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# INEQUALITIES FOR INCREMENTS OF STOCHASTIC PROCESSES AND MODULI OF CONTINUITY

By Endre Csáki[1] and Miklós Csörgő[2]

*Mathematical Institute, Budapest, and Carleton University*

Let $\{\Gamma(t),\ t \in \mathbb{R}\}$ be a Banach space $\mathscr{B}$-valued stochastic process. Let $P$ be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is $P$-almost surely continuous with respect to the norm $\| \ \|$ of $\mathscr{B}$ and that there exists a positive nondecreasing function $\sigma(a)$, $a > 0$, such that $P\{\|\Gamma(t+a) - \Gamma(t)\| \geq x\sigma(a)\} \leq K\exp(-\gamma x^{\beta})$ with some $K, \gamma, \beta > 0$. Then, assuming also that $\sigma(\cdot)$ is a regularly varying function at zero, or at infinity, with a positive exponent, we prove large deviation results for increments like $\sup_{0 \leq t \leq T-a} \sup_{0 \leq s \leq a} \|\Gamma(t+s) - \Gamma(t)\|$, which we then use to establish moduli of continuity and large increment estimates for $\Gamma(\cdot)$. One of the many applications is to prove moduli of continuity estimates for $l^2$-valued Ornstein–Uhlenbeck processes.

**1. Introduction.** The theory of sample path properties of general, nonstationary Gaussian processes based on concepts such as entropy and majorizing measures is now well understood. For an accessible, excellent introduction to these concepts and to the general theory of continuity, boundedness and suprema distributions for real-valued Gaussian processes, we refer to Adler (1990).

Some of this theory can be easily extended to Gaussian processes taking values in more general state spaces. For example, if $X_t = (X_t^1, \dots, X_t^d)$ is an $\mathbb{R}^d$-valued Gaussian process on a metric space $(\mathscr{T}, \rho)$, then $X_t$ is continuous as a function from $(\mathscr{T}, \rho)$ to $(\mathbb{R}^d, \| \ \|)$ if and only if each $X_t^i$ is continuous as a real-valued function on $(\mathscr{T}, \rho)$. However, the problem of the behaviour in distribution of $\sup_{t \in \mathscr{T}} \|X_t\|$, where $\| \ \|$ is the Euclidean norm, is not so simple in general [cf. e.g., the treatment of $\chi^2$ processes in Leadbetter, Lindgren and Rootzén (1983) and Adler (1981)]. The same holds for the distributional behaviour of $\sup_{t, s \in \mathscr{T}, \rho(t,s) \leq a} \|X_t - X_s\|$. If, on the other hand, $X_t = \{X_t^i\}_{i=1}^{\infty} \in l^2$ and each $X_t^i$ is a continuous real-valued Gaussian process on $(\mathscr{T}, \rho)$, then $X_t \in l^2$ is continuous if and only if the $l^2$-norm squared process $\|X_t\| = \chi^2(t) = \sum_{i=1}^{\infty} (X_t^i)^2$ is continuous on $(\mathscr{T}, \rho)$ (cf. e.g., our Lemma 5.1). It may happen of course that neither of these two statements will be easy to establish in some particular cases. For example, continuity of $l^2$-valued Ornstein–Uhlenbeck processes, defined on $(\mathscr{T}, \rho) = (\mathbb{R}, | \ |)$, was treated via that of $\chi^2(t)$ on $(\mathbb{R}, | \ |)$ by Iscoe and McDonald (1986) and directly in $l^2$-norm by Iscoe,

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Marcus, McDonald, Talagrand and Zinn (1990) as a consequence of a corollary due to Fernique [(1990), Théoreme 3.3.3] of Talagrand's (1987) theorem on necessary and sufficient conditions for the continuity of Gaussian processes. The final result along these lines is due to Fernique (1989). We quote a special case of his theorem in our Section 4.

Continuing with the example of $X_t = \{X_t^i\}_{i=1}^{\infty} \in l^2$ as above, it is of interest to study the distributional behaviour of $\sup_{t \in \mathscr{T}} \|X_t\|$ and that of $\sup_{t, s \in \mathscr{T},\ \rho(t, s) \le a} \|X_t - X_s\|$. Iscoe and McDonald (1989) established an upper bound on $P\{\sup_{0 \le t \le T} \|X_t\| > x\}$ and the asymptotics for the given bound as $x \to \infty$, for $l^2$-valued Ornstein–Uhlenbeck processes on $(\mathbb{R}, |\ |)$. As a consequence of our inequalities of Section 2 for increments of Banach space valued, not necessarily Gaussian stochastic processes, we will study the tail behavior of $l^2$-valued Ornstein–Uhlenbeck processes on the real line in terms of increments like $\sup_{0 \le t \le T-a} \sup_{0 \le s \le a} \|X_{t+s} - X_t\|$.

The essence of our approach is the realization that the inequalities of Lemmas 1.1.1 and 1.2.1 for increments of a standard Brownian motion in Csörgő and Révész (1981) [cf. also Lemmas 1 and 1* of Csörgő and Révész (1979)] can be extended to increments of general, nonstationary, not necessarily Gaussian, Banach space valued processes, defined on the real line. We state and prove these inequalities in Section 2.

REMARK 1.1. Concerning the results in Section 2, a referee pointed out that by doing a bit more work, our inequalities (2.5) and (2.7) follow also from the more general methods in a forthcoming book of Ledoux and Talagrand (1991). We thank the referee for sending a copy of the relevant Chapter 11 of this book via the Editor. Nevertheless, we have decided to retain our proofs for the sake of completeness and for illustrating that on occasion, increments of general, Banach space valued processes can be treated directly in much the same fashion, and with much the same tools, as if they were real-valued processes.

The inequalities of Section 2 are established for the sake of studying small and large increments of stochastic processes. This we do in Section 3, where we state our immediate upper bounds for moduli of continuity and large increments of the stochastic processes of Section 2.

We demonstrate the use of our approach by proving the main results of this paper in Sections 4 and 5.

In Section 4 we establish moduli of continuity results for Dawson's (1972) $l^2$-valued Ornstein–Uhlenbeck process in the context of the necessary and sufficient conditions of Fernique (1989) for the almost sure continuity of this process in $l^2$ and compare our moduli of continuity to the one given by Schmuland (1988c).

In Section 5 we prove moduli of continuity estimates for the non-Gaussian $l^2$-norm squared process of Section 4 and compare them to those of others.

Further examples for the use of our general statements are given in Section 6.

**2. Inequalities.** The aim of this section is to show that the inequalities of Lemmas 1.1.1 and 1.2.1 in Csörgő and Révész (1981) [cf. also Lemmas 1 and 1* of Csörgő and Révész (1979)] can be extended to general, not necessarily Gaussian, Banach space valued processes. The following results are also extensions of inequalities in Révész (1985), as well as those of Csáki, Csörgő, Lin and Révész (1990).

LEMMA 2.1. *Let $\mathscr{B}$ be a separable Banach space with norm $\|\ \|$ and let $\{\Gamma(t), -\infty < t < \infty\}$ be a stochastic process with values in $\mathscr{B}$. Let $P$ be the probability measure generated by $\Gamma(\cdot)$. Assume that $\Gamma(\cdot)$ is $P$-almost surely continuous with respect to $\|\ \|$ and that for $|t| \le t_0$, $0 < x^* \le x$ and $0 < h \le h_0$, there exists a positive monotone nondecreasing function $\sigma(h)$ such that*

$$P\{\|\Gamma(t+h) - \Gamma(t)\| \ge x\sigma(h)\} \le K\exp(-\gamma x^\beta) \tag{2.1}$$

*with some $K, \gamma, \beta > 0$. Then*

$$\begin{aligned} P\Bigg\{ \sup_{0\le t\le T-a}\ \sup_{0\le s\le a} \|\Gamma(t+s) - \Gamma(t)\| &\ge x\sigma(a + 1/R) \\ &\quad + 2\sum_{j=0}^{\infty} x_j \sigma(1/2^{r+j+1}) \Bigg\} \\ &\le TR(Ra+1)K\exp(-\gamma x^\beta) + 4TRK\sum_{j=0}^{\infty} 2^j \exp(-\gamma x_j^\beta) \end{aligned} \tag{2.2}$$

*for any $0 \le T \le t_0$, $0 \le a \le T$, $x^* < x$, $x^* < x_j$, $j = 0, 1, \dots,$ and positive integer $r$, where $R = 2^r$ and $a + 1/R \le h_0$.*

PROOF. We follow the proof of Lemma 1.1.1 of Csörgő and Révész (1981). For any positive real number $t$ and integer $r$, put $t_r = [2^r t]/2^r$. We have

$$\begin{aligned} &\|\Gamma(t+s) - \Gamma(t)\| \\ &\quad \le \|\Gamma((t+s)_r) - \Gamma(t_r)\| + \|\Gamma(t+s) - \Gamma((t+s)_r)\| + \|\Gamma(t_r) - \Gamma(t)\| \\ &\quad \le \|((t+s)_r) - \Gamma(t_r)\| + \sum_{j=0}^{\infty} \|\Gamma((t+s)_{r+j+1}) - \Gamma((t+s)_{r+j})\| \\ &\qquad + \sum_{j=0}^{\infty} \|\Gamma(t_{r+j+1}) - \Gamma(t_{r+j})\|, \end{aligned}$$

where in the second inequality the a.s. continuity of $\Gamma(\cdot)$ with respect to $\|\ \|$ is used. Since

$$\sup_{0<s\le a} |(t+s)_r - t_r| \le a + R^{-1},$$

$$\sup_{0<s\le a} |(t+s)_{r+j+1} - (t+s)_{r+j}| \le 2^{-(r+j+1)},$$

from (2.1) we get

$$P\Big\{\sup_{0\le t\le T-a}\sup_{0<s\le a}\|\Gamma((t+s)_r)-\Gamma(t_r)\|\ge x\sigma(a+1/R)\Big\}$$
$$\le KTR(Ra+1)\exp(-\gamma x^{\beta}),$$

$$P\Big\{\sup_{0\le t\le T-a}\sup_{0<s\le a}\|\Gamma((t+s)_{r+j+1})-\Gamma((t+s)_{r+j})\|\ge x_j\sigma(1/2^{r+j+1})\Big\}$$
$$\le KT\exp(-\gamma x_j^{\beta})2^{r+j+1},$$

as well as

$$P\Big\{\sup_{0\le t\le T-a}\sup_{0<s\le a}\|\Gamma(t_{r+j+1})-\Gamma(t_{r+j})\|\ge x_j\sigma(1/2^{r+j+1})\Big\}$$
$$\le KT\exp(-\gamma x_j^{\beta})2^{r+j+1}.$$

Hence (2.2) follows from the last three inequalities. □

LEMMA 2.2. *Let* $\{\Gamma(t), -\infty<t<\infty\}$ *and* $\sigma(h)$ *be as in Lemma* 2.1 *and assume that* $\sigma(\cdot)$ *is a regularly varying function at zero with a positive exponent* $\alpha$, *namely*

$$\sigma(s)=s^{\alpha}L(s),\qquad \alpha>0, \tag{2.3}$$

*where* $L(\cdot)$ *is a slowly varying function at zero, that is, it is measurable, positive and*

$$\lim_{s\downarrow 0}L(\lambda s)/L(s)=1\quad \textit{for all } \lambda>0. \tag{2.4}$$

*Then for any* $\varepsilon>0$, *there exist* $C=C(\varepsilon)>0$ *and* $0<h_0(\varepsilon)<1$ *such that*

$$P\Big\{\sup_{0\le t\le 1-h}\sup_{0\le s\le h}\|\Gamma(t+s)-\Gamma(t)\|>x\sigma(h)\Big\}\le (C/h)\exp(-\gamma x^{\beta}/(1+\varepsilon)) \tag{2.5}$$

*for every* $x\ge x^*$ *and* $0<h\le h_0(\varepsilon)$.

PROOF. For a given slowly varying function at zero, $L(\cdot)$, there exists another slowly varying function at zero, $L^*(\cdot)$, such that

$$\lim_{h\downarrow 0}L^*(h)/L(h)=1$$

and $h^{\alpha/2}L^*(h)$ is an increasing function of $h$ [cf. Corollary 1.2.1 of de Haan (1975) or Lemma 4.1 in Csörgő and Horváth (1990)]. Then [cf. Corollary 1.2.1 of de Haan (1975) or Lemma 4.2 in Csörgő and Horváth (1990)] for any $\varepsilon>0$, there exists $\hat{h}_0(\varepsilon)$ such that

$$L(Kh)/L(h)\le(1+\varepsilon)L^*(Kh)/L^*(h)$$

for all $0 < K < 1$ and $0 < h \le \hat{h}_0(\varepsilon)$. Hence we have

$$\begin{aligned}\sigma(Kh) &= K^\alpha h^\alpha L(Kh) = K^\alpha \sigma(h) L(Kh)/L(h)\\ &\le (1+\varepsilon)K^\alpha\sigma(h)L^*(Kh)/L^*(h)\\ &\le (1+\varepsilon)K^{\alpha/2}\sigma(h)(Kh)^{\alpha/2}L^*(Kh)/h^{\alpha/2}L^*(h)\\ &\le (1+\varepsilon)K^{\alpha/2}\sigma(h).\end{aligned}$$

Moreover, for any fixed $K_0 > 0$, we have

$$\sigma(K_0 h) = K_0^\alpha h^\alpha L(K_0 h) \le (1+\varepsilon)K_0^\alpha h^\alpha L(h) = (1+\varepsilon)K_0^\alpha\sigma(h),$$

if $h$ is small enough, that is, $0 < h \le \tilde{h}_0(\varepsilon)$.

Consider now (2.2) with $a = h$, $T = 1$, $x_j = (j/\gamma + x^\beta)^{1/\beta}$, $j = 0, 1, 2, \dots$, and let $R$ of (2.2) be such that $2R > A/h \ge R$, where $A$ is a positive constant, which is to be specified later on. Then, for $0 < h \le \min(\hat{h}_0(\varepsilon), \tilde{h}(\varepsilon)) = h_0(\varepsilon)$, we have

$$\begin{aligned}x\sigma(h + 1/R) &+ 2\sum_{j=0}^{\infty} x_j \sigma(1/2^{r+j+1})\\ &\le x\sigma(h + 2h/A) + 2\sum_{j=0}^{\infty} x_j\sigma(1/2^{r+j+1})\\ &\le (1+\varepsilon)x(1+2/A)^\alpha\sigma(h) + 2(1+\varepsilon)\sum_{j=0}^{\infty} x_j\sigma(h)/(A2^j)^{\alpha/2}.\end{aligned}$$

However,

$$x_j = (j/\gamma + x^\beta)^{1/\beta} \le (1 \vee 2^{1/\beta - 1})\big((j/\gamma)^{1/\beta} + x\big),$$

and hence we have

$$\begin{aligned}x\sigma(h + 1/R) &+ 2\sum_{j=0}^{\infty} x_j\sigma(1/2^{r+j+1})\\ &\le (1+\varepsilon)x\sigma(h)\left((1+2/A)^\alpha + (2 \vee 2^{1/\beta})\sum_{j=0}^{\infty} 1/(A2^j)^{\alpha/2}\right)\\ &\quad + \big((1+\varepsilon)(2\vee 2^{1/\beta})\sigma(h)/(A^{\alpha/2}\gamma^{1/\beta})\big)\sum_{j=0}^{\infty} j^{1/\beta}/2^{j\alpha/2}\\ &= (1+\varepsilon)x\sigma(h)\big((1+2/A)^\alpha + G/A^{\alpha/2}\big) + (1+\varepsilon)B\sigma(h)/A^{\alpha/2}\\ &\le (1+\varepsilon)x\sigma(h)\big((1+2/A)^\alpha + G/A^{\alpha/2} + B/(x^* A^{\alpha/2})\big)\\ &\le (1+\varepsilon)^2 x\sigma(h),\end{aligned}$$

on taking $A$ large enough, where

$$B = \big((2 \vee 2^{1/\beta})/\gamma^{1/\beta}\big)\sum_{j=0}^{\infty} j^{1/\beta}/2^{j\alpha/2}, \qquad G = (2 \vee 2^{1/\beta})\sum_{j=0}^{\infty} 1/2^{j\alpha/2}.$$

On the other hand,

$$R(Rh+1)K\exp(-\gamma x^{\beta}) + 4RK\sum_{j=0}^{\infty} 2^j \exp\left(-\gamma x_j^{\beta}\right)$$

$$\leq (A(A+1)K/h)\exp(-\gamma x^{\beta}) + (4AK/h)\left(\sum_{j=0}^{\infty} 2^j e^{-j}\right)\exp(-\gamma x^{\beta})$$

$$= (C/h)\exp(-\gamma x^{\beta}).$$

Consequently, by these calculations and (2.2), we obtain

$$P\left\{\sup_{0\leq t\leq 1-h}\ \sup_{0\leq s\leq h} \|\Gamma(t+s)-\Gamma(t)\| \geq (1+\varepsilon)^2 x\sigma(h)\right\}$$

$$\leq (C/h)\exp(-\gamma x^{\beta}),$$

and hence also (2.5). □

Obviously, the inequality (2.5) enables one to study the increments of $\Gamma(\cdot)$ for small $h$ over the interval $(0, 1)$. Also, it can be easily extended to any finite interval $(T_1, T_2)$, $-\infty < T_1 < T_2 < \infty$, as follows.

LEMMA 2.3. *Under the conditions of Lemma* 2.2, *we have*

$$P\left\{\sup_{T_1\leq t\leq T_2-h}\ \sup_{0\leq s\leq h} \|\Gamma(t+s)-\Gamma(t)\| > x\sigma(h)\right\} \tag{2.6}$$

$$\leq (C(T_2-T_1)/h)\exp(-\gamma x^{\beta}/(1+\varepsilon)).$$

REMARK 2.1. It is clear from the proof of Lemma 2.2 that the respective conclusions of Lemmas 2.2 and 2.3 remain true if, instead of assuming that $\sigma(s)$ is a regularly varying function at zero with a positive exponent, we require it to be of the form $s^{\alpha}L(s)$ in a neighbourhood of zero with some $\alpha > 0$, where $L(\cdot)$ is monotone increasing near zero.

The next version of (2.5) is for the sake of studying large increments of $\Gamma(\cdot)$.

LEMMA 2.4. *Let* $\{\Gamma(t), -\infty < t < \infty\}$ *and* $\sigma(s)$ *be as in Lemma* 2.1 *with* $t_0 = h_0 = \infty$. *Assume that* $\sigma(s) = s^{\alpha}\sigma_1(s)$, $s \geq 0$, *for some* $\alpha > 0$, *where* $\sigma_1(s)$ *is a nondecreasing function of* $s$. *Then for any* $\varepsilon > 0$, *there exist* $C = C(\varepsilon) > 0$ *and* $a_0 = a_0(\varepsilon)$ *such that*

$$P\left\{\sup_{0\leq t\leq T-a}\ \sup_{0\leq s\leq a} \|\Gamma(t+s)-\Gamma(t)\| > x\sigma(a)\right\} \tag{2.7}$$

$$\leq (CT/a)\exp(-\gamma x^{\beta}/(1+\varepsilon))$$

*for every* $x \geq x^*$ *and* $T \geq a > a_0$.

The proof of this result is similar to that of Lemma 2.2 and therefore is omitted.

**3. Moduli of continuity and large increments.** Based on our inequalities in Section 2, here we give general results concerning upper moduli of continuity and large increments for stochastic processes.

THEOREM 3.1. *Let* $\{\Gamma(t), -\infty < t < \infty\}$ *and* $\sigma(h)$ *be as in Lemmas* 2.1 *and* 2.2. *Then for any* $-\infty < T_1 < T_2 < \infty$, *we have*

$$\limsup_{h\downarrow 0} \sup_{T_1\le t\le T_2-h} \sup_{0\le s\le h} \frac{\|\Gamma(t+s)-\Gamma(t)\|}{\sigma(h)((1/\gamma)\log(1/h))^{1/\beta}} \le 1 \quad a.s. \tag{3.1}$$

PROOF. Without loss of generality, we take $T_1 = 0$ and $T_2 = 1$. Using Lemma 2.2, the proof of (3.1) is similar to that of the first part of the P. Lévy modulus of continuity for the Wiener process in Csörgő and Révész (1981). We let

$$A_h = \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \|\Gamma(t+s)-\Gamma(t)\|$$

and apply the inequality of (2.5) with $x = (1+\varepsilon)^{2/\beta}((1/\gamma)\log(1/h))^{1/\beta}$, $\varepsilon > 0$. Then

$$\begin{aligned} P&\Big\{A_h/\big(\sigma(h)((1/\gamma)\log(1/h))^{1/\beta}\big) \ge (1+\varepsilon)^{2/\beta}\Big\} \\ &\le (C/h)\exp(-(1+\varepsilon)\log(1/h)) = Ch^{\varepsilon}. \end{aligned}$$

Choose $\Lambda > 1/\varepsilon$ and let $h = h_n = n^{-\Lambda}$. Then

$$\sum_{n=1}^{\infty} P\Big\{A_{h_n}/\big(\sigma(h_n)((1/\gamma)\log(1/h_n))^{1/\beta}\big) \ge (1+\varepsilon)^{2/\beta}\Big\} \le C\sum_{n=1}^{\infty} n^{-\Lambda\varepsilon} < \infty,$$

and it follows from the Borel–Cantelli lemma that

$$\limsup_{n\to\infty} \frac{A_{h_n}}{\sigma(h_n)((1/\gamma)\log(1/h_n))^{1/\beta}} \le (1+\varepsilon)^{2/\beta} \quad \text{a.s.},$$

for all $\varepsilon > 0$. Hence, and because of (2.3), on considering now the case of $h_{n+1} < h < h_n$, we obtain

$$\begin{aligned} &\limsup_{h\downarrow 0} \frac{A_h}{\sigma(h)((1/\gamma)\log(1/h))^{1/\beta}} \\ &\quad\le \limsup_{n\to\infty} \frac{A_{h_n}}{\sigma(h_n)((1/\gamma)\log(1/h_n))^{1/\beta}} \frac{\sigma(h_n)((1/\gamma)\log(1/h_n))^{1/\beta}}{\sigma(h_{n+1})((1/\gamma)\log(1/h_{n+1}))^{1/\beta}} \\ &\quad\le (1+\varepsilon)^{2/\beta} \quad \text{a.s.}, \end{aligned}$$

for all $\varepsilon > 0$. This completes the proof of (3.1). □

REMARK 3.1. In light of Remark 2.1, the conclusion of Theorem 3.1 remains true if $\sigma(\cdot)$ is as postulated there.

THEOREM 3.2. *Let* $\{\Gamma(t), -\infty < t < \infty\}$ *and* $\sigma(s)$ *be as in Lemmas* 2.1 *and* 2.4. *Let* $0 < a_T \le T$ *be a nondecreasing function of* $T$ *for which* $T/a_T$ *is nondecreasing. We define*

$$H_0(T, a_T) = \|\Gamma(T + a_T) - \Gamma(T)\|, \tag{3.2}$$

$$H_1(T, a_T) = \sup_{0 \le s \le a_T} \|\Gamma(t + s) - \Gamma(T)\|, \tag{3.3}$$

$$H_2(T, a_T) = \sup_{0 \le t \le T - a_T} \|\Gamma(t + a_T) - \Gamma(t)\|, \tag{3.4}$$

$$H_3(T, a_T) = \sup_{0 \le t \le T - a_T} \sup_{0 \le s \le a_T} \|\Gamma(t + s) - \Gamma(t)\|, \tag{3.5}$$

*and put*

$$1/g_T(\gamma, \beta) = \sigma(a_T)\big((1/\gamma)(\log(T/a_T) + \log\log T)\big)^{1/\beta}: \tag{3.6}$$

*We have*

$$\limsup_{T \to \infty} g_T(\gamma, \beta) H_i(T, a_T) \le 1 \qquad a.s., i = 0, 1, 2, 3. \tag{3.7}$$

PROOF. Based on Lemma 2.4, the proof of (3.7) is similar to that of (1.2.11) in Csörgő and Révész (1981). We let

$$A(T) = g_T(\gamma, \beta) H_3(T, a_T).$$

By (2.7) we have for any $\varepsilon > 0$,

$$P\{A(T) > (1 + \varepsilon)^{2/\beta}\} \le C\left(\frac{a_T}{T}\right)^{\varepsilon} \frac{1}{(\log T)^{1+\varepsilon}}.$$

Let $T_k = \theta^k$ $(\theta > 1)$. Then

$$\sum_{n=1}^{\infty} P\{A(T_k) \ge (1 + \varepsilon)^{2/\beta}\} < \infty$$

for every $\varepsilon > 0$, $\theta > 1$. Hence, by the Borel–Cantelli lemma,

$$\limsup_{k \to \infty} A(T_k) \le 1 \quad \text{a.s.} \tag{3.8}$$

Also, if $k$ is large enough, then we have

$$1 \le \frac{g_{T_k}(\gamma, \beta)}{g_{T_{k+1}}(\gamma, \beta)} \le \theta. \tag{3.9}$$

On choosing now $\theta$ to be near enough to 1, (3.7) follows from (3.8) and (3.9), because $H_3(T, a_T)$ is nondecreasing and $g_T(\gamma, \beta)$ is nonincreasing in $T$. □

As to the question of establishing lower bounds for the lim sup statements in (3.1) and (3.7), this is a more difficult problem for which there is no immediate general solution. It really depends on the finer structure of the process $\Gamma(\cdot)$. For example, the Slepian lemma and its extensions are efficient tools for resolving such problems for real-valued Gaussian processes. For sharp results along these lines we refer to Theorems 3.1–3.3 of Csáki, Csörgő, Lin and Révész (1991). There are no analogous results available in other cases.

**4. $l^2$-valued Ornstein–Uhlenbeck processes.** Let

$$\{Y(t),\ -\infty < t < \infty\} = \{X_k(t),\ -\infty < t < \infty\}_{k=1}^{\infty}$$

be a sequence of independent Ornstein–Uhlenbeck processes with coefficients $\gamma_k$ and $\lambda_k$, that is, $X_k(\cdot)$ is a stationary, mean zero Gaussian process with

$$EX_k(s)X_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|), \qquad k = 1, 2, \dots, \tag{4.1}$$

where $\gamma_k \geq 0$, $\lambda_k > 0$.

The process $Y(\cdot)$ was introduced by Dawson (1972) as the stationary solution of the infinite array of stochastic differential equations

$$dX_k(t) = -\lambda_k X_k(t)\,dt + (2\gamma_k)^{1/2}\,dW_k(t), \qquad k = 1, 2, \dots, \tag{4.2}$$

where $\{W_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ are independent Wiener processes. The properties of $Y(\cdot)$ have been extensively studied in the literature. Since $EX_k^2(t) = \gamma_k/\lambda_k$, it is clear that for every fixed $t$, $Y(t)$ is almost surely in $l^2$ if and only if

$$E\|Y(t)\|^2 = \sum_{k=1}^{\infty} \gamma_k/\lambda_k = J_0 < \infty.$$

In this section we assume throughout that $Y(\cdot) \in l^2$, that is, that $J_0 < \infty$ and the Banach space $\mathscr{B}$ is identical with $l^2$. Consequently, $\|\ \|$ denotes $l^2$-norm here.

The continuity properties of $Y(\cdot)$ were investigated by Dawson (1972), Iscoe and McDonald (1986, 1989), Schmuland (1987, 1988a, b) and Iscoe, Marcus, McDonald, Talagrand and Zinn (1990), with a final result due to Fernique (1989), which reads as follows: For each $x \in \mathbb{R}^+$, let $K(x) = \{k \in \mathbb{N}: \gamma_k > \lambda_k x\}$ and $\lambda(x) = \sup\{\lambda_k: k \in K(x)\}$. Then $Y(\cdot) \in l^2$ is a.s. continuous if and only if $J_0 < \infty$ and $\int((\log \lambda(x)) \vee 0)\,dx < \infty$. He showed also that

$$\sum_{k=1}^{\infty} (\gamma_k/\lambda_k)(1 + (\log \lambda_k) \vee 0) < \infty \tag{4.3}$$

is a sufficient condition for a.s. $l^2$ continuity of $Y(\cdot)$.

Concerning modulus of continuity, it was shown by Schmuland (1988c) that if

$$\sum_{k=1}^{\infty} \gamma_k/\lambda_k^{1-\varepsilon} < \infty, \tag{4.4}$$

then $Y(\cdot)$ is a.s. $\mu$-Hölder continuous in $l^2$ for any $\mu < \varepsilon/2$.

We now introduce the following notation:

$$\sigma_k^2(h) = E(X_k(h) - X_k(0))^2 = (2\gamma_k/\lambda_k)(1 - \exp(-\lambda_k h)), \tag{4.5}$$

$$\sigma^2(h) = \sum_{k=1}^{\infty} \sigma_k^2(h) = 2\sum_{k=1}^{\infty} (\gamma_k/\lambda_k)(1 - \exp(-\lambda_k h)), \tag{4.6}$$

$$J_1 = \sum_{k=1}^{\infty} \gamma_k, \tag{4.7}$$

$$\gamma^* = \max_k \gamma_k. \tag{4.8}$$

Our main result is the following theorem.

THEOREM 4.1. *Assuming that $Y(\cdot)$ is a.s. continuous in $l^2$ and that $\sigma(h) < \sigma^*(h)$, $0 \le h \le h_0$, where $\sigma(h)$ is defined by* (4.6) *and $\sigma^*(h)$ is regularly varying at zero with a positive exponent, we have*

$$\limsup_{h \downarrow 0} \sup_{0 \le t \le 1-h} \sup_{0 \le s \le h} \frac{\|Y(t+s) - Y(t)\|}{\sigma^*(h)(2\log(1/h))^{1/2}} \le 1 \quad a.s. \tag{4.9}$$

*If, in particular, we have also $J_1 < \infty$, then*

$$\lim_{h \downarrow 0} \sup_{0 \le t \le 1-h} \sup_{0 \le s \le h} \frac{\|Y(t+s) - Y(t)\|}{(2h\gamma^*)^{1/2}(2\log(1/h))^{1/2}} = 1 \quad a.s. \tag{4.10}$$

PROOF. In order to have Theorem 3.1 in terms of $Y(\cdot) \in l^2$, we establish inequality (2.1) for the latter process. It is clear that

$$\begin{aligned} E\left(\exp(\lambda\|Y(h) - Y(0)\|^2)\right) &= E\left(\exp\left(\lambda \sum_{k=1}^{\infty} (X_k(h) - X_k(0))^2\right)\right) \\ &= \prod_{k=1}^{\infty} \left(1 - 2\lambda\sigma_k^2(h)\right)^{-1/2} \quad \text{for } 2\lambda \max_k \sigma_k^2(h) < 1. \end{aligned}$$

By Markov's inequality we get for any $x > 0$ and $0 \le 2\lambda\sigma_k^2(h) \le 1 - \varepsilon$, $k = 1, 2, \dots,$

$$\begin{aligned} P\{\|Y(t+h) - Y(t)\| \ge x\} &= P\{\|Y(h) - Y(0)\|^2 \ge x^2\} \\ &\le \exp(-\lambda x^2) \prod_{k=1}^{\infty} \left(1 - 2\lambda\sigma_k^2(h)\right)^{-1/2} \\ &\le \exp\left(-\lambda x^2 + \lambda\sigma^2(h)/\varepsilon\right) \\ &\le \exp\left(-\lambda x^2 + \lambda(\sigma^*(h))^2/\varepsilon\right), \end{aligned} \tag{4.11}$$

where we used the inequality $1 - u \ge \exp(-u/\varepsilon)$ if $0 \le u \le 1 - \varepsilon$. Now let

$\lambda = (1-\varepsilon)/(2(\sigma^*(h))^2)$. Then for any given $0 < \varepsilon < 1$, there exists $K = K(\varepsilon) > 0$ such that for $x > 0$ we have (2.1) in the following form for $Y(\cdot) \in l^2$:

$$(4.12)\quad P\{\|Y(t+h) - Y(t)\| \geq x\} \leq K \exp\left(-(1-\varepsilon)x^2/\left(2(\sigma^*(h))^2\right)\right).$$

Hence we have also appropriate versions of (2.2) and (2.5). Since $\varepsilon > 0$ is arbitrary, (4.9) follows from Theorem 3.1.

Next, for proving now (4.10), if $J_1 < \infty$, we have $\sigma^2(h) \sim 2hJ_1$ $(h \downarrow 0)$ and, by putting $\lambda = (1-\varepsilon)/(2h\gamma^*)$, from (4.11) we get

$$(4.13)\quad P\{\|Y(t+h) - Y(t)\| \geq x\} \leq K_1 \exp(-(1-\varepsilon)x^2/(2h\gamma^*)).$$

Hence, Theorem 3.1 implies

$$(4.14)\qquad \limsup_{h\downarrow 0} \sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \frac{\|Y(t+s) - Y(t)\|}{\left(4h\gamma^* \log(1/h)\right)^{1/2}} \leq 1 \quad \text{a.s.}$$

On the other hand, there exists $k_0$ such that $\gamma_{k_0} = \gamma^*$ and, obviously,

$$\sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} |X_{k_0}(t+s) - X_{k_0}(t)| \leq \sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \|Y(t+s) - Y(t)\|.$$

We also know that [cf. Theorem 3.1 of Csáki, Csörgő, Lin and Révész (1991)]

$$(4.15)\qquad \lim_{h\downarrow 0} \sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \frac{|X_{k_0}(t+s) - X_{k_0}(t)|}{\left(4\gamma_{k_0} h \log(1/h)\right)^{1/2}} = 1 \quad \text{a.s.}$$

This and (4.14) imply (4.10) and the proof of Theorem 4.1 is now complete. □

REMARK 4.1. A version of Theorem 4.1 with $-\infty < T_1 < T_2 < \infty$ as in Theorem 3.1 also holds true, of course. We do not know how sharp our upper estimate of (4.9) is in general, since we cannot give a lower estimate when $J_1 = \infty$.

We also note that Schmuland's above quoted result follows from Theorem 4.1, since if (4.4) holds true, then

$$\sigma^2(h) = \sum_{k=1}^{\infty} (\gamma_k/\lambda_k)(1 - \exp(-\lambda_k h)) \leq h^{\varepsilon} \sum_{k=1}^{\infty} \left(\gamma_k/\lambda_k^{1-\varepsilon}\right),$$

on account of the inequality $1 - e^{-u} \leq u^{\varepsilon}$ for $u \geq 0$ and $0 < \varepsilon < 1$.

REMARK 4.2. For $Y(\cdot) \in l^2$ there is no analogue of Theorem 3.2, since $\lim_{a\to\infty} \sigma^2(a) = J_0 < \infty$ and hence the assumptions of Theorem 3.2 cannot be satisfied.

**5. $l^2$-norm squared processes.** Here we consider $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ as in Section 4, that is, $Y(\cdot) \in l^2$, which in turn means that we assume $J_0 < \infty$ throughout and study the behavior of the real-valued process

$$\chi^2(t) = \|Y(t)\|^2 = \sum_{k=1}^{\infty} X_k^2(t). \tag{5.1}$$

This process was studied by Iscoe and McDonald (1986, 1989), Schmuland (1988c) and Csörgő and Lin (1990). One of the motivations for studying this process is the next simple observation.

LEMMA 5.1. *$Y(\cdot) \in l^2$ is almost surely continuous if and only if $\chi^2(\cdot) \in \mathbb{R}$ is almost surely continuous.*

PROOF. For $|t| \le T$, we have

$$\begin{aligned} |\chi^2(t+h) - \chi^2(t)| &= \left| \sum_{k=1}^{\infty} (X_k(t+h) - X_k(t))(X_k(t+h) + X_k(t)) \right| \\ &\le 2\|Y(t+h) - Y(t)\| \sup_{|t|\le T} \|Y(t)\|. \end{aligned} \tag{5.2}$$

Hence the continuity of $Y(\cdot) \in l^2$ implies the continuity of $\chi^2(\cdot) \in \mathbb{R}$.

On the other hand, assume that $\chi^2(t)$ is almost surely continuous. Since the Ornstein–Uhlenbeck processes $X_k(t)$ are a.s. continuous processes themselves, there exists a subset $\Omega_0$ of $\Omega$ with $P\{\Omega_0\} = 0$ such that for every $\omega \notin \Omega_0$ and for each $k \ge 1$, $X_k(t, \omega)$ and $\chi^2(t, \omega)$ are continuous real-valued functions in $t$.

Fix the value of $t \in \mathbb{R}$ and $\omega \notin \Omega_0$. Since

$$\chi^2(t, \omega) = \sum_{n=1}^{\infty} X_k^2(t, \omega) < \infty,$$

for any $\varepsilon > 0$ there exists an integer $N = N(t, \omega, \varepsilon)$ such that

$$r_N(t) = \sum_{k=N}^{\infty} X_k^2(t, \omega) < \varepsilon. \tag{5.3}$$

Also, since $\sum_{k=1}^{N-1} X_k^2(\cdot, \omega)$ is continuous at $t$ by definition and $\chi^2(\cdot, \omega)$ is continuous at $t$ by assumption, so is

$$r_N(\cdot, \omega) = \chi^2(\cdot, \omega) - \sum_{k=1}^{N-1} X_k^2(\cdot, \omega).$$

Therefore, there exists a $\delta_0 = \delta_0(t, \omega, \varepsilon) > 0$ such that

$$\sup_{|h|\le\delta_0} r_N(t+h, \omega) \le 2\varepsilon.$$

Clearly,

$$\begin{aligned} &\|Y(t+h) - Y(t)\|^2 \\ &\quad = \sum_{k=1}^{N-1} (X_k(t+h) - X_k(t))^2 + \sum_{k=N}^{\infty} (X_k(t+h) - X_k(t))^2 \\ &\quad \le \sum_{k=1}^{N-1} (X_k(t+h, \omega) - X_k(t, \omega))^2 + 2(r_N(t+h, \omega) + r_N(t, \omega)), \end{aligned}$$

and, on account of $X_k(\cdot, \omega)$ being continuous at $t$ for each $1 \le k \le N-1$, there exists a $\delta_1 = \delta_1(t, \omega, \varepsilon) > 0$ such that we have

$$\sum_{k=1}^{N-1} \left(X_k(t+h, \omega) - X_k(t, \omega)\right)^2 < \varepsilon \quad \text{for } |h| < \delta_1.$$

Combining this with (5.3) gives

$$\|Y(t+h) - Y(t)\|^2 \le 7\varepsilon \quad \text{if } |h| \le \delta = \min(\delta_0, \delta_1)$$

for any given $\varepsilon > 0$, that is, the a.s. continuity of $Y(\cdot) \in l^2$ follows from that of $\chi^2(\cdot)$ in $\mathbb{R}$. This also completes the proof of Lemma 5.1. □

COROLLARY 5.1. *$\chi^2(\cdot) \in \mathbb{R}$ is continuous almost surely if and only if the necessary and sufficient conditions of Fernique* (1989) *for the almost sure continuity of $Y(\cdot) \in l^2$ (quoted in Section* 4) *hold true.*

Concerning moduli of continuity for $\chi^2(\cdot) \in \mathbb{R}$, Schmuland (1988c) proved the following results:

1. If $\sum_{k=1}^{\infty} \gamma_k^2/\lambda_k^{2-\varepsilon} < \infty$ for some $0 < \varepsilon \le 1$, then $\chi^2(\cdot)$ is $\mu$-Hölder continuous for any $\mu < \varepsilon/2$.
2. If $\sum_{k=1}^{\infty} \gamma_k^2/\lambda_k < \infty$ and $\sum_{k=1}^{\infty} \gamma_k^2/\lambda_k^{1-\varepsilon} < \infty$ for some $0 < \varepsilon \le 1$, then $\chi^2(\cdot)$ has Lévy's Hölder modulus $(h \log(1/h))^{1/2}$.

Csörgő and Lin (1990) investigated the problem of moduli of continuity for $\chi^2(\cdot)$ under the condition

$$J_2 = \sum_{k=1}^{\infty} \gamma_k^2/\lambda_k < \infty, \tag{5.4}$$

and in this case they proved the following results:

(i) Let $M = \max_{k \ge 1}(\gamma_k^2/\lambda_k)$ and assume that $T_h \uparrow \infty$ continuously as $h \downarrow 0$. Then

$$\limsup_{h \downarrow 0} \sup_{|t| \le T_h} \sup_{0 \le s \le h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8hM)^{1/2} \log(T_h/h)} \le 1 \quad \text{a.s.} \tag{5.5}$$

(ii) If, in addition, the continuous nondecreasing function $T_h$ is such that $\log T_h/\log(1/h) \to \infty$ as $h \downarrow 0$, then the modulus of continuity of (5.5) is exact, that is, we have equality to 1 there instead of the inequality.

The simple equation

$$\|Y(t) - Y(s)\|^2 = \|Y(t)\|^2 - \|Y(s)\|^2 - 2\langle Y(t) - Y(s), Y(s)\rangle, \tag{5.6}$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product, shows that moduli of continuity estimates in $l^2$ imply similar ones for $\chi^2$ on $\mathbb{R}$ and vice versa. This is utilized, for example, in Schmuland (1988c) for proving his previously mentioned results [cf. (4.4) and results 1 and 2 above, right after Corollary 5.1]. These two problems are not equivalent, however. This is due to the presence of the inner product $\langle \cdot , \cdot \rangle$ as well in (5.6), which may destroy the equivalence of rates obtained by mutual estimation (cf. Corollary 5.2 and the discussion right after).

Here, based on our results in Sections 2 and 3, we establish further moduli of continuity for $\chi^2(\cdot)$. First we introduce some notation.

Let

$$\tilde{\sigma}_k^2(h) = E\left(X_k^2(t+h) - X_k^2(t)\right)^2 = 4(\gamma_k/\lambda_k)^2(1-\exp(-2\lambda_k h)), \tag{5.7}$$

$$\begin{aligned}\tilde{\sigma}^2(h) &= E\left(\chi^2(t+h) - \chi^2(t)\right)^2 \\ &= \sum_{k=1}^{\infty} \tilde{\sigma}_k^2(h) = 4\sum_{k=1}^{\infty} (\gamma_k/\lambda_k)^2(1-\exp(-2\lambda_k h)).\end{aligned} \tag{5.8}$$

THEOREM 5.1. *Assuming the a.s. continuity of* $\chi^2(\cdot)$ *and that* $\tilde{\sigma}(h) \le \tilde{\sigma}^*(h)$, $0 \le h \le h_0$, *where* $\tilde{\sigma}(h)$ *is defined by* (5.8) *and* $\tilde{\sigma}^*(h)$ *is regularly varying at zero with a positive exponent, we have*

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\tilde{\sigma}^*(h)\log(1/h)} \le 1 \quad a.s. \tag{5.9}$$

*Also, if* (5.4) *holds, then with* $M = \max_{k\ge 1}(\gamma_k^2/\lambda_k)$, *we have*

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(8hM)^{1/2}\log(1/h)} \le 1 \quad a.s. \tag{5.10}$$

PROOF. For the sake of proving (5.9), we first show that in the present context we have [cf. also (2.9) of Csörgő and Lin (1990)]

$$\begin{aligned}P\{|\chi^2(t+h) - \chi^2(t)| \ge x\} &\le \exp(-tx)\prod_{k=1}^{\infty}\left(1 - t^2\tilde{\sigma}_k^2(h)\right)^{-1/2} \\ &\le \exp\left(-tx + t^2\tilde{\sigma}^2(h)/(2\varepsilon)\right) \\ &\le \exp\left(-tx + t^2(\tilde{\sigma}^*(h))^2/(2\varepsilon)\right),\end{aligned} \tag{5.11}$$

since $1 - t^2\tilde{\sigma}_k^2(h) \ge \exp(-t^2\tilde{\sigma}_k^2(h)/\varepsilon)$ if $0 \le t^2\tilde{\sigma}_k^2(h) \le 1-\varepsilon$ for a given $0 < \varepsilon < 1$. Let now $t = (1-\varepsilon)/\tilde{\sigma}^*(h)$. Then for any given $0<\varepsilon<1$ and small enough $h$, there exists $K = K(\varepsilon) > 0$ such that for $x > 0$, we have (2.1) in the following form:

$$P\{|\chi^2(t+h) - \chi^2(t)| \ge x\} \le K\exp(-(1-\varepsilon)x/\tilde{\sigma}^*(h)). \tag{5.12}$$

Consequently, appropriate versions of (2.2) and (2.5) follow, rendering also Theorem 3.1 applicable. The latter, in turn, results in (5.9), since $\varepsilon > 0$ is arbitrary.

In order to prove (5.10), we let $t = (1-\varepsilon)/(8hM)^{1/2}$ in (5.11). If $J_2 < \infty$, then it is easy to see that $\tilde{\sigma}^2(h) \sim 8hJ_2$ if $h\downarrow 0$. Consequently, by (5.11), for any given $0 < \varepsilon < 1$ and small enough positive $h$, there exists $K = K(\varepsilon) > 0$ such that for $x > 0$, we have (2.1) in the following form:

$$(5.13)\quad P\{|\chi^2(t+h) - \chi^2(t)| \geq x\} \leq K \exp\left(-(1-\varepsilon)x/(8hM)^{1/2}\right).$$

Hence, appropriate versions of (2.2) and (2.5) hold true, rendering Theorem 3.1 applicable, which then results in (5.10). □

A version of Theorem 5.1 with $-\infty < T_1 < T_2 < \infty$ as in Theorem 3.1 is also true, of course. The result of (5.5) and that of (5.10) are similar, both under $J_2 < \infty$. As stated above, (5.5) is sharp under the additional condition of $\log T_h/\log(1/h) \to \infty$ as $h \downarrow 0$. We do not know how sharp our upper estimates are in Theorem 5.1, since we do not have lower estimates. In some cases however we can expect better rates than given by Theorem 5.1. For example, the inequality (5.2) in combination with Theorem 4.1 yields the following result.

COROLLARY 5.2. *Under the conditions of Theorem* 4.1, *we have*

$$(5.14)\quad \limsup_{h\downarrow 0} \sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{\sigma(h)(2\log(1/h))^{1/2}} \leq 2 \sup_{0\leq t\leq 1} \|Y(t)\| \quad a.s.$$

*and if* $J_1 < \infty$, *then*

$$(5.15)\quad \limsup_{h\downarrow 0} \sup_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s) - \chi^2(t)|}{(4\gamma^* h\log(1/h))^{1/2}} \leq 2 \sup_{0\leq t\leq 1} \|Y(t)\| \quad a.s.,$$

*where* $\sigma(h)$ *is as in* (4.6).

Since we always assume that $Y(\cdot) \in l^2$, that is, that $J_0 = \sum_{k=1}^{\infty}(\gamma_k/\lambda_k) < \infty$, it is easy to see that on assuming also $J_1 = \sum_{k=1}^{\infty}\gamma_k < \infty$ we have $J_2 = \sum_{k=1}^{\infty}(\gamma_k^2/\lambda_k) < \infty$ as well. Thus in case of $J_1 < \infty$, the rate of $(h\log(1/h))^{1/2}$ of (5.15) is better than the rate $h^{1/2}\log(1/h)$ of (5.10). In other cases however it is not easy to compare our results in Theorem 5.1 and Corollary 5.2, respectively, since $\sigma(h)$ and $\tilde{\sigma}(h)$ are not easily comparable in general.

We note also that the above quoted result 2 of Schmuland (1988c) follows from Theorem 5.1 since, if $\sum_{k=1}^{\infty}(\gamma_k^2/\lambda_k^{2-\varepsilon}) < \infty$, then

$$\tilde{\sigma}^2(h) = 4\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^2(1-\exp(-2\gamma_k h)) \leq 4(2h)^{\varepsilon}\sum_{k=1}^{\infty}(\gamma_k^2/\lambda_k^{2-\varepsilon}).$$

Concerning Schmuland's result 2 we can prove Lévy's Hölder modulus rate $(h\log(1/h))^{1/2}$ only under the condition $J_1 < \infty$ as just described above.

**6. Further examples of moduli of continuity and large increments of stochastic processes.** Sections 4 and 5 well demonstrate the validity of our approach as summarized by Lemmas 2.1–2.4 and Theorems 3.1 and 3.2 in that they throw new light on some specific problems which have been studied by many in the contemporary literature. Indeed, as we have already noted in the Introduction, the latter works have served as a source of inspiration for ours.

The premier example of, and another source of inspiration for, our work has been Brownian motion. In particular, Lemmas 2.2 and 2.4 are extensions of Lemmas 1.1.1 and 1.2.1, respectively, in Csörgő and Révész (1981) [cf. Lemmas 1 and 1*, respectively, of Csörgő and Révész (1979)], while Theorems 3.1 and 3.2 correspond to the upper estimation parts of Theorems 1.1.1 (the P. Lévy moduli) and 1.2.1 [Theorem 1 of Csörgő and Révész (1979)], respectively, in Csörgő and Révész (1981).

Here we give further illustrative examples of application of the results in Sections 2 and 3. These examples may very well be treated also by results like those of Jain and Marcus (1978).

EXAMPLE 6.1. Let $\{W_k(t), 0 \le t < \infty\}_{k=1}^{\infty}$ be independent standard Wiener processes and consider $\{G_1(t), 0 \le t < \infty\} = \{\sum_{k=1}^{\infty} a_k W_k(t), 0 \le t < \infty\}$, where the coefficients $a_k$ are real numbers such that

$$A_1 = \sum_{k=1}^{\infty} a_k^2 < \infty. \tag{6.1}$$

The latter condition guarantees the existence of $G_1(t)$ as an almost surely continuous Gaussian process on $\mathbb{R}^1$ by (6.F) (Example 6.2). Indeed, we have $\{G_1(t), 0 \le t < \infty\} =_{\mathscr{D}} \{A_1^{1/2} W(t), 0 \le t < \infty\}$, where $\{W(t), 0 \le t < \infty\}$ is a standard Wiener process. Consequently, with $\sigma^2(h) = EG_1^2(h) = A_1 h$, $\gamma = \frac{1}{2}$ and $\beta = 2$, we have Theorems 3.1 and 3.2, which in this case are also sharp, that is, we have them in the respective forms of Theorems 1.1.1 and 1.2.1. of Csörgő and Révész (1981).

Continuing with this example, we consider next the problem of $\{G_2(t), 0 \le t < \infty\} = \{a_k W_k(t), 0 \le t < \infty\}_{k=1}^{\infty} \in l^2$, with $\| \ \|$ standing for the $l^2$-norm. Since $E(a_k W_k(t))^2 = a_k^2 t$, we have $G_2(t)$ almost surely in $l^2$ for every fixed $t$ if and only if $E\|G_2(t)\|^2 = E(\sum_{k=1}^{\infty} a_k^2 W_k^2(t)) = t\sum_{k=1}^{\infty} a_k^2 < \infty$, that is, if and only if (6.1) holds true. Under the same condition, we have also $E\|G_2(t) - G_2(s)\|^{2m} = \mathscr{O}(|t-s|^m)$ for every positive integer $m$ and hence $G_2(\cdot)$ is almost surely continuous in $l^2$. By Markov's inequality, we get for any $x > 0$ and $0 \le 2\lambda a_k^2 h \le 1 - \varepsilon$, $k = 1, 2 \dots$,

$$P\{\|G_2(t+h) - G_2(t)\| \ge x\} = P\{\|G_2(h)\|^2 \ge x^2\} \le \exp(-\lambda x^2 + \lambda h A_1/\varepsilon).$$

Letting now $\lambda = (1-\varepsilon)/(2ha^*)$, where $a^* = \max_k a_k^2$, we have (2.1) for $G_2(\cdot) \in l^2$ in the following form:

$$P\{\|G_2(t+h) - G_2(t)\| \ge x\} \le K \exp(-(1-\varepsilon)x^2/(2ha^*)). \tag{6.2}$$

Hence we have also appropriate versions of (2.2) and (2.5) and hence Theorem

3.1 yields

$$(6.3)\qquad \limsup_{h\downarrow 0}\ \sup_{0\le t\le 1-h}\ \sup_{0\le s\le h} \frac{\|G_2(t+s)-G_2(t)\|}{(2a^*h\log(1/h))^{1/2}} \le 1 \quad \text{a.s.}$$

On the other hand, there exists $k_0$ such that $a_{k_0}^2 = a^*$ and, obviously,

$$\sup_{0\le t\le 1-h}\ \sup_{0\le s\le h} (a^*)^{1/2}|W_{k_0}(t+s)-W_{k_0}(t)| \le \sup_{0\le t\le 1-h}\ \sup_{0\le s\le h}\|G_2(t+s)-G_2(t)\|.$$

By the P. Lévy modulus of continuity for a standard Brownian motion we have as well

$$(6.4)\quad \limsup_{h\downarrow 0}\ \sup_{0\le t\le 1-h}\ \sup_{0\le s\le h} \frac{(a^*)^{1/2}|W_{k_0}(t+s)-W_{k_0}(t_0)|}{(2a^*h\log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

Hence by (6.3) and (6.4), we have also

$$(6.5)\qquad \limsup_{h\downarrow 0}\ \sup_{0\le t\le 1-h}\ \sup_{0\le s\le h} \frac{\|G_2(t+s)-G_2(t)\|}{(2a^*h\log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

Similar calculations to those resulting in (6.2) yield also an appropriate version of Lemma 2.4 for $G_2(\cdot)$. Hence for the latter process we have also (3.7) with $g_T(\cdot,\cdot)$ of (3.6) now looking like

$$(6.6)\qquad 1/g_T(1/2,2) = \big(2a^*a_T(\log(T/a_T)+\log\log T)\big)^{1/2}.$$

Due to Theorem 1.2.1 of Csörgő and Révész (1981) and an argument similar to that of (6.5), for $G_2(\cdot)$ we also have

$$(6.7)\qquad \limsup_{T\to\infty} g_T(1/2,2)H_i(T,a_T) = 1 \quad \text{a.s., } i=0,1,2,3,$$

where $H_i(T,a_T)$, $i=0,1,2,3$, are as in Theorem 3.2, now defined in terms of $G_2(\cdot)$ and $\|\ \|$ standing for $l^2$-norm. Moreover, if

$$(6.8)\qquad \lim_{T\to\infty} \log(T/a_T)/(\log\log T) = \infty,$$

then $\limsup_{T\to\infty}$ in (6.7) can be replaced by $\lim_{T\to\infty}$.

In a similar vein, if instead of $G_2(\cdot)\in l^2$, we consider $\{G_3(t), 0\le t<\infty\} = \{a_kW_k(t), 0\le t<\infty\}_{k=1}^d \in \mathbb{R}^d$ with the Euclidean norm

$$\|G_3(t)\| = \left(\sum_{k=1}^{d}(a_kW_k(t))^2\right)^{1/2},$$

then, replacing $a^*$ in the above results by $a_d^* = \max_{1\le k\le d} a_k^2$, their respective statements remain true for $G_3(\cdot)$ with $\|\ \|$ standing for the Euclidean norm.

EXAMPLE 6.2. Csáki, Csörgő, Lin and Révész (1991) considered the process $\{X(t), t\in\mathbb{R}\} = \{\sum_{k=1}^{\infty}X_k(t), t\in\mathbb{R}\}$ of infinite series of the Ornstein–Uhlenbeck components of $Y(\cdot)\in l^2$ and showed that this series converges uniformly in every finite interval with probability 1 if and only if

$$\int_0^{1/e} \sigma(u)/\big(u(\log(1/u))^{1/2}\big)\,du < \infty, \tag{6.F}$$

where $\sigma(u)$ is defined by (4.6). This condition figures in a result of Fernique [cf. Corollary 2.5 in Jain and Marcus (1978)], which says that, in general, a real-valued stationary Gaussian process $G(\cdot)$ has almost surely continuous sample paths if and only if (6.F) is satisfied with $\sigma^2(u) = E(G(t+u) - G(t))^2$, which is assumed to be an increasing function in $u > 0$. With $\sigma(\cdot)$ as in (4.6), we have also that $\sigma^2(h) = E|X(t+h) - X(t)|^2$. If we assume as well that $\sigma(\cdot)$ is a regularly varying function at zero with positive exponent, then easy calculations lead us to conclude by Theorem 3.1 that we have

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{|X(t+s) - X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} \le 1 \quad \text{a.s.} \tag{6.9}$$

The same was concluded in Csáki, Csörgő, Lin and Révész (1991) by a less general argument than that of our Theorem 3.1 here, where by using the Slepian lemma we have also shown that the upper estimate of (6.9) is sharp and that we actually have

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{|X(t+s) - X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \tag{6.10}$$

The statements of (4.9) and (6.9) are similar and so are those of (4.10) and (6.10). However, for proving (4.10), we also had to assume that $J_1 < \infty$. Assuming the latter and that $T_h \uparrow \infty$ continuously as $h \to 0$, Csörgő and Lin (1990) proved the following version of (6.10):

$$\limsup_{h\downarrow 0} \sup_{0\le t\le T_h-h} \sup_{0\le s\le h} \frac{|X(t+s) - X(t)|}{(2h\Gamma_1)^{1/2}(2\log(T_h/h))^{1/2}} = 1 \quad \text{a.s.} \tag{6.11}$$

By (4.2), the above process $\{X(t), t \in \mathbb{R}\}$ has the representation

$$\begin{aligned} X(t) &= \sum_{k=1}^{\infty} X_k(t) \\ &= \sum_{k=1}^{\infty} \int_{-\infty}^{t} \exp(-\lambda_k|t-s|)(2\gamma_k)^{1/2}\,dW_k(s), \quad t \in \mathbb{R}. \end{aligned} \tag{6.12}$$

The latter form suggests the study of

$$G_4(t) = \int_0^{\infty}\int_{-\infty}^{t} \exp(-\lambda(x)|t-s|)(2\gamma(x))^{1/2} W(ds, dx), \qquad t \in \mathbb{R}, \tag{6.13}$$

where $\gamma(\cdot)$ and $\lambda(\cdot)$ are positive continuous functions on $[0,\infty)$ and $\{W(s,x), -\infty < s < \infty, 0 \le x < \infty\}$ is a two-parameter Wiener process [cf. e.g., Sections 1.10–1.15 and supplementary remarks on Chapter 1 in Csörgő and Révész (1981)]. For further background on $G_4(\cdot)$, we refer to Csörgő and Lin (1989). Clearly $G_4(t)$ is almost surely finite for each fixed $t$ if $\int_0^\infty(\gamma(x)/\lambda(x))\,dx < \infty$. We have also

$$(6.14)\quad EG_4(t) = 0,\quad EG_4(t)G_4(s) = \int_0^\infty (\gamma(x)/\lambda(x))\exp(-\lambda(x)|t-s|)\,dx,$$

and this mean zero, real-valued stationary Gaussian process has almost surely continuous sample paths if and only if (4.17) is satisfied with $\sigma(\cdot)$ defined by

$$(6.15)\quad \begin{aligned}\sigma^2(u) &= E(G_4(t+u) - G_4(t))^2\\ &= 2\int_0^\infty (\gamma(x)/\lambda(x))(1 - \exp(-\lambda(x)u))\,dx, \qquad u > 0.\end{aligned}$$

If we assume also that this $\sigma(\cdot)$ is a regularly varying function at zero with a positive exponent, then easy calculations lead us to conclude again by Theorem 3.1, just as in the case of the $X(\cdot)$ process of this example, that we have (6.9) for $G_4(\cdot)$ as well. Since for any $a < b < c < d$, we have also

$$E(G_4(b) - G_4(a))(G_4(d) - G_4(c)) \le 0,$$

by Theorem 3.1 of Csáki, Csörgő, Lin and Révész (1991) (a Slepian lemma argument) we conclude that

$$(6.16)\qquad \liminf_{h\downarrow 0}\ \sup_{0\le t\le 1-h}\ \sup_{0\le s\le h} \frac{|G_4(t+s) - G_4(t)|}{\sigma(h)(2\log(1/h))^{1/2}} \ge 1 \quad \text{a.s.},$$

and hence we have (6.10) for $G_4(\cdot)$ as well.

EXAMPLE 6.3. Let $\{W(t), t \ge 0\}$ be a standard Wiener process and let $L(a,t)$ be its local time at $a \in \mathbb{R}$ up to time $t \in [0,\infty)$, jointly continuous in the pair $(a,t) \in \mathbb{R} \times [0,\infty)$. Let

$$T_u = \inf\{t\colon t \ge 0, L(0,t) \ge u\}, \qquad u \ge 0,$$

and consider the two parameter process

$$\mathscr{L}(a,u) = L(a,T_u) - L(0,T_u) = L(a,T_u) - u.$$

It is well known that $\mathscr{L}(a,u)$ has the moment generating function

$$(6.17)\quad E\exp(v\mathscr{L}(a,u)) = \exp(-vu + vu/(1-2|a|v)), \qquad |v| \le 1/(2|a|),$$

[cf. Itô and McKean (1965), Problem 4, pages 73–74, or Bass and Griffin (1985)] and hence we have

$$(6.18)\qquad E\mathscr{L}(a,u) = 0, \qquad E\mathscr{L}^2(a,u) = 4|a|u.$$

Also, $\{\mathscr{L}(a,u), u \ge 0\}$ is a strictly stationary process of independent increments in $u$ for any fixed $a \in \mathbb{R}$. In this example, we assume $a \in \mathbb{R}$ to be fixed and, for convenience, positive. By (6.17), for $\lambda > 0$, we have

$$\begin{aligned}(6.19)\quad & E(\exp(\lambda|\mathscr{L}(a,u) - \mathscr{L}(a,0)|)) \\ &= E(\exp(\lambda|\mathscr{L}(a,u)|)) \\ &\le E(\exp(\lambda(L(a,T_u) - u))) + E(\exp(-\lambda(L(a,T_u) - u))) \\ &= \exp\left(-\lambda u + \frac{\lambda u}{1-2a\lambda}\right) + \exp\left(\lambda u - \frac{\lambda u}{1-2a\lambda}\right) \\ &= 2\exp\left(\frac{2\lambda^2 au}{1-2a\lambda}\right) \le 2\exp(4\lambda^2 au),\end{aligned}$$

where the last inequality holds if $\lambda \le 1/(4a)$. Hence, by Markov's exponential inequality, we obtain

$$\begin{aligned}(6.20)\quad P\{|\mathscr{L}(a,u+h) - \mathscr{L}(a,u)| \ge x\} &= P\{|\mathscr{L}(a,h) - \mathscr{L}(a,0)| \ge x\} \\ &\le \exp(-\lambda x)(2\exp(4\lambda^2 ah)).\end{aligned}$$

On choosing now $\lambda = x/(8ah)$, with some positive constant $C$ we have

$$(6.21)\quad P\{|\mathscr{L}(a,u+h) - \mathscr{L}(a,u)| \ge x\} \le C\exp(-x^2/(2(4ah)))$$

for $0 < x < 2h$ if $h$ is small enough. Thus in (6.21), we have (2.1) for $\mathscr{L}(a,\cdot)$ with $\gamma = \frac{1}{2}$, $\beta = 2$ and $\sigma^2(h) = E\mathscr{L}^2(a,h) = 4ah$. Hence the appropriate versions of (2.2) and (2.5) hold true for $\mathscr{L}(a,\cdot)$ and by Theorem 3.1, we get

$$(6.22)\quad \limsup_{h\downarrow 0} \sup_{0\le u\le 1-h} \sup_{0\le s\le h} \frac{|\mathscr{L}(a,u+s) - \mathscr{L}(a,u)|}{(8ah\log(1/h))^{1/2}} \le 1 \quad \text{a.s.}$$

We also have the appropriate form of (2.7) for $\mathscr{L}(a,\cdot)$ and hence, by Theorem 3.2, with $0 \le a_T \le T$, a nondecreasing function of $T$ for which $T/a_T$ is nondecreasing, we obtain

$$\begin{aligned}(6.23)\quad & \limsup_{T\to\infty} \sup_{0\le u\le T-a_T} \sup_{0\le s\le a_T} \frac{|\mathscr{L}(a,u+s) - \mathscr{L}(a,u)|}{(8aa_T(\log(T/a_T) + \log\log T))^{1/2}} \\ &\le 1 \quad \text{a.s.}\end{aligned}$$

as well as the other corresponding statements of Theorem 3.2 for $\mathscr{L}(a,\cdot)$.

We do not have a lower bound for the lim sup statement of (6.22). However, under further assumptions on the growth of $a_T$, that of (6.23) is sharp. This can be seen as follows. In Csáki, Csörgő, Földes and Révész (1989), where we study $\mathscr{L}(\cdot,\cdot)$ as a two-parameter process, we remark also that, on account of $\{\mathscr{L}(a,u), u \ge 0\}$ being a strictly stationary process of independent increments and having also a finite moment generating function in a neighbourhood of zero, the Komlós, Major and Tusnády (1975) theorem implies that, on a rich enough probability space, the process $\{\mathscr{L}(a,u)/(2a^{1/2}), u \ge 0\}$ can be approximated by a Wiener process $\{W_a(u), u \ge 0\}$ for any fixed $a \ne 0$ with the a.s. rate of $\mathscr{O}(\log u)$. Hence, if we assume also that $a_T/\log T \to \infty$ as $T \to \infty$ then, by Theorem 1.2.1 of Csörgő and Révész (1981), (6.23) holds true with $=1$ a.s.

instead of $\leq 1$ a.s. Moreover, if condition (6.8) is also satisfied, then in (6.23) and as well as in the other corresponding statements of Theorem 3.2 for $\mathscr{L}(a, \cdot)$, we have $\lim_{T\to\infty}$ instead of $\limsup_{T\to\infty}$ and $= 1$ a.s. instead of less than or equal to 1 a.s.

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13-15
H-1053 BUDAPEST
HUNGARY

DEPARTMENT OF MATHEMATICS
AND STATISTICS
CARLETON UNIVERSITY
OTTAWA, ONTARIO
CANADA K1S 5B6