresting case $m = 1$. The proof does not extend to $m > 1$. Indeed, a key step in their proof is the following: If $r \in L_k$, for $k \geq 1$, and if the spectral density exists and $H_j(x)$ is the Hermite polynomial of order $j \geq k$, then the pth moment of $t^{-1/2}\int_0^t H_j(X(s))\,ds$ converges to the corresponding moment of the normal distribution with mean 0 and variance $2j!\int_0^\infty r^j(t)\,dt$. The complex supporting calculation does not directly extend to the moment of the corresponding functional which we consider in the proof of Theorem 4.1, namely,

$$t^{-1/2}\int_0^t \sum_{i=1}^m H_{j_i}(X_i(s))\,ds,$$

where $(j_1, \dots, j_m)$ are nonnegative integers satisfying $j_1 + \cdots + j_m \geq 2$. Our proof is based on a technique previously used by the author [Berman (1970)] and which appears to be simpler than the method of moments.

The reader should note that the scaling factor $t^{-3/2}$ in (1.6) is the same for all dimensions $m \geq 1$.

2. Preliminary formulas. We begin with an elementary inequality.

LEMMA 2.1. *Let $q > 0$ be arbitrary. If x and y are such that, for some $c > 0$,*

$$x > c \quad \text{and} \quad |y| < c/2, \tag{2.1}$$

then

$$|(x-y)^{-q} - x^{-q}| \leq |y| q (c/2)^{-q-1}. \tag{2.2}$$

PROOF. Elementary calculus yields

$$|(x-y)^{-q} - x^{-q}| = q\left|\int_0^y (x-z)^{-q-1}\,dz\right|,$$

and (2.2) then follows from (2.1). □

LEMMA 2.2. *Let (ξ, η) have a bivariate normal distribution with means 0, and put $\sigma_1^2 = \operatorname{Var}\xi$, $\sigma_2^2 = \operatorname{Var}\eta$ and $\sigma_{12} = \operatorname{Cov}(\xi, \eta)$. Then*

$$E(\cos\xi\cos\eta) = \exp\left[-\tfrac{1}{2}(\sigma_1^2 + \sigma_2^2)\right]\cosh\sigma_{12}, \tag{2.3}$$

$$E(\sin\xi\sin\eta) = \exp\left[-\tfrac{1}{2}(\sigma_1^2 + \sigma_2^2)\right]\sinh\sigma_{12}. \tag{2.4}$$

PROOF. These formulas are based on the complex exponential representations of sine and cosine and on the formula for the normal characteristic function. □

The formula for the bivariate normal density yields the following lemma.

LEMMA 2.3. *Let* (ξ, η) *have a bivariate normal distribution with standard marginal distributions and correlation* ρ. *Then, for* $a > 0$, $b > 0$,

$$(2.5)\quad \begin{aligned} &E\left\{\exp\left[-\frac{1}{2}\left(\frac{\xi^2}{a}+\frac{\eta^2}{b}\right)\right]\right\} \\ &\quad = (1-\rho^2)^{-1/2}\left[\left(\frac{1}{a}+\frac{1}{1-\rho^2}\right)\left(\frac{1}{b}+\frac{1}{1-\rho^2}\right)-\frac{\rho^2}{(1-\rho^2)^2}\right]^{-1/2}. \end{aligned}$$

LEMMA 2.4. *For arbitrary* $0 < d < 1$ *and* $m \geq 1$, *there exists* $K = K(d, m) < \infty$ *such that*

$$(2.6)\quad \begin{aligned} &\Bigg|(1-\rho^2)^{-m/2}\left[\left(\frac{1}{a}+\frac{1}{1-\rho^2}\right)\left(\frac{1}{b}+\frac{1}{1-\rho^2}\right)-\frac{\rho^2}{(1-\rho^2)^2}\right]^{-m/2} \\ &\qquad -\left[\left(\frac{1+a}{a}\right)\left(\frac{1+b}{b}\right)\right]^{-m/2}\Bigg| \leq K\rho^2, \end{aligned}$$

for all $a > 0$ *and* $b > 0$ *and for all* ρ *such that* $\rho^2/(1-\rho^2)^2 < d/2$.

PROOF. We repeatedly apply Lemma 2.1 to the factors in (2.6). For $x = 1$, $y = \rho^2$ and $q = m/2$, we obtain

$$|(1-\rho^2)^{-m/2} - 1| \leq (m/2)\rho^2(2/d)^{(m/2)+1}.$$

For $x = (a^{-1} + (1-\rho^2)^{-1})(b^{-1} + (1-\rho^2)^{-1}) \geq 1$ and $y = \rho^2/(1-\rho^2)^2$, we obtain

$$\begin{aligned} &\Bigg|\left[\left(\frac{1}{a}+\frac{1}{1-\rho^2}\right)\left(\frac{1}{b}+\frac{1}{1-\rho^2}\right)-\frac{\rho^2}{(1-\rho^2)^2}\right]^{-m/2} \\ &\qquad -\left[\left(\frac{1}{a}+\frac{1}{1-\rho^2}\right)\left(\frac{1}{b}+\frac{1}{1-\rho^2}\right)\right]^{-m/2}\Bigg| \\ &\quad \leq \frac{m}{2}\frac{\rho^2}{(1-\rho^2)^2}\left(\frac{2}{d}\right)^{m/2+1}. \end{aligned}$$

For $x = (1+a)/a$ and $y = \rho^2/(1-\rho^2)$, we obtain

$$\begin{aligned} &\left|\left(\frac{1}{a}+\frac{1}{1-\rho^2}\right)^{-m/2} - \left(\frac{1}{a}+1\right)^{-m/2}\right| \\ &\quad = \left|\left(\frac{1+a}{a}+\frac{\rho^2}{1-\rho^2}\right)^{-m/2} - \left(\frac{1+a}{a}\right)^{-m/2}\right| \\ &\quad \leq \frac{m}{2}\frac{\rho^2}{1-\rho^2}\left(\frac{2}{d}\right)^{m/2+1}, \end{aligned}$$

with an analogous inequality with b in the place of a. Here we have also used the elementary inequalities $\rho^2 \le \rho^2/(1-\rho^2) \le \rho^2/(1-\rho^2)^2$, for $|\rho| \le 1$. The assertion of the lemma now follows from the inequality $\rho^2/(1-\rho^2)^2 \le 4\rho^2/(2-d)^2$, for $\rho^2 < d/2$. □

LEMMA 2.5. *Let $\phi(z)$ be the standard normal density function. Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$ be an m-component vector process. Then, for every $t > 0$, $\varepsilon > 0$, the expression*

$$\varepsilon^{-m} \int_0^t \int_0^t \sum_{j=1}^m \phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon}\right) ds\, ds' \tag{2.7}$$

is equal to

$$(2\pi)^{-m} \int_{R^m} \left[\left(\int_0^t \sin(\mathbf{u}, \mathbf{X}(s))\, ds \right)^2 + \left(\int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\, ds \right)^2 \right] e^{-\varepsilon^2 |\mathbf{u}|^2/2}\, d\mathbf{u}, \tag{2.8}$$

where $(\mathbf{u}, \mathbf{x})$ is the inner product in R^m and $|\mathbf{u}|^2 = (\mathbf{u}, \mathbf{u})$.

PROOF. The sum of the squares in the brackets in the integrand in (2.8) is equal to

$$\int_0^t \int_0^t \exp[i(\mathbf{u}, \mathbf{X}(s) - \mathbf{X}(s'))]\, ds\, ds'.$$

Substitute this in (2.8), interchange the order of integration and then apply the formula for the normal characteristic function. This leads to (2.7). □

LEMMA 2.6. *Let $\mathbf{X}(t)$ be a vector process whose m components are independent copies of a stationary Gaussian process $X(t)$, and let $h(x)$ be a real Borel function such that*

$$\int_{-\infty}^{\infty} h^2(x)\phi(x)\, dx < \infty \tag{2.9}$$

and

$$\int_{-\infty}^{\infty} x h(x)\phi(x)\, dx = 0. \tag{2.10}$$

For every $t > 0$, we have

$$\operatorname{Var}\left(\int_0^t \sum_{j=1}^m h(X_j(s))\, ds \right) = 2\int_0^t (t-s)\left\{ \left(E[h(X(0))h(X(s))] \right)^m - \left(E[h(X(0))] \right)^{2m} \right\} ds. \tag{2.11}$$

If $r \in L_2$, *then*

$$\begin{aligned} &\operatorname{Var}\left(t^{-1/2}\int_0^t \prod_{j=1}^m h(X_j(s))\,ds\right) \\ (2.12)\qquad &\to 2\int_0^\infty \left\{(E[h(X(0))h(X(s))])^m - (E[h(X(0))])^{2m}\right\} ds, \end{aligned}$$

for $t \to \infty$, *and the right-hand member of* (2.12) *is finite.*

PROOF. The formula (2.11) is a consequence of Fubini's theorem, stationarity and the independence of the component processes.

Let $H_n(x)$ be the Hermite polynomial of order n, for $n \geq 0$, and put

$$h_n = \int_{-\infty}^\infty h(x)H_n(x)\phi(x)\,dx, \tag{2.13}$$

the nth Hermite coefficient of h. Then, under (2.9), h has the expansion

$$h(x) = \sum_{n=0}^\infty \frac{h_n}{n!}H_n(x). \tag{2.14}$$

By (2.10) and the fact $H_1(x) = x$, we have

$$h_1 = 0. \tag{2.15}$$

Let $\phi(x, y; \rho)$ be the standard bivarite normal density; then, by the well-known diagonal expansion in Hermite polynomials [see Cramér (1946), page 290],

$$\phi(x, y; \rho) = \phi(x)\phi(y)\sum_{n=0}^\infty \frac{1}{n!}H_n(x)H_n(y)\rho^n, \tag{2.16}$$

the right-hand member of (2.11), divided by t, is equal to

$$\frac{2}{t}\int_0^t (t-s)\left\{\left(h_0^2 + \sum_{n=2}^\infty \frac{h_n^2}{n!}r^n(s)\right)^m - h_0^{2m}\right\} ds. \tag{2.17}$$

For arbitrary T, $0 < T < t$, the portion of (2.17) corresponding to the integral over the subdomain $[0, T]$ converges, for $t \to \infty$, to

$$2\int_0^T \left\{\left(h_0^2 + \sum_{n=2}^\infty \frac{h_n^2}{n!}r^n(s)\right)^m - h_0^{2m}\right\} ds. \tag{2.18}$$

The remaining portion of (2.17) is at most equal to two times

$$\int_T^\infty \left|\left(h_0^2 + \sum_{n=2}^\infty \frac{h_n^2}{n!}r^n(s)\right)^m - h_0^{2m}\right| ds. \tag{2.19}$$

Bessel's equation and the definition (2.13) imply

$$\sum_{n=2}^{\infty} \frac{h_n^2}{n!} \leq \int_{-\infty}^{\infty} h^2(x)\phi(x)\,dx. \tag{2.20}$$

Furthermore, since $r \in L_2$, the corresponding Fourier transform f in L_2 is, in fact, the spectral density and, thus, also belongs to L_1. Therefore, by the Riemann–Lebesgue theorem, $r(t) \to 0$ for $t \to \infty$. Hence, the term

$$\sum_{n=2}^{\infty} \frac{h_n^2}{n!} r^n(s)$$

in (2.19), which is dominated by

$$r^2(s)\int_{-\infty}^{\infty} h^2(x)\phi(x)\,dx,$$

can be made arbitrarily small uniformly in s by choosing T, the lower limit of integration, sufficiently large. By a simple application of the law of the mean, we see that (2.19) is at most equal to a constant times $\int_T^\infty r^2(s)\,ds$. The latter can be made small by choosing T sufficiently large.

We have now proved that the variance in (2.12) converges to a finite limit which is equal to (2.18) with $T = \infty$. By the Hermite expansion leading to (2.17), the right-hand member of (2.12) is also equal to the same limit:

$$\begin{aligned}&\int_0^\infty \left\{ \left(E[h(X(0))h(X(s))]\right)^m - \left(E[h(X(0))]\right)^{2m} \right\} ds \\ &= \int_0^\infty \left\{ \left(h_0^2 + \sum_{n=2}^{\infty} \frac{h_n^2}{n!} r^n(s) \right)^m - h_0^{2m} \right\} ds.\end{aligned} \tag{2.21}$$

This proves (2.12). □

3. Approximation of a stationary Gaussian process by one whose covariance has compact support. The results here are refinements and extensions of those in Berman (1970). Let $r(t)$ be the covariance function of a stationary Gaussian process such that $r(0) = 1$ and $r \in L_2$. It has the well-known representation

$$r(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda)\,d\lambda, \tag{3.1}$$

where f is the spectral density and where $f \in L_1 \cap L_2$. Let $b(t)$ be a nonnegative bounded measurable function with support $[-\frac{1}{2}, \frac{1}{2}]$ and such that

$$\int_{-1/2}^{1/2} b^2(t)\,dt = 1.$$

Put

$$\rho(t) = \int_{-1/2}^{1/2} b(t+s)b(s)\,ds.$$

Then $\rho(t)$ is even, positive definite and has support in $[-1,1]$. It has the spectral representation

$$\rho(t) = \int_{-\infty}^{\infty} e^{i\lambda t} q(\lambda)\,d\lambda,$$

where

$$q(\lambda) = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} e^{i\lambda s} b(s)\,ds\right|^2.$$

Put

$$(q_n * f)(\lambda) = \int_{-\infty}^{\infty} f\left(\lambda + \frac{y}{n}\right) q(y)\,dy, \tag{3.2}$$

for $n \geq 1$, and put

$$r_n(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\left[f(\lambda)(q_n * f)(\lambda)\right]^{1/2} d\lambda. \tag{3.3}$$

LEMMA 3.1.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} q(y) \int_{-\infty}^{\infty} \left| f\left(\lambda + \frac{y}{n}\right) - f(\lambda)\right| d\lambda\,dy = 0. \tag{3.4}$$

PROOF. The inner integral in (3.4) is bounded by $2\int_{-\infty}^{\infty} f(\lambda)\,d\lambda$, uniformly in y; hence it suffices to prove that the inner integral converges to 0 for each fixed y because $q(y)$ is actually a density function.

For arbitrary $T > 0$, the inner integral is at most equal to the sum of the terms

$$2\int_{|\lambda| > T - y/n} f(\lambda)\,d\lambda \tag{3.5}$$

and

$$\int_{-T}^{T} \left| f\left(\lambda + \frac{y}{n}\right) - f(\lambda)\right| d\lambda. \tag{3.6}$$

The term (3.5) can be made arbitrarily small by choosing T and n sufficiently large. By the Cauchy–Schwarz inequality, (3.6) is at most

$$(2T)^{1/2}\left\{\int_{-\infty}^{\infty} \left| f\left(\lambda + \frac{y}{n}\right) - f(\lambda)\right|^2 d\lambda\right\}^{1/2},$$

which, by Parseval's theorem, is equal to the square root of

$$\frac{T}{\pi}\int_{-\infty}^{\infty} |e^{ity/n} - 1|^2 r^2(t)\,dt,$$

which converges to 0 for $n \to \infty$. □

LEMMA 3.2. *$\lim_{n\to\infty} r_n(t) = r(t)$ uniformly in t.*

PROOF. The proof is done by the same calculation as in Berman [(1970), page 723]. Lemma 3.1 fills a gap in the latter proof concerning the convergence of $(q_n * f)(\lambda)$ to $f(\lambda)$. □

LEMMA 3.3. *$\lim_{n\to\infty} \int_{-\infty}^{\infty} |r_n(t) - r(t)|^2\, dt = 0$.*

PROOF. The proof is by the same calculation as in Berman [(1970), page 723]. □

Let $X(t)$ be a real stationary Gaussian process with mean 0 and the covariance function $r(t)$ with the representation (3.1). Let $W_1(\lambda)$ and $W_2(\lambda)$ be independent standard Brownian motions and put $W(\lambda) = \frac{1}{2}W_1(\lambda) + \frac{1}{2}iW_2(\lambda)$. Then $X(t)$ has the stochastic integral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}(f(\lambda))^{1/2} W(d\lambda). \tag{3.7}$$

Define

$$Y_n(t) = \int_{-\infty}^{\infty} e^{i\lambda t}[(q_n * f)(\lambda)]^{1/2} W(d\lambda) \tag{3.8}$$

with respect to the same Brownian motion on the same space. Then $(X(t), Y_n(t))$ is a bivariate stationary Gaussian process with mean vector $(0, 0)$ and

$$\begin{aligned} EY_n(s)Y_n(t+s) &= \int_{-\infty}^{\infty} e^{i\lambda t}(q_n * f)(\lambda)\, d\lambda \\ &= r(t)\rho(t/n) \end{aligned} \tag{3.9}$$

(which follows from the convolution formula) and, by (3.3), (3.7) and (3.8),

$$EX(0)Y_n(t) = r_n(t). \tag{3.10}$$

By (3.9), the support of the covariance function of $Y_n(t)$ is contained in $[-n, n]$. Furthermore, the process $Y_n(t)$ converges to $X(t)$ in the sense

$$\lim_{n\to\infty} E(X(t) - Y_n(t))^2 = 0,$$

uniformly in t.

We remark that there does not seem to be a version of this construction in the case of a discrete-time stationary Gaussian process because the spectral density has support in $[-\pi, \pi]$, and the convolution operation enlarges the domain of support.

4. A central limit theorem for a class of integral functionals of a stationary vector Gaussian process. Let $\mathbf{X}(t)$ be the vector process whose components $X_j(t)$, $j = 1, \dots, m$, are independent copies of $X(t)$. The following theorem is the main result of this section.

THEOREM 4.1. *Let $h(x)$ be a function satisfying* (2.9) *and* (2.10), *and suppose $r \in L_2$. Then*

$$t^{-1/2}\left\{\int_0^t \prod_{j=1}^m h(X_j(s))\,ds - th_0^m\right\} \tag{4.1}$$

has, for $t \to \infty$, a limiting normal distribution with mean 0 *and variance*

$$2\int_0^\infty \left\{(E[h(X(0))h(X(s))])^m - h_0^{2m}\right\} ds, \tag{4.2}$$

where h_0 is defined as $Eh(X(0))$.

PROOF. We begin the proof with that of a variant of the theorem. For an arbitrary m-tuple $(n_1, \dots, n_m)$ of nonnegative integers such that

$$\nu = n_1 + \cdots + n_m \geq 2, \tag{4.3}$$

consider the functional

$$t^{-1/2}\int_0^t \sum_{j=1}^m H_{n_j}(X_j(s))\,ds, \tag{4.4}$$

where $H_n(x)$ is the Hermite polynomial of order n. It is a consequence of the well-known orthogonality property of the Hermite polynomials and of the expansion (2.16) that

$$\operatorname{Cov}\left[H_k(X_i(s)), H_l(X_j(s'))\right] = \delta_{ij}\delta_{kl}k!r^k(s - s'), \tag{4.5}$$

$i, j = 1, \dots, m$, $k, l \geq 0$, and where δ is the Kronecker δ. From this it follows that the random variable (4.4) has mean 0 and variance

$$\prod_{j=1}^m n_j!(2/t)\int_0^t (t - s)(r(s))^\nu\,ds. \tag{4.6}$$

By applying the preceding results to the process $Y_n(t)$ defined by (3.8), we find that for the independent copies $Y_{j,n}(t)$, $j = 1, \dots, m$, the functional

$$t^{-1/2}\int_0^t \prod_{j=1}^m H_{n_j}(Y_{j,n}(s))\,ds \tag{4.7}$$

has mean 0 and variance

$$\prod_{j=1}^m n_j!(2/t)\int_0^t (t - s)(r(s)\rho(s/n))^\nu\,ds. \tag{4.8}$$

Since $Y_{j,n}(s)$ has, for each n, a covariance with support $[-n, n]$, the vector process $(Y_{j,n}(s))$ is $2n$-independent, that is, parts of the process separated by at least $2n$ time units are independent, and so the real process $\prod_{j=1}^m H_{n_j}(Y_{j,n}(s))$ has the same property. A classical theorem of Hoeffding and Robbins (1948) asserts the validity of the central limit theorem for the partial sums in a stationary $2n$-dependent sequence of random variables under the condition of the existence of the third moment of the marginal distribution. Their proof extends to the integral of a stationary, continuous parameter process. In our case, the third moment exists so that the central limit theorem holds for the functional (4.7), for $t \to \infty$ and n fixed. The mean of the limiting normal distribution is 0 and the variance is the limit of (4.8), which, by dominated convergence, is equal to

$$2\prod_{j=1}^{m} n_j! \int_0^\infty (r(s)\rho(s/n))^\nu \, ds. \tag{4.9}$$

Our next step is to estimate the variance of the difference between the functionals (4.4) and (4.7).

LEMMA 4.1. *Let $(X_j(t), Y_{j,n}(t))$ $j = 1, \dots, m$, be independent copies of the bivariate process $(X(t), Y_n(t))$, defined at the end of Section 3. Then, for every $t > 0$ and $n \geq 1$,*

$$\operatorname{Var}\left\{t^{-1/2}\int_0^t \prod_{j=1}^{m} H_{n_j}(X_j(s))\,ds - t^{-1/2}\int_0^t \sum_{j=1}^{m} H_{n_j}(Y_{j,n}(s))\,ds\right\}$$

$$\leq 2\nu \prod_{j=1}^{m} n_j! \left\{\int_0^\infty |r(s) - r_n(s)|^2\,ds \int_0^\infty \left[2r^2(s) + 2r_n^2(s)\right] ds\right\}^{1/2}. \tag{4.10}$$

PROOF. By the reasoning used to establish (2.11), the left-hand member of (4.10) is equal to

$$\begin{aligned}(2/t)\int_0^t (t-s)\Bigg\{&\prod_{j=1}^{m} E\left[H_{n_j}(X(0))H_{n_j}(X(s))\right]\\ &-2\prod_{j=1}^{m} E\left[H_{n_j}(X(0))H_{n_j}(Y_n(s))\right]\\ &+\prod_{j=1}^{m} E\left[H_{n_j}(Y_n(0))H_{n_j}(Y_n(s))\right]\Bigg\}\,ds,\end{aligned}$$

which, by (4.5), (3.9) and (3.10), is equal to $\prod n_j!$ times

$$(2/t)\int_0^t (t-s)\left\{[r(s)]^\nu - 2[r_n(s)]^\nu + [r(s)\rho(s/n)]^\nu\right\} ds. \tag{4.11}$$

The expression (4.11) is at most equal to

$$2\int_0^\infty \left|[r(s)]^\nu - [r_n(s)]^\nu\right| ds \tag{4.12}$$

plus

$$2\int_0^\infty \left|[r_n(s)]^\nu - [r(s)\rho(s/n)]^\nu\right| ds. \tag{4.13}$$

From the elementary identity $A^\nu - B^\nu = (A - B)(A^{\nu-1} + A^{\nu-2}B + \cdots + AB^{\nu-2} + B^{\nu-1})$, we deduce

$$|A^\nu - B^\nu| \le \nu|A - B|(|A| + |B|), \tag{4.14}$$

for $|A| \le 1$ and $|B| \le 1$. It follows that (4.11) is at most equal to

$$2\nu\int_0^\infty |r(s) - r_n(s)|(|r(s)| + |r_n(s)|)\, ds,$$

which, by an application of the Cauchy–Schwarz inequality, is at most equal to

$$2\nu\left\{\int_0^\infty |r(s) - r_n(s)|^2\, ds \int_0^\infty \left[2r^2(s) + 2r_n^2(s)\right] ds\right\}^{1/2}. \qquad \square$$

Next we show that the functional (4.4) has a limiting normal distribution with mean 0 and variance

$$2\prod_{j=1}^{n} n_j! \int_0^\infty (r(s))^\nu\, ds. \tag{4.15}$$

Let $J(t)$ and $J_n(t)$ represent the functionals (4.4) and (4.7), respectively. By the limit theorem for (4.7), sketched following (4.8), $J_n(t)$ has, for $t \to \infty$ and then $n \to \infty$, a limiting normal distribution with mean 0 and variance equal to the limit of (4.9), for $n \to \infty$, namely, (4.15). Next we note that the right-hand member of (4.10) converges to 0 for $n \to \infty$; this is a consequence of Lemma 3.3. It follows from Lemma 4.1 that $\sup_t \mathrm{Var}(J(t) - J_n(t)) \to 0$, for $n \to \infty$, and, as a consequence, $J(t)$ has, for $t \to \infty$, the same limiting distribution as does $J_n(t)$, for $t \to \infty$ and then $n \to \infty$. The last assertion can be verified by using the general result of Dynkin [1988), Lemma 1.1].

Finally, we show how the validity of the central limit theorem for the functional (4.4) implies its validity for the functional (4.1). For $m \ge 1$, the function $h(x_1) \cdots h(x_m)$ has the Hermite expansion

$$\prod_{j=1}^{m} h(x_j) = h_0^m + \sum_{n_1 + \cdots + n_m \ge 2} \prod_{j=1}^{m} \left[\frac{h_{n_j} H_{n_j}(x_j)}{n_j!}\right], \tag{4.16}$$

where the summation is over all $(n_1, \ldots, n_m)$ satisfying $n_1 + \cdots n_m \ge 2$, and where (h_n) is defined by (2.13) and $h_1 = 0$ by (2.15). If $h(x)$ is a function with a finite Hermite expansion, that is, $h_n = 0$ for all but finitely many n, then the product expansion (4.16) has finitely many terms. It follows that the random variable (4.1) is a linear combination of the random variables (4.4), for various m-tuples $(n_1, \ldots, n_m)$. Furthermore, for distinct $(n_1, \ldots, n_m)$ and $(n'_1, \ldots, n'_m)$, the corresponding random variables (4.4) are orthogonal; this is a consequence of (4.5). From this orthogonality and the validity of the central limit for each of the terms (4.4), we deduce the statement of Theorem 4.1 for a function $h(x)$ with a finite expansion. Furthermore, (2.21) indicates that (4.2) is the variance of the limiting distribution. The statement of the theorem for the case of a general function h now follows by the well-known L_2-approximation of $\prod_{j=1}^m h(x_j) - h_0^m$ by the partial sums of the series in (4.16). $\square$

As an application of Theorem 4.1, with $h(x) = \exp(-\frac{1}{2}x^2)$, and Lemmas 2.3 and 2.4, we obtain the following corollary.

COROLLARY 4.1. *The random variable*

$$t^{-1/2}\left[\int_0^t \exp\left(-\tfrac{1}{2}|\mathbf{X}(s)|^2\right) ds - t2^{-m/2}\right] \tag{4.17}$$

has, for $t \to \infty$, a limiting normal distribution with mean 0 *and variance*

$$2\int_0^\infty \left\{ \left(1 - r^2(s)\right)^{-m/2} \left[\left(1 + \frac{1}{1 - r^2(s)}\right)^2 - \frac{r^2(s)}{\left(1 - r^2(s)\right)^2} \right]^{-m/2} - 2^{-m} \right\} ds, \tag{4.18}$$

where the integral is, by simple algebra, equal to that in (1.7).

5. Proof of Theorem 1.1. In this section we use the symbol "Lim" to denote the limit operation $\varepsilon \to 0$, $t \to \infty$, restricted by (1.5):

$$\mathrm{Lim} = \lim_{\substack{\varepsilon \to 0,\, t \to \infty, \\ B(\varepsilon)t^{-1/2} \to 0}} . \tag{5.1}$$

The first step in the completion of the Proof of Theorem 1.1 is the following lemma.

LEMMA 5.1. *Under the condition $r \in L_2$,*

$$\mathrm{Lim}\, t^{-3/2} \int_{R^m} E\left(\int_0^t \sin(u, \mathbf{X}(s))\, ds\right)^2 e^{-\varepsilon^2|\mathbf{u}|^2/2}\, d\mathbf{u} = 0. \tag{5.2}$$

PROOF. Apply Lemma 2.2 to $\xi = (\mathbf{u}, \mathbf{X}(s))$ and $\eta = (\mathbf{u}, \mathbf{X}(s'))$, for $s \neq s'$; here $\operatorname{Var} \xi = \operatorname{Var} \eta = |\mathbf{u}|^2$ and $\operatorname{Cov}(\xi, \eta) = |\mathbf{u}|^2 r(s - s')$ so that the expression under the limit sign in (5.2) is equal to

$$2t^{-3/2} \int_0^t (t - s) \int_{R^m} \exp\left[-|\mathbf{u}|^2\left(1 + \frac{\varepsilon^2}{2}\right)\right] \sinh\left(|\mathbf{u}|^2 r(s)\right) d\mathbf{u}\, ds, \tag{5.3}$$

which, by the elementary formula $\int_{-\infty}^{\infty} \exp(-y^2 c)\, dy = (\pi/c)^{1/2}$, is equal to

$$t^{-3/2}\pi^{m/2} \int_0^t (t - s) \left[\left(\frac{\varepsilon^2}{2} + 1 - r(s)\right)^{-m/2} - \left(\frac{\varepsilon^2}{2} + 1 + r(s)\right)^{-m/2} \right] ds. \tag{5.4}$$

By the argument following (2.20), $r(t) \to 0$ for $t \to \infty$; hence $\inf_s r(s) > -1$, and, for any $t > 0$, $\sup(r(s): |s| > t) < 1$. It follows that, for any T, $1 < T < t$, the portion of (5.4) corresponding to the integral over the subdomain $[0, T]$ is at most equal to

$$t^{-1/2}\pi^{m/2}\left\{B(\varepsilon) + \int_1^T\left(\frac{\varepsilon^2}{2} + 1 - \sup_{1<s<T} r(s)\right)^{-m/2} ds + \int_0^T\left(\frac{\varepsilon^2}{2} + 1 + \inf_s r(s)\right)^{-m/2} ds\right\},$$

and the latter converges to 0 under the limit operation (5.1).

Now we consider the portion of (5.4) corresponding to the domain $[T, t]$. If T is sufficiently large, then $|r(s)|$ in the integrand may be assumed to be arbitrarily small. Thus, by an argument similar to that used in the proof of Lemma 4.1, we see that the expression (5.4), with $\int_T^t$ in the place of $\int_0^t$, is at most equal to a constant times $t^{-1/2}\int_T^t|r(s)|\,ds$. By the Cauchy–Schwarz inequality, the latter is at most equal to $(\int_T^\infty r^2(s)\,ds)^{1/2}$, which is small for large T. □

LEMMA 5.2. *Under the condition $r \in L_2$ and the limiting operation* (5.1), *the random variable*

$$(5.5)\qquad t^{-3/2}\int_{R^m}\left[\left(\int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\,ds\right)^2 - t^2 e^{-|\mathbf{u}|^2}\right]\exp\left(-\frac{\varepsilon^2|\mathbf{u}|^2}{2}\right)d\mathbf{u}$$

has the same limiting distribution as

$$(5.6)\qquad 2t^{-1/2}\int_{R^m}\left[\int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\,ds - te^{-|\mathbf{u}|^2/2}\right]\exp\left(-\tfrac{1}{2}(1+\varepsilon^2)|\mathbf{u}|^2\right)d\mathbf{u}.$$

PROOF. By the simple identity $A^2 - B^2 = 2B(A - B) + (A - B)^2$, with

$$A = \int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\,ds, \qquad B = te^{-|\mathbf{u}|^2/2},$$

we see that (5.5) is equal to (5.6) plus

$$(5.7)\qquad t^{-3/2}\int_{R^m}\left[\int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\,ds - te^{-|\mathbf{u}|^2/2}\right]^2 \exp\left(-\frac{\varepsilon^2|\mathbf{u}|^2}{2}\right)d\mathbf{u}.$$

Thus, for the proof of the lemma, it suffices to show that

$$(5.8)\qquad t^{-3/2}\int_{R^m} E\left[\int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\,ds - te^{-|\mathbf{u}|^2/2}\right]^2 \exp\left(\frac{\varepsilon^2|\mathbf{u}|^2}{2}\right)d\mathbf{u}$$

converges to 0 under (5.1).

Lemma 2.2 implies that (5.8) is equal to

$$t^{-3/2}\int_{R^m}\left[2\int_0^t(t-s)\left[\cosh\left(|\mathbf{u}|^2 r(s)\right)-1\right]ds\right]\exp\left(-|\mathbf{u}|^2\left(1+\frac{\varepsilon^2}{2}\right)\right)d\mathbf{u},$$

which, by a change of order of integration and the formula

$$\int_{R^m}\cosh(|\mathbf{u}|^2 r)e^{-c|\mathbf{u}|^2}\,d\mathbf{u}=\tfrac{1}{2}\pi^{m/2}\left[(c-r)^{-m/2}+(c+r)^{-m/2}\right],$$

for $c>|r|$, is at most equal to

$$\pi^{m/2}t^{-3/2}\int_0^t(t-s)\left[\left(\frac{\varepsilon^2}{2}+1-r(s)\right)^{-m/2}-\left(\frac{\varepsilon^2}{2}+1+r(s)\right)^{-m/2}\right]ds. \tag{5.9}$$

The expression (5.9) is identical with (5.4). As shown in the proof of Lemma 5.1, the latter converges to 0; hence, the expression (5.8) also converges to 0. □

LEMMA 5.3. *The random variable* (5.6) *is representable in the form*

$$2t^{-1/2}\left(\frac{2\pi}{1+\varepsilon^2}\right)^{m/2}\left[\int_0^t\exp\left(-\frac{|\mathbf{X}(s)|^2}{2(1+\varepsilon^2)}\right)ds-t\left(\frac{1+\varepsilon^2}{2+\varepsilon^2}\right)^{m/2}\right]. \tag{5.10}$$

PROOF. This is a consequence of the application of the formula

$$\int_{R^m}\cos(\mathbf{u},\mathbf{x})\exp\left(-\frac{1}{2}c|\mathbf{u}|^2\right)d\mathbf{u}=\int_{R^m}\exp[i(\mathbf{u},\mathbf{x})]\exp\left(-\frac{1}{2}c|\mathbf{u}|^2\right)d\mathbf{u}$$
$$=\left(\frac{2\pi}{c}\right)^{m/2}\exp\left(-\frac{|\mathbf{x}|^2}{2c}\right),$$

for $c>0$ and $x\in R^m$. □

LEMMA 5.4. *Consider the difference between the random variable in the bracket in* (5.10) *and the one obtained from it by putting* $\varepsilon=0$:

$$\int_0^t\left[\exp\left(-\frac{|\mathbf{X}(s)|^2}{2(1+\varepsilon^2)}\right)-\exp\left(-\frac{1}{2}|\mathbf{X}(s)|^2\right)\right]ds-t\left[\left(\frac{1+\varepsilon^2}{2+\varepsilon^2}\right)^{m/2}-2^{-m/2}\right]. \tag{5.11}$$

The variance of this difference is equal to

$$
\begin{aligned}
2\int_0^t (t-s)\Bigg\{ & (1-r^2(s))^{-m/2} \\
& \times\left[\left(\frac{1}{1+\varepsilon^2}+\frac{1}{1-r^2(s)}\right)^2-\frac{r^2(s)}{(1-r^2(s))^2}\right]^{-m/2} \\
& +(1-r^2(s))^{-m/2}\left[\left(1+\frac{1}{1-r^2(s)}\right)^2-\frac{r^2(s)}{(1-r^2(s))^2}\right]^{-m/2} \\
& -2(1-r^2(s))^{-m/2}\Bigg[\left(\frac{1}{1+\varepsilon^2}+\frac{1}{1-r^2(s)}\right)\left(1+\frac{1}{1-r^2(s)}\right) \\
& \qquad -\frac{r^2(s)}{(1-r^2(s))^2}\Bigg]^{-m/2}\Bigg\}\,ds \\
-t^2 & \left[\left(\frac{1+\varepsilon^2}{2+\varepsilon^2}\right)^{m/2}-2^{-m/2}\right]^2.
\end{aligned}
\tag{5.12}
$$

PROOF. A simple calculation shows that the expression (5.11) has the expected value 0; hence, the variance of (5.11) is equal to

$$
\begin{aligned}
2\int_0^t (t-s)E\Bigg\{ & \left[\exp\left(-\frac{|\mathbf{X}(s)|^2}{2(1+\varepsilon^2)}\right)-\exp\left(-\frac{1}{2}|\mathbf{X}(s)|^2\right)\right] \\
& \times\left[\exp\left(-\frac{|\mathbf{X}(0)|^2}{2(1+\varepsilon^2)}\right)-\exp\left(-\frac{1}{2}|\mathbf{X}(0)|^2\right)\right]\Bigg\}\,ds \\
-t^2 & \left[\left(\frac{1+\varepsilon^2}{2+\varepsilon^2}\right)^{m/2}-2^{-m/2}\right]^2.
\end{aligned}
$$

The formula (5.12) is obtained from this by several applications of Lemma 2.3 with $\xi = X_j(0)$ and $\eta = X_j(s)$, for $j = 1, \dots, m$, and $a = 1, 1+\varepsilon^2$ and $b = 1, 1+\varepsilon^2$. □

LEMMA 5.5. *The expression* (5.12), *divided by* t, *converges to* 0 *under the double limit operation* $t \to \infty$, $\varepsilon \to 0$.

PROOF. By simple algebra and the identity

$$\frac{1}{1+\varepsilon^2}+\frac{1}{1-r^2(s)}=\frac{2+\varepsilon^2}{1+\varepsilon^2}+\frac{r^2(s)}{1-r^2(s)},$$

the variance (5.12) is equal to the sum of

$$2\int_0^t (t-s)\left\{\left(1-r^2(s)\right)^{-m/2}\left[\left(\frac{2+\varepsilon^2}{1+\varepsilon^2}+\frac{r^2(s)}{1-r^2(s)}\right)^2 - \frac{r^2(s)}{\left(1-r^2(s)\right)^2}\right]^{-m/2} - \left(\frac{2+\varepsilon^2}{1+\varepsilon^2}\right)^{-m}\right\} ds \tag{5.13}$$

and

$$2\int_0^t (t-s)\left\{\left(1-r^2(s)\right)^{-m/2}\left[\left(2+\frac{r^2(s)}{1-r^2(s)}\right)^2 - \frac{r^2(s)}{\left(1-r^2(s)\right)^2}\right]^{-m/2} - 2^{-m}\right\} ds, \tag{5.14}$$

minus twice the term

$$2\int_0^t (t-s)\left\{\left(1-r^2(s)\right)^{-m/2}\left[\left(\frac{2+\varepsilon^2}{1+\varepsilon^2}+\frac{r^2(s)}{1-r^2(s)}\right)\times\left(2+\frac{r^2(s)}{1-r^2(s)}\right) - \frac{r^2(s)}{\left(1-r^2(s)\right)^2}\right]^{-m/2} -2^{-m/2}\left(\frac{2+\varepsilon^2}{1+\varepsilon^2}\right)^{-m/2}\right\} ds. \tag{5.15}$$

Let us now divide each of the preceding expressions by t and, for arbitrary fixed $T > 0$, consider only those portions of each integral from 0 to T. Then, under the limit operation $t \to \infty$, $\varepsilon \to 0$, each of the corresponding portions of (5.13), (5.14) and (5.15) converges to the common limit

$$2\int_0^T \left\{\left(1-r^2(s)\right)^{-m/2}\left[\left(2+\frac{r^2(s)}{1-r^2(s)}\right)^2 - \frac{r^2(s)}{\left(1-r^2(s)\right)^2}\right]^{-m/2} - 2^{-m}\right\} ds.$$

Hence, the corresponding sum of (5.13) and (5.14), minus twice (5.15), converges to 0.

In order to complete the proof, we will show that each of the terms (5.13), (5.14) and (5.15), after division by t and the replacement of the lower limit of integration by T, can be made arbitrarily small by choosing T sufficiently large and then letting $t \to \infty$, uniformly in ε. For example, let us estimate the corresponding version of (5.13):

$$2\int_T^t \left(1 - \frac{s}{t}\right)\left\{(1 - r^2(s))^{-m/2}\left[\left(\frac{2+\varepsilon^2}{1+\varepsilon^2} + \frac{r^2(s)}{1-r^2(s)}\right)^2 - \frac{r^2(s)}{(1-r^2(s))^2}\right]^{-m/2} - \left(\frac{2+\varepsilon^2}{1+\varepsilon^2}\right)^{-m}\right\} ds.$$

By Lemma 2.4, this integral has a bound of the form $K\int_T^\infty r^2(s)\,ds$, where K is a constant that does not depend on ε or t. [For the application of the lemma, it is necessary to use the simple identity preceding (5.13).] The integral is small if T is large. The same reasoning applies to the other terms, (5.14) and (5.15). □

We now complete the proof of Theorem 1.1. By Lemma 2.5, the random variable (1.6) is equal to the sum of

$$(5.16) \qquad t^{-3/2}(2\pi)^{-m}\int_{R^m}\left(\int_0^t \sin(\mathbf{u}, \mathbf{X}(s))\,ds\right)^2 e^{-\varepsilon^2|\mathbf{u}|^2/2}\,d\mathbf{u}$$

and

$$(5.17) \quad t^{-3/2}\left[(2\pi)^{-m}\int_{R^m}\left[\left(\int_0^t \cos(\mathbf{u}, \mathbf{X}(s))\,ds\right)^2 - t^2 e^{-|\mathbf{u}|^2}\right] e^{-\varepsilon^2|\mathbf{u}|^2/2}\,d\mathbf{u}\right].$$

By Lemma 5.1, the random variable (5.16) converges to 0 in probability. By Lemmas 5.2–5.5, the random variable (5.17) has the same limiting distribution as the random variable (4.17), times $2(2\pi)^{-m/2}$. An application of Corollary 4.1 now completes the proof. □

6. Negligibility of the contribution to (1.2) of points (s, s') in any neighborhood of the diagonal.

THEOREM 6.1. *For every $T > 0$, the conclusion of Theorem* 1.1 *is not changed if the random variable* (1.2) *in* (1.6) *is replaced by*

$$(6.1) \qquad \varepsilon^{-m}\iint_{\substack{0 \le s, s' \le t,\\ |s-s'| > T}} \prod_{j=1}^m \phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon}\right) ds\,ds'.$$

PROOF. First we show that the complementary integral

$$\varepsilon^{-m} \iint_{\substack{0 \le s, s' \le t, \\ |s-s'| \le T}} \prod_{j=1}^{m} \phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon} \right) ds\, ds' \tag{6.2}$$

has an expected value of order t, for $t \to \infty$. For simplicity and without loss of generality, we take $T = 1$. Then, by stationarity, the expected value of (6.2) is at most equal to twice the quantity

$$\varepsilon^{-m} \int_0^t \int_0^1 \prod_{j=1}^{m} E\phi\left(\frac{X_j(s) - X_j(s')}{\varepsilon} \right) ds\, ds'. \tag{6.3}$$

Since $X_j(s) - X_j(s')$ has a normal distribution with mean 0 and variance $2(1 - r(s - s'))$, the expression (6.3) is equal to

$$(2\pi)^{m/2} \int_0^t \int_0^1 \left[2(1 - r(s - s')) + \varepsilon^2\right]^{-m/2} ds'\, ds. \tag{6.4}$$

For arbitrary $T > 0$, write the integral (6.4) as the sum of

$$(2\pi)^{-m/2} \int_0^T \int_0^1 \left[2(1 - r(s - s')) + \varepsilon^2\right]^{-m/2} ds'\, ds \tag{6.5}$$

and

$$(2\pi)^{-m/2} \int_T^t \int_0^1 \left[2(1 - r(s - s')) + \varepsilon^2\right]^{-m/2} ds'\, ds. \tag{6.6}$$

Under the limit operation (1.5), the integral (6.5) is of smaller order than $t^{1/2}$. On the other hand, since $r(t) \to 0$ for $t \to \infty$, the integrand in (6.6) can be made arbitrarily close to $(2 + \varepsilon^2)^{-m/2}$, uniformly on $(s, s') \in [T, t] \times [0, 1]$, by choosing T sufficiently large. Hence, the expression (6.6) is approximately equal to

$$(2\pi)^{-m/2} t 2^{-m/2},$$

for $t \to \infty$, $\varepsilon \to 0$.

The random variable (6.2) is the difference between the random variables (1.2) and (6.1). Since it is a nonnegative random variable whose expected value is of order t, and since the factor $t^{-3/2}$ multiplies it in (1.6), this random variable may be ignored in the conclusion of Theorem 1.1, so that (6.1) may be used in the place of (1.2). □

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