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OPERATOR EXPONENTS OF PROBABILITY MEASURES AND LIE SEMIGROUPS[1]

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Dedicated to my teacher and master, Professor Kazimierz Urbanik, on the occasion of his sixtieth birthday

A notion of U-exponents of a probability measure on a linear space is introduced. These are bounded linear operators and it is shown that the set of all U-exponents forms a Lie wedge for full measures on finite-dimensional spaces. This allows the construction of U-exponents commuting with the symmetry group of a measure in question. Then the set of all commuting exponents is described and elliptically symmetric measures are characterized in terms of their Fourier transforms. Also, self-decomposable measures are identified among those which are operator-self-decomposable. Finally, S-exponents of infinitely divisible measures are discussed.

The theory of Lie groups and their Lie algebras is very well established and has many applications in different branches of mathematics and physics. Notions and first results for Lie *semigroups* are already in the fundamental book by Hille and Phillips [(1957), in the next-to-last chapter]. During the last 10 years or so many papers have appeared on Lie semigroups; compare references in Hilgert and Hofmann (1986a, b) or the recent monograph by Hilgert, Hofmann and Lawson (1989). In many of these, differential geometry and geometric control theory are used as examples where such theory is needed. This paper shows that tangent spaces to some operator semigroups and groups are very natural and basic tools in the central limit problem in probability theory on vector spaces. It seems that there are no results in Lie semigroup theory to be applied directly in the context of probability measures. One can hope that both fields will benefit from this new direction of investigation, Lie semigroup theory and probability theory. Here tangent spaces (sets) are used to characterize self-decomposable measures among those which are operator-self-decomposable (cf. Proposition 1.6). Furthermore, the presentation of results in this note emphasizes the probabilistic origin of the problem in question.

1. Notation, definitions and results. Let X be a Banach space and $(\mathscr{P}(X), *)$ denote the convolution semigroup of probability measures on X

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with the topology of weak convergence. A measure $\mu \in \mathscr{P}(X)$ is said to be *full* or *genuine* on X if its support is not contained in any proper hyperplane. By End(X) and Aut(X) we mean the Banach algebra of all bounded linear operators on X with the *operator norm topology* and the group of all invertible operators on X, respectively. For $\mu \in \mathscr{P}(X)$ and $A \in \mathrm{End}(X)$, $A\mu$ denotes the image of μ by A, that is, $A\mu$ is the probability distribution of $A\xi$ whenever μ is a probability distribution of X-valued random variable ξ. Then

$$A(\mu * \nu) = A\mu * A\nu, \qquad (AB)\mu = A(B\mu),$$

for $A, B \in \mathrm{End}(X)$ and $\mu, \nu \in \mathscr{P}(X)$. Urbanik (1972) introduced the notion of a *decomposability semigroup* $\mathbf{D}(\mu)$ of a probability measure μ, namely,

$$(1.1) \qquad \mathbf{D}(\mu) := \{A \in \mathrm{End}(X) \colon \mu = A\mu * \mu_A, \text{ for some } \mu_A \in \mathscr{P}(X)\}.$$

It is obvious that $\mathbf{D}(\mu)$ is a semigroup containing the zero 0 and the identity I. Furthermore, $\mathbf{D}(\mu)$ is closed in End(X); compare the proof of Proposition 1.2(b). [The norm topology is always taken in End(X).]

We say that $Q \in \mathrm{End}(X)$ is a *U-exponent* of a measure $\mu \in \mathscr{P}(X)$ if, for each $t \geq 0$, there exists $\nu_{t,Q} \in \mathscr{P}(X)$ such that

$$(1.2) \qquad \mu = e^{tQ}\mu * \nu_{t,Q}.$$

In other words, Q is a U-exponent of μ iff $\mathbf{D}(\mu)$ contains the one-parameter semigroup e^{tQ}, $t \geq 0$. Of course, $Q = 0$ is a U-exponent for all $\mu \in \mathscr{P}(X)$. On the other hand, each full operator-self-decomposable μ admits a U-exponent Q with the property $\lim_{t\to\infty} e^{tQ} = 0$ (in norm topology) [cf. Urbanik (1978)]. Recall here that μ is called *operator-self-decomposable* (or a *Lévy measure*) if μ is the limit distribution of the sequence

$$(1.3) \qquad A_n(\xi_1 + \xi_2 + \cdots + \xi_n) + x_n, \qquad n \geq 1,$$

where $x_n \in X$, $A_n \in \mathrm{Aut}(X)$, the ξ_n's are X-valued independent random variables such that the triangular array $\{A_n\xi_j \colon 1 \leq j \leq n,\, n \in \mathbf{N}\}$ is uniformly infinitesimal and

$$(1.4) \qquad \mathrm{sem}\{A_m A_n^{-1} \colon 1 \leq n \leq m, \in \mathbf{N}\} \quad \text{is compact in } \mathrm{End}(X).$$

[sem(F) denotes the smallest closed semigroup spanned by the family F. If $\dim X < \infty$ or $A_n = a_n I$, $a_n \in \mathbf{R}^+$, then (1.4) can be omitted.] Compare Urbanik (1978). The limits of (1.3) with $A_n = a_n I$ are called *self-decomposable measures* (or *Lévy class L distributions*); compare Loève (1963) and Jurek and Vervaat (1983). Thus a full μ is operator-self-decomposable if and only if μ admits a nonzero U-exponent Q with $\lim_{t\to\infty} e^{tQ} = 0$, while μ is self-decomposable if and only if μ admits $Q = -I$ as its U-exponent [cf. Urbanik (1978)].

There are also other instances where some purely probabilistic properties of measures can be expressed in terms of algebraic properties of their decomposability semigroups [cf. the proofs of Corollary 1.3 and Proposition 1.4, and (2.2)].

In combination with decomposability semigroups $\mathbf{D}(\mu)$ it is natural to consider *symmetry semigroups* $\mathbf{A}(\mu)$ consisting of those $A \in \mathbf{D}(\mu)$ for which

$\mu_A = \delta(a)$ (point mass measure concentrated at $a \in X$) in (1.1) [cf. Billingsley (1966), Sharpe (1969) and Urbanik (1972, 1975) for $X = \mathbf{R}^k$ and, more generally, Urbanik (1978) for X a Banach space]. They are also closed subsemigroups of $\mathbf{D}(\mu)$ and always contain I. In fact, we have the following proposition.

PROPOSITION 1.1. *$\mathbf{A}_0(\mu) := \mathbf{A}(\mu) \cap \mathrm{Aut}(X)$ is the largest subgroup of the decomposability semigroup $\mathbf{D}(\mu)$.*

First we are going to describe the set $E_U(\mu)$ of all U-exponents of a given measure μ. After that we are able to characterize self-decomposable measures among those which are operator-self-decomposable (cf. Proposition 1.6). Our main tool is the notion of *tangent space*. Recall that the tangent space $\mathscr{T}(\mathbf{H})$ at the identity of a given subset $\mathbf{H}$ of $\mathrm{End}(X)$ consists of those $A \in \mathrm{End}(X)$ such that

$$\lim_{n\to\infty} d_n^{-1}(G_n - I) = A \quad \text{for some } G_n \in \mathbf{H} \text{ and } 0 < d_n \downarrow 0 \tag{1.5}$$

[cf. Hille and Phillips (1957), Definition 24.14.1. and Lemma 24.14.2].

PROPOSITION 1.2. *For a probability measure μ on a Banach space X, its set $E_U(\mu)$ of all U-exponents has the following properties:*

(a) $0 \in E_U(\mu)$, $\alpha E_U(\mu) = E_U(\mu)$ *for* $\alpha > 0$.
(b) $\overline{E_U(\mu)} = E_U(\mu)$ *(closure in the operator norm topology).*
(c) $E_U(\mu) + E_U(\mu) = E_U(\mu)$.
(d) $AE_U(\mu)A^{-1} = E_U(\mu)$ *whenever* $A \in \mathbf{A}_0(\mu)$.
(e) $E_U(\mu) \cap (-E_U(\mu)) = \mathscr{T}(\mathbf{A}(\mu)) \cap (-\mathscr{T}(\mathbf{A}(\mu)))$ *and it is the largest linear subspace contained in* $E_U(\mu)$, *while* $E_U(\mu) - E_U(\mu)$ *is the smallest linear space containing* $E_U(\mu)$.

We specify the preceding results for the particular case $X = \mathbf{R}^k$ and the full Borel measures on $\mathbf{R}^k$. In this case, $\mathbf{A}(\mu) = \mathbf{A}_0(\mu)$ is a compact subgroup of $\mathrm{Aut}(\mathbf{R}^k)$ [cf. Billingsley (1966), Sharpe (1969) or Urbanik (1972)].

COROLLARY 1.3. *If μ is a full measure on $\mathbf{R}^k$, then the following hold:*

(a) $E_U(\mu) \cap (-E_U(\mu)) = \mathscr{T}(\mathbf{A}(\mu))$ *and is a Lie algebra.*
(b) $AE_U(\mu)A^{-1} = E_U(\mu)$, *for all* $A \in \mathbf{A}(\mu)$.
(c) $e^{\mathrm{ad}_A} E_U(\mu) = E_U(\mu)$, *for all* $A \in \mathscr{T}(\mathbf{A}(\mu))$, *where* $\mathrm{ad}_A(B) := AB - BA$ *for* $B \in \mathrm{End}(\mathbf{R}^k)$.

REMARK. Subsets W of a completely normable vector space L with properties (a), (b) and (c) in Proposition 1.1 are called *wedges*. If W also possesses property (c) of Corollary 1.3 for $A \in W \cap (-W)$, then it is called a *Lie wedge* [cf. Hilgert and Hofmann (1986a), Definition 0.5]. Hilgert and Hofmann proved that Lie wedges are always tangent spaces of some local semigroups [see

Hilgert and Hofmann (1986a), Corollary 5.7]. Here we investigate tangent spaces (sets) of some closed subsemigroups of End(X) associated with probability measures by more elementary methods.

PROPOSITION 1.4. *Let μ be a full measure on $\mathbf{R}^k$. Then the following hold*:

(a) $E_U(\mu) = \mathscr{T}(\mathbf{D}(\mu) \cap \{A\colon 0 < \det A \le 1\})$.
(b) *For a given U-exponent Q*,

$$Q_c := \int_{\mathbf{A}(\mu)} gQg^{-1}H(dg) \in E_U(\mu)$$

and commutes with elements from $\mathbf{A}(\mu)$. [*Here H is the Haar probability measure on* $\mathbf{A}(\mu)$.]

Let $E_{cU}(\mu)$ denote the set of all U-exponents commuting with the symmetry semigroup $\mathbf{A}(\mu)$ and let

$$c\mathbf{D}(\mu) := \{A \in \mathbf{D}(\mu)\colon AB = BA \text{ for each } B \in \mathbf{A}(\mu)\}.$$

Of course, $c\mathbf{D}(\mu)$ is a closed subsemigroup of $\mathbf{D}(\mu)$ and $I \in c\mathbf{D}(\mu)$.

COROLLARY 1.5. *For a full measure μ on $\mathbf{R}^k$ we have*

$$E_{cU}(\mu) = \mathscr{T}(c\mathbf{D}(\mu) \cap \{A\colon 0 < \det A \le 1\}).$$

Now let us return to the question when $-I \in E_U(\mu)$, that is, when μ is a limit distribution in (1.3) with $A_n = a_n I$, $a_n \in \mathbf{R}^+$. In such a case μ is said to be a *self-decomposable measure* [cf. Loève (1963), Section 23, and Jurek and Vervaat (1983)].

PROPOSITION 1.6. *Suppose* {0} *and $\mathbf{R}^k$ are the only invariant subspaces of $\mathbf{R}^k$ with respect to the symmetry semigroup $\mathbf{A}(\mu)$. If μ is full and $\mathscr{T}(\mathbf{D}(\mu)) \setminus \mathscr{T}(\mathbf{A}(\mu)) \neq \emptyset$ then μ is self-decomposable, that is, $-I \in E_U(\mu)$.*

Moreover, for a nondegenerate self-decomposable μ we have $-I \in \mathscr{T}(\mathbf{D}(\mu)) \setminus \mathscr{T}(\mathbf{A}(\mu))$.

A measure μ on $\mathbf{R}^k$ is said to be *elliptically symmetric* if its symmetry semigroup $\mathbf{A}(\mu)$ is conjugate to the full orthogonal group $\mathscr{O} = \mathscr{O}(k, \mathbf{R})$, so that $\mathbf{A}(\mu) = W^{-1}\mathscr{O}W$ for some positive definite and symmetric W.

COROLLARY 1.7. *A measure μ on $\mathbf{R}^k$ is elliptically symmetric with $\mathscr{T}(\mathbf{D}(\mu)) \setminus \mathscr{T}(\mathbf{A}(\mu)) \neq \emptyset$ if and only if there exists a probability measure ρ on $(0, \infty)$ such that, for $y \in \mathbf{R}^k$*,

$$\log \hat{\nu}(y) = i(y, a) - c_1\|Wy\|^2 + c_2\Gamma\left(\frac{k}{2}\right)\int_0^\infty \sum_{j=1}^\infty (-1)^j\left(j!2j\Gamma\left(j + \frac{k}{2}\right)\right)^{-1}$$

$$\times\left(\|Wy\|\frac{s}{2}\right)^{2j} / \log(1 + s^2)\rho(ds),$$

where $c_i \geq 0$, $c_1 + c_2 > 0$ and W is a positive definite, invertible and symmetric matrix.

Now we will illustrate the notion of U-exponents by the example of Gaussian measure on $\mathbf{R}^k$.

EXAMPLE. Let γ_S be a zero-mean Gaussian measure on $\mathbf{R}^k$ with covariance operator S. Then

$$E_U(\gamma_S) = \{Q \in \operatorname{End}(\mathbf{R}^k): QS + SQ^* \leq 0\}.$$

Let us return again to sequences of the form (1.3). Their limit distributions are called *operator-stable measures* if the uniform infinitesimality of the triangular array is replaced by the assumption that the ξ_n's are identically distributed. Sharpe (1969), for $X = \mathbf{R}^k$, and Krakowiak (1979), for the Banach space case, proved that a full measure ν is operator-stable if and only if there exists an operator B and, for each $t > 0$, a $b_t \in X$ such that

$$\nu^{*t} = t^B \nu * \delta(b_t). \tag{1.6}$$

The convolution power ν^{*t} is well-defined because operator-stable measures are infinitely divisible. The operators B in (1.6) are called *S-exponents* of ν. Let $E_S(\nu)$ denote the set of all S-exponents of ν and $E_{cS}(\nu)$ consists of all S-exponents commuting with the symmetry semigroup $\mathbf{A}(\nu)$. Commuting S-exponents play an essential role in the construction of the invariant norm in Hudson, Jurek and Veeh (1986). Finally, from (1.6) we obtain

$$\nu = \nu^{*t} * \nu^{*(1-t)} = e^{(-B)s}\nu * \nu^{*(1-e^{-s})} * \delta(b_{e^{-s}}),$$

for $0 < t \leq 1$ and $s = -\log t$. Hence,

$$-E_S(\nu) \subset E_U(\nu), \tag{1.7}$$

for infinitely divisible ν. The following characterizes S-exponents in a form slightly different from that in Holmes, Hudson and Mason (1982) and Hudson, Jurek and Veeh (1986).

COROLLARY 1.8. *Let ν be a full measure on $\mathbf{R}^k$ and let B and B_c be S-exponents of ν, where B_c is a commuting exponent. Then the following hold*:

(a) $E_S(\nu) = B + \mathscr{T}(\mathbf{A}(\nu) \cap \{A: \det A = 1\})$.

(b) $E_{cS}(\nu) = B_c + \mathscr{T}(C\mathbf{A}(\nu) \cap \{A: \det A = 1\})$, *where $C\mathbf{A}(\nu)$ is the center of the group $\mathbf{A}(\nu)$.*

The previous results allow us to restrict ourselves to subgroups of the special linear group $\mathrm{SL}(k, \mathbf{R})$, which has been investigated extensively [cf. Lang (1975)].

2. Proofs. In order to make this paper more accessible, we quote the following fundamental fact [cf. Hille and Phillips (1957), Lemma 24.14.4].

LEMMA 2.1. *If* $\mathbf{H}$ *is a subsemigroup of* $\mathrm{End}(X)$ *then* exp *maps* $\mathscr{T}(\mathbf{H})$ *into* $\overline{\mathbf{H}}$, *where* $\exp A := \lim_{n\to\infty}(I + n^{-1}A)^n$.

PROOF. Let $h = \lim_{n\to\infty} d_n^{-1}(h_n - I)$ for some $h_n \in \mathbf{H}$ and $0 < d_n \downarrow 0$. Taking $n_k := \lfloor d_k^{-1} \rfloor$, i.e., n_k is the integer part of d_k^{-1}) and $g_k := n_k(h_k - I) - h$, we get $n_k d_k \to 1$ and $g_k \to 0$. Since $h_k = n_k^{-1}(g_k + h) + I$, we obtain $h_k^{n_k} = (I + n_k^{-1}(g_k + h)^{n_k} \in \mathbf{H}$ (semigroup property) and hence $h_k^{n_k} \to \exp h \in \overline{\mathbf{H}}$, which completes the proof. □

REMARK. One can show that $\exp A = \sum_{n=0}^{\infty} A^n/n!$ [convergence in $\mathrm{End}(X)$].

Now, using the notion of tangent space and Lemma 2.1, we obtain the identity

$$E_U(\mu) = \mathscr{T}(\mathbf{D}(\mu)). \tag{2.1}$$

For further reference, we quote the following equivalence between one statement in terms of measures and another in terms of groups:

$$\mu \text{ is full in } \mathbf{R}^k \text{ iff } \mathbf{A}(\mu) \text{ is a compact subgroup of } \mathrm{Aut}(\mathbf{R}^k), \tag{2.2}$$

[cf. Billingsley (1966), Sharpe (1969) and Urbanik (1972, 1975 and 1978)]. Also compare Jurek (1981) for counterexamples in infinite-dimensional linear spaces.

PROOF OF PROPOSITION 1.1. It is obvious that $\mathbf{A}_0(\mu)$ forms a subgroup in $\mathbf{D}(\mu)$. On the other hand, if A and A^{-1} are in $\mathbf{D}(\mu)$, then $A \in \mathbf{A}_0(\mu)$ because of Proposition 1.3 in Urbanik (1972). [This argument is extended in the proof of Proposition 1.2(e).] Thus $\mathbf{A}_0(\mu)$ is the largest subgroup of $\mathbf{D}(\mu)$. □

PROOF OF PROPOSITION 1.2. Part (a) is obvious.

(b) Let $Q_n \in E_U(\mu)$. Then $\mu = e^{tQ_n}\mu * \nu_{t,Q_n}$, for all $t \geq 0$ and $n \in \mathbf{N}$ and for some $\nu_{t,Q_n} \in \mathscr{P}(X)$. If $Q_n \to Q$ in $\mathrm{End}(X)$, then $e^{tQ_n}\mu \Rightarrow e^{tQ}\mu$ as $n \to \infty$, by Theorem 5.5 in Billingsley (1968). [Here "$\Rightarrow$" denotes weak convergence in $\mathscr{P}(X)$.] Theorem 2.1 of Chapter III in Parthasarathy (1967) gives that $\nu_{t,Q_n}, n \in \mathbf{N}$ is conditionally compact in $\mathscr{P}(X)$. Hence $\mu = e^{tQ}\mu * \nu_{t,Q}$ for any limit point $\nu_{t,Q}$ of $\{\mu_{t,Q_n}: n \in \mathbf{N}\}$. (In fact, $\nu_{t,Q_n} \Rightarrow \nu_{t,Q}$ whenever the Fourier transform $\hat{\mu}$ does not vanish.) Thus part (b) is proved.

(c) Let $Q_1, Q_2 \in E_U(\mu)$. Then Lemma 2.1 together with (a) and (2.1) implies that $\exp(tQ_1)$ and $\exp(tQ_2)$ are in $\mathbf{D}(\mu)$, for all $t \geq 0$. From

$$t^{-1}[\exp(tQ_1)\exp(tQ_2) - I] \to Q_1 + Q_2, \quad \text{as } t \to 0,$$

it follows that $Q_1 + Q_2 \in E_U(\mu)$, so $E_U(\mu) + E_U(\mu) \subset E_U(\mu)$. The opposite inclusion follows from the fact that $0 \in E_U(\mu)$.

(d) Let $\mu = A\mu * \delta(a)$ and $A \in \mathrm{Aut}(X)$, $a \in X$. If $Q \in E_U(\mu)$, then $\mu = e^{tQ}\mu * \nu_{t,Q} = e^{tQ}A\mu * \nu_{t,Q} * \delta(e^{tQ}a)$. This combines with $\mu = A^{-1}\mu * \delta(-A^{-1}a)$ into $\mu = A^{-1}e^{tQ}A\mu * A^{-1}\nu_{t,Q} * \delta(A^{-1}e^{tQ}a - A^{-1}a)$. Hence $A^{-1}QA \in E_U(\mu)$, so $A^{-1}E_U(\mu)A \subset E_U(\mu)$, which completes the proof of part (d).

(e) By (a) and (c) we have that $E_U(\mu) \cap (-E_U(\mu))$ is a linear subspace of $E_U(\mu)$. It is clear that it is the largest subspace in $E_U(\mu)$. Similarly, $E_U(\mu) - E_U(\mu)$ is the smallest linear space containing $E_U(\mu)$. The inclusion $\mathscr{T}(\mathbf{A}(\mu)) \cap (-\mathscr{T}(\mathbf{A}(\mu))) \subset \mathrm{E}_U(\mu) \cap (-\mathrm{E}_U(\mu))$ follows from (2.1). Conversely, if A and $-A$ are from $E_U(\mu)$, then (2.1) and Lemma 2.1 imply that both e^{tA} and e^{-tA} are in $\mathbf{D}(\mu)$, for $t \geq 0$. Consequently, by (1.1) there exist $\nu_{t,A}$ and $\nu_{t,-A}$ in $\mathscr{P}(X)$ such that

$$\mu = e^{tA}\mu * \nu_{t,A} = e^{tA}\left(e^{-tA}\mu * \nu_{t,-A}\right) * \nu_{t,A}$$
$$= \mu * e^{tA}\nu_{t,-A} * \nu_{t,A}.$$

Hence $|\hat{\mu}| \leq |\hat{\mu}|\,|\hat{\nu}_{t,A}|$, where again $\hat{\mu}$ denotes the Fourier transform of μ. Thus $|\hat{\nu}_{t,A}| = 1$, in some open neighborhood of zero in the topological dual X'. Consequently, $|\hat{\nu}_{t,A}| \equiv 1$ and $\nu_{t,A} = \delta(a_{t,A})$, for some $a_{t,A} \in X$. Similarly, $\nu_{t,-A} = \delta(a_{t,-A})$. So e^{tA} and e^{-tA} are in $\mathbf{A}(\mu)$, for $t \geq 0$, which is equivalent to $\pm A$ being in $\mathscr{T}(\mathbf{A}(\mu))$. Thus identity (e) is proved. [The preceding is an extension of the arguments of Urbanik (1972), Proposition 1.3.] □

PROOF OF COROLLARY 1.3. In view of (2.2) we have that $\mathbf{A}(\mu)$ is a group, so $\mathbf{A}(\mu) = \mathbf{A}_0(\mu)$; therefore (a) and (b) follow from (d) and (e) of Proposition 1.2. To see (c), note that for the adjoint representation ad_A we have $\exp(\mathrm{ad}_A)(B) = e^{-A}Be^{A}$ [cf. Lang (1975), Lemma on page 144]. By Lemma 2.1, we have $\exp \mathscr{T}(\mathbf{A}(\mu)) \subset \mathbf{A}(\mu)$, so (c) is a consequence of (b). □

PROOF OF PROPOSITION 1.4. (a) By Proposition 1.1 in Urbanik (1972), $\mathbf{D}(\mu)$ is a compact subsemigroup for full $\mu \in \mathscr{P}(\mathbf{R}^k)$. So for any $Q \in E_U(\mu)$,

$$\sup_{s \geq 0} \det e^{sQ} = \sup_{s \geq 0} e^{s\,\mathrm{trace}\,Q} < \infty,$$

that is, trace $Q \leq 0$ and $0 < \det e^{sQ} \leq 1$. Therefore, $Q \in \mathscr{T}(\mathbf{D}(\mu) \cap \{A\colon 0 < \det A \leq 1\})$, which gives (a) because of (2.1).

(b) Fullness of μ gives that $\mathbf{A}(\mu)$ is a compact group [cf. (2.2)] and therefore $\mathbf{A}(\mu)$ carries a Haar probability measure H. Since Q_c is a limit of the sequence $\sum_{k=1}^{m}\alpha_k g_k Q g_k^{-1}$, $m \geq 1$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$ and $g_k \in \mathbf{A}(\mu)$, we conclude by (a), (b) and (c) of Proposition 1.2 that $Q_c \in E_U(\mu)$. This completes the proof of Proposition 1.4. □

PROOF OF COROLLARY 1.5. In view of Lemma 2.1 and the definition of tangent spaces, we have $E_{cU}(\mu) = \mathscr{T}(c\mathbf{D}(\mu))$. The rest of the proof is similar to that of Proposition 1.4(a). □

PROOF OF PROPOSITION 1.6. By (2.2), $\mathbf{A}(\mu)$ is a compact subgroup of $\mathrm{Aut}(\mathbf{R}^k)$. Furthermore, $\mathbf{A}(\mu) = WO_0W^{-1}$ for some positive definite self-adjoint W and closed subgroup O_0 of the full orthogonal group, by the classical Fenchel result [cf. Billingsley (1966)]. Since $\mathbf{A}(W^{-1}\mu) = O_0$, $\mathbf{D}(W^{-1}\mu) = W^{-1}\mathbf{D}(\mu)W$ and $W^{-1}\mu$ is full, we may and do assume without loss of generality that $\mathbf{A}(\mu) = O_0$. Let $Q \in \mathscr{T}(\mathbf{D}(\mu)) \setminus \mathscr{T}(\mathbf{A}(\mu))$ and Q_c be the commuting exponent from Proposition 1.4(b). Since $A^* = A^{-1}$ for $A \in \mathbf{A}(\mu)$ (O_0 consists of orthogonal matrices), also Q_c^* commutes with $\mathbf{A}(\mu)$ and so does $Q_cQ_c^*$. By Schur's lemma [cf. Lang (1975), page 362] we conclude $Q_cQ_c^* = \lambda^2 I$ and $Q_c^*Q_c = \rho^2 I$ for some real λ and ρ. Hence, $\lambda^2 Q_c = (Q_cQ_c^*)Q_c = Q_c(Q_c^*Q_c) = \rho^2 Q_c$ and $(\lambda^2 - \rho^2)Q_c = 0$. Consequently, $Q_cQ_c^* = Q_c^*Q_c$ and both Q_c, Q_c^* commute with $\mathbf{A}(\mu)$. Once again by Schur's Lemma we get $Q_c = \lambda I$. Now $\mathbf{D}(\mu)$ is compact and $e^{t\lambda}I \in \mathbf{D}(\mu)$, for $t \geq 0$, so $\lambda \leq 0$. Furthermore, we infer from

$$k\lambda = trQ_c = \int_{\mathbf{A}(\mu)} tr(gQg^{-1})H(dg) = trQ$$

that $\lambda < 0$, because otherwise $\det e^{tQ} = 1$, $e^{tQ} \in \mathbf{D}(\mu)$, so $e^{tQ} \in \mathbf{A}(\mu)$ by Proposition 1.4 in Urbanik (1972), and finally $Q \notin \mathscr{T}(\mathbf{D}(\mu)) \setminus \mathscr{T}(\mathbf{A}(\mu))$, contradicting our assumption. So, by (a) of Proposition 1.2 we obtain $-I \in E_U(\mu)$.

Conversely, $-I \in E_U(\mu)$ for self-decomposable μ. If $-I \in \mathscr{T}(\mathbf{A}(\mu))$, then $e^{-t}I \in \mathbf{A}(\mu)$, so $|\hat{\mu}(y)| = |\hat{\mu}(e^{-t}y)| = |\hat{\mu}(e^{-nt}y)| \to 1$ as $n \to \infty$, which gives that μ is concentrated at one single point. Thus the proof is complete. □

PROOF OF COROLLARY 1.7. By Proposition 1.6, μ is self-decomposable, more specifically a translation of a symmetric self-decomposable measure, since $-I \in \mathbf{A}(\mu)$. From this and Theorem 7.2 in Jurek and Vervaat (1983), we conclude

$$\log \hat{\mu}(y) = i(y,a) - (y, Dy) + c_2\int_{\mathbf{R}^k \setminus \{0\}}\int_0^1 [\cos t(y,s) - 1]t^{-1}\,dt\,m(dx)/\log(1 + \|x\|^2),$$

for a probability measure m on $\mathbf{R}^k \setminus \{0\}$ and a real $c_2 \geq 0$.

Assume $\mathbf{A}(\mu) = \mathscr{O}$. Uniqueness of the Gaussian and Poissonian parts in the Lévy–Khintchine formula for infinitely divisible measures implies

$$ADA^* = D \quad \text{and} \quad Am(\cdot) = m(\cdot) \quad \text{for all } A \in \mathscr{O}.$$

From Schur's lemma we obtain $D = c_1 I$ for some $c_1 \geq 0$. To solve the preceding measure equation, let us define measures H_k and ρ on the unit sphere S^{k-1} and the positive half-line as images of m under the mappings $x \to x/\|x\|$ and $x \to \|x\|$, respectively. Then H_k is the Haar probability measure on the homogeneous space $S^{k-1} = O(k,\mathbf{R})/O(k-1,\mathbf{R})$ and its Fourier transform is given by

$$\hat{H}_k(y) = \Gamma(k/2)\sum_{j=0}^{\infty}(-1)^j(j!\Gamma(j+k/2))^{-1}(\|y\|/2)^{2j}.$$

Note that $\hat{H}_1(y) = \cos|y|$ and that $\hat{H}_k$, for $k \geq 2$, can be expressed in terms of Bessel functions. Finally, $\mathbf{R}^k \setminus \{0\} \cong S^{k-1} \times \mathbf{R}^+$ and $(u, t) \to u \cdot t$ is a product of two independent random variables under m, which gives

$$m(F) = \int_0^\infty H_k(t^{-1}F)\rho(dt), \quad \text{for Borel subsets } F.$$

Substituting this into the formula for $\hat{\mu}$ and performing some integration, we get Corollary 1.7 with $W = I$.

The general case $\mathbf{A}(\mu) = W^{-1}\mathscr{O}W$ is reduced to the previous one by the following observations:

$$\mathbf{A}(W\mu) = \mathscr{O}, \qquad \mathbf{D}(W\mu) = W\mathbf{D}(\mu)W^{-1},$$
$$\mathscr{T}(\mathbf{D}(W\mu)) = W\mathscr{T}(\mathbf{D}(\mu))W^{-1}.$$

Thus the proof is complete. □

Proof of the Example. Note that $A\gamma_S$ is also a Gaussian measure with covariance operator ASA^*. Furthermore, if $\gamma_S = \nu_1 * \nu_2$, then both ν_1 and ν_2 are Gaussian measures [cf. Cramér's theorem in Loève (1963)]. Consequently,

$$\mathbf{D}(\gamma_S) = \{A \in \mathrm{End}(\mathbf{R}^k) \colon ASA^* \leq S\}.$$

So, $Q \in E_U(\gamma_S)$ iff $S - e^{tQ}Se^{TQ*} \geq 0$, for all $t \geq 0$. Differentiation with respect to t gives $-(e^{tQ}QSe^{tQ*} + e^{tQ}SQ^*e^{tQ*}) \geq 0$, for all $t \geq 0$. In particular, $QS + SQ^* \leq 0$. Conversely, if $QS + SQ^* \leq 0$, then $-(e^{tQ}(QS + SQ^*)e^{tQ*}x, x) \geq 0$ for all $t \geq 0$ and $x \in \mathbf{R}^k$. Consequently,

$$\frac{d}{dt}[(Sx, x) - (e^{tQ}Se^{tQ*}x, x)] \geq 0,$$

for all $x \in \mathbf{R}^k$ and $t \geq 0$, so $S - e^{tQ}Se^{tQ*} \geq 0$, for $t \geq 0$. Hence, $e^{tQ} \in \mathbf{D}(\gamma_S)$, so $Q \in E_U(\gamma_S)$. □

Proof of Corollary 1.8. Since $\mathbf{A}(\nu)$ is a compact subgroup of $\mathrm{Aut}(\mathbf{R}^k)$, $\mathscr{T}(\mathbf{A}(\nu)) = \mathscr{T}(\mathbf{A}(\nu) \cap \{A\colon \det A = 1\})$ and (a) and (b) follow from Holmes, Hudson and Mason (1982) and Hudson, Jurek and Veeh (1986), respectively. □

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REFERENCES

BILLINGSELY, P. (1966). Convergence of types in k-spaces. *Z. Wahrsch. Verw. Gebiete* **5** 175–179.

BILLINGSELY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

HILGERT, J. and HOFMANN, K. H. (1986a). On Sophus Lie's fundamental theorem. *J. Funct. Anal.* **67** 293–319.

HILGERT, J. and HOFMANN, K. H. (1986b). On the automorphisms of cones and wedges. *Geom. Dedicata* **21** 205–217.

HILGERT, J., HOFMANN, K. H. and LAWSON, J. D. (1989). *Lie Groups, Convex Cones, and Semigroups*. Oxford Univ. Press, New York.

HILLE, E. and PHILLIPS, R. S. (1957). *Functional Analysis and Semi-Groups*. Amer. Math. Soc., Providence, R.I.

HOLMES, J. P., HUDSON, W. N. and MASON, J. D. (1982). Operator-stable laws: Multiple exponents and elliptical symmetry. *Ann. Probab.* **10** 602–612.

HUDSON, W. N., JUREK, Z. J. and VEEH, J. A. (1986). The symmetry group and exponents of operator-stable probability measures. *Ann. Probab.* **14** 1014–1023.

JUREK, Z. J. (1981). Convergence of types, self-decomposability and stability on measures on linear spaces. *Probability on Banach Spaces III. Lecture Notes in Math.* **860** 257–267. Springer, New York.

JUREK, Z. J. and VERVAAT, W. (1983). An integral representation for self-decomposable Banach space valued random variables. *Z. Wahrsch. Verw. Gebiete* **62** 247–272.

KRAKOWIAK, W. (1979). Operator-stable probability measures on Banach spaces. *Colloq. Math.* **41** 313–326.

LANG, S. (1975). *$SL_2(R)$*. Addison-Wesley, Reading, Mass.

LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

PARTHASARATHY, K. R. (1967). Probability measures on metric spaces. Academic, New York.

SHARPE, M. (1969). Operator-stable probability measures on vector groups. *Trans. Amer. Math. Soc.* **136** 51–65.

URBANIK, K. (1972). Lévy probability measures on Euclidean spaces. *Studia Math.* **44** 119–148.

URBANIK, K. (1975). Decomposability properties of probability measures. *Sankhyā Ser. A* **37** 530–537.

URBANIK, K. (1978). Lévy probability measures on Banach spaces. *Studia Math.* **63** 283–308.

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