The Annals of Probability
1992, Vol. 20, No. 2, 825–854

ON THE POSITION OF A RANDOM WALK AT THE TIME OF FIRST EXIT FROM A SPHERE

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Let T_r be the first time a sum S_n of nondegenerate i.i.d. random vectors leaves the sphere of radius r. The spheres are determined by some given norm on $\mathbb{R}^d$ which need not be the Euclidean norm. As a particular case of our results, we obtain, for mean-zero random vectors and each $0 < p < \infty$ and $0 \le q < \infty$, necessary and sufficient conditions on the distribution of the summands to have $E(\|S_{T_r}\| - r)^p = O(r^q)$ as $r \to \infty$. We also characterize tightness of the family $\{\|S_{T_r}\| - r\}$ and obtain other related results on the rate of growth of $\|S_{T_r}\|$. In particular, we obtain a simple necessary and sufficient condition for $\|S_{T_r}\|/r \to_p 1$.

1. Introduction and statement of main results. We study the position of a d-dimensional random walk at the time of first exit from a sphere of radius r. In particular, for mean-zero walks, we obtain necessary and sufficient conditions for the moments of the overshoot to be bounded independently of r. One interesting aspect of this work is that the spheres need not be based upon the usual Euclidean norm. Indeed all of our results hold for an arbitrary norm on $\mathbb{R}^d$.

Let $X, X_1, X_2, \dots$ be nondegenerate i.i.d. $\mathbb{R}^d$-valued random vectors with distribution function F and set $S_n = \sum_{j=1}^n X_j$. Let $\| \ \|$ be some given norm on $\mathbb{R}^d$ (we reserve the symbol $\| \ \|_E$ for the usual Euclidean norm), and let $T_r = \min\{n: \|S_n\| > r\}$. Broadly speaking, all of our results concern the rate of growth of $\|S_{T_r}\|$ relative to r, as influenced by the distribution of X. The relevant influence of the distribution is measured by four fundamental functions: $G(\lambda) = P(\|X\| > \lambda)$, $M(\lambda) = \lambda^{-1}\|E(X; \|X\| \le \lambda)\|$, $K(\lambda) = \lambda^{-2}E(\|X\|^2; \|X\| \le \lambda)$, and $h(\lambda) = M(\lambda) + G(\lambda) + K(\lambda)$. It is also convenient to introduce $Q(\lambda) = G(\lambda) + K(\lambda) = \lambda^{-2}E(\|X\| \wedge \lambda)^2$. The importance of these functions can be seen from the estimate

$$ET_r \approx h(r)^{-1} \tag{1.1}$$

of Pruitt [10]. (See also the paragraph following Lemma 2.3.) Here, and elsewhere in this paper, two variable quantities A and B are related by $\approx$ and said to be *comparable*, if there are constants c and C such that $cA \le B \le CA$.

Received October 1989; revised November 1990.
[1]Supported in part by NSF Grant DMS-87-00928.
[2]Supported in part by NSF Grant DMS-89-00503.
AMS 1980 *subject classifications*. Primary 60J15, 60G50; secondary 60K05.
Key words and phrases. Random walk, overshoot, multivariate renewal theory, moments.

In Section 3 we begin our investigation by studying the size of $\|S_{T_r}\|$ relative to powers of r. For example, we give necessary and sufficient conditions for $\sup_{r\geq 1}(E\|S_{T_r}\|^p/r^q) < \infty$ for all $0 < p \leq q < \infty$ (Theorem 3.3). The case $p = q$ is the most interesting and can be viewed as giving conditions for $E(\|S_{T_r}\| - r)^p = O(r^p)$. This leads naturally to more precise results on the overshoot $\|S_{T_r}\| - r$. As a first step in this direction, we show in Theorem 3.5 that

$$\frac{\|S_{T_r}\| - r}{r} \to_p 0 \tag{1.2}$$

if and only if

$$\lim_{r\to\infty}\left(\frac{G(r)}{h(r)}\right) = 0. \tag{1.3}$$

Our remaining results, the principal ones of this paper, all deal with stronger conditions than (1.2). Before describing these results, we would like to discuss the role of the three terms in the function h. Each measures aspects of the distribution function which have differing effects on the rate of growth of $\|S_{T_r}\|$. In the relatively crude measures of this growth discussed above, only the large jumps, controlled by G/h, are important. On a finer scale, diffusion and drift, associated respectively with K and M, become important. When M is the dominant influence (e.g., when the X_i have *nonzero* mean), the study of $\|S_{T_r}\|$ is very closely related to renewal theory, and there is a sizable literature, even in higher dimensions. (See, e.g., [1] and [5], and the references cited there.) We will focus on the phenomena which arise when M does not dominate, that is, when $h \approx Q$. This occurs, for example, when X is symmetric or, by Lemma 2.1, when $EX = 0$ and $X \in L^2$, a case studied in dimension $d = 1$ for the one-sided overshoot by Lai [9].

In order to state our main results, we need to introduce an exit condition which we shall refer to as condition (E);

(E) The family $\dfrac{S_{T_r}}{\|S_{T_r}\|}$ has no subsequential limit supported on a closed half space.

We shall always, in the following, assume that the X_i are *genuinely d-dimensional*, that is, have probability distribution not supported in any affine hyperplane. For L^2 random variables, this is equivalent to assuming a nonsingular covariance matrix. Finally, recall that a random vector X is said to belong to weak L^p (WL^p) if $\sup_{r>0} r^pG(r) < \infty$.

THEOREM 1.1. *For genuinely d-dimensional random vectors and* $0 < q < p \wedge 2$, *the following are equivalent*:

$$E\big(\|S_{T_r}\| - r\big)^p = O(r^q) \quad \text{as } r\to\infty \quad \text{and} \quad (\mathrm{E}), \tag{1.4}$$

$$X \in WL^{2+p-q} \quad \text{and} \quad EX = 0. \tag{1.5}$$

The case $2 \le q < p$ is less interesting and is dealt with in Theorem 6.2. Next, we consider the limiting case of Theorem 1.1 as $q \to 0$. Note that if we formally set $q = 0$ in (1.4) and (1.5), the result is not quite right.

THEOREM 1.2. *For genuinely d-dimensional random vectors and* $0 < p < \infty$, *the following are equivalent*:

$$\sup_{0<r<\infty} E\big(\|S_{T_r}\| - r\big)^p < \infty \quad \textit{and} \quad (\mathrm{E}), \tag{1.6}$$

$$E\|X\|^{2+p} < \infty \quad \textit{and} \quad EX = 0. \tag{1.7}$$

We also settle the limiting case of this latter result as $p \to 0$.

THEOREM 1.3. *For genuinely d-dimensional random vectors, the following are equivalent*:

$$\big\{\|S_{T_r}\| - r\big\}_{r\ge 0} \quad \textit{is tight} \quad \textit{and} \quad (\mathrm{E}), \tag{1.8}$$

$$E\|X\|^{2} < \infty \quad \textit{and} \quad EX = 0. \tag{1.9}$$

Several remarks are in order about these results, each of which states the equivalence of a probabilistic and an analytic condition.

REMARK 1.4. It is perhaps not immediately apparent that all of these results deal with the situation where $h \approx Q$. We could have assumed this as a side condition, but there is no need since conditions (1.4)–(1.9) all imply $h \approx Q$ (recall $EX = 0$ and $X \in L^2$ imply $h \approx Q$).

REMARK 1.5. If we assume $EX = 0$ as a side condition, then it may be deduced from Theorem 1.2 that for $0 < p < \infty$,

$$\sup_{r>0} E\big(\|S_{T_r}\| - r\big)^p < \infty \quad \text{iff} \quad E\|X\|^{2+p} < \infty, \tag{1.10}$$

thus giving the necessary and sufficient condition mentioned in the opening paragraph. The reason we have chosen not to phrase our result in this form is that Theorem 1.2 is more general. In addition, assuming a finite first moment as a side condition seems unnatural for $0 < p < 1$ in (1.6).

REMARK 1.6. It is well known that the behavior of the one-sided overshoot in one dimension depends heavily upon whether $EX = 0$ or $EX \neq 0$. Since we have not placed any side conditions on our random vectors, one expects that some condition needs to be added to the various bounds on the overshoot in (1.4), (1.6) and (1.8) to ensure the validity of our results. This is the role played by condition (E). It is a probabilistic condition which ensures that M does not dominate (see Proposition 4.5). Observe that in dimension 1 it simply states that the walk exits from both ends of the interval with positive probability;

more precisely,

$$\liminf_{r \to \infty} \left(P\left(S_{T_r} > 0\right) \wedge P\left(S_{T_r} < 0\right) \right) > 0.$$

It is an interesting problem to determine an analytic condition equivalent to (E). It is not sufficient for example to assume $EX = 0$, even in dimension 1 (see Example 7.1). If (1.2) is assumed, then we are able to show that (E) is equivalent to $h \approx Q$ in dimension $d = 1$, thus confirming that in some sense (E) is the right condition in our setting. For this and the higher-dimensional analog, see Corollary 4.8. If condition (E) is omitted in (1.4), (1.6) and (1.8), then Theorems 1.1–1.3 all break down. This is a consequence of standard results on the overshoot for nonnegative random variables; see, for example, [7].

REMARK 1.7. It is clear that the analytic conditions (1.5), (1.7) and (1.9) are all independent of the norm. This is because all norms on $\mathbb{R}^d$ are comparable. Note, however, that this is not obvious for probabilistic conditions (1.4), (1.6) and (1.8) since the exit time T_r depends on the norm. Indeed we do not see how to prove this independence directly.

REMARK 1.8. In dimension $d = 1$ the implication (1.7) $\Rightarrow$ (1.6) follows from known results. In [9] Lai proves this for integer p and the overshoot of a one-sided boundary. The two-sided result (1.6) is an immediate consequence for such p. Other values of p may be handled by using the results of [2]. Lai's method is to reduce to the case of nonnegative variables by passing to the ladder height process. This approach, of course, does not extend directly to higher dimensions.

The plan of the paper is as follows. Section 2 is devoted to notation and some preliminary results. In Section 3 we study the rate of growth of the moments of $\|S_{T_r}\|$ relative to powers of r, and in Section 6 we study the rate of growth of the moments of $\|S_{T_r}\| - r$ relative to powers of r. Section 6 also contains the proofs of Theorems 1.1–1.3. In Sections 4 and 5 we collect various subsidiary results according to topic. Section 4 contains results pertaining to condition (E), while Section 5 concerns bounds on the occupation times of annuli. Finally, Section 7 presents two examples which show that various hypotheses cannot be weakened.

2. Notation and preliminary results. Recall that the symbol $\| \ \|$ stands for an arbitrary given norm on $\mathbb{R}^d$, and $\| \ \|_E$ stands for the Euclidean norm. Since we are working in finite dimensions, these norms are comparable, that is, we have

$$\rho^{-1}\|x\| \le \|x\|_E \le \rho\|x\| \tag{2.1}$$

for some constant $\rho \ge 1$. We use the symbol $\langle x, y \rangle$ for the usual inner product of vectors x and y. The symbols $B(x; r)$ and $B_E(x; r)$ denote the open balls centered at x of radius r based upon the given and Euclidean norms respectively. Their boundaries will be denoted by $\partial B(x; r)$ and $\partial B_E(x; r)$, their complements by $B(x; r)^c$ and $B_E(x; r)^c$ and their closures by $\overline{B(x; r)}$ and $\overline{B_E(x; r)}$. The Euclidean unit sphere will be denoted by S^{d-1}.

Given $x \in \mathbb{R}^d$, $\theta \in S^{d-1}$ and $\beta \in [0, 1]$, let $\Gamma_x(\theta, \beta) = \{y\colon \langle y - x, \theta\rangle \geq \beta\|y - x\|_E\}$. Thus $\Gamma_x(\theta, \beta)$ is a closed solid cone with vertex at x and axis pointing in the direction θ. The number β is the cosine of the angle at the vertex between the axis and any generator of the cone. For $v \geq 0$ let

$$\Gamma_x(\theta, \beta, v) = \Gamma_x(\theta, \beta) \cap \{y\colon \|x - y\| > v\}. \tag{2.2}$$

The functions G, M, K, Q and h associated with the distribution of the random vector X have been defined above in Section 1. We shall often use the following elementary properties of these functions which the reader may easily verify (also see [10] and [11]): Q is nonincreasing and $v^2Q(v)$ is nondecreasing; h satisfies a doubling condition,

$$\frac{1}{c}h(v) \leq h(2v) \leq ch(v), \qquad v > 0, \tag{2.3}$$

for some constant c. (Throughout the rest of this paper, c will stand for a constant which may change from line to line.) Q also satisfies (2.3). One further useful property is that if $G(v)/Q(v) \to 0$, then Q is regularly varying with exponent -2; for this see [7]. When the norm is the usual Euclidean norm, we will denote the functions by G_E, M_E, K_E, Q_E and h_E respectively.

LEMMA 2.1. *Suppose* $EX = 0$ *and* $\lim_{r\to\infty}(G(r)/Q(r)) = 0$. *Then* $\lim_{r\to\infty}(Q(r)/h(r)) = 1$.

PROOF. For $r > 0$,

$$\begin{aligned}
\|EX1(\|X\| \leq r)\| &= \|EX1(\|X\| > r)\| \\
&\leq E\|X\|1(\|X\| > r) \\
&= rG(r) + \int_{u=r}^{\infty} G(u)\,du \\
&\leq rG(r) + \sup_{u\geq r}\frac{G(u)}{Q(u)}\int_{u=r}^{\infty} Q(u)\,du \\
&\sim rG(r) + \sup_{u\geq r}\frac{G(u)}{Q(u)}rQ(r),
\end{aligned}$$

since Q is regularly varying with exponent -2; see [7]. Thus $M(r)/Q(r) \to 0$. Hence

$$\liminf_{r\to\infty}\frac{Q(r)}{h(r)} \geq 1.$$

Since $Q(r) \leq h(r)$ for all r, this proves the lemma. $\square$

LEMMA 2.2. *We have* $E(Q(\|X\|))^{-1} < \infty$ *if and only if* $E\|X\|^2 < \infty$.

PROOF. If $E\|X\|^2 < \infty$, then $Q(\lambda) \sim (E\|X\|^2)\lambda^{-2}$ as $\lambda \to \infty$, hence $E(Q(\|X\|))^{-1} < \infty$.

Now assume $E(Q(\|X\|))^{-1} < \infty$. Then $E[K(\|X\|)^{-1}; \|X\| > c] < \infty$ for some $c > 0$. Let f be the right-continuous inverse of the tail probability function of $X1_{[\|X\| > c]}$. Since

$$\int_0^1 f^2(t)\left(\int_t^1 f^2(u)\,du\right)^{-1} dt < \infty,$$

we have by the dominated convergence theorem,

$$\int_a^b f^2(t)\left(\int_t^1 f^2(u)\,du\right)^{-1} dt \to 0$$

as $0 < a \le b \to 0$. But

$$\int_a^b f^2(t)\left(\int_t^1 f^2(u)\,du\right)^{-1} dt \ge \left(1 - \frac{\int_b^1 f^2(u)\,du}{\int_a^1 f^2(u)\,du}\right).$$

Thus

$$\left(\int_b^1 f^2(u)\,du\right)\left(\int_a^1 f^2(u)\,du\right)^{-1} \to 1$$

as $0 < a \le b \to 0$. From this we may conclude

$$E\|X\|^2 = c^2 P(\|X\| \le c) + \int_0^1 f^2(u)\,du < \infty. \qquad \square$$

An analog of the above argument for series is given in [12], Exercise 11, page 79.

LEMMA 2.3. *There exists a constant* $c \ge 1$ *such that for all* $r > 0$,

$$c^{-1}h(r) \le h_E(r) \le ch(r).$$

PROOF. Fix $r > 0$. Then

$$\begin{aligned} Q(r) &= r^{-2}E(\|X\| \wedge r)^2 \\ &= \rho^2 r^{-2} E\left(\rho^{-1}\|X\| \wedge \rho^{-1}r\right)^2 \\ &\le \rho^2 r^{-2} E\left(\|X\|_E \wedge \rho^{-1}r\right)^2 \\ &= Q_E\left(\rho^{-1}r\right) \\ &\le \rho^2 Q_E(r), \end{aligned}$$

since $v^2 Q_E(v)$ is nondecreasing. Thus

$$Q(r) \le \rho^2 Q_E(r).$$

We now consider two cases:

Case 1: $M(r) \geq 2\rho^3 Q(r)$. Then

$$M(r) = \|r^{-1}EX1(\|X\|_E \leq r) + r^{-1}(EX1(\|X\| \leq r) - EX1(\|X\|_E \leq r))\|$$
$$\leq \rho M_E(r) + r^{-1}E\|X\|1_A,$$

where

$$A = (\|X\| \leq r) \triangle (\|X\|_E \leq r)$$

and $\triangle$ denotes symmetric difference. But $A \subseteq (\rho^{-1}r < \|X\| \leq \rho r)$, hence

$$\begin{aligned} r^{-1}E\|X\|1_A &\leq \rho G(\rho^{-1}r) \\ &\leq \rho Q(\rho^{-1}r) \\ &\leq \rho^3 Q(r) \\ &\leq \tfrac{1}{2}M(r). \end{aligned}$$

Hence $M(r) \leq \rho M_E(r) + \frac{1}{2}M(r)$, that is, $M(r) \leq 2\rho M_E(r)$. Thus

$$\begin{aligned} h(r) &= Q(r) + M(r) \\ &\leq \rho^2 Q_E(r) + 2\rho M_E(r) \\ &\leq (\rho \vee 2)^2 h_E(r). \end{aligned}$$

Case 2: $M(r) < 2\rho^3 Q(r)$. Then

$$\begin{aligned} h(r) &= Q(r) + M(r) \\ &\leq (2\rho^3 + 1)Q(r) \\ &\leq (2\rho^3 + 1)\rho^2 Q_E(r) \\ &\leq (2\rho^3 + 1)\rho^2 h_E(r). \end{aligned}$$

Thus if we take $c = \max\{(\rho \vee 2)^2, (2\rho^3 + 1)\rho^2\}$, then for any $r > 0$, $h(r) \leq c h_E(r)$. By interchanging $\|\cdot\|$ and $\|\cdot\|_E$ we obtain (for the same c)

$$h_E(r) \leq c h(r). \qquad \square$$

Observe that (1.1) for an arbitrary norm can now be easily deduced from the corresponding result for the Euclidean norm (Theorem 1 of [10]). This is because

$$B_E(x; \rho^{-1}r) \subseteq B(x; r) \subseteq B_E(x; \rho r)$$

and h satisfies (2.3).

Next we use (1.1) to obtain some growth rates on ET_r relative to powers of r.

LEMMA 2.4. (i) *If* $X \in WL^t$, $0 < t < 1$; *or* $X \in WL^t$, $1 < t < 2$, *and* $EX = 0$; *or* $X \in L^t$, $t = 1$, *and* $EX = 0$, *then* $\liminf_{r \to \infty} ET_r/r^t > 0$.

(ii) *If* $X \in L^2$ *and* $EX = 0$, *then*

$$\liminf_{r\to\infty} \frac{ET_r}{r^2} > 0.$$

PROOF. If $0 < t < 2$ and $X \in WL^t$, then

$$Q(r) = r^{-2}\int_0^r 2uG(u)\,du = O(r^{-t}).$$

If $0 < t < 1$ and $X \in WL^t$, then

$$\begin{aligned} M(r) &\le r^{-1}\int_{\|x\|\le r} \|x\|\,dF(x) \\ &\le r^{-1}\int_{u=0}^{r} G(u)\,du \\ &= O(r^{-t}), \end{aligned}$$

which, by (1.1), proves the first part of (i). If $EX = 0$, then

$$\begin{aligned} M(r) &= r^{-1}\left\|\int_{\|x\|>r} x\,dF(x)\right\| \\ &\le r^{-1}\int_{\|x\|>r} \|x\|\,dF(x) \\ &= G(r) + r^{-1}\int_r^\infty G(u)\,du. \end{aligned}$$

Thus if $1 < t < 2$ and $X \in WL^t$, the second part of (i) follows from $G(u) = O(u^{-t})$. If $t = 1$ and $X \in L^1$, then we use $E\|X\| = \int_0^\infty G(u)\,du < \infty$ to obtain the result. Part (ii) follows similarly. □

We will need two of Wald's identities, which we list here for ease of reference; let S_n be a sum of i.i.d. random vectors X_i with mean $EX_i = \mu$ and finite variance $E\|X_i - \mu\|_E^2 = \sigma^2$. Let $\{F_n\}_{n\ge 1}$ be an increasing sequence of σ-fields such that $F_n \supseteq \sigma(X_1, X_2, \dots, X_n)$ and such that $\sigma(X_{n+1})$ and F_n are independent for $n > 1$. If T is a stopping time relative to F_n and $ET < \infty$, then

$$ES_T = \mu ET \tag{2.4}$$

and

$$\operatorname{Var}(S_T) = \sigma^2 ET. \tag{2.5}$$

A convenient reference for both results is the textbook of Chow and Teicher [3]. The first identity is Theorem 1 on page 137 and the second is Theorem 3 on page 139. (While these results are stated for the one-dimensional case, they carry over to higher dimensions with essentially the same proofs.)

The remainder of this section is devoted to preliminary geometric results.

LEMMA 2.5. *There exists a covering of* $\mathbb{R}^d$ *by a finite number of closed cones each with vertex at the origin, say* $\{\Gamma_j\}_{j=1}^m$, *such that for some constant* $\alpha > 0$ *and every* $1 \le j \le m$,

$$\|x + y\| \ge \|x\| + \alpha\|y\| \quad \text{if } x, y \in \Gamma_j.$$

PROOF. Fix $x, y \in \mathbb{R}^d$. Let $h \in (\mathbb{R}^d, \|\ \|)^*$ satisfy $\|h\| = 1$ and $h(x) = \|x\|$. Then

$$\begin{aligned}\|x + y\| &\ge h(x + y)\\ &= \|x\| + h(y)\\ &= \|x\| + h\left(y - \frac{x\|y\|_E}{\|x\|_E} + \frac{x\|y\|_E}{\|x\|_E}\right)\\ &= \|x\| + \frac{\|x\|\,\|y\|_E}{\|x\|_E} + h\left(y - \frac{x\|y\|_E}{\|x\|_E}\right)\\ &\ge \|x\| + \rho^{-2}\|y\| + h\left(y - \frac{x\|y\|_E}{\|x\|_E}\right).\end{aligned}$$

Observe that

$$\begin{aligned}\left|h\left(y - \frac{x\|y\|_E}{\|x\|_E}\right)\right| &\le \left\|y - \frac{x\|y\|_E}{\|x\|_E}\right\|\\ &\le \rho\left\|y - \frac{x\|y\|_E}{\|x\|_E}\right\|_E\\ &= \rho\left(\|y\|_E^2 - 2\langle x, y\rangle\frac{\|y\|_E}{\|x\|_E} + \|y\|_E^2\right)^{1/2}\\ &= 2^{1/2}\rho\|y\|_E\left(1 - \frac{\langle x, y\rangle}{\|x\|_E\|y\|_E}\right)^{1/2}\\ &\le 2^{1/2}\rho^2\|y\|\left(1 - \frac{\langle x, y\rangle}{\|x\|_E\|y\|_E}\right)^{1/2}.\end{aligned}$$

Hence

$$\|x + y\| \ge \|x\| + \left(\rho^{-2} - 2^{1/2}\rho^2\left(1 - \frac{\langle x, y\rangle}{\|x\|_E\|y\|_E}\right)^{1/2}\right)\|y\|. \tag{2.6}$$

Now choose $\beta \in (0, 1)$ so that for all $\theta \in S^{d-1}$, if $x, y \in \Gamma_0(\theta, \beta)$, then

$$\langle x, y\rangle \ge \left(1 - (8\rho^8)^{-1}\right)\|x\|_E\|y\|_E.$$

In particular, by (2.6)

$$\|x + y\| \ge \|x\| + (2\rho^2)^{-1}\|y\|.$$

It is now easy to produce the desired cover of the form $\{\Gamma(\theta_j, \beta)\}_{j=1}^m$ by means of a compactness argument. □

LEMMA 2.6. *For $x \in B(0;1)$ let $d(x)$ denote the Euclidean distance to $\partial B(0;1)$. Then for any $\beta \in (0,1]$ there is a $\theta = \theta_x \in S^{d-1}$ so that*

$$\Gamma_x(\theta, \beta, \rho\beta^{-1}d(x)) \subseteq B(0;1)^c. \tag{2.7}$$

PROOF. Fix $z \in \partial B(0;1)$ so that $d(x) = \|x - z\|_E$. Let $\theta = (z-x)/\|z-x\|_E$. Then the linear functional $\phi(y) = \langle y, \theta\rangle$ satisfies

$$\phi(y) \le \phi(z), \qquad y \in B(0;1).$$

This follows from a general result in convex analysis (see, e.g., [8], pages 87–89). But it is then easy to check that if $y \in \Gamma_x(\theta, \beta)$ satisfies $\|y - x\| > \rho\beta^{-1}d(x)$, then $\phi(y) > \phi(z)$. This implies (2.7). □

REMARK 2.7. We shall use the result of Lemma 2.6 in a different form. Fix $\beta \in (0,1]$ and let $\gamma = \rho^2\beta^{-1}$. Then for any $r > 0$, any $0 \le u \le r$, and any x with $r - u \le \|x\| < r$, we claim

$$\Gamma_x(\theta, \beta, \gamma u) \subseteq B(0;r)^c \quad \text{for some } \theta \in S^{d-1}.$$

If $\|x\| < r$, then this follows directly from Lemma 2.6 and the observation

$$d(r^{-1}x) \le r^{-1}\left\| x - r\frac{x}{\|x\|}\right\|_E \le \rho r^{-1}\left\| x - \frac{r}{\|x\|}x\right\| \le \rho r^{-1}u.$$

If $\|x\| = r$ $(u = 0)$, then we obtain the desired conclusion by taking for θ the outward unit normal to any hyperplane which supports $B(0;r)$ at x.

LEMMA 2.8. *Let w be a Borel probability measure which is genuinely d-dimensional, that is, $w(H_0) < 1$ for any hyperplane H_0, and which has mean 0, that is, $\int x_j w(dx) = 0$, $j = 1, 2, \dots, d$, where x_j is the jth coordinate function. Then the origin belongs to the interior of the convex hull of the support of w.*

PROOF. Let C denote the convex hull of $S = \text{support}(w)$. If the assertion were false, we would find $\phi \in (\mathbb{R}^d)^*$, $\phi \ne 0$, such that $\phi \ge 0$ on C. (See [8], Corollary 4B, page 15.) But

$$\int \phi(x)w(dx) = 0,$$

implying that ϕ vanishes w-a.e. Since ϕ is continuous, ϕ must vanish identically on S, and so $w(\{x: \phi(x) = 0\}) = 1$. This contradicts the fact that w is genuinely d-dimensional. □

3. The rate of growth of $\|S_{T_r}\|$. Let P^x denote the probability measure under which the random walk starts at $x \in \mathbb{R}^d$; thus $P^x(S_0 = x) = 1$. We will write P for P^0. For Borel subsets A of $\mathbb{R}^d$ and $r > 0$ let

$$U_r(A) = \sum_{k=0}^{\infty} P(\|S_k\|^* \le r,\ S_k \in A)$$

where $\|S_k\|^* = \max_{0 \le j \le k} \|S_j\|$. Thus $U_r(A)$ is the expected number of visits to A before time T_r. The following proposition is basic to our approach.

PROPOSITION 3.1. *For any $r > 0$ and any $\lambda \ge 0$,*

$$P\big(\|S_{T_r}\| - r > \lambda\big) = \int_{\|x\| \le r} P(\|X + x\| > r + \lambda)\, dU_r(x).$$

PROOF. This is a straightforward computation:

$$\begin{aligned} P&\big(\|S_{T_r}\| - r > \lambda\big) \\ &= \sum_{k=1}^{\infty} P\big(\|S_{T_r}\| > r + \lambda,\ T_r = k\big) \\ &= \sum_{k=1}^{\infty} P(\|S_k\| > r + \lambda,\ \|S_{k-1}\|^* \le r) \\ &= \sum_{k=1}^{\infty} \int_{\|x\| \le r} P\big(\|S_k\| > r + \lambda,\ \|S_{k-1}\|^* \le r,\ S_{k-1} \in dx\big) \\ &= \sum_{k=1}^{\infty} \int_{\|x\| \le r} P(\|X + x\| > r + \lambda) P\big(\|S_{k-1}\|^* \le r,\ S_{k-1} \in dx\big) \\ &= \int_{\|x\| \le r} P(\|X + x\| > r + \lambda)\, dU_r(x). \end{aligned}$$

□

Since $\int_{\|x\| \le r} dU_r(x) = ET_r$, we obtain:

COROLLARY 3.2. *For $\lambda > 1$ we have*

$$P(\|X\| > (\lambda + 1)r) ET_r \le P\left(\frac{\|S_{T_r}\|}{r} > \lambda\right) \le P(\|X\| > (\lambda - 1)r) ET_r.$$

This result is quite useful in conjunction with (1.1) and leads immediately to results on the rate of growth of $\|S_{T_r}\|$.

THEOREM 3.3. *For $0 < p \le q < \infty$ we have*

$$\sup_{r \ge 1} E \frac{\|S_{T_r}\|^p}{r^q} < \infty \quad \textit{iff} \quad \sup_{r \ge 1} \frac{1}{r^q h(r)} \int_r^{\infty} \lambda^{p-1} G(\lambda)\, d\lambda < \infty.$$

We also have that $\{\|S_{T_r}\|^p/r^q\}_{r\geq 1}$ *is uniformly integrable if and only if*

$$\lim_{\xi\to\infty}\sup_{r\geq 1}\frac{1}{r^q h(r)}\int_{(\xi r)^{q/p}}^{\infty}\lambda^{p-1}G(\lambda)\,d\lambda = 0.$$

PROOF. We have by Corollary 3.2 that

$$\begin{aligned}\frac{E\|S_{T_r}\|^p}{r^q} &= \frac{1}{r^{q-p}}\int_0^\infty p\lambda^{p-1}P\left(\frac{\|S_{T_r}\|}{r} > \lambda\right)d\lambda\\ &\leq \frac{2^p}{r^{q-p}} + r^{p-q}p\int_2^\infty \lambda^{p-1}P\left(\frac{\|S_{T_r}\|}{r} > \lambda\right)d\lambda\\ &\leq 2^p r^{p-q} + r^{p-q}p\int_2^\infty \lambda^{p-1}P\left(\frac{\|X\|}{r} > \lambda/2\right)d\lambda\, ET_r\\ &= 2^p r^{p-q} + pET_r 2^p r^{-q}\int_r^\infty \lambda^{p-1}P(\|X\| > \lambda)\,d\lambda.\end{aligned}$$

Similarly,

$$\frac{E\|S_{T_r}\|^p}{r^q} \geq \frac{pET_r}{2^p}r^{-q}\int_{2r}^\infty \lambda^{p-1}P(\|X\| > \lambda)\,d\lambda.$$

Thus the first statement of the theorem follows from (1.1) and (2.3). The statement about uniform integrability is proved by a similar estimation of $E(\|S_{T_r}\|^p/r^q;\ \|S_{T_r}\|^p > (\xi r)^q)$ for large ξ. □

As an immediate consequence of Corollary 3.2 and (2.3), we also have what may be regarded as the limiting case of Theorem 3.3 as $p = q \to 0$.

THEOREM 3.4. $\{\|S_{T_r}\|/r\}_{r\geq 1}$ *is tight if and only if*

$$\lim_{\xi\to\infty}\sup_{r\geq 1}\frac{G(\xi r)}{h(r)} = 0.$$

The final result of this section gives the necessary and sufficient condition for $\|S_{T_r}\|/r \to_p 1$ as mentioned in Section 1.

THEOREM 3.5. *We have*

$$\frac{\|S_{T_r}\| - r}{r} \to_p 0 \quad \textit{iff} \quad \lim_{r\to\infty}\frac{G(r)}{h(r)} = 0.$$

PROOF. By Proposition 3.1, for any $\varepsilon > 0$ we have

$$P\big(\|S_{T_r}\| - r > \varepsilon r\big) = \int_{\|x\| \le r} P(\|X + x\| > (1 + \varepsilon)r)\, dU_r(x)$$

$$\le \int_{\|x\| \le r} P(\|X\| > \varepsilon r)\, dU_r(x) = G(\varepsilon r) ET_r.$$

On the other hand,

$$P\big(\|S_{T_r}\| - r > r\big) = \int_{\|x\| \le r} P(\|X + x\| > 2r)\, dU_r(x)$$

$$\ge P(\|X\| > 3r) ET_r = G(3r) ET_r.$$

The desired result now follows from (1.1) and (2.3). □

4. Results related to condition (E). For the rest of this paper we will be primarily interested in stronger conditions than (1.2). In this section we will study condition (E) and in particular obtain an analytic characterization of it when (1.2) holds.

The first result is a technical strengthening of condition (E) which will be used later. The proof, involving a straightforward compactness argument, is omitted.

LEMMA 4.1. *Assume* (E); *then there exist* $\beta > 0$ *and* $\eta > 0$ *such that*

$$\liminf_{r \to \infty} \inf_{\theta \in S^{d-1}} P\big(S_{T_r} \in \Gamma_0(\theta, \beta)\big) \ge \eta. \tag{4.1}$$

We shall occasionally need a stronger exit condition than (4.1).

LEMMA 4.2. *Suppose* $EX = 0$ *and* $E\|X\|^2 < \infty$. *Then for every* $\beta \in [0, 1)$ *there exists* $\eta > 0$ *such that* (4.1) *holds.*

We will only sketch the proof since it is a standard application of an invariance principle.

Let Γ be the covariance matrix of X and $B(t)$ be Brownian motion with covariance matrix Γ; that is, B has all the usual properties of Brownian motion on $\mathbb{R}^d$ except that $B(1)$ is normally distributed with mean 0 and covariance matrix Γ. Let $Y_r(t) = (1/r)\sum_{j=1}^{[r^2 t]} X_j$. By combining Theorems 2.2.6 (page 168) and 1.6.5 (page 31) of [6], it is not difficult to show that

$$Y_r(\cdot) \to B(\cdot) \quad \text{as } r \to \infty, \quad \text{weakly on } D([0, \infty)).$$

For any path $\omega \in D([0, \infty))$, let $\tau_\omega = \inf\{t > 0 \colon \|\omega(t)\| > 1\}$. For $f \in C(\partial B(0; 1))$, let $\tilde{f}(r\theta) = f(\theta)$ be its radial extension. Then

$$E\tilde{f}\big(Y_r(\tau_{Y_r})\big) \to Ef\big(B(\tau_B)\big).$$

But $E\tilde{f}(Y_r(\tau_{Y_r})) = Ef(S_{T_r}/\|S_{T_r}\|)$. Thus the law of $S_{T_r}/\|S_{T_r}\|$ converges weakly to the exit distribution (harmonic measure) of B on $\partial B(0;1)$. It then suffices to show that this harmonic measure has a strictly positive density relative to surface measure a.e., or, by making an orthogonal linear change of variables, that ordinary harmonic measure (i.e., with respect to standard Brownian motion) has this property on the image of $B(0;1)$. Since the latter domain is easily shown to be Lipschitz, the desired result follows from the well-known result of Dahlberg [4].

Since condition (E) plays an important role in our results, the remainder of this section is devoted to obtaining a better understanding of it and its relationship to the overshoot. We begin by showing that, in dimension 1, bounds on the overshoot can imply (E).

PROPOSITION 4.3. *If* $d = 1$, $EX = 0$ *and* $E(|S_{T_r}| - r) = o(r)$, *then*

$$P(S_{T_r} > 0) \to \tfrac{1}{2}. \tag{4.2}$$

PROOF. Since $ES_{T_r} = 0$ by (2.4), we have

$$E(S_{T_r}; S_{T_r} > 0) = -E(S_{T_r}; S_{T_r} < 0).$$

Thus

$$E|S_{T_r}| = 2E(S_{T_r}; S_{T_r} > 0) \geq 2rP(S_{T_r} > 0).$$

Hence

$$\frac{E(|S_{T_r}| - r)}{r} \geq 2P(S_{T_r} > 0) - 1,$$

from which we have that

$$\limsup_{r\to\infty} P(S_{T_r} > 0) \leq \tfrac{1}{2}.$$

The same argument shows

$$\limsup_{r\to\infty} P(S_{T_r} < 0) \leq \tfrac{1}{2},$$

which proves (4.2). □

We obtain the following result by a similar use of Wald's identity.

PROPOSITION 4.4. *Suppose* $d = 1$, $EX = 0$, *and* X *is bounded below. Then*

$$\liminf_{r\to\infty} P(S_{T_r} < 0) \geq \tfrac{1}{2}.$$

PROOF. We have

$$1 \leq \frac{E|S_{T_r}|}{r} = \frac{2}{r}E(|S_{T_r}|; S_{T_r} < 0) \leq 2\frac{r + \|X1(X<0)\|_\infty}{r}P(S_{T_r} < 0),$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm. The desired result now follows by letting $r \to \infty$. □

In higher dimensions, $E\|S_{T_r}\| - r = o(r)$ does not imply condition (E). As illustrated by Example 7.2 below, this can be due to the fact that $S_{T_r}/\|S_{T_r}\|$ may have weak limit points which are not genuinely d-dimensional. To prevent this, we will later sometimes impose the following strong nondegeneracy condition:

$$\liminf_{r\to\infty} \frac{E(\langle\theta, X\rangle^2; \|X\| \le r)}{E(\|X\|^2; \|X\| \le r)} > 0 \quad \text{for all } \theta \in S^{d-1}. \tag{4.3}$$

Although we will not need it, we should point out that (4.3) is equivalent to the seemingly stronger

$$\liminf_{r\to\infty} \inf_{\theta\in S^{d-1}} \frac{E(\langle\theta, X\rangle^2; \|X\| \le r)}{E(\|X\|^2; \|X\| \le r)} > 0.$$

This is proved by a compactness argument. It is also interesting to note that $E(|S_{T_r}| - r)^p = o(r^p)$ does not imply (E) even in dimension 1 if $0 < p < 1$. This can be seen from Example 7.1.

We now come to our first result which directly links condition (E) to the analytic condition $h \approx Q$.

PROPOSITION 4.5. *Assume* (E) *holds. Then* $h \approx Q$.

PROOF. Assume there is a sequence $r_k \to \infty$ such that

$$\frac{Q(r_k)}{h(r_k)} \to 0. \tag{4.4}$$

Let H_k be the half space given by

$$H_k = \{x: \langle x, EX1(\|X\| \le r_k)\rangle < 0\}.$$

We claim that

$$P(S_{T_{r_k}} \in H_k) \to 0, \tag{4.5}$$

which combined with Lemma 4.1 proves that (E) fails. To prove (4.5), we begin by observing that for any $\zeta > 0$,

$$\begin{aligned} P(S_{T_{r_k}} \in H_k) &\le P\left(T_{r_k} > \frac{\zeta}{h(r_k)}\right) + P\left(S_{T_{r_k}} \in H_k, T_{r_k} \le \frac{\zeta}{h(r_k)}\right) \\ &\le P\left(T_{r_k} > \frac{\zeta}{h(r_k)}\right) + P\left(\|S_n\| > r_k, S_n \in H_k \text{ some } n \le \frac{\zeta}{h(r_k)}\right) \\ &\le P\left(T_{r_k} > \frac{\zeta}{h(r_k)}\right) + P\left(\|U_n\| > r_k, U_n \in H_k \text{ some } n \le \frac{\zeta}{h(r_k)}\right) \\ &\quad + P\left(\|X_n\| > r_k \text{ some } n \le \frac{\zeta}{h(r_k)}\right) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned}$$

where

$$U_n = \sum_{i=1}^{n} X_i 1(\|X_i\| \le r_k).$$

Letting $[\cdot]$ denote the greatest integer function, we have

$$\begin{aligned} \mathrm{I} &= P\big(\|S_{[\zeta/h(r_k)]}\|^* \le r_k\big) \\ &\le \frac{c}{\zeta} \end{aligned}$$

by (1.2) of [10]. Next

$$\begin{aligned} \mathrm{III} &\le \frac{\zeta}{h(r_k)} P(\|X\| > r_k) \\ &\le \zeta \frac{Q(r_k)}{h(r_k)} \to 0 \end{aligned}$$

by (4.4). Finally, by definition of H_k and Doob's inequality for nonnegative submartingales,

$$\begin{aligned} \mathrm{II} &\le P\left(\|U_n\|_E > r_k \rho^{-1}, U_n \in H_k \text{ some } n \le \frac{\zeta}{h(r_k)} \right) \\ &\le P\left(\|U_n - EU_n\|_E > r_k \rho^{-1} \text{ some } n \le \frac{\zeta}{h(r_k)} \right) \\ &\le \frac{\zeta}{h(r_k)} \rho^2 \frac{E\|X\|_E^2 1(\|X\| \le r_k)}{r_k^2} \\ &\le \frac{\zeta}{h(r_k)} \rho^4 \frac{E\|X\|^2 1(\|X\| \le r_k)}{r_k^2} \\ &\le \rho^4 \frac{\zeta Q(r_k)}{h(r_k)} \to 0 \end{aligned}$$

by (4.4). Combining the estimates for I, II and III, we have for any $\zeta > 0$,

$$\limsup_{k \to \infty} P\big(S_{T_{r_k}} \in H_k\big) \le \frac{c}{\zeta}.$$

Letting $\zeta \to \infty$ proves (4.5) and completes the proof. □

The following result, in dimension $d = 1$, can be viewed in part as a converse to Proposition 4.5 when (1.2) holds (see Corollary 4.8 for this and the higher-dimensional analog). We have not formulated it this way since the stated result is interesting in its own right, and will be used in this form in the proofs in Section 6. Note that it relates an analytic condition (4.7) which is just slightly weaker than (1.9) (see Remark 4.7 below) to a probabilistic condition (4.6) which should be contrasted with (1.8). By comparing with Theorem 3.5, it also illustrates the effect that assuming condition (E) can have.

THEOREM 4.6. *The following two statements are equivalent*:

$$\frac{\|S_{T_r}\| - r}{r} \to_p 0 \quad \text{and} \quad \text{(E) holds}, \tag{4.6}$$

$$EX = 0, \qquad \lim_{r\to\infty} \frac{G(r)}{Q(r)} = 0 \quad \text{and} \quad \text{(4.3) holds}. \tag{4.7}$$

PROOF. Suppose (4.6) holds; then combining Theorem 3.5 and Proposition 4.5, we have

$$\lim_{r\to\infty} \frac{G(r)}{Q(r)} = 0. \tag{4.8}$$

It then follows that $E\|X\| < \infty$; see [7]. If $\mu = EX$ were not 0, the one-dimensional random walk $\langle \mu, S_n \rangle$ would drift to $+\infty$, contradicting (E). To prove (4.6) $\Rightarrow$ (4.3), first note that (4.6) and Lemma 4.1 give

$$\liminf_{r\to\infty} \inf_{\theta \in S^{d-1}} E\left(\frac{\langle \theta, S_{T_r} \rangle^2}{r^2}; \|S_{T_r}\| \le 2r \right) > 0.$$

Since $\|X_j\| \le 3r$ on $\{\|S_{T_r}\| \le 2r\}$ for $j = 1, \dots, T_r$, there is a constant $c > 0$ such that for every $\theta \in S^{d-1}$ and all r large,

$$E\left(\sum_{j=1}^{T_r} \langle \theta, X_j \rangle 1(\|X_j\| \le 3r) \right)^2 \ge cr^2. \tag{4.9}$$

Next, by (4.8), Lemma 2.1, (1.1) and (2.3), we have

$$\frac{1}{r} \|E(X; \|X\| \le 3r)\| ET_r \to 0 \quad \text{as } r \to \infty.$$

It then follows from (4.9), (2.4) and (2.5) that for large r,

$$E(\langle \theta, X \rangle^2; \|X\| \le 3r) ET_r \ge cr^2.$$

Finally we deduce (4.3) from (1.1) and (2.3).

To prove the converse, (4.7) $\Rightarrow$ (4.6), first note that (1.2) holds, by Theorem 3.5, so it is enough to show that (E) holds. For this it is enough to show that for any given $\theta \in S^{d-1}$ there exists $\delta > 0$ such that

$$\liminf_{r\to\infty} P(\langle \theta, S_{T_r} \rangle > \delta r) > 0. \tag{4.10}$$

Define $\hat{X}_j$ by

$$\hat{X}_j = \begin{cases} X_j, & \text{if } \|X_j\| \le 3r \\ 3r\dfrac{X_j}{\|X_j\|}, & \text{if } \|X_j\| > 3r \end{cases}$$

and set $\hat{S}_n = \sum_{j=1}^n \hat{X}_j$. Note that S_n and $\hat{S}_n$ have the same exit time from $B(0;r)$ and

$$P\left(\hat{S}_{T_r} \ne S_{T_r}\right) \le P\left(\|S_{T_r}\| > 2r\right) \to 0 \tag{4.11}$$

by (1.2). Thus it suffices to prove (4.10) with S_{T_r} replaced by $\hat{S}_{T_r}$. Observe also that by (1.2), $\hat{S}_{T_r}/r$ and $S_{T_r}/\|S_{T_r}\|$ have the same weak limit points. Now by (2.4),

$$\frac{\|E\hat{S}_{T_r}\|}{r} \le (3G(3r) + 3M(3r))ET_r \to 0 \tag{4.12}$$

by (1.1) and Lemma 2.1. Thus by (2.5) and (4.12),

$$\begin{aligned} \frac{E\langle\theta, \hat{S}_{T_r}\rangle^2}{r^2} &\ge \frac{\operatorname{Var}(\langle\theta, \hat{X}\rangle)}{r^2}ET_r = \frac{E\langle\theta, \hat{X}\rangle^2}{r^2}ET_r + o(1) \\ &\ge c\frac{E(\langle\theta, X\rangle^2; \|X\| \le 3r)}{r^2h(r)} + o(1) \end{aligned}$$

as $r \to \infty$. Hence by Lemma 2.1, (2.3) and (4.3), there is a constant $c > 0$ so that

$$E\frac{\langle\theta, \hat{S}_{T_r}\rangle^2}{r^2} \ge c$$

for all r sufficiently large. Since also $\|\hat{S}_{T_r}\|/r \le 4$, it follows from a well-known reverse Chebyshev inequality (see, e.g., [7], page 152) that for each $\theta \in S^{d-1}$ there exists a $\delta > 0$ such that

$$\liminf_{r\to\infty} P\left(|\langle\theta, \hat{S}_{T_r}\rangle| > \delta r\right) > 0. \tag{4.13}$$

But by (4.12) any weak limit point Y of $\hat{S}_{T_r}/r$ must satisfy $EY = 0$. This together with (4.11) and (4.13) forces (4.10) to hold, since if not there would exist a weak limit Y and a $\theta \in S^{d-1}$ such that $\langle Y, \theta\rangle$ is nondegenerate, $E\langle Y, \theta\rangle = 0$ and $P(\langle Y, \theta\rangle > 0) = 0$. □

REMARK 4.7. Recall that a one-dimensional random variable X belongs to the domain of attraction of the normal ($X \in D(2)$) if and only if $\lim_{r\to\infty}(G(r)/Q(r)) = 0$. Thus, in dimension $d = 1$, Theorem 4.6 asserts that $X \in D(2)$, $EX = 0$ is equivalent to (4.6). In higher dimensions (4.7) is equivalent to the existence of a sequence a_n such that S_n/a_n is tight and all subsequential limits are genuinely d-dimensional Gaussian.

COROLLARY 4.8. *In the presence of condition* (1.2), *condition* (E) *is equivalent to* $h \approx Q$ *and* (4.3).

This follows from Proposition 4.5, Theorems 4.6 and 3.5 and the remark in Section 1 that M dominates when $EX \neq 0$.

COROLLARY 4.9. *Suppose* (4.6) *holds. Then every subsequential weak limit of* $S_{T_r}/\|S_{T_r}\|$ *is genuinely d-dimensional and has mean* 0.

This follows from the proof of Theorem 4.6.

We have characterized (1.2) analytically in Theorem 3.5 and (1.2) together with (E) in Theorem 4.6. It is an interesting and natural problem to characterize (E) analytically by itself.

5. Bounds on occupation times.

For $0 \le a < b \le r$ let $U_r(a,b)$ denote the expected occupation time of the annulus $a \le \|x\| \le b$ by S_n prior to time T_r. Thus

$$U_r(a,b) = \int_{a \le \|x\| \le b} dU_r(x).$$

In this section we obtain upper and lower bounds on this and related quantities.

PROPOSITION 5.1. *Assume* (E); *then for every* $r > 0$ *and every* $0 \le u \le r$, *we have*

$$U_r(r-u,r) \le \frac{cP(\exists\, n < T_r, \|S_n\| \in [r-u,r])}{h(u)}$$

where c is independent of u and r.

PROOF. For notational convenience, assume that the random walk is given as the canonical process on sequence space $(\mathbb{R}^d)^\infty$; thus $S_k(w) = w(k)$ for $w \in (\mathbb{R}^d)^\infty$. Fix $r > 0$ and $0 \le u \le r$. Choose $\beta > 0$ as in Lemma 4.1. Let $\gamma = \rho^2\beta^{-1}$ as in Remark 2.7. Let

$$\tau = \inf\{n \ge 0\colon r-u \le \|S_n\| \le r\},$$

$$\sigma = \inf\{n \ge 0\colon \|S_n - S_0\| > \gamma u\}$$

and define

$$\tau_1 = \tau,$$

$$\sigma_1 = (\tau \circ \theta_{\tau_1}) + \tau_1,$$

$$\tau_{k+1} = (\tau \circ \theta_{\sigma_k}) + \sigma_k,$$

$$\sigma_{k+1} = (\sigma \circ \theta_{\tau_{k+1}}) + \tau_{k+1},$$

where θ_n are the usual shift operators defined by $(\theta_n w)(k) = w(k+n)$. Observe that

$$\begin{aligned}
U_r(r-u,r) &= E\sum_{n=0}^{T_r} 1_{[r-u,r]}(\|S_n\|) \\
&= E\sum_{k=1}^{\infty}\sum_{\tau_k\wedge T_r}^{\sigma_k\wedge T_r} 1_{[r-u,r]}(\|S_n\|) \\
&\le \sum_{k=1}^{\infty} E\left(\sum_{\tau_k}^{\sigma_k} 1_{[r-u,r]}(\|S_n\|);\, T_r > \tau_k\right) \\
&\le \sum_{k=1}^{\infty} E(\sigma_k - \tau_k;\, T_r > \tau_k) \\
&= \sum_{k=1}^{\infty} E\big(E^{S_{\tau_k}}\sigma;\, T_r > \tau_k\big) \qquad \text{(strong Markov property)} \\
&= \sum_{k=1}^{\infty} ET_{\gamma u} P(T_r > \tau_k).
\end{aligned}$$

Now again by the Markov property, for any $j \ge 1$,

$$\begin{aligned}
P(T_r > \tau_{j+1}) &\le P(T_r > \sigma_j) \\
&= E\big(P^{S_{\tau_j}}(T_r > \sigma);\, T_r > \tau_j\big).
\end{aligned}$$

But if $r - u \le \|x\| \le r$,

$$\begin{aligned}
P^x(T_r > \sigma) &= 1 - P^x(T_r \le \sigma) \\
&\le 1 - P^x(\|S_\sigma\| > r) \\
&\le 1 - P^x(S_\sigma \in \Gamma_x(\theta,\beta,\gamma u)),
\end{aligned}$$

where θ is chosen as in Remark 2.7. Now by Lemma 4.1 there exists $\eta > 0$ such that

$$P^x(S_\sigma \in \Gamma_x(\theta,\beta,\gamma u)) = P^0\big(S_{T_{\gamma u}} \in \Gamma_0(\theta,\beta)\big) \ge \eta,$$

provided u is sufficiently large. In that case

$$P(T_r > \tau_{j+1}) \le (1-\eta)P(T_r > \tau_j).$$

Hence for u sufficiently large, say $u \ge u_0$,

$$\begin{aligned}
U_r(r-u,r) &\le ET_{\gamma u} P(T_r > \tau_1)\sum_{k=1}^{\infty}(1-\eta)^{k-1} \\
&= \frac{P(T_r > \tau_1)}{\eta} ET_{\gamma u} \\
&\le \frac{cP(T_r > \tau_1)}{h(u)}
\end{aligned}$$

by (1.1) and (2.3). Finally, if $0 \le u \le u_0$, then a slight modification of the above argument gives

$$\begin{aligned} U_r(r-u,r) &\le \frac{cP(T_r > \tau_1)}{h(u_0)} \\ &\le c\left(\sup_{0\le u\le u_0} \frac{h(u)}{h(u_0)}\right)\frac{P(T_r>\tau_1)}{h(u)} \\ &= c\frac{P(T_r>\tau_1)}{h(u)}. \end{aligned}$$ □

LEMMA 5.2. *Assume $EX = 0$, $E\|X\|^2 < \infty$ and that $\{\|S_{T_r}\| - r\}_{r>0}$ is tight. Fix $\beta \in [0,1)$. Then for all $\theta \in S^{d-1}$ and $0 \le u \le r$,*

$$\int_{\substack{\|x\|\in[r-u,r]\\ x\in\Gamma_0(\theta,\beta)}} dU_r(x) > \frac{c}{h(u)},$$

where $c > 0$ depends on β but not on θ, u or r, provided u and $r-u$ are sufficiently large.

PROOF. Fix $\beta < \beta_1 < 1$ and let $\Gamma_1 = \Gamma_0(\theta, \beta_1)$. By Lemma 4.2 there exists $\delta > 0$ such that

$$P\big(S_{T_{r-2u/3}} \in \Gamma_1\big) \ge \delta,$$

provided $r - 2u/3$ is sufficiently large. By tightness,

$$P\left(\|S_{T_{r-2u/3}}\| - \left(r - \frac{2u}{3}\right) \ge \frac{u}{3}\right) \le \frac{\delta}{2},$$

provided $r - 2u/3$ and $u/3$ are sufficiently large. Hence if u and $r-u$ are large,

$$(5.1)\quad P\left(S_n \in \Gamma_1 \text{ and } \|S_n\| \in \left(r - \frac{2u}{3}, r - \frac{u}{3}\right) \text{ for some } n < T_r\right) \ge \frac{\delta}{2}.$$

Let

$$A = \left\{x \in \Gamma_1 \colon \|x\| \in \left(r - \frac{2u}{3}, r - \frac{u}{3}\right)\right\}$$

and

$$\Gamma_0(\theta,\beta; r-u, r) = \Gamma_0(\theta,\beta) \cap \{y \colon r - u \le \|y\| \le r\}.$$

Then for $x \in A$ a little geometry shows that for some $\zeta > 0$, independent of r and u,

$$B(x;\zeta u) \subseteq \Gamma_0(\theta,\beta; r-u, r).$$

Set $T_A = \inf\{n: S_n \in A\}$. Then by the Markov property,

$$\begin{aligned}
\int_{\substack{\|x\|\in[r-u,r]\\ x\in\Gamma_0(\theta,\beta)}} dU_r(x)
&\geq \int_A \sum_{n=1}^{\infty} P^x(S_n \in \Gamma_0(\theta,\beta; r-u, r); n < T_r) P(S_{T_A} \in dx; T_A < T_r)\\
&\geq \int_A \sum_{n=1}^{\infty} P^x(S_n \in B(x; \zeta u); n < T_r) P(S_{T_A} \in dx; T_A < T_r)\\
&\geq ET_{\zeta u} P(T_A < T_r) \geq \frac{c}{h(u)}
\end{aligned}$$

by (1.1), (2.3) and (5.1). □

REMARK 5.3. The tightness assumption may be omitted in Lemma 5.2 since, as we show later in the proof of Theorem 1.3, it follows from the other assumptions.

If we had Theorem 1.3 at our disposal, the next proposition would be an immediate consequence of Lemma 5.2, since the hypotheses are equivalent. However, in order to prove Theorem 1.3 we need the following result.

PROPOSITION 5.4. *Assume* $\{\|S_{T_r}\| - r\}_{r>0}$ *is tight and* (E) *holds. Let* $\theta_0 \in S^{d-1}$ *and* $\beta_0 \in [0,1)$ *be given. Then there exist a sequence* $R_n \to \infty$ *and constants* $c > 0$ *and* $u_0 > 0$ *so that*

$$\int_{\substack{\|x\|\in[R_n-u,R_n]\\ x\in\Gamma_0(\theta_0,\beta_0)}} dU_{R_n}(x) \geq \frac{c}{h(u)}, \qquad u_0 \leq u \leq \frac{R_n}{2}.$$

PROOF. Choose β_1, β_2 such that $\beta_0 < \beta_1 < \beta_2 < 1$, and let $\Gamma_1 = \Gamma_0(\theta, \beta_1)$. We will construct the sequence R_n so that for some $\delta > 0$ and all n,

$$P(T_{R_n} = \tau_n) \geq \delta, \tag{5.2}$$

where

$$\tau_n = \min\{n: S_n \notin D_n\}$$

and

$$D_n = B\left(0; \frac{R_n}{2}\right) \cup \left(\Gamma_1 \cap \overline{B(0; R_n)}\right).$$

As in the proof of Lemma 5.2, this together with tightness shows that for u and $R_n - u$ sufficiently large and $u \leq R_n/2$,

$$P(S_n \in A \text{ some } n < T_{R_n}) \geq \frac{\delta}{2},$$

where $A = \{x \in \Gamma_1: \|x\| \in (R_n - 2u/3, R_n - u/3)\}$ as before. The proof of the proposition is then completed exactly as in Lemma 5.2.

To prove (5.2), let $r'_n \uparrow \infty$ be such that $S_{T_{r'_n}}/\|S_{T_{r'_n}}\| \to_d Y$, for some $\partial B(0;1)$-valued random variable Y. Note that $S_{T_{r'_n}}/r'_n \to_d Y$ also, by the tightness assumption. Since condition (E) holds, Y is genuinely d-dimensional. Also $EY = 0$ by Corollary 4.9. Let w be the probability distribution of Y and S the support of w. Then by Lemma 2.8 the convex hull of S contains an open neighborhood of 0. Choose a vector $v_0 \neq 0$ in this neighborhood such that $v_0/\|v_0\| = \theta_0$.

There is a finite collection $\theta_1, \dots, \theta_m \in S \subseteq \partial B(0;1)$, and $p_j \geq 0$, $j = 1, 2, \dots, m$, with $\sum_{j=1}^m p_j = 1$, such that $v_0 = \sum_{j=1}^m p_j \theta_j$. By considering rational number approximations to the p_j, it is easy to see that there are positive integers $\alpha_1, \dots, \alpha_m$ and M, and a vector v (close to v_0) so that $v \in \Gamma_0(\theta_0, \beta_2)$, $\sum_{j=1}^m (\alpha_j/M) = 1$ and $\sum_{j=1}^m (\alpha_j/M)\theta_j = v$. To simplify the notation, repeat each θ_j, α_j times so that we have

$$\frac{1}{M} \sum_{j=1}^{M} \theta_j = v. \tag{5.3}$$

Let $r_n = 2Mr'_n$.

CLAIM. There exists $p > 0$ so that $P(S_k \in \Gamma_0(\theta_0, \beta_2)$ and $\|S_k\| \geq (\|v\|/4) r_n$ for some $k < T_{r_n}) \geq p$.

To see this, by (5.3) and continuity we may choose neighborhoods U_i of θ_i such that $U_i \subseteq B(0;2)$ and such that if $\zeta_i \in U_i$ are any vectors, $i = 1, 2, \dots, M$, then

$$\sum_{j=1}^{M} \zeta_j \in \Gamma_0(\theta_0, \beta_2) \quad \text{and} \quad \frac{1}{2}\|v\| < \left\| \frac{1}{M} \sum_{j=1}^{M} \zeta_j \right\| < 2\|v\|.$$

Since the θ_i belong to the support of w, there is a number $\xi > 0$ such that

$$P\big(S_{T_{r'_n}}/r'_n \in U_i\big) \geq \xi, \qquad i = 1, 2, \dots, M, \tag{5.4}$$

for all large n. By omitting the first few terms in the sequence r'_n, if necessary, we may assume (5.4) holds for all n.

For a fixed n define stopping times $\mu_0, \mu_1, \dots, \mu_M$ by $\mu_0 = 0$, $\mu_j = T_{r'_n} \circ \theta_{\mu_{j-1}} + \mu_{j-1}$ for $1 \leq j \leq M$. On the event

$$E = \left\{ \left(S_{\mu_j} - S_{\mu_{j-1}}\right)/r'_n \in U_j,\ j = 1, 2, \dots, M \right\},$$

we have $S_{\mu_M} \in \Gamma_0(\theta_0, \beta_2)$ and $\|S_{\mu_M}\|/r_n \geq \|v\|/4$. Also, for any $j \leq \mu_M$, $\|S_j\| < 2r'_n M = r_n$. Thus $\mu_M < T_{r_n}$ on E. The claim follows since by independence, $P(E) \geq \xi^M$.

To complete the proof of (5.2), we may assume β_2 has been chosen close enough to 1 so that for some $\alpha > 0$,

$$x, y \in \Gamma_0(\theta_0, \beta_2) \Rightarrow \|x + y\| \geq \|x\| + \alpha\|y\| \tag{5.5}$$

(see Lemma 2.5). Choose A so large that if $x \in \Gamma_0(\theta_0, \beta_2)$ and $\|x\| \geq Ar_n/2 - r_n$, then

$$B(x, r_n) \subseteq \Gamma_0(\theta_0, \beta_1). \tag{5.6}$$

We will show that if $R_n = Ar_n$ and $L = [4A/\alpha\|v\|] + 1$, then for all n, (5.2) holds with $\delta = p^L$, where p is as in the previous claim. To see this, define for fixed n a sequence of stopping times $\nu_0, \nu_1, \dots,$ as follows; recalling (2.2), let

$$\nu = \inf\left\{k\colon S_k - S_0 \in \Gamma_0\left(\theta_0, \beta_2, \frac{\|v\|}{4} r_n\right)\right\}, \qquad \inf \phi = +\infty,$$

then set $\nu_0 = 0$ and for $j \geq 1$,

$$\nu_j = \begin{cases} \nu \circ \theta_{\nu_{j-1}} + \nu_{j-1}, & \text{on } \{\nu_{j-1} < \infty\} \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $\Gamma_0(\theta_0, \beta_2)$ is closed under vector addition, we have that $S_{\nu_j} \in \Gamma_0(\theta_0, \beta_2)$ for any j such that $\nu_j < \infty$. It then follows from (5.5) that $T_{R_n} \leq \nu_L$.

Let $E_0 = \{\nu_j < T_{r_n} \circ \theta_{\nu_{j-1}} + \nu_{j-1},\ j = 1, \dots, L\}$. Then by the strong Markov property $P(E_0) \geq p^L$. Now $\nu_L < \infty$ on E_0. Also on E_0, if $\|S_k\| \geq R_n/2$ for some $k < \nu_L$, then $\|S_{\nu_j}\| \geq R_n/2 - r_n$, where $0 \leq j < L$ satisfies $\nu_j \leq k < \nu_{j-1}$. Thus $S_k \in \Gamma_0(\theta_0, \beta_1)$ by (5.6). Hence

$$\begin{aligned} E_0 &\subseteq \left\{S_k \in \left(B\left(0; \frac{R_n}{2}\right) \cup \Gamma_0(\theta_0, \beta_1)\right) \text{ all } k < \nu_L; \nu_L < \infty\right\} \\ &\subseteq \{T_{R_n} = \tau_n\}, \end{aligned}$$

which completes the proof of (5.2). □

6. The rate of growth of $(\|S_{T_r}\| - r)$.

We begin with an important lemma which enables us to estimate the moments of $\|S_{T_r}\| - r$.

LEMMA 6.1. *For any* $p > 0$,

$$\begin{aligned} E\big(\|S_{T_r}\| - r\big)^p \leq{}& E(\|X\|^p U_r(r - \|X\|, r); \|X\| \leq r) \\ &+ ET_r E(\|X\|^p; \|X\| > r). \end{aligned} \tag{6.1}$$

On the other hand, let Γ *be any cone with vertex at* 0 *such that*

$$\|x + y\| \geq \|x\| + \alpha\|y\|, \qquad x, y \in \Gamma.$$

Then

$$\begin{aligned} E\big(\|S_{T_r}\| - r\big)^p \geq{}& \int_0^r p\lambda^{p-1} P(\alpha\|X\| > 2\lambda,\ X \in \Gamma) \int_{\substack{\|x\| \in [r-\lambda, r] \\ x \in \Gamma}} dU_r(x)\, d\lambda \\ &+ ET_r \int_r^\infty p\lambda^{p-1} P(\|X\| > 3\lambda)\, d\lambda. \end{aligned} \tag{6.2}$$

PROOF. Integrating the result of Proposition 3.1, we have

$$E\big(\|S_{T_r}\| - r\big)^p = \int_{\|x\| \leq r} \int_{\lambda = 0}^\infty p\lambda^{p-1} P(\|X + x\| > \lambda + r)\, d\lambda\, dU_r(x). \tag{6.3}$$

Letting $L(x) = P(\|X\| \le x)$, we have

$$\begin{aligned}
E\big(\|S_{T_r}\| - r\big)^p &\le \int_{\|x\| \le r} \int_{\lambda=0}^{\infty} p\lambda^{p-1} P(\|X\| > \lambda + r - \|x\|)\, d\lambda\, dU_r(x) \\
&= \int_{\|x\| \le r} \int_{\lambda=0}^{\infty} p\lambda^{p-1} \int_{u > \lambda + r - \|x\|} dL(u)\, d\lambda\, dU_r(x) \\
&= \int_{\|x\| \le r} \int_{u > r - \|x\|} \int_{\lambda=0}^{u - r + \|x\|} p\lambda^{p-1}\, d\lambda\, dL(u)\, dU_r(x) \\
&\le \int_{\|x\| \le r} \int_{u > r - \|x\|} u^p\, dL(u)\, dU_r(x) \\
&= \int_{u=0}^{r} u^p \int_{r - u < \|x\| \le r} dU_r(x)\, dL(u) \\
&\quad + \int_{u > r} u^p \int_{0 \le \|x\| \le r} dU_r(x)\, dL(u) \\
&\le \int_{u=0}^{r} u^p U_r(r - u, r)\, dL(u) + ET_r \int_{u > r} u^p\, dL(u),
\end{aligned}$$

which is (6.1).

For (6.2), we divide the range of the λ integration in (6.3) into two parts: $[0, r]$ and (r, ∞). The latter integral becomes

$$\begin{aligned}
&\int_{\|x\| \le r} \int_{r}^{\infty} p\lambda^{p-1} P(\|X + x\| > \lambda + r)\, d\lambda\, dU_r(x) \\
&\qquad \ge \int_{\|x\| \le r} \int_{r}^{\infty} p\lambda^{p-1} P(\|X\| > \lambda + r + \|x\|)\, d\lambda\, dU_r(x),
\end{aligned}$$

which dominates the last term in (6.2).

For the other range of integration, we have

$$\begin{aligned}
&\int_{\|x\| \le r} \int_{0}^{r} p\lambda^{p-1} P(\|X + x\| > \lambda + r)\, d\lambda\, dU_r(x) \\
&\qquad \ge \int_{\substack{\|x\| \le r \\ x \in \Gamma}} \int_{0}^{r} p\lambda^{p-1} P(\alpha\|X\| > \lambda + r - \|x\|,\ X \in \Gamma)\, d\lambda\, dU_r(x) \\
&\qquad \ge \int_{\substack{\|x\| \le r \\ x \in \Gamma}} \int_{\lambda = r - \|x\|}^{r} p\lambda^{p-1} P(\alpha\|X\| > 2\lambda,\ X \in \Gamma)\, d\lambda\, dU_r(x) \\
&\qquad \ge \int_{\lambda=0}^{r} p\lambda^{p-1} P(\alpha\|X\| > 2\lambda,\ X \in \Gamma) \int_{\substack{\|x\| \in [r - \lambda, r] \\ x \in \Gamma}} dU_r(x)\, d\lambda. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 1.1. First assume that (1.4) holds. Since this implies (4.6) we have that $EX = 0$ by Theorem 4.6. Next observe that

$$ET_r \int_r^{2r} p\lambda^{p-1} P(\|X\| > 3\lambda)\, d\lambda \geq (2^p - 1) ET_r r^p P(\|X\| > 6r),$$

thus by (6.2),

$$\limsup_{r\to\infty} ET_r r^{p-q} P(\|X\| > 6r) < \infty. \tag{6.4}$$

Since $EX = 0$, we have by Lemma 2.4,

$$\liminf_{r\to\infty} \frac{ET_r}{r} > 0.$$

Thus by (6.4), $X \in WL^{1+p-q}$. If $1 + (p-q) < 2$, then by Lemma 2.4,

$$\liminf_{r\to\infty} \frac{ET_r}{r^{1+p-q}} > 0.$$

Thus by (6.4), $X \in WL^{1+2(p-q)}$. Iterating this procedure, we obtain $X \in WL^{1+m(p-q)}$, where

$$m = \min\{j: 1 + j(p-q) \geq 2\}.$$

If $1 + m(p-q) > 2$, then $X \in L^2$. If $1 + m(p-q) = 2$, then since $1 + (m - \frac{1}{2})(p-q) < 2$, we can use the above argument to get $X \in WL^{1+(m+1/2)(p-q)}$ and hence again $X \in L^2$. Now use Lemma 2.4(ii) together with (6.4) to obtain $X \in WL^{2+p-q}$.

Now assume (1.5). Since $q < p$ we have $X \in L^2$, hence (E) follows from Lemma 4.2. Furthermore, when $X \in L^2$ and $EX = 0$, it is easily seen (using Lemma 2.1, for example) that $h(u) \sim E\|X\|^2 u^{-2}$. It then follows from (6.1), (1.1) and Proposition 5.1 that

$$E\big(\|S_{T_r}\| - r\big)^p \leq cE(\|X\|^{2+p}; \|X\| \leq r) + cr^2 E(\|X\|^p; \|X\| > r). \tag{6.5}$$

A straightforward calculation shows that the right-hand side is $O(r^q)$ as $r \to \infty$. □

Before moving on to the proof of Theorem 1.2, we will give the analog of Theorem 1.1 for $2 \leq q < p$. Since the proof works equally well for $q = p$, we include this case also, although it can be easily deduced from Theorem 3.3.

THEOREM 6.2. *If $2 \leq q \leq p$, then the following are equivalent*:

(i) $E(\|S_{T_r}\| - r)^p = O(r^q)$ *and* (E) *holds.*
(ii) $E(\|S_{T_r}\| - r)^p = o(r^2)$ *and* (E) *holds.*
(iii) $X \in L^p$ *and* $EX = 0$.

PROOF. We have by (6.2),

$$E\big(\|S_{T_r}\| - r\big)^p \geq ET_r \int_{\lambda=r}^{\infty} p\lambda^{p-1} P(\|X\| > 3\lambda)\, d\lambda.$$

Thus if $E(\|S_{T_r}\| - r)^p < \infty$ for any r, then $X \in L^p$. Since $p \geq 2$, $\mu = EX$ exists. If $\mu \neq 0$, then $\langle \mu, S_n \rangle$ would drift to $+\infty$, forcing (E) to fail. Hence each of the first two conditions implies the third.

Now assume $X \in L^p$ and $EX = 0$. Since $p \geq 2$, (E) holds by Lemma 4.2 and we have as in the proof of Theorem 1.1,

$$E\big(\|S_{T_r}\| - r\big)^p \leq cE(\|X\|^{2+p}; \|X\| \leq r) + cr^2 E(\|X\|^p; \|X\| > r) = o(r^2).$$

Since $q \geq 2$, this completes the proof. □

Proof of Theorem 1.2. If $X \in L^{2+p}$ and $EX = 0$, then (E) holds by Lemma 4.2. Hence, as in the proof of Theorem 1.1, we have (6.5) holds. Then

$$E\big(\|S_{T_r}\| - r\big)^p \leq cE\|X\|^{2+p} < \infty.$$

On the other hand, if (1.6) holds, then $\|S_{T_r}\| - r$ is tight. Also by Theorem 1.1, $X \in L^2$ and $EX = 0$. Now let $\{\Gamma_j\}_{j=1}^m$ be the covering introduced in Lemma 2.5. Then by Lemma 5.2 there is a constant c so that

$$\int_{\substack{\|x\| \in [r-\lambda, r] \\ x \in \Gamma_j}} dU_r(x) \geq \frac{c}{h(\lambda)},$$

provided λ and $r - \lambda$ are sufficiently large. Thus by (6.2) there is a λ_0 so that for all large r,

$$\begin{aligned} E\big(\|S_{T_r}\| - r\big)^p &\geq \frac{1}{m}\sum_{j=1}^m \int_{\lambda_0}^{r-\lambda_0} P\big(\alpha\|X\| > 2\lambda, X \in \Gamma_j\big)\frac{c}{h(\lambda)} p\lambda^{p-1}\, d\lambda \\ &\geq c\int_{\lambda_0}^{r-\lambda_0} P\left(\|X\| > \frac{2}{\alpha}\lambda\right)\lambda^{p+1}\, d\lambda, \end{aligned}$$

since $h(\lambda) \sim \lambda^{-2}E\|X\|^2$. Letting $r \to \infty$, we conclude $E\|X\|^{2+p} < \infty$. □

Proof of Theorem 1.3. If $X \in L^2$ and $EX = 0$, then (E) holds by Lemma 4.2. Recalling that $L(x) = P(\|X\| \leq x)$, by Proposition 3.1,

$$\begin{aligned} P\big(\|S_{T_r}\| - r > \lambda\big) &\leq \int_{\|x\| \leq r} P(\|X\| > \lambda + r - \|x\|)\, dU_r(x) \\ &= \int_{\|x\| \leq r}\int_{u > \lambda + r - \|x\|} dL(u)\, dU_r(x) \\ &= \int_{u=\lambda}^{r+\lambda}\int_{\|x\| \in (\lambda + r - u, r]} dU_r(x)\, dL(u) \\ &\quad + \int_{u > r+\lambda}\int_{\|x\| \leq r} dU_r(x)\, dL(u) \\ &\leq c\int_{\lambda}^{r+\lambda}\frac{dL(u)}{h(u-\lambda)} + cr^2 P(\|X\| > r + \lambda), \end{aligned}$$

the last inequality following from Proposition 5.1.

Now since $h \geq Q$ and Q is nonincreasing, we obtain

$$\sup_r P\big(\|S_{T_r}\| - r > \lambda\big) \leq c\int_{u=\lambda}^{\infty} \frac{dL(u)}{Q(u)} + c\sup_r r^2 P(\|X\| > r + \lambda).$$

But $X \in L^2$ implies $Q(u) \sim u^{-2}E\|X\|^2$, which means

$$\int_{u=\lambda}^{\infty} \frac{dL(u)}{Q(u)} \to 0 \quad \text{as } \lambda \to \infty.$$

Also, $X \in L^2$ implies

$$\begin{aligned}\sup_r r^2 P(\|X\| > r + \lambda) &\leq \sup_r \frac{r^2}{(r+\lambda)^2} E\big(\|X\|^2; \|X\| > r + \lambda\big)\\ &\leq E\big(\|X\|^2; \|X\| > \lambda\big) \to 0\end{aligned}$$

as $\lambda \to \infty$, which completes the proof of the implication (1.9) $\Rightarrow$ (1.8).

For the proof of the reverse implication, (1.8) $\Rightarrow$ (1.9), fix any cone $\Gamma = \Gamma(\theta_0, \beta_0)$ among the covering $\{\Gamma_j\}$ of Lemma 2.5. Let $R_n \uparrow \infty$, u_0 and c be as in Proposition 5.4 for the chosen Γ. Let $L_\Gamma(u) = P(\alpha\|X\| \leq u,\ X \in \Gamma)$. Then by Proposition 3.1,

$$\begin{aligned}&P\big(\|S_{T_{R_n}}\| - R_n > \lambda\big)\\
&\quad\geq \int_{\substack{\|x\| \leq R_n\\ x \in \Gamma}} P\big(\|X + x\| > \lambda + R_n,\ X \in \Gamma\big)\, dU_{R_n}(x)\\
&\quad\geq \int_{\substack{\|x\| \leq R_n\\ x \in \Gamma}} P\big(\alpha\|X\| > \lambda + R_n - \|x\|,\ X \in \Gamma\big)\, dU_{R_n}(x)\\
&\quad= \int_{\substack{\|x\| \leq R_n\\ x \in \Gamma}} \int_{u > \lambda + R_n - \|x\|} dL_\Gamma(u)\, dU_{R_n}(x)\\
&\quad\geq \int_{u=\lambda}^{R_n+\lambda} \int_{\substack{\|x\| \in (\lambda + R_n - u,\, R_n]\\ x \in \Gamma}} dU_{R_n}(x)\, dL_\Gamma(u)\\
&\quad\geq c\int_{u=2\lambda}^{(R_n+\lambda)/2} \frac{dL_\Gamma(u)}{h(u-\lambda)}\end{aligned}$$

for $\lambda > u_0$ by Proposition 5.4. Letting $n \to \infty$, we obtain

$$1 \geq c\int_{u=2\lambda}^{\infty} \frac{dL_\Gamma(u)}{h(u-\lambda)}. \tag{6.6}$$

Now $v^2Q(v)$ is nondecreasing, hence $Q(u-\lambda) \leq 4Q(u)$ for $u \geq 2\lambda$. Thus Proposition 4.5 and (6.6) imply that for $\lambda > u_0$,

$$1 \geq c\int_{u=2\lambda}^{\infty} \frac{dL_\Gamma(u)}{Q(u)}. \tag{6.7}$$

Now recall that there is a covering of $\mathbb{R}^d$ by cones $\Gamma_1, \dots, \Gamma_m$ such that (6.7) holds for each cone. One may then apply the above argument to each cone, provided λ exceeds the largest u_0 from the applications of Proposition 5.4 to each of the various cones. Summing the estimates obtained, we have for all large enough λ,

$$1 \geq \frac{1}{m}\sum_{j=1}^{m} c\int_{u=2\lambda}^{\infty} \frac{dL_{\Gamma_j}(u)}{Q(u)} \geq c\int_{2\lambda}^{\infty} \frac{P(\alpha\|X\| \in du)}{Q(u)}.$$

We conclude that $EQ(\alpha\|X\|)^{-1} < \infty$, and hence $EQ(\|X\|)^{-1} < \infty$. Hence $X \in L^2$ by Lemma 2.2. Finally, $EX = 0$ by Theorem 4.6. □

7. Examples. The first example shows that condition (E) cannot be omitted in (1.8) or in (4.6), even if mean 0 is assumed as a side condition. It also shows that $E(|S_{T_r}| - r)^p = o(r^p)$ for $0 < p < 1$ does not imply (E); cf. Proposition 4.3.

EXAMPLE 7.1. Let $d = 1$. Let X be bounded below, have mean 0 and satisfy

$$P(X > \lambda) = \frac{1}{\lambda(\log \lambda)^2}, \qquad \lambda \geq 3.$$

Then it is easy to check that for large r,

$$\begin{aligned} M(r) &= r^{-1}|E(X; |X| \leq r)| = r^{-1}|E(X; |X| > r)| \\ &= r^{-1}\int_r^{\infty} P(X > x)\, dx + P(X > r) \sim (r \log r)^{-1}, \end{aligned}$$

whereas $Q(r) \sim 2(r \log^2 r)^{-1}$. By Theorem 3.5, $(|S_{T_r}| - r)/r \to_p 0$. But $X \notin D(2)$, so (E) cannot be omitted from (4.6).

We claim that in fact $\{|S_{T_r}| - r\}_{r \geq 0}$ is tight. Since X is bounded below this follows if we can show $\lim_{r \to \infty} P(S_{T_r} < 0) = 1$.

Let $\hat{X}_j$, $\hat{S}_n = \sum_{j=1}^{n} \hat{X}_j$, be the truncations introduced above just before statement (4.11). Then by (2.5),

$$\operatorname{Var}\left(\frac{\hat{S}_{T_r}}{r}\right) = \frac{\operatorname{Var}(\hat{X})}{r^2} ET_r \leq \frac{cQ(3r)}{h(r)} \to 0.$$

Thus any weak limit point of $\hat{S}_{T_r}/r$, hence of S_{T_r}/r, must be concentrated on $+1$ or -1. In fact, the latter must be true by Proposition 4.4. Since $X \notin L^2$, condition (E) cannot be omitted in (1.8).

It is also interesting to note that $r^{-p}E(|S_{T_r}| - r)^p \to 0$ for $0 < p < 1$ since $\{r^{-p}|S_{T_r}|^p\}_{r \geq 1}$ is uniformly integrable by Theorem 3.3. Thus the conditions $EX = 0$ and $E(|S_{T_r}| - r)^p = o(r^p)$ do not imply (E) when $0 < p < 1$, in contrast to the case $p \geq 1$ (see Proposition 4.3).

The next example shows that condition (4.3) cannot be omitted in (4.7).

EXAMPLE 7.2. Let $d = 2$ and $\| \ \|$ be the box norm on $\mathbb{R}^2$. Let $X = (X^{(1)}, X^{(2)})$, where $EX^{(2)} = EX^{(1)} = 0$, $E|X^{(2)}|^2 < \infty$, $E|X^{(1)}|^2 = \infty$, but $X^{(1)} \in D(2)$. Let $T_r^{(i)}$, $i = 1, 2$, denote the exit times for the one-dimensional random walks based on $X^{(i)}$, and $h^{(i)}$ the corresponding h-functions. It is easy to check, using Lemma 2.1, that for any $\varepsilon > 0$,

$$\lim_{r \to \infty} \frac{h^{(1)}(r)}{h^{(2)}(\varepsilon r)} = \infty.$$

It then follows from (1.2) in [10] that $\lim_{r \to \infty} P(T_r^{(1)} < T_{\varepsilon r}^{(2)}) = 1$, hence (E) does not hold.

Also $E(\|S_{T_r}\| - r)^p = o(r^p)$ for all $0 < p < 2$ because

$$\left(\|S_{T_r}\| - r\right)^p \le \left(\left| \sum_{i=1}^{T_r^{(1)}} X_i^{(1)} \right| - r \right)^p + \left(\left| \sum_{i=1}^{T_r^{(2)}} X_i^{(2)} \right| - r \right)^p,$$

and the right-hand side of this inequality is $o(r^p)$ by Theorem 4.6 and the uniform integrability result in Theorem 3.3. Thus the result of Proposition 4.3 does not extend to higher dimensions.

Acknowledgments. The authors would like to thank S. Lally for bringing this problem area to their attention and W. Pruitt for suggesting Example 7.1. We would also like to thank the referee for his careful reading of the manuscript.

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