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A DISTRIBUTIONAL FORM OF LITTLE'S LAW IN HEAVY TRAFFIC

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Consider a single-server queue with units served in order of arrival for which we can define a stationary distribution (equilibrium distribution) of the vector of the waiting time and the queue size. Denote this vector by $(w(\rho), l(\rho))$, where $\rho < 1$ is the traffic intensity in the system when it is in equilibrium and λ_ρ is the intensity of the arrival stream to this system. Szczotka has shown under some conditions that $(1-\rho)(l(\rho) - \lambda_\rho w(\rho)) \to_p 0$ as $\rho \uparrow 1$ (in heavy traffic). Here we will show under some conditions that $\sqrt{1-\rho}\,(l(\rho) - \lambda_\rho w(\rho)) \to_D bN\sqrt{M}$ as $\rho \uparrow 1$, where N and M are mutually independent random variables such that N has the standard normal distribution and M has an exponential distribution while b is a known constant.

1. Introduction. In this paper we consider a single-server queue with units served in the order of arrival. Thus the single-server queueing system is generated by the generic sequence $(\mathbf{v}, \mathbf{u}) = \{(v_k, u_k),\ k \geq 1\}$, where v_k, $k \geq 1$, represents the service time of the kth unit and u_k, $k \geq 1$, represents the interarrival time between the kth and $(k+1)$th units. Let w_k, $k \geq 1$, denote the waiting time of the kth unit and let l_k, $k \geq 1$, denote the number of units in the queueing system at the moment of the kth arrival, including the kth unit. Throughout the paper, $\mathscr{L}(X)$ denotes the distribution of the random variable or random element X and the symbol $\Rightarrow$ denotes the weak convergence of probability measures. Hereafter, by the "stationary distribution of waiting time" and the "stationary joint distribution of waiting time and queue size" we mean the limiting distributions, in the weak convergence sense, of the sequences of distributions

$$\frac{1}{k}\sum_{i=1}^{k}\mathscr{L}(w_i) \quad \text{and} \quad \frac{1}{k}\sum_{i=1}^{k}\mathscr{L}(w_i, l_i), \qquad k \geq 1,$$

respectively. Obviously, this definition agrees with that when the sequences of distributions $\mathscr{L}(w_k)$ and $\mathscr{L}(w_k, l_k)$, $k \geq 1$, are weakly convergent.

Now we describe the class of queues for which the stationary distribution of waiting time and the stationary joint distribution of waiting time and queue size exist and we give their form.

Let $\mathbf{Y} = \{Y_k,\ k \geq 1\}$ be a discrete time process with values in R^m, $m \geq 1$, and $\mathbf{Y}_n =_{\text{df}} \{Y_{n+k},\ k \geq 1\}$, $n \geq 1$. $\mathbf{Y}$ is said to be either (i) weakly asymptotically

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stationary or (ii) weakly asymptotically stationary in the mean if there exists an R^m-valued process $\mathbf{Y}^0 = \{Y_k^0, k \geq 1\}$ such that $\mathscr{L}(\mathbf{Y}_n) \Rightarrow \mathscr{L}(\mathbf{Y}^0)$ in case (i) and $(1/n)\sum_{i=1}^n \mathscr{L}(\mathbf{Y}_i) \Rightarrow \mathscr{L}(\mathbf{Y}^0)$ in case (ii). In both cases, the process $\mathbf{Y}^0$ is stationary and is called the stationary representation of $\mathbf{Y}$. By $\mathbf{Y}^* = \{Y_k^*, -\infty < k < \infty\}$ we denote the two-sided stationary extension of $\mathbf{Y}^0$. According to the preceding, a sequence $(\mathbf{v}^0, \mathbf{u}^0) = \{(v_k^0, u_k^0), k \geq 1\}$ denotes a stationary representation of $(\mathbf{v}, \mathbf{u})$ (if it exists in any sense) and $(\mathbf{v}^*, \mathbf{u}^*) = \{(v_k^*, u_k^*), -\infty < k < \infty\}$ denotes the two-sided stationary extension of $(\mathbf{v}^0, \mathbf{u}^0)$. Furthermore, let $S_0 = 0$, $S_k = \sum_{i=1}^k (v_i - u_i)$, for $k \geq 1$, and $S_0^* = 0$, $S_k^* = \sum_{i=k+1}^0 (v_i^* - u_i^*)$, for $k < 0$.

We say that $(\mathbf{v}, \mathbf{u})$ satisfies condition AB or condition AB in the mean if

$$\lim_{k \to \infty} \limsup_{n \to \infty} P\Big\{ \max_{k \leq j \leq n} (S_n - S_{n-j}) > 0 \Big\} = 0$$

or

$$\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{i=k}^n P\Big\{ \max_{k \leq j \leq i} (S_i - S_{i-j}) > 0 \Big\} = 0,$$

respectively.

In [10] (Proposition 2), it is shown that if $\mathbf{v} - \mathbf{u}$ is weakly asymptotically stationary or weakly asymptotically stationary in the mean and $S_{-k}^* \to -\infty$ a.e. as $k \to \infty$, then for any initial condition $w_1 \geq 0$ the following hold:

$$\lim_{k \to \infty} \limsup_{n \to \infty} P\Big\{ \max\Big(S_n + w_1, \max_{k \leq j \leq n} (S_n - S_{n-j}) \Big) > 0 \Big\} = 0$$

or

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{i=k}^n P\Big\{ \max\Big(S_i + w_1, \max_{k \leq j \leq i} (S_i - S_{i-j}) \Big) > 0 \Big\} = 0,$$

respectively. Hence the strengthened version of Theorem 1 from [6] (see also [10], Theorem 1 or the closely related result in [2], Chapter 1, page 28) says the following: Let $(\mathbf{v}, \mathbf{u})$ be either (i) weakly asymptotically stationary or (ii) weakly asymptotically stationary in the mean and let it satisfy condition AB or condition AB in the mean, respectively. Furthermore, in both cases let $(\mathbf{v}^0, \mathbf{u}^0)$ be such that $S_{-k}^* \to -\infty$ a.e. as $k \to \infty$. Then $(\mathbf{w}, \mathbf{v}, \mathbf{u}) = \{(w_k, v_k, u_k), k \geq 1\}$ is weakly asymptotically stationary in case (i) and weakly asymptotically stationary in the mean in case (ii) and in both cases the two-sided stationary extension of a stationary representation $(\mathbf{w}^0, \mathbf{v}^0, \mathbf{u}^0)$ has the form $(\mathbf{w}^*, \mathbf{v}^*, \mathbf{u}^*) = \{(w_k^*, v_k^*, u_k^*), -\infty < k < \infty\}$, where

$$w_k^* = \sup_{j \leq k} \sum_{i=j+1}^k (v_i^* - u_i^*), \qquad -\infty < k < \infty, \tag{1}$$

and $\{(v_k^*, u_k^*), -\infty < k < \infty\}$ is a two-sided stationary extension of a stationary representation $(\mathbf{v}^0, \mathbf{u}^0)$. (Throughout this paper sums of the type $\sum_{i=j}^k$, where $k < j$, are taken as zero or the zero vector.) Moreover, under additional assumptions (a_1) and (a_2) from [7], the sequence $(\mathbf{l}, \mathbf{w}, \mathbf{v}, \mathbf{u}) = \{(l_k, w_k, v_k, u_k),$

$k \geq 1\}$ is weakly asymptotically stationary in case (i) and weakly asymptotically stationary in the mean in case (ii) and

$$
\begin{aligned}
&P\{w_1^0 \geq x, l_1^0 > k\} \\
(2) \qquad &= P\{w_{k+1}^* \geq x, w_1^* + v_1^* > u_1^* + u_2^* + \cdots + u_k^*\} \\
&= P\{w_0^* \geq x, w_{-k}^* + v_{-k}^* > u_{-k}^* + u_{-k+1}^* + \cdots + u_{-1}^*\},
\end{aligned}
$$

for $k \geq 1$ and $x \geq 0$.

Hereafter, for clarity we write (w, l) instead of (w_1^0, l_1^0).

Now we will introduce the concept of families of queueing systems, for which the heavy traffic limits are to be obtained. First, let us notice that the distribution of (w, l) depends on $(\mathbf{v}^*, \mathbf{u}^*)$. However, under some conditions on $(\mathbf{v}^*, \mathbf{u}^*)$ and in the heavy traffic situation, that is, in the case $a =_{\mathrm{df}} E(v_1^* - u_1^*) \uparrow 0$, we have $w \to_p \infty$ as $a \uparrow 0$ while $l \to_p \infty$ if $a \uparrow 0$ and $0 < \lim_{a \to 0} Eu_1^* < \infty$. Hereafter, while considering an approximation of (w, l) in heavy traffic, when writing $a \uparrow 0$, we have in mind the conditions $a \uparrow 0$ and $Eu_1^* \to 1/\lambda$, $0 < \lambda < \infty$. However, in this case we parameterize (w, l) and λ by a writing $(w(a), l(a))$ and λ_a.

Reference [9] gives general conditions under which $|a|w(a) \to_D M$ as $a \uparrow 0$, where M is exponentially distributed and [7] gives conditions for $|a|(l(a) - \lambda_a w(a)) \to_p 0$ as $a \uparrow 0$. This result is close to the results obtained by Glynn and Whitt [3], Keilson and Servi [4], Whitt [11] and Szczotka [8, 9] (other references are cited in these sources). Here we will show under some conditions that $\sqrt{|a|}\,(l(a) - \lambda_a w(a)) \to_D bN\sqrt{M}$, as $a \uparrow 0$, where N and M are mutually independent random variables such that N has the standard normal distribution, M has an exponential distribution and b is some known constant.

2. Main results. The main investigation tool in this paper is the theory of weak convergence of probability measures on metric spaces. So all main notions appearing here can be found in [1]. Furthermore, we use some special notation, for example, $D[0, \infty)$, which denotes the space of all right-continuous real-valued functions on $[0, \infty)$ with the limit from the left. This space is considered with the Stone topology metrized by Lindvall's metric d (see [5]). We assume also that subspaces of a metric space are endowed with the product topology. If X, X_n, $n \geq 1$, are random elements of a metric space, then as in [1] we assume the notation $X_n \to X$ a.e., $X_n \to_p X$ and $X_n \to_D X$ for almost sure convergence, convergence in probability and convergence in distribution of $\{X_n\}$ to X, respectively. In the following, by an m-dimensional Wiener process, $m \geq 1$, we mean an m-dimensional Gaussian process $(\mathscr{W}_1, \mathscr{W}_2, \dots, \mathscr{W}_m)$ being a random element of $D^m[0, \infty)$ and such that $\mathscr{W}_i$, $1 \leq i \leq m$, are Wiener processes with $E\mathscr{W}_i(t)\mathscr{W}_j(s) = E\mathscr{W}_j(t)\mathscr{W}_i(s) = \sigma_{i,j} \min(t, s)$, $s, t \geq 0$, $1 \leq i, j \leq m$, $\sigma_{i,i} = 1$. Notice that if $(\mathscr{W}_1, \mathscr{W}_2)$ is a two-dimensional Wiener process, then for any positive numbers t_1 and t_2 we have $t_1^2 - 2t_1t_2\sigma_{1,2} + t_2^2 \geq 0$, where $\sigma_{1,2} = E\mathscr{W}_1(1)\mathscr{W}_2(1)$. Moreover, if $\mathscr{W}_1 = \mathscr{W}_2$ (in this situation $\sigma_{1,2} = 1$) and if $t_1 = t_2$, then $t_1^2 + 2t_1t_2\sigma_{1,2} + t_2^2 = (t_1 - t_2)^2 = 0$.

Let us consider the family of random vectors $(w(a), l(a))$, $a < 0$, such that, for each $a < 0$, the random vector $(w(a), l(a))$ is defined in (2) by a stationary sequence $(\mathbf{v}^*(a), \mathbf{u}^*(a)) = \{(v_k^*(a), u_k^*(a)),\ -\infty < k < \infty\}$ of pairs of nonnegative random variables $v_k^*(a), u_k^*(a)$ satisfying the following conditions:

$$a = Ev_1^*(a) - Eu_1^*(a), \qquad a \uparrow 0,$$

(3) $$\sum_{i=1}^{k} (v_{-i}^*(a) - u_{-i}^*(a)) \to -\infty \quad \text{a.e. as } k \to \infty, \text{ for each } a < 0.$$

Let us introduce the following notation:

$$\mu_a = (Ev_1^*(a))^{-1}, \qquad \lambda_a = (Eu_1^*(a))^{-1}, \qquad c = |a|,$$

$$V_a(t) = \sqrt{c} \sum_{i=1}^{[t/c]} (v_{-i}^*(a) - \mu_a^{-1}),$$

$$\tilde{V}_a(t) = \sqrt{c} \sum_{i=1}^{[t/c]} (v_{-i+1}^*(a) - \mu_a^{-1}), \qquad t \geq 0,$$

$$U_a(t) = \sqrt{c} \sum_{i=1}^{[t/c]} (u_{-i}^*(a) - \lambda_a^{-1}),$$

$$\tilde{U}_a(t) = \sqrt{c} \sum_{i=1}^{[t/c]} (u_{-i+1}^*(a) - \lambda_a^{-1}), \qquad t \geq 0,$$

$$X_k^*(a) = v_k^*(a) - u_k^*(a), \qquad -\infty < k < \infty.$$

(Throughout this paper, square brackets are used exclusively to denote the integer part.)

The main result of the paper is the following theorem.

THEOREM 1. *Let the following conditions hold*:

(4) $$\sup_{a<0} E(v_1^*(a) - u_1^*(a))^2 < \infty;$$

(5) $$\lambda_a \to \lambda, \quad \text{as } a \uparrow 0,\ 0 < \lambda < \infty;$$

there exist finite and positive numbers σ_1, σ_2 *such that*

(6) $$\left(\left(\frac{1}{\sigma_1} V_a, \frac{1}{\sigma_2} U_a\right), \left(\frac{1}{\sigma_1} V_b, \frac{1}{\sigma_2} U_b\right)\right) \to_D (\mathscr{W}_1, \mathscr{W}_2), \quad \text{as } a \uparrow 0,$$

where $b = a^2$, *while* $\mathscr{W}_1 = (\mathscr{W}_{1,1}, \mathscr{W}_{1,2})$ *and* $\mathscr{W}_2 = (\mathscr{W}_{2,1}, \mathscr{W}_{2,2})$ *are independent two-dimensional Wiener processes. Then under*

$$\sigma^2 =_{df} \sigma_1^2 - 2\sigma_1\sigma_2\sigma_{1,2} + \sigma_2^2 > 0,$$

where $\sigma_{1,2} = E\mathscr{W}_1(1)\mathscr{W}_2(1)$, *we have the following convergence*:

$$\sqrt{|a|}\,(l(a) - \lambda_a w(a)) \to_D \lambda^{3/2}\sigma_1 N\sqrt{M}, \quad \text{as } a \uparrow 0,$$

where the random variables N *and* M *are mutually independent*, N *has the standard normal distribution and* $P\{M > x\} = \exp(-2x/\sigma^2)$, *for* $x \geq 0$.

Before proving Theorem 1 we prove two lemmas. The first gives sufficient conditions under which condition (6) of Theorem 1 holds and the second lemma will be needed to prove Theorem 1.

LEMMA 1. *Let* $\{Y_{n,k},\ k \geq 1, n \geq 1\}$ *be an array of random vectors in* R^m *such that for each* $n \geq 1$, $\{Y_{n,k},\ k \geq 1\}$ *is a stationary sequence of random vectors in* R^m *being* $\varphi_n = \{\varphi_n(k),\ k \geq 1\}$*-mixing, that is*,

$$\sup_{A \in F_k(n),\ B \in F^{k+i}(n)} |P(A \cap B) - P(A)P(B)| \leq \varphi_n(i),$$

where $F_k(n)$ *and* $F^{k+i}(n)$ *are* σ*-fields generated by* $\{Y_{n,j},\ j \leq k\}$ *and* $\{Y_{n,j},\ j \geq k+i\}$, *respectively, and* $\varphi_n(i) \to 0$ *as* $i \to \infty$. *Furthermore, let*

$$S_{n,k} = \frac{1}{\sqrt{k}} \sum_{j=1}^{[kt]} Y_{n,j}, \qquad t \geq 0, n, k \geq 1,$$

and let $\{k_n\}$ *and* $\{r_n\}$ *be sequences of positive numbers tending to infinity such that* $k_n/r_n \to 0$ *and* $\varphi_n(k_n) \to 0$ *as* $n \to \infty$. *If* $S_{n,k_n} \to_D \xi$ *and* $S_{n,k_n} \to_D \xi$ *as* $n \to \infty$, *where* $\xi = (\sigma_1\mathscr{W}_1, \sigma_2\mathscr{W}_2, \ldots, \sigma_m\mathscr{W}_m)$, σ_i, $1 \leq i \leq m$, *are finite and positive numbers while* $(\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_m)$ *is an m-dimensional Wiener process, then* $(S_{n,k_n}, S_{n,r_n}) \to_D (\xi_1, \xi_2)$ *as* $n \to \infty$, *where*

$$\xi_i = (\sigma_1\mathscr{W}_{i,1}, \sigma_2\mathscr{W}_{i,2}, \ldots, \sigma_m\mathscr{W}_{i,m}), \qquad i = 1, 2,$$

while $(\mathscr{W}_{1,1}, \mathscr{W}_{1,2}, \ldots, \mathscr{W}_{1,m})$ *and* $(\mathscr{W}_{2,1}, \mathscr{W}_{2,2}, \ldots, \mathscr{W}_{2,m})$ *are independent m-dimensional Wiener processes*.

PROOF. Notice that

$$S_{n,r_n}(t) = \sqrt{\frac{k_n}{r_n}}\, S_{n,k_n}(2t) + \frac{1}{\sqrt{r_n}} \sum_{j=[2k_n t]+1}^{[r_n t]} Y_{n,j}, \qquad t \geq 0.$$

By the mixing condition it follows that, for any Borel sets A and B in R^m, we have

$$\left| P\left\{ S_{n,k_n}(t) \in A, \quad \frac{1}{\sqrt{r_n}} \sum_{i=[2k_n t]+1}^{[r_n t]} Y_{n,j} \in B \right\} \right.$$
$$\left. - P\{S_{n,k_n}(t) \in A\} P\left\{ \frac{1}{\sqrt{r_n}} \sum_{j=[2k_n t]+1}^{[r_n t]} Y_{n,j} \in B \right\} \right| \leq \varphi_n(k_n).$$

Hence, and by $\sqrt{k_n/r_n}\, S_{n,k}(2t) \to_p \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in R^m, we have

$$(7) \qquad \big(S_{n,k_n}(t), S_{n,r_n}(t)\big) \to_D \big(\xi_1(t), \xi_2(t)\big), \quad \text{as } n \to \infty.$$

Now let us notice that for any $t_2 > t_1$ and for sufficiently large n, that is, such that $r_n t_1 - k_n t_2 > 1$ we have

$$\begin{aligned}&\big|P\{S_{n,k_n}(t_2) - S_{n,k_n}(t_1) \in A,\, S_{n,r_n}(t_2) - S_{n,r_n}(t_1) \in B\}\\ &\quad -P\{S_{n,k_n}(t_2) - S_{n,k_n}(t_1) \in A\}P\{S_{n,r_n}(t_2) - S_{n,r_n}(t_1) \in B\}\big| \le \varphi_n(k_n).\end{aligned}$$

Hence and from (7) we have the weak convergence of two-dimensional distributions of (S_{n,k_n}, S_{n,r_n}) to (ξ_1, ξ_2). In a similar way we show the weak convergence of any finite-dimensional distributions of these processes. This together with the tightness of (S_{n,k_n}, S_{n,r_n}), which follows from the tightness of $\{S_{n,k_n}\}$ and $\{S_{n,r_n}\}$, gives the assertion of Lemma 1. □

NOTE 1. It is worth noticing that the assertion of Lemma 1 holds when $Y_{n,1}, Y_{n,2}, \dots$ are i.i.d. random vectors in R^m for each $n \ge 1$, with the zero vector of expectation, positive and finite variances of each coordinate $Y^i_{n,1}$ of $Y_{n,1}$ and with finite

$$\sup_{n\ge 1} \sum_{i=1}^{m} E|Y^i_{n,1} - EY^i_{n,1}|^{2+\delta}, \quad \text{for some } \delta > 0.$$

NOTE 2. Lemma 1 can be generalized to the situation when the limiting process is a process with independent increments.

LEMMA 2. *Let* (X, Y) *and* (X_n, Y_n), $n \ge 1$, *be random elements of the product space* $D[0,\infty) \times S$, *where* S *is a Polish metric space and let* X *have almost all continuous sample paths and* $P\{\sup_{0\le t<\infty} X_n(t) < \infty\} = P\{\sup_{0\le t<\infty} X(t) < \infty\} = 1$. *Furthermore, let* $(X_n, Y_n) \to_D (X, Y)$ *as* $n \to \infty$ *and*

$$(8) \qquad \lim_{k\to\infty} \limsup_{n\to\infty} P\Big\{\sup_{0\le t<\infty} X_n(t) - \sup_{0\le t\le k} X_n(t) > 0\Big\} = 0.$$

Then $(\sup_{0\le t<\infty} X_n(t), Y_n) \to_D (\sup_{0\le t<\infty} X(t), Y)$ *as* $n \to \infty$.

PROOF. For any closed set F in $R \times S$ and any $k \ge 1$, we have

$$(9) \qquad \begin{aligned}&P\Big\{\Big(\sup_{0\le t<\infty} X_n(t), Y_n\Big) \in F\Big\}\\ &\quad\le P\Big\{\Big(\sup_{0\le t<\infty} X_n(t), Y_n\Big) \in F,\ \sup_{0\le t<\infty} X_n(t) = \sup_{0\le t\le k} X_n(t)\Big\}\\ &\qquad + P\Big\{\sup_{0\le t<\infty} X_n(t) - \sup_{0\le t\le k} X_n(t) > 0\Big\}\\ &\quad\le P\Big\{\Big(\sup_{0\le t\le k} X_n(t), Y_n\Big) \in F\Big\} + P\Big\{\sup_{0\le t<\infty} X_n(t) - \sup_{0\le t<\infty} X_n(t) > 0\Big\}.\end{aligned}$$

The measurability of the mapping

$$D[0,\infty) \times S \ni (x,y) \to \Big(\sup_{0\le t\le k} x(t), y \Big) \in R \times S,$$

its continuity on $C[0,\infty) \times S$, and the convergences $(X_n, Y_n) \to_D (X, Y)$ as $n \to \infty$ and $\sup_{0\le t\le k} X(t) \to \sup_{0\le t<\infty} X(t)$ a.e. as $k \to \infty$ give

$$\limsup_{k\to\infty} \limsup_{n\to\infty} P\Big\{\Big(\sup_{0\le t\le k} X_n(t), Y_n \Big) \in F\Big\} \le \limsup_{k\to\infty} P\Big\{\Big(\sup_{0\le t\le k} X(t), Y \Big) \in F\Big\}$$

$$\le P\Big\{\Big(\sup_{0\le t<\infty} X(t), Y \Big) \in F\Big\}.$$

This, in view of (9) and (8) and Theorem 2.1 from [1], gives the assertion of Lemma 2. □

Proof of Theorem 1. In a similar way as we obtained (2) in [7], we can obtain

$$\begin{aligned} P\{(l(a) - \lambda_a w(a)) > x\} &= P\{l(a) > [x + \lambda_a w(a)]\} \\ &= P\{w^*_{-c(x,a)} + v^*_{-c(x,a)} \\ &\qquad > u^*_{-c(x,a)} + u^*_{-c(x,a)+1} + \cdots + u^*_{-1}\}, \end{aligned}$$

for $x \ge -\lambda_a w^*(a)$, where $c(x,a) = [x + \lambda_a w^*(a)]$. For simplicity and clarity, we drop the mark of dependence of $w_0^*, w_k^*, v_k^*, u_k^*, X_k^*$, $-\infty < k < \infty$, on the parameter a.

Let us denote

$$\theta(x,a) = \left[\frac{1}{c} \big(\sqrt{c}\, x + \lambda_a c w_0^* \big) \right] \quad \text{and} \quad z(x,a) = \sqrt{c}\, x + \lambda_a c w_0^*.$$

Hereafter, for simplicity and clarity, we drop the mark of dependence of θ and z on the parameters x and a. Then we have

$$\begin{aligned} P\{\sqrt{c}\,(l(a) - \lambda_a w(a)) > x\} &= P\{l(a) > \theta\} \\ &= P\{w^*_{-\theta} + v^*_{-\theta} > u^*_{-\theta} + u^*_{-\theta+1} + \cdots + u^*_{-1}\} \\ &= P\Big\{ (w^*_{-\theta} - w_0^*) - (\lambda_a^{-1}\theta - w_0^*) + v^*_{-\theta} \\ &\qquad > \sum_{i=1}^{\theta} (u^*_{-i} - \lambda_a^{-1}) \Big\}. \end{aligned} \tag{10}$$

Hereafter, we denote $S_{j,k} = \sum_{i=j+1}^{k} X_i^*$, $-\infty < k, j < \infty$.

Then, for any integer k, $k \geq 0$ (k may be a random variable), we have the following relations:

$$(11)\quad \begin{aligned} w_0^* &= \max\Big(\max_{-k \leq j \leq 0} S_{j,0},\ \sup_{j \leq -k} S_{j,0} \Big) \\ &= \max\Big(\max_{-k \leq j \leq 0} S_{j,0},\ S_{-k,0} + \sup_{j \leq -k} S_{j,-k} \Big), \end{aligned}$$

$$(12)\quad w_0^* - |S_{-k,0}| \leq \max_{-k \leq j \leq 0} S_{j,0} + \sup_{j \leq -k} S_{j,-k} \leq 2w_0^* + |S_{-k,0}|,$$

$$(13)\quad \begin{aligned} w_0^* - w_{-k}^* &= \max\Big(\max_{-k \leq j \leq 0} S_{j,0} - w_{-k}^*,\ S_{-k,0} \Big) \\ &= \max\Big(2 \max_{-k \leq j \leq 0} S_{j,0} - \Big(\max_{-k \leq j \leq 0} S_{j,0} + \sup_{j \leq -k} S_{j,-k} \Big),\ S_{-k,0} \Big) \end{aligned}$$

and

$$(14)\quad \max_{-k \leq j \leq 0} S_{j,0} = \max_{0 \leq j \leq k} \sum_{i=1}^{j} X_{-i+1}^* = \sup_{0 \leq t \leq 1} \sum_{i=1}^{[kt]} X_{-i+1}^*.$$

Furthermore, we have the following relations:

$$(15)\quad \sqrt{c} \sum_{i=1}^{\theta} \left(u_{-i}^* - \lambda_a^{-1} \right) = U_a(z),$$

$$(16)\quad \sqrt{c}\, S_{-\theta,0} = \sqrt{c} \sum_{i=1}^{\theta} X_{-i+1}^* = \tilde{V}_a(z) - \tilde{U}_a(z) - c^{3/2}\theta,$$

$$(17)\quad \begin{aligned} \sqrt{c} \max_{-\theta \leq j \leq 0} S_{j,0} &= \sqrt{c} \sup_{0 \leq t \leq 1} \sum_{i=1}^{[\theta t]} X_{-i+1}^* \\ &= \sup_{0 \leq t \leq 1} \left(\tilde{V}_a(zt) - \tilde{U}_a(zt) - c^{3/2}[\theta t] \right), \end{aligned}$$

$$(18)\quad c w_0^* = \sup_{0 \leq t < \infty} \left(\tilde{V}_b(t) - \tilde{U}_b(t) - a^2 \left[\frac{t}{a^2} \right] \right), \quad \text{where } b = a^2,$$

$$(19)\quad \sqrt{c v_{-\theta}^*} \leq \sqrt{c}\, \mu_a^{-1} + \omega_1(\xi_a),$$

where $\xi_a(t) = \tilde{V}_a(zt)$, $t \geq 0$, and ω_b, with $b \geq 0$, maps $D[0,\infty)$ into R as $\omega_b(x) = \sup_{0 \leq t \leq b} |x(t) - x(t-)|$, for x belonging to $D[0,\infty)$.

Now let us notice that

$$(20)\quad \sup_{0 \leq t \leq b} |V_a(t) - \tilde{V}_a(t)| \leq \sqrt{c}\, v_0^* + \omega_b(V_a) \quad \text{and}$$

$$(21)\quad \sup_{0 \leq t \leq b} |U_a(t) - \tilde{U}_a(t)| \leq \sqrt{c}\, u_0^* + \omega_b(U_a).$$

Since ω_b, $b \geq 0$, is measurable and continuous on the set of continuous functions, that is, on $C[0,\infty)$ (see, e.g., [8], Property 1), then by assumption (6) we get $\omega_b(V_a) \to_p 0$ and $\omega_b(U_a) \to_p 0$ as $a \uparrow 0$, for any $b \geq 0$. That and the inequalities (20) and (21) give

$$d\left(V_a, \tilde{V}_a\right) \to_p 0 \quad \text{and} \quad d\left(U_a, \tilde{U}_a\right) \to_p 0, \quad \text{as } a \uparrow 0. \tag{22}$$

Further, by assumption (6) we get

$$\left(\left(\frac{1}{\sigma_1}\tilde{V}_a, \frac{1}{\sigma_2}\tilde{U}_a\right), \left(\frac{1}{\sigma_1}\tilde{V}_b, \frac{1}{\sigma_2}\tilde{U}_b\right)\right) \to_D (\mathscr{W}_1, \mathscr{W}_2), \quad \text{as } a \uparrow 0, \text{ where } b = a^2,$$

and next

$$\left(\left(\frac{1}{\sigma_1}\tilde{V}_a, \frac{1}{\sigma_2}\tilde{U}_a\right), \tilde{V}_b - \tilde{U}_b\right) \to_D (\mathscr{W}_1, \sigma_1\mathscr{W}_{2,1} - \sigma_2\mathscr{W}_{2,2}), \quad \text{as } a \uparrow 0, \text{ where } b = a^2, \tag{23}$$

where $\mathscr{W}_1 = (\mathscr{W}_{1,1}, \mathscr{W}_{1,2})$ and $\mathscr{W}_2 = (\mathscr{W}_{2,1}, \mathscr{W}_{2,2})$ are mutually independent two-dimensional Wiener processes.

Now let

$$X(t) = \sigma_1\mathscr{W}_{2,1}(t) - \sigma_2\mathscr{W}_{2,2}(t) - t, \qquad t \geq 0, Y = \mathscr{W}_1,$$

$$X_n(t) = \tilde{V}_{b_n}(t) - \tilde{U}_{b_n}(t) - b_n\left[\frac{t}{b_n}\right], \qquad t \geq 0, \text{ where } b_n = a_n^2,$$

$$Y_n = \left(\frac{1}{\sigma_1}V_{a_n}, \frac{1}{\sigma_2}U_{a_n}\right), \quad \text{for } a_n \uparrow 0 \text{ as } n \to \infty.$$

Of course $(X_n, Y_n) \to_D (X, Y)$ as $n \to \infty$. Moreover, $X(t) \to -\infty$ a.e. as $t \to \infty$, while the convergence $X_n(t) \to -\infty$ a.e. as $t \to \infty$, for each $n \geq 1$, follows from assumption (3). Thus we have

$$P\left\{\sup_{0 \leq t < \infty} X_n(t) < \infty\right\} = P\left\{\sup_{0 \leq t < \infty} X(t) < \infty\right\} = 1 \quad \text{and} \quad X_n(0) = 0 \quad \text{a.e.}$$

Since $(\mathbf{v}^*(a), \mathbf{u}^*(a))$ is stationary for each $a < 0$, X_n has stationary increments for each $n \geq 1$. This implies that X_n satisfies condition (5) in [9]. Independently of this, condition (4) in Theorem 1 and Lemma 1 from [9] give the tightness of the sequence $\{\sup_{0 \leq t < \infty} X_n(t),\ n \geq 1\}$ which, in view of Lemma 3 in [9], gives (8). Hence, and by Lemma 2, we get

$$\left(cw_0^*, \frac{1}{\sigma_1}\tilde{V}_a, \frac{1}{\sigma_2}\tilde{U}_a\right) \to_D (M, \mathscr{W}_{1,1}, \mathscr{W}_{1,2}), \quad \text{as } a \uparrow 0, \tag{24}$$

where M and $(\mathscr{W}_{1,1}, \mathscr{W}_{1,2})$ are mutually independent and

$$M = \sup_{0 \leq t < \infty} \left(\sigma_1\mathscr{W}_{2,1}(t) - \sigma_2\mathscr{W}_{2,2}(t) - t\right).$$

But $(1/\sigma)(\sigma_1\mathscr{W}_{2,1} - \sigma_2\mathscr{W}_{2,2})$ is distributed as a one-dimensional Wiener process $\mathscr{W}$, which gives that $(1/\sigma)M$ is distributed as $\sup_{0 \leq t < \infty}(\mathscr{W}(t) - t/\sigma)$. Hence, M is exponentially distributed and $P\{M > x\} = \exp(-2x/\sigma^2)$, for $x \geq 0$. Now using the relations (16)–(19) and (24), then the random time change and properties of ω_b and, finally, the continuous mapping theorem (see [1], theorem 5.1) we get the convergences

$$\sqrt{c}\, v^*_{-\theta} \to_p 0, \quad \text{as } a \uparrow 0, \tag{25}$$

$$\begin{aligned}&\Big((cw_0^*), \sqrt{c}\, S_{-\theta,0}, \sqrt{c} \max_{-\theta \le j \le 0} S_{j,0}\Big)\\ &\quad \to_D \Big(M, \sigma_1 \mathscr{W}_{1,1}(\lambda M) - \sigma_2 \mathscr{W}_{1,2}(\lambda M),\\ &\qquad \sup_{0 \le t \le 1} \big(\sigma_1 \mathscr{W}_{1,1}(\lambda M t) - \sigma_2 \mathscr{W}_{1,2}(\lambda M t)\big)\Big), \quad \text{as } a \uparrow 0.\end{aligned} \tag{26}$$

Now the inequalities in (12) and the convergence (26) lead to

$$\sqrt{c} \max_{-\theta \le j \le 0} S_{j,0} + \sqrt{c} \sup_{j \le -\theta} S_{j,-\theta} \to_p \infty, \quad \text{as } a \uparrow 0, \tag{27}$$

which in view of (13), (26), (27), (22) and the continuous mapping theorem gives the convergence

$$\begin{aligned}&\big(\sqrt{c}\,(w_0^* - w^*_{-\theta}), U_a(z)\big)\\ &\quad \to_D \big(\sigma_1 \mathscr{W}_{1,1}(\lambda M) - \sigma_2 \mathscr{W}_{1,2}(\lambda M), \sigma_2 \mathscr{W}_{1,2}(\lambda M)\big), \quad \text{as } a \uparrow 0.\end{aligned}$$

Finally, using (10) and then the convergences (25), (26) and the relation

$$\lambda_a^{-1} x - \sqrt{c} \le \sqrt{c}\,\big(\lambda_a^{-1}\theta - w_0^*\big) \le \lambda_a^{-1} x$$

as well as the continuous mapping theorem, we get the convergence

$$\begin{aligned}P\big\{\sqrt{c}\,(l(a) - \lambda_a w(a)) > x\big\} &\to P\{\sigma_1 \mathscr{W}_{1,1}(\lambda M) < -\lambda x\}\\ &= P\big\{\sigma_1 \mathscr{W}_{1,1}(\lambda M) > \lambda^{-1} x\big\}, \quad \text{as } a \uparrow 0.\end{aligned}$$

But $\lambda \sigma_1 \mathscr{W}_{1,1}(\lambda M)$ is distributed as $\lambda^{3/2}\sigma_1 \sqrt{M}\, N$, which finishes the proof of Theorem 1. □

NOTE 3. Theorem 1 holds for GI/GI/1 queueing systems with generating sequences $(\mathbf{v}(a), \mathbf{u}(a)) = \{(v_k(a), u_k(a)),\ k \ge 1\}$, $a < 0$, such that $a = Ev_1(a) - Eu_1(a) \uparrow 0$, $\lambda_a = (Eu_1(a))^{-1} \to \lambda$, $0 < \lambda < \infty$, $\operatorname{Var} v_1(a) \to \sigma_1^2$, $\operatorname{Var} u_1(a) \to \sigma_2^2$, where $0 < \sigma_1, \sigma_2 < \infty$ and $\sigma = \sigma_1^2 + \sigma_2^2$, and

$$\sup_{a<0} \Big(E(v_1(a))^{2+\delta} + E(u_1(a))^{2+\delta}\Big) < \infty, \quad \text{for some } \delta > 0.$$

In this situation the Wiener processes $\mathscr{W}_{1,1}$ and $\mathscr{W}_{1,2}$ in (6) are independent, that is, $\sigma_{1,2} = 0$.

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