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A NECESSARY CONDITION FOR MAKING MONEY FROM FAIR GAMES

By Harry Kesten[1] and Gregory F. Lawler[2]

Cornell University and Duke University

Let $X_1, X_2, \ldots$ be independent random variables such that X_j has distribution $F_{\sigma(j)}$, where $\sigma(j) = 1$ or 2, and the distributions F_i have mean 0. Assume that F_i has a finite q_ith moment for some $1 < q_i < 2$. Let $S_n = \sum_{j=1}^n X_j$. We show that if $q_1 + q_2 > 3$, then $\limsup P\{S_n > 0\} > 0$ and $\limsup P\{S_n < 0\} > 0$ for *each* sequence $\{\sigma(j)\}$ of ones and twos.

1. Introduction. Let $X_1, X_2, \ldots$ be independent random variables with $E(X_j) = 0$ and let $S_n = \sum_{j=1}^n X_j$. If the X_j all have the same distribution, it is known [1] that S_n is recurrent, that is, for every $\varepsilon > 0$,

$$P\{|S_n| \leq \varepsilon \text{ i.o.}\} = 1. \tag{1}$$

It is easy to see that (1) can fail to hold if we remove the assumption that the X_j are identically distributed. Likewise (1) can fail to hold if we do not assume that $E(X_j)$ exists, even if the distribution of X_j is symmetric about the origin. By (1), it is impossible for $S_n \to \infty$ w.p. 1 if the X_j are i.i.d. mean 0. However, there are examples of mean-zero i.i.d. X_j such that $S_n \to \infty$ in probability (see, e.g., [2]). This last condition is easily seen to be equivalent to

$$P\{S_n > 0\} \to 1.$$

While such examples exist, it can be shown that there are no such examples such that $E(|X_j|^q) < \infty$ for some $q > 1$ (see Proposition 2).

In this paper we consider the case where the X_j are not identically distributed, but rather can have a finite number of possible distributions. Let $F_1, \ldots, F_p$ be nontrivial mean-zero distribution functions on $\mathbb{R}$ with moment of order q_i, that is,

$$F_i(0) < 1, \qquad \int x\, dF_i(x) = 0, \qquad \int |x|^{q_i}\, dF_i(x) < \infty, \tag{2}$$

and let $\{Y_{i,j}\}_{i=1,\ldots,p;\ j=1,2,\ldots}$ be independent random variables with $Y_{i,j}$ having distribution F_i. Let $\sigma: \{1, 2, \ldots\} \to \{1, \ldots, p\}$ be any sequence, $X_j =$

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$Y_{\sigma(j),j}$ and

$$S_n = \sum_{j=1}^{n} X_j.$$

In [2] examples were constructed with $p = 2$ and $q_1 + q_2 < 3$ in which $S_n \to \infty$ w.p.1. Such examples were also constructed for general $p < \infty$ under the condition

$$\sum_{\varnothing \leq S \in \{1,\dots,p\}} (|S| - 1) \prod_{i \in S} \frac{2 - q_i}{q_i - 1} > 1. \tag{3}$$

Note that (3) reduces to $q_i < 2 - 1/p$ when all the q_i are equal. Here we consider the converse question: Under what moment conditions can one find an example such that S_n is transient? It was conjectured in [2] that (3), in the weak form with a greater than or equal sign, is necessary for $S_n \to \infty$ w.p.1 This paper proves this result for $p = 2$. Note that the previous paragraph discusses the case $p = 1$.

Suppose that w.p.1 $S_n \to \infty$. Then clearly

$$\lim_{n\to\infty} P\{S_n > 0\} = 1. \tag{4}$$

We will assume (2) and the weaker condition (4) and see what restriction this puts on the q_i. Fix the sequence σ and let

$$N_i(n) = \#\{j \leq n : \sigma(j) = i\}.$$

Then S_n has the distribution of

$$\sum_{j=1}^{N_1(n)} Y_{1,y} + \sum_{j=1}^{N_2(n)} Y_{2,j} + \cdots + \sum_{j=1}^{N_p(n)} Y_{p,j}.$$

We will assume that $N_i(n) \to \infty$ for each i (otherwise we can ignore the ith distribution). Suppose $q_p = 2$, that is, F_p has a finite variance. Then by the central limit theorem,

$$\lim_{n\to\infty} P\left\{ \sum_{j=1}^{N_p(n)} Y_{p,j} < 0 \right\} = \frac{1}{2}. \tag{5}$$

Since

$$P\{S_n \leq 0\} \geq P\left\{ \sum_{i=1}^{p-1} \sum_{j=1}^{N_i(n)} Y_{i,j} \leq 0 \right\} P\left\{ \sum_{j=1}^{N_p(n)} Y_{p,j} \leq 0 \right\},$$

(4) and (5) imply

$$\lim_{n\to\infty} P\left\{ \sum_{i=1}^{p-1} \sum_{j=1}^{N_i(n)} Y_{i,j} \leq 0 \right\} = 0.$$

In other words, the random walk which does not take steps from the distribution F_p also satisfies (4). We will therefore concentrate here on the case with $\int x^2 \, dF_i(x) = \infty$ for each i. The case with $p = 2$ and $\int x^2 \, dF_i(x) < \infty$ for one i is easily treated by the above observation and Proposition 1 below.

THEOREM 1. *Suppose* $p = 2$, (2) *holds*, $\int x^2\, dF_1(x) = \int x^2\, dF_2(x) = \infty$ *and*

$$\lim_{n\to\infty} P\{S_n > 0\} = 1. \tag{6}$$

Then

$$q_1 + q_2 \le 3. \tag{7}$$

We conjecture that $S_n \to \infty$ w.p.1 actually implies that $q_1 + q_2 < 3$, but we do not have a proof. It is also interesting to ask whether or not there exist two distributions satisfying (2) and (4) with $1 < q_1, q_2 < 2$ and $q_1 + q_2 = 3$. In the case of general p, we expect that a similar theorem as above will hold, possibly with the weak form of (3) being the correct condition. Note that the form of the theorem in the abstract is just the contrapositive of the version here.

There is a similar problem which we call the "control problem" where we allow the $\sigma(j)$ to be random variables measurable with respect to $\{S_n: 0 \le n < j\}$. Examples were given by Rogozin and Foss [4] of transient walks with $p = 2$ and $q_1 + q_2 < 3$. In these examples, one selected the distribution at step n based on whether $S_n > 0$ or $S_n \le 0$. We conjecture that one cannot do any better in the control problem than in the problem with nonrandom σ, that is, if $q_1 + q_2 \ge 3$, then the walk will visit some fixed interval $[-L, L]$ infinitely often. While we have no proof of this, it has been shown [2] that if $q_1 = q_2 = \cdots = q_p = 2$, then such an interval $[-L, L]$ does exist.

The proof of the theorem consists of a sequence of reductions. We first show that we may assume $\text{supp}(F_i) \cap (0, \infty)$ is a single point. We next find a necessary and sufficient condition for (6) to hold *in a given strategy*, that is, choice of σ (see Lemma 2). This will allow us to restrict the F_i further to distributions with

$$\text{supp}(F_i) \cap (-\infty, 0) \subset \{-a_{i,j}\} \tag{8}$$

for some sequences $0 < a_{i,1} < a_{i,2} < \cdots$ with $a_{i,j+1}/a_{i,j} \to \infty$. It is then shown that the necessary and sufficient condition for (6) is more or less equivalent to convergence to 0 of two simple sequences of ratios involving specially selected atoms of the distributions. The final step is to show that the atoms can be chosen to satisfy these conditions and (2) only if (7) holds.

2. Proof.

Step I: *Reduction of* $\text{supp}(F_i) \in (0, \infty)$. Let $\tilde{F}_1$ be the mean-zero distribution which agrees with F_1 on $(-\infty, 0]$, but which puts all the mass of F_1 in $(0, \infty)$ on a single point. Since $\tilde{F}_1$ has mean 0, this point must be the conditional expectation

$$b := \frac{\int_{x>0} x\, dF_1(x)}{\int_{x>0} dF_1(x)},$$

and the mass there must be

$$p := \int_{x>0} dF_1(x).$$

Let $Y_1, Y_2, \dots$ be independent random variables with distribution F_1 and $Z_1, Z_2, \dots$ independent with distribution F_2. Also assume that the $\{Y_i\}$ and $\{Z_i\}$ are independent. Let $U_k = \sum_{i=1}^k Y_i$ and $V_k = \sum_{i=1}^k Z_i$. Then S_n has the distribution of

$$U_{N_1(n)} + V_{N_2(n)},$$

where $N_i(n)$ is the number of $k \le n$ with X_k having distribution F_i. (We tacitly assume that $N_i(n) \to \infty$, $i = 1, 2$, for otherwise S_n is recurrent by [1].) Similarly, let $\tilde{Y}_i$ have distribution $\tilde{F}_1$ and define $\tilde{U}_k$, $\tilde{S}_n$ in the obvious way. We wish to show that

$$P\{S_n \le 0\} \to 0 \quad \text{implies } P\{\tilde{S}_n \le 0\} \to 0. \tag{9}$$

Since

$$P\{S_n \le 0\} = \int P\{V_{N_2(n)} \in dv\} P\{U_{N_1(n)} \le -v\},$$

and similarly for $\tilde{S}_n$, it clearly suffices to prove that there exists a $C > 0$ such that uniformly in v,

$$P\{U_N \le -v\} \ge CP\{\tilde{U}_N \le -v\} - o_N(1). \tag{10}$$

We decompose the Y_i into their positive and negative parts, that is, we write

$$U_N = \sum_{i=1}^N Y_i^+ - \sum_{i=1}^N Y_i^-.$$

Let M be the number of Y_i which are strictly positive, so that $\sum_{i=1}^N Y_i^+$ contains exactly M nonzero terms. If we replace these strictly positive terms by b, we obtain

$$Mb - \sum_{i=1}^N Y_i^-,$$

which clearly has the distribution of $\tilde{U}_N$. By conditioning on M and all the Y_i^-, we easily see that (10) will follow if we show there exists a $C > 0$ such that uniformly in w,

$$P\left\{\sum_{i=1}^M Y_i^+ \le w \middle| Y_i > 0, 1 \le i \le M\right\} \ge CP\{Mb \le w\} - o_M(1). \tag{11}$$

This needs no proof if $Mb > w$, while for $Mb \le w$ it suffices to prove there exists a $C > 0$ such that for all sufficiently large M,

$$P\left\{\sum_{i=1}^M Y_i^+ \le Mb \middle| Y_i > 0, 1 \le i \le M\right\} \ge C. \tag{12}$$

Note that under the condition $\{Y_i > 0, 1 \le i \le M\}$, the Y_i^+ are i.i.d., each with the conditional distribution of Y_1, given $Y_1 > 0$. This conditional distribution has mean b. Thus (12) is a special case of the following lemma giving a bound for the probability of a sum of i.i.d. random variables to lie on one side of its mean.

LEMMA 1. *Let $W, W_1, W_2, \dots$ be i.i.d. random variables. Assume that there exists a $c < \infty$ with $P\{W < -c\} = 0$ and $EW = b < \infty$. Then if $Z_M = \sum_{j=1}^M W_j$,*

$$\liminf_{M\to\infty} P\{Z_M \le Mb\} > 0. \tag{13}$$

PROOF. We assume $b = 0$, and without loss of generality we may assume that W does not have compact support, for otherwise by the central limit theorem,

$$P\{Z_M \le 0\} \to 1/2.$$

Assume (13) fails and fix $M_1 < M_2 < \cdots$ such that

$$P\{Z_{M_k} \le 0\} \to 0.$$

Restrict M to this subsequence $\{M_k\}$ in the argument below. Write $W = X + V$, where X is bounded, $E(X) = 0$, $d := P\{X \ne 0\} > 0$, $P\{X \ne 0, V \ne 0\} = 0$ (this typically requires randomization at the atoms of W), and define X_j and V_j similarly. Let $\tilde{X}_1, \tilde{X}_2, \dots$ be i.i.d. with the distribution of X given $X \ne 0$, and let A_M be the event that $X_j = 0$ for at least $(1 - d/2)M$ values of $j \le M$. Then for any $D < \infty$,

$$\begin{aligned} P\{Z_M \le 0\} &\ge P\left\{\sum_{j=1}^M X_j \le -D\sqrt{M},\ \sum_{j=1}^M V_j \le D\sqrt{M}\right\} \\ &= \sum_{S\subset\{1,\dots,M\}} P\Bigg\{\sum_{j=1}^M X_j \le -D\sqrt{M},\ \sum_{j=1}^M V_j \le D\sqrt{M}, \\ &\qquad\qquad \text{and } V_j = 0,\ X_j \ne 0 \text{ if and only if } j \in S\Bigg\} \\ &\ge P\left\{\sum_{j=1}^M V_j \le D\sqrt{M}\right\} P\left\{\sup_{dM/2\le N\le M} \sum_{j=1}^N \tilde{X}_j \le -D\sqrt{M}\right\} - P(A_M). \end{aligned}$$

Note that $P(A_M) \to 0$ and by the invariance principle,

$$\liminf_{M\to\infty} P\left\{\sup_{dM/2\le N\le M} \sum_{j=1}^N \tilde{X}_j \le -D\sqrt{M}\right\} > 0.$$

Therefore, $P\{\sum_{j=1}^{M} V_j \le D\sqrt{M}\} \to 0$. Also, if $0 < D, C < \infty$,

$$P\{Z_M \le D\sqrt{M}\} \le P\left\{\sum_{j=1}^{M} V_j \le (D + C)\sqrt{M}\right\} + P\left\{\sum_{j=1}^{M} X_j \le -C\sqrt{M}\right\},$$

so by taking $C \to \infty$, we can see that $P\{Z_M \le D\sqrt{M}\} \to 0$. Finally, if $U_1, U_2, \dots$ are independent random variables, independent of $W_1, W_2, \dots$, each with a uniform distribution on $[-1, 1]$, then a similar argument shows that

$$P\left\{\sum_{j=1}^{M} (W_j + U_j) \le 0\right\} \to 0.$$

Hence, without loss of generality, we may assume that W has a continuous distribution.

Now choose y_k such that

$$P\{W \le y_k\}^{M_k} = \max\left\{P\{Z_{M_k} \le 0\}^{1/2}, M_k^{-1}\right\}. \tag{14}$$

Then $P\{W \le y_k\}^{M_k} \to 0$ and $P\{W \le y_k\} \to 1$ and $y_k \to \infty$. The first relation implies

$$M_k P\{W > y_k\} \to \infty.$$

Let $W_{j,k}$, $j = 1, 2, \dots$, be i.i.d. with the conditional distribution of W given $W \le y_k$. Then

$$\begin{aligned} P\{Z_{M_k} \le 0\} &\ge P\{Z_{M_k} \le 0; W_j \le y_k, j = 1, \dots, M_k\} \\ &= P\{W \le y_k\}^{M_k} P\left\{\sum_{j=1}^{M_k} W_{j,k} \le 0\right\}. \end{aligned}$$

We will show that

$$\liminf_{k \to \infty} P\left\{\sum_{j=1}^{M_k} W_{j,k} \le 0\right\} \ge \frac{1}{2}, \tag{15}$$

which implies for all large k,

$$P\{Z_{M_k} \le 0\} \ge \tfrac{1}{4} P\{W \le y_k\}^{M_k},$$

contradicting (14).

Let

$$\begin{aligned} m_k = EW_{j,k} &= [P\{W \le y_k\}]^{-1} E(WI\{W \le y_k\}) \\ &= -[P\{W \le y_k\}]^{-1} E(WI\{W > y_k\}). \end{aligned}$$

Then

$$-m_k \sim E(WI\{W > y_k\}) \ge y_k P\{W > y_k\},$$

and hence for large k,

$$\frac{y_k}{M_k|m_k|} \leq \frac{2}{M_k P\{W > y_k\}} \to 0. \tag{16}$$

Suppose for a given k,

$$M_k m_k^2 \geq 2\operatorname{Var}(W_{j,k}). \tag{17}$$

For any such k, Chebyshev's inequality shows that

$$\begin{aligned} P\left\{\sum_{j=1}^{M_k} W_{j,k} > 0\right\} &= P\left\{\sum_{j=1}^{M_k} (W_{j,k} - EW_{j,k}) > -M_k m_k\right\} \\ &\leq \frac{M_k \operatorname{Var}(W_{j,k})}{M_k^2 m_k^2} \leq \frac{1}{2}. \end{aligned}$$

If all sufficiently large k satisfy (17) we are finished. Otherwise, consider the subsequence of M_k, which we still denote by M_k, such that

$$M_k|m_k|^2 < 2\operatorname{Var}(W_{i,k}).$$

Along this subsequence we have [using (16)]

$$\frac{M_k E|W_{j,k} - m_k|^3}{\left[M_k \operatorname{Var}(W_{j,k})\right]^{3/2}} \leq \frac{M_k(y_k + c)\operatorname{Var}(W_{j,k})}{\left[M_k \operatorname{Var}(W_{j,k})\right]^{3/2}} \to 0.$$

Thus, by Lyapunov's theorem (see [3], Exercise 8.10.17),

$$P\left\{\sum_{j=1}^{M_k} W_{j,k} \leq 0\right\} \geq P\left\{\sum_{j=1}^{M_k} (W_{j,k} - EW_{j,k}) \leq 0\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x^2/2}\, dx = \frac{1}{2}.$$

This then gives (15) and hence the lemma. □

Lemma 1 proves (12) and allows us to replace F_1 with $\tilde{F}_1$. After F_1 has been replaced we can replace F_2 with $\tilde{F}_2$, defined in the obvious way. We then have

$$P\{\tilde{S}_n \geq 0\} \to 1$$

for the same strategy as before, but with $\operatorname{supp}(F_i) \cap (0, \infty)$ being one point. We drop the tildes and assume from now on that

$$\operatorname{supp}(F_i) \cap (0, \infty) = \{b_i\}. \tag{18}$$

Step II: *A necessary and sufficient condition for* $P\{S_n \geq 0\} \to 1$ *under* (18). We begin with an analog of Lemma 1 which gives a bound for $P\{\sum_{j=1}^{M} W_i \geq Mb + \text{some positive quantity}\}$. For each $\varepsilon > 0$ and $i = 1, 2$, define $L_i(n, \varepsilon)$ to be the largest number satisfying

$$F_i(-L_i(n, \varepsilon)\, -) \leq \frac{\varepsilon}{n} \leq F_i(-L_i(n, \varepsilon)). \tag{19}$$

It is necessary to be specific about randomization if there is an atom at $-L_i(n,\varepsilon)$. Let

$$\frac{\varepsilon}{n} = \theta\left[F_i(-L_i(n,\varepsilon)) - F_i(-L_i(n,\varepsilon)-)\right] + F_i(-L_i(n,\varepsilon)-), \qquad 0 \le \theta \le 1. \tag{20}$$

We then think of a fraction θ of the atom giving rise to values on the left of $-L_i(n,\varepsilon)$ and a fraction $(1-\theta)$ giving rise to values to the right of $-L_i(n,\varepsilon)$. Accordingly, we define the truncated first and second moments,

$$\mu_i(n,\varepsilon) = \int_{x > -L_i(n,\varepsilon)} x\, dF_i(x) - (1-\theta)L_i(n,\varepsilon)\left[F_i(-L_i(n,\varepsilon)) - F_i(-L_i(n,\varepsilon)-)\right],$$

$$s_i^2(n,\varepsilon) = \int_{x > -L_i(n,\varepsilon)} x^2\, dF_i(x) + (1-\theta)L_i(n,\varepsilon)^2\left[F(-L_i(n,\varepsilon)) - F_i(-L_1(n,\varepsilon)-)\right].$$

Note that $\mu_i(n,\varepsilon) \ge 0$ and $\mu_i(n,\varepsilon) \to 0$ as $n \to \infty$. Also, $s_i^2(n,\varepsilon) < \infty$ by (18).

The next lemma is stated in a stronger form than needed here; the stronger form will be used in a forthcoming paper by Kesten and Maller. We shall apply the lemma in this paper with $W_j = -Y_j$, where $Y_1, Y_2, \dots$ are i.i.d. with distribution F_i, $i = 1,2$. These variables Y_j are bounded below by $-b_i$ and therefore satisfy (22) with $B_1 = b_i$. If these Y_j have an infinite second moment then they also satisfy (23) with this B_1 and some B_2. Then we obtain for each $\varepsilon > 0$, $k, l < \infty$, that for sufficiently large M,

$$P\left\{\sum_{j=1}^{M} Y_j \le M\mu_i(M,\varepsilon) - k\sqrt{M}\,s_i(M,\varepsilon) - lL_i(M,\varepsilon)\right\} \ge C(k,l,\varepsilon). \tag{21}$$

If Y_j has a finite second moment, then (21) follows easily from the central limit theorem, since $L_i(M,\varepsilon) = o(\sqrt{M})$ in this case and the Y_j have zero mean.

LEMMA 2. *For all $k, l, \varepsilon, B_1, B_2 > 0$, there exist constants $C = C(k,l,\varepsilon,B_1,B_2) > 0$ and $n_0 = n_0(k,l,\varepsilon,B_1,B_2) < \infty$, with the following property: Let $W_j^{(n)}$, $j = 1,\dots,n$, be i.i.d. with distribution function $G^{(n)}$. Let $L(n,\delta)$ be any $(1-\delta/n)$-quantile of $G^{(n)}$, that is,*

$$G^{(n)}(L(n,\delta)-) \le 1 - \frac{\delta}{n} \le G^{(n)}(L(n,\delta)),$$

and define the truncated moments

$$\mu(n,\delta) = \int_{x<L(n,\delta)} x\, dG^{(n)}(x) + \left[1 - \frac{\delta}{n} - G^{(n)}(L(n,\delta)-)\right]L(n,\delta),$$

$$s^2(n,\delta) = \int_{x<L(n,\delta)} x^2\, dG^{(n)}(x) + \left[1 - \frac{\delta}{n} - G^{(n)}(L(n,\delta)-)\right]L^2(n,\delta).$$

Assume that $G^{(n)}$ satisfies the following inequalities:

$$\int_{x \le 0} |x|^3 \, dG^{(n)}(x) \le B_1^3, \tag{22}$$

$$1 - G^{(n)}(B_2) \le \frac{1}{16} \quad \textit{and} \quad s(n, \delta) \ge 4(B_1 + B_2). \tag{23}$$

Then for all $\varepsilon/4 \le \delta \le 4\varepsilon$ and $n \ge n_0$,

$$P\left\{ \sum_{j=1}^{n} W_j^{(n)} \ge n\mu(n, \delta) + k\sqrt{n}\, s(n, \delta) + lL(n, \delta) \right\} \ge C. \tag{24}$$

PROOF. Let $W_{j,n}$, $j = 1, 2, \dots, n$, be i.i.d. with distribution

$$P\{W_{j,n} \le x\} = \left(1 - \frac{\delta}{n}\right)^{-1} G^{(n)}(x), \qquad x < L(n, \delta),$$
$$P\{W_{j,n} \le L(n, \delta)\} = 1.$$

Except for the randomization at L this is the conditional distribution of $W_1^{(n)}$ given $W_1 \le L$. Note that

$$EW_{j,n} = \left(1 - \frac{\delta}{n}\right)^{-1} \mu(n, \delta). \tag{25}$$

Also, by Schwarz and Jensen's inequalities,

$$\begin{aligned} |EW_{j,n}| &\le \left(1 - \frac{\delta}{n}\right)^{-1} \left\{ B_1 + \int_{0 \le x \le B_2} x \, dG^{(n)}(x) \right. \\ &\qquad \left. + s(n, \delta)\left[1 - G^{(n)}(B_2)\right]^{1/2} \right\} \\ &\le \frac{1}{\sqrt{2}} s(n, \delta). \end{aligned} \tag{26}$$

The second inequality follows from (23) for n sufficiently large [i.e., for all $n \ge n_0(\varepsilon)$—in the remainder of this proof we will say "for large n" to mean for all $n \ge n_0(k, l, \varepsilon, B_1, B_2)$]. Thus for large n,

$$\operatorname{Var}(W_{j,n}) = \left(1 - \frac{\delta}{n}\right)^{-1} s^2(n, \delta) - |EW_{j,n}|^2 \ge \frac{1}{2} s^2(n, \delta). \tag{27}$$

Also,

$$\begin{aligned} E|W_{j,n} - EW_{j,n}|^3 &\le 4\left[E|W_{j,n}^-|^3 + E|W_{j,n}^+ - EW_{j,n}|^3 \right] \\ &\le 4\left[B_1^3 + \left(L + |EW_{j,n}|\right) \operatorname{Var}(W_{j,n}) \right] \\ &\le 4\left[B_1^3 + L \operatorname{Var}(W_{j,n}) + 2^{-1/2} s(n, \delta) \operatorname{Var}(W_{j,n}) \right]. \end{aligned}$$

Therefore, by (23) and (27), there exists a $D = D(\varepsilon, B_1, B_2) > 0$ such that for all large n,

$$\frac{nE|W_{j,n} - EW_{j,n}|^3}{\left[n\operatorname{Var}(W_{j,n})\right]^{3/2}} \le \frac{D}{\sqrt{n}}\left[1 + \frac{L(n,\delta)}{s(n,\delta)}\right]. \tag{28}$$

By the Berry–Esseen theorem ([3], Theorem 16.5.2), there exists an $\alpha = \alpha(k)$ such that if $X_1, \dots, X_s$ are independent random variables with

$$\frac{\sum_{j=1}^{s} E|X_j - EX_j|^3}{\left[\sum_{j=1}^{s}\operatorname{Var}(X_j)\right]^{3/2}} \le \alpha,$$

then

$$\sup_{|x|\le 2k} |\Gamma(x) - \Phi(x)| \le \frac{1}{2}\Phi(x),$$

where Φ is the standard normal distribution and Γ is the distribution function of

$$\frac{\sum_{j=1}^{s}(X_j - EX_j)}{\left[\sum_{j=1}^{s}\operatorname{Var}(X_j)\right]^{1/2}}.$$

We now split the argument into two cases.

Case (i): $L(n,\delta) \le (\alpha/2D)\sqrt{n}\, s(n,\delta)$. In this case the left-hand side of (24) is at least

$$\begin{aligned} &P\left\{W_j^{(n)} > L(n,\delta) \text{ for exactly } l \text{ values of } j \le n\right\} \\ &\quad\times P\left\{\sum_{j=1}^{n-l} W_{j,n} \ge n\mu(n,\delta) + k\sqrt{n}\, s(n,\delta)\right\}. \end{aligned} \tag{29}$$

In the first factor we take into the account the randomization when $W_j^{(n)} = L(n,\delta)$, in which case we count $W_j^{(n)}$ as being strictly greater than $L(n,\delta)$ with conditional probability

$$\frac{G^{(n)}(L(n,\delta)) - (1 - \delta/n)}{G^{(n)}(L(n,\delta)) - G^{(n)}(L(n,\delta)-)}.$$

Thus the first factor equals

$$\binom{n}{l}\left(\frac{\delta}{n}\right)^l\left(1 - \frac{\delta}{n}\right)^{n-l} \to \frac{e^{-\delta}\delta^l}{l!} \ge \frac{e^{-4\varepsilon}(\varepsilon/4)^l}{l!}.$$

It is easily seen that this factor is at least C_0 for some $C_0 = C_0(l,\varepsilon) > 0$ and all $n \ge l$. By (25), for large n,

$$\begin{aligned} n\mu(n,\delta) - (n-l)EW_{1,n} &= (l-\delta)EW_{1,n} \\ &\le 2(l+\delta)s(n,\delta). \end{aligned} \tag{30}$$

Therefore, by (27), the second factor in (29) is at least

$$P\left\{\frac{\sum_{j=1}^{n-l}(W_{j,n}-EW_{j,n})}{[(n-l)\mathrm{Var}(W_{1,n})]^{1/2}}\geq 1.5k\right\},$$

at least for large n. For large n, the right-hand side of (28) is less than α and hence by the Berry–Esseen theorem is bounded below by $\Phi(-2k)/2$ for large n. This finishes case (i).

Case (ii): $L(n,\delta)\geq(\alpha/2D)\sqrt{n}\,s(n,\delta)$. Let r be the smallest integer larger than

$$l+\frac{2D(k+4)}{\alpha}.$$

Then as in (29), the left-hand side of (24) is at least

$$P\{W_{j,n}>L(n,\delta)\text{ for exactly } r \text{ values of } j\leq n\}$$
$$\times P\left\{\sum_{j=1}^{n-r}W_{j,n}\geq n\mu(n,\delta)+k\sqrt{n}\,s(n,\delta)+(l-r)L(n,\delta)\right\}.$$

As before, the first term is bounded below for all n by some $C_0=C_0(\varepsilon,r)>0$, and by the choice of r the second term is bounded below by

$$P\left\{\sum_{j=1}^{n-r}W_{j,n}\geq n\mu(n,\delta)-4\sqrt{n}\,s(n,\delta)\right\}.$$

As in (30), for large n this is greater than

$$P\left\{\sum_{j=1}^{n-r}(W_{j,n}-EW_{j,n})\geq -2\sqrt{n}\,s(n,\delta)\right\}.$$

Finally, since

$$s^2(n,\delta)\geq\left(1-\frac{\delta}{n}\right)E(W_{1,n}^2)\geq\left(1-\frac{\delta}{n}\right)\mathrm{Var}(W_{1,n}),$$

Chebyshev's inequality shows that the last probability is at least $1/2$ for n sufficiently large. This completes case (ii) and finishes the proof of the lemma. □

PROPOSITION 1. *Under* (18), *a necessary and sufficient condition for* $P\{S_n>0\}\to 1$ *is*

$$(31)\qquad \forall\,\varepsilon>0,\qquad \frac{N_1(n)s_1^2+L_1^2+N_2(n)s_2^2+L_2^2}{[N_1(n)\mu_1]^2+[N_2(n)\mu_2]^2}\to 0,$$

where $N_i(n)$ *is the number of* $k\leq n$ *with* X_k *having distribution* F_i *and* $L_i=L_i(N_i(n),\varepsilon)$, $\mu_i=\mu_i(N_i(n),\varepsilon)$, $s_i^2=s_i^2(N_i(n),\varepsilon)$.

PROOF. Assume (31) holds. Let $\{Y_{i,j}\}$, $i = 1, 2,\ j = 1, 2, \dots,$ be independent random variables with $Y_{i,j}$ having distribution F_i. Then S_n has the distribution of

$$\sum_{j=1}^{N_1(n)} Y_{1,j} + \sum_{j=1}^{N_2(n)} Y_{2,j}.$$

For each $\varepsilon > 0$,

$$P\left\{\sum_{j=1}^{N_i(n)} Y_{i,j} \neq \sum_{j=1}^{N_i(n)} Y_{i,j} I[Y_{i,j} \geq -L_i]\right\} \leq N_i(n)\frac{\varepsilon}{N_i(n)} = \varepsilon,$$

where $L_i = L_i(N_i(n), \varepsilon)$ and again we use an appropriate randomization if $Y_{i,j} = -L_i$. By Chebyshev,

$$\begin{aligned}
&P\left\{\sum_{j=1}^{N_1(n)} Y_{1,j} I[Y_{1,j} \geq -L_1] + \sum_{j=1}^{N_2(n)} Y_{2,j} I[Y_{2,j} \geq -L_2] \leq 0\right\} \\
&\quad = P\left\{\sum_{j=1}^{N_1(n)} \left(Y_{1,j} I[Y_{1,j} \geq -L_1] - \mu_1\right) + \sum_{j=1}^{N_2(n)} \left(Y_{2,j} I[Y_{2,j} \geq -L_2] - \mu_2\right)\right. \\
&\qquad\qquad \left. \leq -N_1(n)\mu_1 - N_2(n)\mu_2\right\} \\
&\quad \leq \frac{N_1(n)s_1^2 + N_2(n)s_2^2}{[N_1(n)\mu_1 + N_2(n)\mu_2]^2},
\end{aligned}$$

which goes to 0 by (31) since $\mu_i \geq 0$. This proves sufficiency.

To prove the necessity, suppose for some $\varepsilon > 0$ that the ratio in (31) is greater than $\eta > 0$. Then

$$\begin{aligned}
N_1\mu_1 + N_2\mu_2 &\leq \sqrt{2}\left([N_1\mu_1]^2 + [N_2\mu_2]^2\right)^{1/2} \\
&\leq \sqrt{\frac{2}{\eta}}\left[N_1(n)s_1^2 + L_1^2 + N_2(n)s_2^2 + L_2^2\right]^{1/2} \\
&\leq \sqrt{\frac{2}{\eta}}\left[\sqrt{N_1(n)}\, s_1 + L_1 + \sqrt{N_2(n)}\, s_2 + L_2\right],
\end{aligned}$$

and hence

$$\begin{aligned}
P\{S_n \leq 0\} \geq P\Bigg\{&\sum_{j=1}^{N_1(n)} Y_{1,j} \leq N_1(n)\mu_1 - \sqrt{\frac{2}{\eta}}\sqrt{N_1(n)}\, s_1 - \sqrt{\frac{2}{\eta}} L_1, \\
&\sum_{j=1}^{N_2(n)} Y_{2,j} \leq N_2(n)\mu_2 - \sqrt{\frac{2}{\eta}}\sqrt{N_2(n)}\, s_2 - \sqrt{\frac{2}{\eta}} L_2\Bigg\}.
\end{aligned}$$

By (21), this implies for n sufficiently large that

$$P\{S_n \le 0\} \ge \left[C\left(\sqrt{\frac{2}{\eta}}, \sqrt{\frac{2}{\eta}}, \varepsilon\right)\right]^2 > 0.$$

The necessity of (31) is then immediate. □

To get a feeling for the condition (31), we will describe what sort of moment conditions it implies for one distribution.

Proposition 2. *Let $X, X_1, X_2, \dots$ be i.i.d. mean-zero random variables with distribution function F and suppose*

$$\lim_{n \to \infty} P\left\{\sum_{j=1}^{n} X_j > 0\right\} = 1.$$

Then $E|X^-|^q = \infty$ for every $q > 1$.

Proof. Assume $P\{\sum_{j=1}^n X_j > 0\} \to 1$. Let $V_1, V_2, \dots$ be independent standard normal random variables independent of $X_1, X_2, \dots$. Then an argument as in Lemma 1 shows that $P\{\sum_{j=1}^n (X_j + V_j) > 0\} \to 1$. We may then apply the result of Step I and combine all of the mass of the distribution of $X_j + V_j$ on the positive axis to a single point. Hence, without loss of generality, we may assume that $F(x)$ is continuous and strictly increasing for $x < 0$, and that the distribution on the positive axis concentrates on a single $x > 0$. By (31) applied with $F_1 = F_2 = F$ we see that for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{L(n, \varepsilon)}{n\mu(n, \varepsilon)} = 0.$$

An examination of the proof shows that this holds uniformly for $1/2 \le \varepsilon \le 2$ and so

$$\lim_{a \to 0} \frac{aL(a)}{\int_{-\infty}^{-L(a)} y\, dF(y)} = 0,$$

where now $L = -F^{-1}$. The substitution $b = F(y)$ in the denominator changes the integral to $-\int_0^a L(b)\, db$, and hence if

$$g(a) = \log \int_0^a L(b)\, db,$$

$g'(a) = o(a^{-1})$ as $a \downarrow 0$. It is easy to see that this implies for every $\delta > 0$,

$$\int_0^a L(b)\, db \ge c(\delta)|a|^{\delta}$$

for some $c(\delta) > 0$. However, it is easy to see that if $E|X^-|^q < \infty$, for some

$q > 1$, then $a^{1/q}L(a) \to 0$ and hence

$$\int_0^a L(b)\, db \le O(a^{(q-1)/q}). \qquad \square$$

Rather than work directly with (31), which has a separate requirement for each $\varepsilon > 0$, we want to work with a single sequence. Note that $L_i(N, \varepsilon)$ increases as ε decreases. Consequently, $s_i^2(N, \varepsilon)$ increases and $\mu_i(N, \varepsilon)$ decreases as ε decreases. It is therefore easy to see that (31) holds if and only if there exists a sequence $\varepsilon_n \to 0$ for which

$$(32) \qquad \frac{\sum_{i=1}^2 [N_i(n)s_i^2(N_i(n), \varepsilon_n) + L_i^2(N_i(n), \varepsilon_n)]}{[N_1(n)\mu_1(N_1(n), \varepsilon_1)]^2 + [N_2(n)\mu_2(N_2(n), \varepsilon_2)]^2} \to 0.$$

From now on we will assume that $\varepsilon_n \to 0$ has been fixed such that (32) holds.

Step III: Replacement of F_i by a discrete distribution. Let $0 = a_0 < a_1 < a_2 < \cdots$ be any sequence of real numbers increasing to ∞. For any $\{a_l\}$, we can define the distribution obtained by "pushing the mass of F_1 onto $\{-a_l\}$." We do this in a unique way by preserving the "mass and mean of the interval $[-a_{l+1}, -a_l)$," that is, we take the mass from $[-a_{l+1}, -a_l)$ and give mass α to $-a_{l+1}$ and mass β to $-a_l$, where

$$\alpha + \beta = \int_{[-a_{l+1}, a_l)} dF_1(x)$$

and

$$-\alpha a_{l+1} - \beta a_l = \int_{[-a_{l+1}, -a_l)} x\, dF_1(x).$$

Let $-b \in [-a_{l+1}, -a_l)$ be such that

$$\alpha = \int_{[-a_{l+1}, -b)} dF_1(x),$$

with an appropriate randomization at $-b$ if necessary. We think of the mass in $[-a_{l+1}, -b)$ as being shoved to the left (away from the origin) and the mass in $[-b, -a_l)$ as being shoved to the right.

Now assume that F_i are chosen satisfying (2), (6) and (18). Then also (32) must hold. What we will show is that we can find a sequence $\{a_l\}$ with $a_{l+1}/a_l \to \infty$ such that the distribution derived by pushing the mass of F_1 on $\{-a_l\}$ satisfies (2) and (32) and hence (6). Note that (18) is unchanged.

Let $\{a_l\}$ be a sequence such that a_{l+1}/a_l increases with l. We will consider what happens to the quantities $L_1(N, \varepsilon)$, $s_1^2(N, \varepsilon)$ and $\mu_1(N, \varepsilon)$ after moving the mass of F_1 onto $\{-a_l\}$. Assume that

$$-a_{l+1} \le -L_1(N, \varepsilon) < -a_l$$

and write $\tilde{L}$, $\tilde{s}$ and $\tilde{\mu}$ for the quantities after modification. The total mass of each interval $[-a_{l+1}, -a_l)$ has been distributed over $-a_l$ and $-a_{l+1}$. Therefore, $\tilde{L}_1(N, \varepsilon) = -a_l$ or $-a_{l+1}$. Thus

$$\frac{\tilde{L}}{L} \le \frac{a_{l+1}}{a_l}.$$

Also, no mass in $[-a_{j+1}, -a_j)$ goes further than $-a_{j+1}$ and the mass in $[-\tilde{L}, \infty)$ for $\tilde{F}_1$, the modified distribution, is the same as in $[-L, \infty)$ for F_1. Therefore,

$$\begin{aligned}\tilde{s}_1^2(N, \varepsilon) &= \int_{x \ge -\tilde{L}_1(N,\varepsilon)} x^2 \, d\tilde{F}_1(x) \\ &\le \left(\frac{a_{l+1}}{a_l}\right)^2 \int_{x \ge -L_1(N,\varepsilon)} x^2 \, dF_1(x) = \left(\frac{a_{l+1}}{a_l}\right)^2 s_1^2(N, \varepsilon)\end{aligned}$$

and

$$\begin{aligned}\int |x|^{q_1} \, d\tilde{F}_1(x) &= O(1) + \sum_{j=0}^{\infty} \int_{[-a_{j+1}, -a_j)} |x|^{q_1} \, d\tilde{F}_1(x) \\ &\le O(1) + \sum_{j=1}^{\infty} \int_{[-a_{j+1}, -a_j)} |x|^{q_1} \left(\frac{a_{j+1}}{a_j}\right)^{q_1} dF_1(x).\end{aligned}$$

Since (2) holds, we can find some sequence with $a_{j+1}/a_j \uparrow \infty$ so slowly that the last sum is finite. In addition, we can let a_{j+1}/a_j increase so slowly that we may replace $N_1 s_1^2 + L_1^2$ in the numerator of (32) by

$$N_1(n)\tilde{s}_1^2(N_1(n), \varepsilon_n) + \tilde{L}_1^2(N_1(n), \varepsilon_n).$$

As far as the denominator goes, we claim that

$$\tilde{\mu}_1(N, \varepsilon) \ge \mu_1(N, \varepsilon), \tag{33}$$

and hence for this choice of $\{a_j\}$, (32) holds for the modified distribution, with the same $N_i(n)$ and same ε_n. To prove (33), we need only consider the two cases $L \le b$ and $L > b$ separately (with appropriate modification if b is an atom). In the first case (32) is easy to verify since all the mass in $[-L, -a_l)$ is moved toward the origin. In the latter case all the mass in $[-a_{l+1}, -L]$ gets moved away from the origin, so $-\tilde{\mu}_1(N, \varepsilon) \le -\mu_1(N, \varepsilon)$.

In a similar way we can modify F_2 to a discrete $\tilde{F}_2$ with atoms of $-b_1 > -b_2 > \cdots$ on the negative side and $b_{j+1}/b_j \uparrow \infty$. We assume from now on that these changes have been made and we drop the tildes. Thus F_1 has atoms at $a > 0$, 0, $-a_1$, $-a_2$, $\ldots$ with mass at $-a_j$ denoted p_j, and F_2 has atoms at $b > 0$, 0, $-b_1$, $-b_2$, $\ldots$ with mass at $-b_j$ denoted q_j.

Step IV: A necessary condition in terms of special atoms of F_i. We assume that the F_i have the form given at the end of the last step and satisfy (6) and deduce a weakened form of (32). First, we replace ε_n in (32) by two sequences $\overline{\eta}_i(n) \ge \varepsilon_n$ which have the property that $\overline{\eta}_i$ does not change between jump points of L_i. Let

$$\zeta_i(n) = \sup\{\varepsilon_j : j \geq N_i(n)\}.$$

Find sequences $\eta_i(k) \downarrow 0$ such that

$$\eta_1(k) \geq \max\left\{\left(\sum_{l=k}^{\infty} p_l\right)^{1/2}, \zeta_1\left(\left(\sum_{j=k}^{\infty} p_j\right)^{-1/2}\right)\right\},$$

$$\eta_2(k) \geq \max\left\{\left(\sum_{l=k}^{\infty} q_l\right)^{1/2}, \zeta_2\left(\left(\sum_{j=k}^{\infty} q_j\right)^{-1/2}\right)\right\},$$

and such that

$$(34) \qquad \frac{\eta_i(k)\sum_{l=k}^{\infty} a_l p_l}{\sum_{j=k}^{\infty} p_l} \uparrow \infty, \qquad \frac{\eta_2(k)\sum_{l=k}^{\infty} b_l q_l}{\sum_{l=k}^{\infty} q_l} \uparrow \infty.$$

It is easy to verify that such η_i can be found. Let

$$M_1(k) = \frac{\eta_1(k)}{\sum_{l=k}^{\infty} p_l}, \qquad M_2(k) = \frac{\eta_2(k)}{\sum_{l=k}^{\infty} q_l}.$$

Then $M_i(k) \uparrow \infty$ and

$$L_1(M, \eta_1(k)) = a_k \quad \text{for } M_1(k) \leq M < M_1(k+1),$$
$$L_2(M, \eta_2(k)) = b_k \quad \text{for } M_2(k) \leq M < M_2(k+1).$$

We let

$$\bar{\eta}_i(n) = \eta_i(k) \quad \text{for } M_i(k) \leq N_i(n) < M_i(k+1).$$

Note that since ζ_1 is decreasing and $n \geq N_1(n)$, if $M_1(k) \leq N_1(n) < M_1(k+1)$,

$$\varepsilon_n \leq \zeta_1(n) \leq \zeta_1(M_1(k)) = \zeta_1\left(\eta_1(k)\left(\sum_{l=k}^{\infty} p_l\right)^{-1}\right)$$

$$\leq \zeta_1\left(\left(\sum_{l=k}^{\infty} p_l\right)^{-1/2}\right) \leq \eta_1(k) = \bar{\eta}_1(n).$$

Similarly, $\bar{\eta}_2(n) \geq \varepsilon_n$.

By the monotonicity properties of L, s and μ, (32) holds with $\bar{\eta}_i(n)$ substituted for ε_n, that is, for every $\delta > 0$ we have for all large n,

$$(35) \quad \delta[N_1(n)\mu_1]^2 - N_1(n)s_1^2 - L_1^2 + \delta[N_2(n)\mu_2]^2 - N_2(n)s_2^2 - L_2^2 > 0,$$

where $\mu_i = \mu_i(N_i(n), \bar{\eta}_i(n))$, $s_i^2 = s_i^2(N_i(n), \bar{\eta}_i(n))$ and $L_i = L_i(N_i(n), \bar{\eta}_i(n))$. Define $\bar{H}_i$ by

$$\bar{H}_i(M) = \tfrac{1}{4}[M\mu_i(M, \eta_i(k))]^2 - Ms_i^2(M, \eta_i(k)) - L_i^2(M, \eta_i(k))$$

for $M_i(k) \le M < M_i(k+1)$. Then by (35) for large n,

$$\overline{H}_1(N_1(n)) + \overline{H}_2(N_2(n)) > 0.$$

$\overline{H}_i$ has a slightly unpleasant definition because of the randomization. Let $\eta = \eta_1$ for the time being. If

$$M = \frac{\eta(k)}{\theta p_k + \sum_{l=k+1}^{\infty} p_l} \quad \text{for some } 0 < \theta \le 1,$$

then

$$\begin{aligned} L_1(M, \eta(k)) &= a_k, \\ s_1^2(M, \eta(k)) &\ge \sum_{j<k} p_j a_j^2 + (1-\theta) p_k a_k^2, \\ \mu_1(M, \eta(k)) &= \sum_{j=k+1}^{\infty} p_j a_j + \theta p_k a_k. \end{aligned}$$

We are going to replace $M s_1^2(M, \eta(k)) + L_1^2(M, \eta(k))$ by $M \sum_{j \le k} p_j a_j^2$ and $\mu_1(M, \eta(k))$ by $\sum_{j=k+1}^{\infty} p_j a_j$ for $M \in [M_1(k), M_1(k+1))$. To see that this is permissible, note that for large k,

$$\begin{aligned} M\mu_1(M, \eta(k)) &= M\theta p_k a_k + M \sum_{j=k+1}^{\infty} p_j a_j \\ &= \frac{\theta p_k a_k \eta(k)}{\theta p_k + \sum_{j=k+1}^{\infty} p_j} + M \sum_{j=k+1}^{\infty} p_j a_j \\ &\le a_k + M \sum_{j=k+1}^{\infty} p_j a_j \\ &= L_1(M, \eta(k)) + M \sum_{j=k+1}^{\infty} p_j a_j. \end{aligned}$$

Hence

$$\frac{1}{4}[M\mu_1(M, \eta(k))]^2 \le \frac{1}{2} L_1^2(M, \eta(k)) + \frac{1}{2}\left[M \sum_{j=k+1}^{\infty} p_j a_j\right]^2.$$

In addition, for large k,

$$\begin{aligned} M(1-\theta)p_k + 1 &= \frac{\eta(k)(1-\theta)p_k}{\theta p_k + \sum_{j=k+1}^{\infty} p_j} + 1 \\ &= \frac{\eta(k)(1-\theta)p_k + \theta p_k + \sum_{j=k+1}^{\infty} p_j}{\theta p_k + \sum_{j=k+1}^{\infty} p_j} \\ &\ge \frac{\eta(k) p_k}{\theta p_k + \sum_{j=k+1}^{\infty} p_j} = M p_k. \end{aligned}$$

Therefore,

$$Ms_1^2(M,\eta(k)) + L_1^2(M,\eta(k)) \geq M\sum_{j\leq k} p_j a_j^2.$$

If we define H_1 by

$$H_1(M) = \left[M\sum_{j=k+1}^{\infty} p_j a_j\right]^2 - M\sum_{j\leq k} p_j a_j^2, \qquad M_1(k) \leq M < M_1(k+1),$$

then for large k,

$$\begin{aligned}\overline{H}_1(M) &\leq \frac{1}{2}L_1^2(M,\eta(k)) + \frac{1}{2}\left[M\sum_{j=k+1}^{\infty} p_j a_j\right]^2 \\ &\quad - Ms_1^2(M,\eta(k)) - L_1^2(M,\eta(k)) \\ &\leq \frac{1}{2}\left[M\sum_{j=k+1}^{\infty} p_j a_j\right]^2 - \frac{1}{2}\left[Ms_1^2(M,\eta(k)) + L_1^2(M,\eta(k))\right] \\ &\leq \frac{1}{2}\left[M\sum_{j=k+1}^{\infty} p_j a_j\right]^2 - \frac{1}{2}M\sum_{j\leq k} p_j a_j^2 = \frac{1}{2}H_1(M).\end{aligned}$$

If we define H_2 similarly,

$$H_2(M) = \left[M\sum_{j=k+1}^{\infty} q_j b_j\right]^2 - M\sum_{j\leq k} q_j b_j^2, \qquad M_2(k) \leq M < M_2(k+1),$$

then for n sufficiently large,

$$H_1(N_1(n)) + H_2(N_2(n)) \geq 2\overline{H}_1(N_1(n)) + 2\overline{H}_2(N_2(n)) > 0. \tag{36}$$

This is close to the desired form of the necessary condition. Note that on the interval $[M_i(k), M_i(k+1))$, H_i is the simple quadratic function of M,

$$H_i(M) = AM^2 - BM$$

with coefficients

$$A = A_i(k) = \begin{cases} \left(\sum_{j=k+1}^{\infty} p_j a_j\right)^2, & \text{if } i = 1, \\ \left(\sum_{j=k+1}^{\infty} q_j b_j\right)^2, & \text{if } i = 2, \end{cases}$$

$$B = B_i(k) = \begin{cases} \sum_{j=1}^{k} p_j a_j^2, & \text{if } i = 1, \\ \sum_{j=1}^{k} q_j b_j^2, & \text{if } i = 2. \end{cases}$$

This function of M has a minimum value of $-B^2/4A$ at $M = B/2A$. (For convenience we will now consider M to be a continuous variable. It is easily checked that this makes no significant difference.) It is of course possible that $B_i(k)/(2A_i(k))$ lies outside $[M_i(k), M_i(k+1))$, so that this minimum is not "realized." The idea of the following argument is to look for intervals $[M_i(k), M_i(k+1))$ which actually do contain the corresponding minimum at $B_i(k)/(2A_i(k))$. This will occur for some n when $N_i(n) = B_i(k)/(2A_i(k))$. At this n, H_i will be quite small and by (36) this will have to be compensated for by $H_{3-i}(N_{3-i}(n))$. Even though we have little knowledge of $N_{3-i}(n)$, which depends of course on the strategy $\{\sigma(j)\}$, we will be able to give an estimate on the maximal height of $H_{3-i}(N_{3-i}(n))$ at all n's before $N_i(n)$ reaches $B_i(k)/(2A_i(k))$. Inequality (36) says this maximum has to exceed $B_i^2(k)/(4A_i(k))$. At the end of this step, we will have the final form of the necessary condition for (6).

We define k to be a *J-index of type i* if

$$p_k a_k \geq \sum_{j=k+1}^{\infty} p_j a_j, \qquad i = 1,$$

$$q_k b_k \geq \sum_{j=k+1}^{\infty} q_j b_j, \qquad i = 2.$$

[We call this a J-index because $H_i(\cdot)$ has a big jump at $M_i(k)$ if k is a J-index.] First, we show that there are infinitely many J-indices. Note that if $q_1 + q_2 > 3$ and $q_1, q_2 < 2$, then $q_1, q_2 > 1$. Hence we will assume from this point on that $q_1, q_2 > 1$.

LEMMA 3. *If F_i has a $(1+\varepsilon)$ moment for some $\varepsilon > 0$, then there exist infinitely many J-indices of type i.*

PROOF. Assume not, say for $i = 1$ all indices $j \geq k$ are not J-indices of type 1. Then

$$\sum_{l=j}^{\infty} p_l a_l > \frac{1}{2} \sum_{l=j-1}^{\infty} p_l a_l > \cdots > \left(\frac{1}{2}\right)^{j-k} \sum_{l=k}^{\infty} p_l a_l.$$

However,

$$\sum_{l=j}^{\infty} p_l a_l \leq a_j^{-\varepsilon} \sum_{l=j}^{\infty} p_l a_l^{1+\varepsilon} \leq a_j^{-\varepsilon} C.$$

Since $a_{j+1}/a_j \to \infty$, $a_j \geq \lambda^j$ eventually for any prescribed λ. Therefore for any λ and j sufficiently large,

$$C \geq \lambda^{j\varepsilon} 2^{k-j} \sum_{l=k}^{\infty} p_l a_l,$$

which is impossible if $\lambda^{\varepsilon} > 2$. □

Let $j_1(i) < j_2(i) < \cdots$ be the successive J-indices of type i. We will call the interval $[M_i(j_l(i)), M_i(j_{l+1}(i)))$ *special* if

$$\max\{H_i(M): M_i(j_l(i)) \le M < M_i(j_{l+1}(i))\} > H_i(M_i(j_l(i))). \tag{37}$$

In this case we attach a *special index* k to this interval. It is the smallest index $k \in [j_l(i), j_{l+1}(i))$ for which $H_i(\cdot)$ is not decreasing on the whole interval $[M_i(k), M_i(k+1))$. We prove below that each special interval has a unique special index. Note that k special means

$$M_i(k) \le \frac{B_i(k)}{2A_i(k)} < M_i(k+1),$$

since the minimum occurs at $B_i(k)/2A_i(k)$.

LEMMA 4. (i) *For any* r,

$$H_i(M_i(r)) \le H_i(M_i(r)-),$$

that is, at the jumps, H_i jumps downward.

(ii) *For* $r_1 < r_2$,

$$-\frac{B_i(r_1)^2}{4A_i(r_1)} > -\frac{B_i(r_2)^2}{4A_i(r_2)},$$

that is, the potential minimum of $H_i(\cdot)$ on $[M_i(r), M_i(r+1))$ decreases as r increases.

(iii) *On $[M_i(r), M_i(r+1))$, $H_i(\cdot)$ takes its maximum at $M_i(r)$ or $M_i(r+1)-$. Indeed, on $[M_i(r), M_i(r+1))$, the behavior of H_i is one of the following: increasing, decreasing, or first decreasing and then increasing.*

(iv) *If j_r is a sufficiently large J-index of type i, and k is the first special index of type i which is greater than or equal to j_r, then*

$$H_i(M) \le H_i(M_i(j_r)) \quad \text{for } M_i(j_r) \le M \le \frac{B_i(k)}{2A_i(k)},$$

and the minimum of H_i over $[M_i(j_r), M_i(k+1))$ is $-B_i(k)^2/4A_i(k)$. [See Figure 1 for a "typical" graph of $H_i(\cdot)$.]

(v) *There are infinitely many special intervals of each type, and each special interval has a special index attached to it.*

PROOF. We will assume $i = 1$ (an identical argument works for $i = 2$). Note that $A_1(r)$ decreases and $B_1(r)$ increases as r increases. Hence

$$\begin{aligned} H_1(M_1(r)) &= [M_1(r)]^2 A_1(r) - M_1(r) B_1(r) \\ &\le [M_1(r)]^2 A_1(r-1) - M_1(r) B_1(r-1) = H_1(M_1(r)-). \end{aligned}$$

This gives part (i). Part (ii) is immediate from the monotonicity of A_1 and B_1, and part (iii) follows from the fact that H_1 is a quadratic function on $[M_i(r), M_i(r+1))$.

To prove part (iv), let j_r be a J-index of type 1 and $[M_1(j_l), M_1(j_{l+1}))$ the first special interval in $[M_1(j_r), \infty)$. Then by definition

$$\max\{H_1(M): M_1(j_s) \le M < M_1(j_{s+1})\} = H_1(M_1(j_s)) \quad \text{for } r \le s < l.$$

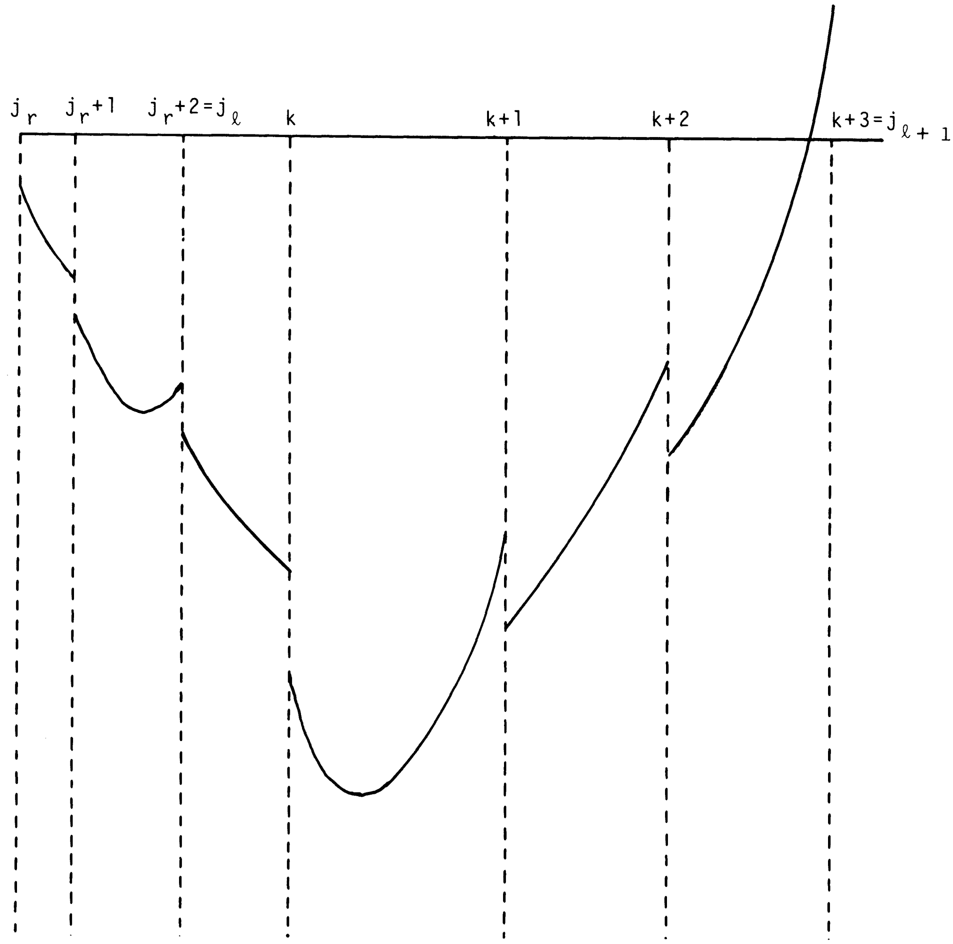

FIG. 1. *Illustration of the graph of $H_1(M)$ with some indices used in part* (iv) *marked. In this figure the J-indices are j_r, $j_l = j_{r+1}$, and j_{l+1}. The interval $[j_r, j_l)$ is not a special interval but the interval $[j_l, j_{l+1})$ is a special interval with special index k.* [*For convenience we label the indices by j rather than the proper notation $M(j)$.*]

Also, by part (i), for $r \leq s - 1 < l$,

$$H_1(M_1(j_s)) \leq H_1(M_1(j_s) -) \leq H_1(M_1(j_{s-1})).$$

Hence

(38) $$H_1(M) \leq H_1(M_1(j_r)) \quad \text{for } M_1(j_r) \leq M < M_1(j_l),$$

and $H_1(\cdot)$ jumps down at $M = M_1(j_l)$. Next, we check the behavior of $H_1(\cdot)$ on the interval $[M_1(j_l), M_i(j_{l+1}))$. First, note that for any J-index j of type 1 and

$M_1(j) \le M < M_1(j+1)$

$$
\begin{aligned}
\frac{d}{dM} H_1(M) &= 2M\Big(\sum_{r \ge j+1} p_r a_r\Big)^2 - \sum_{r \le j} p_r a_r^2 \\
&\le 2M\Big(\sum_{r \ge j+1} p_r a_r\Big)^2 - p_j a_j^2 \\
&\le 2M\Big(\sum_{r \ge j+1} p_r a_r\Big)^2 - \Big(\sum_{r \ge j} p_r\Big)^{-1} (p_j a_j)^2 \\
&\le 2M\Big(\sum_{r \ge j+1} p_r a_r\Big)^2 - \Big(\sum_{r \ge j} p_r\Big)^{-1} \Big(\sum_{r \ge j+1} p_r a_r\Big)^2 .
\end{aligned} \tag{39}
$$

This is negative for $M = M_1(j) = \eta_1(j)[\Sigma_{r \ge j} p_r]^{-1}$ as soon as $2\eta_1(j) < 1$. Thus, if j is a sufficiently large J-index, then the quadratic function $H_1(\cdot)$ is decreasing immediately to the right of $M_1(j)$. It will continue to decrease until it goes through a minimum or until it reaches a jump point. If $H_1(M)$ reaches a jump point $M_1(k)$ and is still decreasing on the left, that is,

$$\left(\frac{dH_1(M)}{dM}\right)_{M_1(k)-} = 2M_1(k)A_1(k-1) - B_1(k-1) \le 0,$$

then by the monotonicity of A_1 and B_1, the right-hand derivative will also be negative (regardless of whether or not k is a J-index) and hence H_1 will start to decrease after the jump. Thus H_1 will decrease on successive intervals $[M_1(j), M_1(j+1))$, $j = j_l, j_l + 1, \dots,$ until the first k such that H_1 has a turning point in $[M_1(k), M_1(k+1))$. This is the special index attached to the special interval $[M_1(j_l), M_1(j_{l+1}))$. Note that there must be such a special index, for otherwise the argument would show that H_1 is decreasing on $[M_1(j_l), M_1(j_{l+1}))$ which contradicts the fact that this is a special interval. Since $H_1(\cdot)$ is decreasing on $[M_1(j_l), B_1(k)/2A_1(k)]$ this gives part (iv).

We have shown above that every special interval has a special index, so to prove part (v) it remains to show that there are infinitely many special indices. Assume that there are only a finite number; then from some J-index j_0 of type 1 on

$$H_1(M) \le H_1(M_1(j_0)) < 0, \qquad M \ge M_1(j_0).$$

The first inequality follows as in (38), while the second inequality follows as in (39) from

$$
\begin{aligned}
H_1(M_1(j_0)) &= M_1(j_0)[A_1(j_0)M_1(j_0) - B_1(j_0)] \\
&\le M_1(j_0)\left[M_1(j_0) - \Big(\sum_{r \ge j_0} p_r\Big)^{-1}\right]\Big(\sum_{r \ge j_0+1} p_r a_r\Big)^2 \\
&= M_1(j_0)(\eta_1(j_0) - 1)\Big(\sum_{r \ge j_0} p_r\Big)^{-1}\Big(\sum_{r \ge j_0+1} p_r a_r\Big)^2 \\
&< 0,
\end{aligned} \tag{40}
$$

at least if $\eta_1(j_0) < 1$, which is true if j_0 is chosen large enough. But if $H_1(M) < 0$ eventually, then $H_1(N_1(n)) < 0$ eventually. By (36) this implies $H_2(N_2(n)) > 0$ eventually. Since $N_2(n)$ increases by jumps of size 1, this implies $H_2(M) > 0$ eventually. This is impossible, since the calculation in (40) shows that for all large j which are J-indices of type 2,

$$H_2(M_2(j)) < 0.$$

This completes the proof of the lemma. □

For a special index K of type i, let

$$\nu(i, K) = \left\lfloor \frac{B_i(K)}{2A_i(K)} \right\rfloor$$

be the value of M for which $H_i(M)$ (essentially) reaches its corresponding minimum. Let $\rho_i(K)$ be the smallest value of n for which $N_i(n) = \nu(i, K)$, that is, the first time at which our strategy really reaches this minimum. We call the $\rho_i(K)$ *special minima*. Blocks of special minima of one type alternate with blocks of special minima of the other type. The last special minima in these blocks will play a special role. To isolate these, write $K_i(0) < K_i(1) < \cdots$ for all the special indices of type i. Corresponding to these indices are the special minima $\rho_i(K_i(0)) < \rho_i(K_i(1)) < \cdots$. We start with some (sufficiently large) special index of type 1, $K_1(p(0))$, with special minimum $\rho_1(K_1(p(0)))$. Let $\rho_2(K_2(q(0)))$ be the smallest special minimum of type 2 greater than $\rho_1(K_1(\rho(0)))$, and let $\rho_1(K_1(p(1)))$ be the largest special minimum of type 1 less than $\rho_2(K_2(q(0)))$. Thus

$$\rho_1(K_1(p(0))) \le \rho_1(K_1(p(1))) < \rho_2(K_2(q(0))) < \rho_1(K_1(p(1) + 1)).$$

Similarly, $\rho_2(K_2(q(1)))$ will be the largest special minimum of type 2 less than $\rho_1(K_1(p(1) + 1))$, that is,

$$\rho_2(K_2(q(0))) \le \rho_2(K_2(q(1))) < \rho_1(K_1(p(1) + 1)) < \rho_2(K_2(q(1) + 1)).$$

Next we let $\rho_1(K(p(2)))$ be the largest special minimum of type 1 less than $\rho_2(K_2(q(1) + 1))$, and continue in the obvious way. We have for $i \ge 1$,

$$\rho_2(K_2(q(i))) < \rho_1(K_1(p(i) + 1)) \le \rho_1(K_1(p(i + 1))) < \rho_2(K_2(q(i) + 1)),$$

and for $i \ge 0$,

$$\rho_1(K_1(p(i + 1))) < \rho_2(K_2(q(i) + 1)) \le \rho_2(K_2(q(i + 1))) < \rho_1(K_1(p(i + 1) + 1)).$$

By (36),

$$H_1(N_1(\rho_1(K_1(p(j))))) + H_2(N_2(\rho_1(K_1(p(j))))) > 0 \tag{41}$$

and

$$H_1(N_1(\rho_1(K_1(p(j))))) \sim -\frac{B_1^2(K_1(p(j)))}{4A_1(K_1(p(j)))}. \tag{42}$$

Since the next special minimum of type 2 after $\rho_1(K_1(p(j)))$ is $\rho_2(K_2(q(j-1)+1))$,

$$N_2(n) \le \frac{B_2(K_2(q(j-1)+1))}{2A_2(K_2(q(j-1)+1))} \quad \text{for } n \le \rho_1(K_1(p(j))).$$

Therefore,

$$\begin{aligned} &H_2(N_2(\rho_1(K_1(p(j))))) \\ &\quad \le \max\left\{H_2(M) \colon M \le \frac{B_2(K_2(q(j-1)+1))}{2A_2(K_2(q(j-1)+1))}\right\}. \end{aligned} \tag{43}$$

We turn to the calculation of this maximum. We need the following notation. Let K be a special index of type i. We denote the special interval that it belongs to by $[M_i(\tilde{L}_i(K)), M_i(L_i(K)))$. Thus $\tilde{L}_i(K)$ $[L_i(K)]$ is the last (first) J-index of type i that is less than or equal to K (greater than K). Note that neither $\tilde{L}_i(K)$ nor $L_i(K)$ has to be a special index.

LEMMA 5. *Let $K' < K < K''$ be three successive special indices of type i. Then*:

(i)

$$\max\left\{H_i(M) \colon M \le \frac{B_i(K'')}{2A_i(K'')}\right\} \le \begin{cases} 4\left[\eta_1(L_1(K))a_{L_1(K)}\right]^2, & i=1, \\ 4\left[\eta_2(L_2(K))b_{L_2(K)}\right]^2, & i=2. \end{cases}$$

(ii) *For any fixed $\delta > 0$, if K is sufficiently large*,

$$A_1(K) \le \left[\frac{a_{L_1(K)}}{a_{L_1(K')}}\right]^{\delta}\left[p_{L_1(K)}a_{L_1(K)}\right]^2,$$

$$A_2(K) \le \left[\frac{b_{L_2(K)}}{b_{L_2(K')}}\right]^{\delta}\left[q_{L_2(K)}b_{L_2(K)}\right]^2.$$

PROOF. We will do the case $i = 1$. Since $[\tilde{L}_1(K''), L_1(K''))$ can, by definition, contain only one special index, we have $K < \tilde{L}_1(K'')$ and

$$K < L_1(K) \le \tilde{L}_1(K'') \le K'' < L_1(K'').$$

Since K and K'' are successive special indices, there is no special index between the indices $L_1(K)$ and K''. Therefore, by Lemma 4(iv),

$$\max\left\{H_1(M) \colon M \le \frac{B_1(K'')}{2A_1(K'')}\right\} = \max\{H_1(M) \colon M \le M_1(L_1(K))\}.$$

On the intervals $[M_1(r), M_1(r+1))$, $H_1(\cdot)$ takes its maximum at one of the endpoints, and it jumps downward at each $M_1(r)$. Thus

$$\begin{aligned}\max\{H_1(M): M \le L_1(K)\} &= \max\{H_1(M_1(r)\,-): r \le L_1(K)\}\\ &= \max\{M_1(r)^2 A_1(r-1)\\ &\qquad -M_1(r)B_1(r-1): r \le L_1(K)\}\\ &\le \max\{M_1(r)^2 A_1(r-1): r \le L_1(K)\}\\ &= \max\left\{\left[\frac{\eta_1(r)}{\sum_{j=r}^{\infty} p_j}\sum_{j=r}^{\infty} p_j a_j\right]^2 : r \le L_1(K)\right\}.\end{aligned}$$

By (34), the last term above equals

$$\left[\frac{\eta_1(L_1(K))\sum_{j=L_1(K)}^{\infty} p_j a_j}{\sum_{j=L_1(K)}^{\infty} p_j}\right]^2.$$

Since $L_1(K)$ is a J-index,

$$\sum_{j=L_1(K)}^{\infty} p_j a_j \le 2p_{L_1(K)} a_{L_1(K)}.$$

Combining this with the above, we get

$$\max\left\{H_1(M): M \le \frac{B_1(K'')}{2A_1(K'')}\right\} \le 4\big[\eta_1(L_1(K))a_{L_1(K)}\big]^2,$$

which gives part (i).

To prove part (ii), note that if $(r+1)$ is not a J-index, then

$$\begin{aligned}[A_1(r)]^{1/2} &= \sum_{j=r+1}^{\infty} p_j a_j = p_{r+1}a_{r+1} + \sum_{j=r+2}^{\infty} p_j a_j \le 2\sum_{j=r+2}^{\infty} p_j a_j\\ &= 2[A_1(r+1)]^{1/2}.\end{aligned}$$

Since $K+1,\dots, L_1(K)-1$ are not J-indices but $L_1(K)$ is a J-index, we have

$$\begin{aligned}[A_1(K)]^{1/2} &\le 2^{L_1(K)-K-1}[A_1(L_1(K)-1)]^{1/2}\\ &= 2^{L_1(K)-K-1}\sum_{j=L_1(K)}^{\infty} p_j a_j\\ &\le 2^{L_1(K)-K} p_{L_1(K)} a_{L_1(K)}.\end{aligned}$$

Since $a_{r+1}/a_r \to \infty$, we have $4 \le (a_{r+1}/a_r)^{\delta}$ for all sufficiently large r. Thus, for sufficiently large K,

$$A_1(K) \le \left(\frac{a_{L_1(K)}}{a_K}\right)^{\delta}[p_{L_1(K)}a_{L_1(K)}]^2.$$

Finally, $L_1(K') \le K$. Thus $a_K \ge a_{L_1(K')}$, which completes the proof. $\square$

From (41)–(43), we get for j sufficiently large,

$$4\left[\eta_2(L_2(K_2(q(j-1))))b_{L_2(K_2(q(j-1)))}\right]^2 \ge \frac{B_1^2(K_1(p(j)))}{8A_1(K_1(p(j)))},$$

and hence

$$\frac{B_1(K_1(p(j)))}{\left[A_1(K_1(p(j)))\right]^{1/2} b_{L_2(K_2(q(j-1)))}} \to 0.$$

Since $K_1(p(j-1)) < K_1(p(j))$ implies $L_1(K_1(p(j-1))) \le K_1(p(j))$, we get

$$\begin{aligned} B_1(K_1(p(j))) &= \sum_{l=1}^{K_1(p(j))} p_l a_l^2 \\ &\ge p_{L_1(K_1(p(j-1)))}\left[a_{L_1(K_1(p(j-1)))}\right]^2, \end{aligned}$$

which implies finally from Lemma 5(ii) that for every $\delta > 0$,

$$\frac{p_{L_1(K_1(p(j-1)))}\left[a_{L_1(K_1(p(j-1)))}\right]^2}{\left[\dfrac{a_{L_1(K_1(p(j)))}}{a_{L_1(K_1(p(j-1)))}}\right]^\delta p_{L_1(K_1(p(j)))} a_{L_1(K_1(p(j)))} b_{L_2(K_2(q(j-1)))}} \to 0.$$

Similarly, after interchanging the role of $i = 1$ and $i = 2$ [and noting that the last $K_1(p(\cdot))$ before $K_2(q(j))$ is $K_1(p(j))$ rather than $K_1(p(j-1))$], we obtain

$$\frac{q_{L_2(K_2(q(j-1)))}\left[b_{L_2(K_2(q(j-1)))}\right]^2}{\left[\dfrac{b_{L_2(K_2(q(j)))}}{b_{L_2(K_2(q(j-1)))}}\right]^\delta q_{L_2(K_2(q(j)))} b_{L_2(K_2(q(j)))} a_{L_1(K_1(p(j)))}} \to 0.$$

If we let

$$\begin{aligned} P(j) &= p_{L_1(K_1(p(j)))}, & Q(j) &= q_{L_2(K_2(q(j)))}, \\ C(j) &= a_{L_1(K_1(p(j)))}, & D(j) &= b_{L_2(K_2(q(j)))}, \end{aligned}$$

then we have shown that for every $\delta > 0$,

$$\frac{P(j-1)[C(j-1)]^{2+\delta}}{P(j)[C(j)]^{1+\delta} D(j-1)} \to 0$$

and

$$\frac{Q(j-1)[D(j-1)]^{2+\delta}}{Q(j)D(j)^{1+\delta} C(j)} \to 0.$$

In addition, if we assume (2), then for j sufficiently large

$$P(j)[C(j)]^{q_1} \le 1, \qquad Q(j)[D(j)]^{q_2} \le 1.$$

Step V: A lemma on sequences. We have reduced Theorem 1 to the following lemma about sequences.

LEMMA 6. *Let $P(j), Q(j), C(j), D(j)$ be sequences of strictly positive numbers with $D(j) \to \infty$ such that for every $\delta > 0$,*

$$\frac{P(j-1)[C(j-1)]^{2+\delta}}{P(j)[C(j)]^{1+\delta}D(j-1)} \to 0 \tag{44}$$

and

$$\frac{Q(j-1)[D(j-1)]^{2+\delta}}{Q(j)D(j)^{1+\delta}C(j)} \to 0. \tag{45}$$

Suppose also that for some $1 < q_1, q_2 < 2$, for j sufficiently large,

$$P(j)[C(j)]^{q_1} \le 1 \quad \textit{and} \quad Q(j)[D(j)]^{q_2} \le 1. \tag{46}$$

Then

$$\left(\frac{2-q_1}{q_1-1}\right)\left(\frac{2-q_2}{q_2-1}\right) \ge 1,$$

that is, $q_1 + q_2 \le 3$.

PROOF. Assume (44)–(46) hold for some $1 < q_1, q_2 < 2$ with $q_1 + q_2 > 3$ and find $0 < \delta < q_i - 1$ with

$$\left(\frac{2-q_1+\delta}{q_1-1-\delta}\right)\left(\frac{2-q_2+\delta}{q_2-1-\delta}\right) < 1. \tag{47}$$

Let

$$\Gamma(j) = P(j)C(j)^{2+\delta},$$

$$\Delta(j) = \left[Q(j)D(j)^{1+\delta}\right]^{-1}.$$

Then by (46), $\Gamma(j) \le C(j)^{2-q_1+\delta}$ and $\Delta(j) \ge D(j)^{q_2-1-\delta}$. If we multiply (44) and (45) we obtain

$$\frac{\Gamma(j-1)\Delta(j)}{\Gamma(j)\Delta(j-1)} \to 0,$$

from which it follows that $\Delta(j) = o(\Gamma(j))$. Going back to (44), we get

$$\frac{\Gamma(j-1)C(j)}{\Gamma(j)D(j-1)} \to 0.$$

But for large j,

$$\frac{C(j)}{D(j-1)} \geq \frac{\Gamma(j)^{1/(2-q_1+\delta)}}{\Delta(j-1)^{1/(q_2-1-\delta)}} \geq \frac{\Gamma(j)^{1/(2-q_1+\delta)}}{\Gamma(j-1)^{(1/q_2-1-\delta)}}.$$

Thus

$$\Gamma(j)^{(q_1-1-\delta)/(2-q_1+\delta)}\Gamma(j-1)^{-(2-q_2+\delta)/(q_2-1-\delta)} \to 0$$

or

$$\Gamma(j) = o\Big([\Gamma(j-1)]^{((2-q_1+\delta)/(q_1-1-\delta))((2-q_2+\delta)/(q_2-1-\delta))}\Big).$$

In view of (47), this clearly implies that $\Gamma(j)$ is bounded as $j \to \infty$ and hence that $\Delta(j) \to 0$. But this is impossible since

$$Q(j)D(j)^{1+\delta} = o\big(Q(j)D(j)^{q_2}\big) = o(1),$$

and hence $\Delta(j) \to \infty$. □

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DEPARTMENT OF MATHEMATICS
WHITE HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

DEPARTMENT OF MATHEMATICS
DUKE UNIVERSITY
DURHAM, NORTH CAROLINA 27706