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ON A MAXIMUM SEQUENCE IN A CRITICAL MULTITYPE BRANCHING PROCESS[1]

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Let $\{Z_n\}$ be a p type positively regular nonsingular critical branching process with mean matrix M. If v is a right eigenvector of M for the eigenvalue 1 and $Y_n = Z_n \cdot v$, and if $M_n = \max_{0 \le j \le n} Y_j$, then it is shown that under second moments $(\log n)^{-1} E_{\mathbf{i}} M_n \to \mathbf{i} \cdot \mathbf{v}$, where $E_{\mathbf{i}}$ denotes starting with $Z_0 = \mathbf{i}$ and $\cdot$ denotes inner product. This is an extension of the result for the single type case obtained by Athreya in 1988.

1. Introduction. In a recent paper [1] it was shown that if $\{Z_n\}_0^\infty$ is a single type critical branching process such that $E_1 Z_1^2 < \infty$, then $(\log n)^{-1} E_i M_n \to i$ as $n \to \infty$, where E_i refers to starting with $Z_0 = i$ and $M_n \equiv \max_{0 \le j \le n} Z_j$. In the same paper the multitype analogue (Theorem 2 of [1]) was stated (with a misprint) without a proof and with a claim that the method of proof carries over fully. It turns out that the argument for the lower bound (proof of Proposition 4 of [1]) does not go over easily. Whereas in [1] there was only one random walk involved, here there are finitely many random walks with the number of summands in each being random. The purpose of this paper is to give a complete proof of the correct version of the multitype case. A preprint of Spataru [5] addresses this problem and proves a weaker result, which is stated in Remark 2.

2. The main result.

THEOREM 1. *Let* $\mathbf{Z}_n = (Z_{n1}, Z_{n2}, \dots, Z_{np})$ *be a* p *type positively regular nonsingular critical branching process with mean matrix* M *and finite second moments. Let* $\mathbf{u} = (u_1, u_2, \dots, u_p)$ *and* $\mathbf{v} = (v_1, v_2, \dots, v_p)$ *be nonnegative vectors such that* $M\mathbf{v} = \mathbf{v}$, $\mathbf{u}'M = \mathbf{u}'$, $\sum_1^p u_i = 1$, $\sum_1^p u_i v_i = 1$. *Let* $M_n = \max_{0 \le j \le n} \sum_1^p v_i Z_{ji}$. *Then*

$$(1) \qquad (\log n)^{-1} E_{\mathbf{i}} M_n \to \mathbf{i} \cdot \mathbf{v},$$

where $E_{\mathbf{i}}$ *denotes starting with* $\mathbf{Z}_0 = \mathbf{i}$ *and* $\cdot$ *denotes inner product.*

REMARK 1. This theorem is the same as Theorem 2 of [1] except that there $\mathbf{u}$ denotes the right and $\mathbf{v}$ the left eigenvector (which is an error).

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REMARK 2. Spataru [5] has proved the following weaker form of Theorem 1:

$$(\mathbf{i} \cdot \mathbf{v}) \min_{1 \le j \le p} v_j \le \liminf_n (\log n)^{-1} E_{\mathbf{i}} M_n \le \limsup_n (\log n)^{-1} E_{\mathbf{i}} M_n \le \mathbf{i} \cdot \mathbf{v}.$$

The improvement of this result obtained in the present paper is via a better estimate which simplifies the proof even in the single type case.

3. The proof. We shall need the following facts about p type critical positively regular, nonsingular branching processes. For proof, see Athreya and Ney ([2], Chapter V).

Let $\mathbf{Z}_n = (Z_{n1}, Z_{n2}, \dots, Z_{np})$ denote the population size vector of the nth generation. Assume throughout that $E_i Z_{1j}^2 < \infty$ for all $1 \le i, j \le p$, where E_i denotes starting with $Z_{0i} = 1$, $Z_{0j} = 0$ for $j \ne i$. Let $\mathbf{u}, \mathbf{v}, M$ be as in Theorem 1. Then the following hold:

1. $\{Y_n \equiv \mathbf{v} \cdot \mathbf{Z}_n,\ n \ge 0\}$ is a nonnegative martingale.
2. For each $\mathbf{i}$, there are constants $\lambda_{\mathbf{i}}$ and $\mu_{\mathbf{i}}$ such that, as $n \to \infty$: (a) $nP_{\mathbf{i}}(Y_n > 0) \to \lambda_{\mathbf{i}}$ and (b) $n^{-1} E_{\mathbf{i}} Y_n^2 \to \mu_{\mathbf{i}}$.
3. For any $\mathbf{i} \ne 0$ and any $\varepsilon > 0$,

$$\lim_{n \to \infty} P_{\mathbf{i}}\big(|Z_{nj} Y_n^{-1} - u_j| > \varepsilon \quad \text{for some } 1 \le j \le p | Y_n > 0\big) = 0.$$

4. There is a λ in $(0, \infty)$ such that, for all $\mathbf{i} \ne 0$,

$$\lim_n \sup_{x > 0} \big| P_{\mathbf{i}}(Y_n \le nx | Y_n > 0) - (1 - e^{-\lambda x}) \big| = 0.$$

We now start the proof with the upper bound.

PROPOSITION 1. *For any* $\mathbf{i}$,

$$\limsup_n (\log n)^{-1} E_{\mathbf{i}} M_n \le \mathbf{i} \cdot \mathbf{v}. \tag{2}$$

PROOF. For $0 < x < \infty$, let S_x denote the stopping time

$$S_x = \min\{\inf\{r: r \ge 0, Y_r \ge x\}, n\}.$$

Then,

$$P_{\mathbf{i}}(M_n \ge x) = P_{\mathbf{i}}\big(Y_{S_x} \ge x\big) \le x^{-1} E_{\mathbf{i}} Y_{S_x} = x^{-1} E_{\mathbf{i}} Y_0$$

(by Doob's optimal sampling theorem). So, for $n > \mathbf{i} \cdot \mathbf{v}$,

$$\int_{\mathbf{i} \cdot \mathbf{v}}^{n} P(M_n \ge x)\, dn \le (EY_0)(\log n - \log \mathbf{i} \cdot \mathbf{v}).$$

Also, $\int_n^\infty P(M_n \ge x)\, dx \le (EY_n^2) n^{-1}$ (by Doob's inequality).

Now, by fact 2(b),

$$\sup_n \int_n^{\infty} P(M_n \geq x)\, dx < \infty.$$

Since $E_{\mathbf{i}} M_n = \mathbf{i} \cdot \mathbf{v} + \int_{\mathbf{i}\cdot\mathbf{v}}^{\infty} P(M_n \geq x)\, dx$ and $EY_0 = \mathbf{i} \cdot \mathbf{v}$, (2) follows. □

REMARK 3. The preceding proof is due to Spataru [5]. More generally, Pakes [3] has shown that, for any nonnegative martingale $\{Y_n\}$ for which $EY_n \log Y_n \to \infty$, one has

$$\limsup_n (EM_n)(EY_n \log Y_n)^{-1} \leq 1.$$

For our Y_n it turns out (see Lemma 3) that under finite second moments $E_{\mathbf{i}} Y_n \log Y_n \sim (\mathbf{i} \cdot \mathbf{v}) \log n$. In the case of infinite second moments, one could still get an upper growth rate for EM_n if one could get a rate for $EY_n \log Y_n$.

PROPOSITION 2.

$$\liminf_n (\log n)^{-1} E_{\mathbf{i}} M_n \geq \mathbf{i} \cdot \mathbf{v}. \tag{3}$$

We need the following lemmas.

LEMMA 1. *Let $\{X_i\}_1^{\infty}$ be i.i.d. r.v.'s with $EX_1 = 0$, $\sigma^2 = EX_1^2 < \infty$. Then, for each $\varepsilon > 0$, $nP(|\overline{X}_n| > \varepsilon) \to 0$, where $\overline{X}_n = n^{-1}(X_1 + X_2 + \cdots + X_n)$.*

LEMMA 2. *Let $\{X_i\}_1^{\infty}$ be i.i.d. nonnegative r.v.'s with $EX_1^2 < \infty$ and $EX_1 \leq 1$. Then, for any $\rho > 1$, $nE\psi(\overline{X}_n) \to 0$, where $\psi(x) \equiv x(\log \rho^{-1}x)^+$.*

PROOF OF LEMMA 1. This is known in the literature. See, for example, [4], page 286. □

PROOF OF LEMMA 2. Since, for $x > \rho > 1$,

$$\psi(x) = x \log(\rho^{-1}x) = \int_{\rho}^{x} \left(\log(\rho^{-1}y) + 1\right) dy,$$

we have

$$nE\psi(\overline{X}_n) = \int_{\rho}^{\infty} \left(\log(\rho^{-1}y) + 1\right) nP(\overline{X}_n > y)\, dy.$$

For $y > \rho > 1$,

$$\begin{aligned} nP(\overline{X}_n > y) &\leq nP\left(|\overline{X}_n - EX_1| \geq (y - EX_1)\right) \\ &\leq nV(\overline{X}_n)(y - EX_1)^{-2} \leq V(X_1)(y-1)^{-2}. \end{aligned}$$

Since $(\log(\rho^{-1}y) + 1)(y-1)^{-2}$ is integrable in (ρ, ∞), it follows from Lemma 1 and the dominated convergence theorem that $nE\psi(\overline{X}_n) \to 0$. □

In what follows we write

$$\phi(n, F, p) \equiv nE\big(\psi(\bar{X}_n)\big), \tag{4}$$

where $\psi(x) = x(\log(\rho^{-1}x))^+$ and the $\{X_i\}$ are i.i.d. r.v.'s with c.d.f. F.

LEMMA 3. *Let $\{Y_n\colon n \geq 0\}$ be a nonnegative martingale. Fix $n \geq 1$, $l > 1$. Let $T = T_{n,l}$ be a stopping time defined by*

$$T = \begin{cases} \min\{r\colon 1 \leq r \leq n,\ Y_r = 0 \text{ or } Y_r \geq l\}, \\ n, \quad \text{if } 0 < Y_r < l \text{ for } 1 \leq r \leq n. \end{cases}$$

Then

$$E(Y_T\colon Y_T \geq l) \geq E(Y_n\colon Y_n \geq l). \tag{5}$$

PROOF. This is Proposition 3 in [1]. □

PROOF OF PROPOSITION 2. Fix integers l and n and a number $\rho > 1$. Then, by Lemma 3,

$$\begin{aligned} E(Y_n\colon Y_n \geq l) &\leq E(Y_T\colon Y_T \geq l) \\ &\leq E(Y_T\colon Y_T \geq l,\ Y_T \leq \rho l) + E(Y_T\colon Y_T > \rho l) \\ &= a_{nl} + b_{nl}, \quad \text{say}. \end{aligned}$$

Note that $a_{nl} \leq \rho l P(Y_T \geq l) \leq \rho l P(M_n \geq l)$. Thus,

$$\rho \sum_1^\infty P(M_n \geq l) + \sum_1^\infty l^{-1} b_{nl} \geq \sum_1^\infty l^{-1} E(Y_n\colon Y_n \geq l).$$

Since $\rho > 1$ is arbitrary, $EM_n = \sum_1^\infty P(M_n \geq l)$ and $\sum_1^\infty l^{-1}E(Y_n\colon\ Y_n \geq l) \sim EY_n \log Y_n$, it is enough to show that

$$(\log n)^{-1} E_{\mathbf{i}} Y_n \log Y_n \to \mathbf{i} \cdot \mathbf{v}, \tag{6}$$

$$(\log n)^{-1} \sum_1^\infty l^{-1} b_{nl} \to 0. \tag{7}$$

To establish (6) we use the following lemma.

LEMMA 4. $E_{\mathbf{i}} Y_n \log Y_n - \mathbf{i} \cdot \mathbf{v} \log n \to \lambda_{\mathbf{i}} \lambda \int_0^\infty x(\log x) e^{-\lambda x}\, dx$.

PROOF. Since $E_{\mathbf{i}} Y_n = E_{\mathbf{i}} Y_0 = \mathbf{i} \cdot \mathbf{v}$,

$$\begin{aligned} E_{\mathbf{i}} Y_n \log Y_n - \mathbf{i} \cdot \mathbf{v} \log n &= E_{\mathbf{i}} Y_n \log\big(n^{-1} Y_n\big) \\ &= E_{\mathbf{i}}\big(n^{-1} Y_n (\log n^{-1} Y_n) | Y_n > 0\big) n P_{\mathbf{i}}(Y_n > 0). \end{aligned}$$

By fact 2(a), $nP_{\mathbf{i}}(Y_n > 0) \to \lambda_{\mathbf{i}}$ and by facts 2(b) and 4,

$$\big\{n^{-1} Y_n \log\big(n^{-1} Y_n\big) | Y_n > 0\big\}$$

are uniformly integrable and $\{n^{-1}Y_n | Y_n > 0\}$ converges in distribution to a r.v. with c.d.f. $1 - e^{\lambda x}$ for $x > 0$. This yields the lemma. □

Clearly, (6) is implied by Lemma 4.
Next,

$$b_{n,l} \le \sum_{r=1}^{n} E(Y_T: Y_T > \rho l, T = r) \le \sum_{r=1}^{n} E(Y_r: Y_{r-1} < l, Y_r > \rho l).$$

So,

$$\sum_{l=1}^{\infty} l^{-1} b_{nl} \le \sum_{r=1}^{n} \sum_{l=1}^{\infty} l^{-1} E(Y_r: Y_r > \rho l, 0 < Y_{r-1} < l)$$

$$= \sum_{r=1}^{n} E\left(Y_r I(Y_{r-1} > 0) \left(\sum_{l=1}^{\infty} l^{-1} I\left(Y_{r-1} < l < \rho^{-1} Y_r\right) \right) \right),$$

implying

$$\sum_{l=1}^{\infty} l^{-1} b_{nl} \le \sum_{r=1}^{n} E\left(Y_r I(Y_{r-1} > 0) \left(\log \frac{Y_r}{\rho Y_{r-1}} \right)^{+} \right). \tag{8}$$

By the branching property, we can write

$$Y_r = \sum_{i=1}^{p} \sum_{j=1}^{Z_{(r-1)i}} X_{ij},$$

where $\{X_{ij},\ j = 1, 2, \dots,\ i = 1, 2, \dots, p\}$ are independent random variables and are independent of $Z_0, Z_1, \dots, Z_{r-1}$, and, for each j, X_{ij} has the same distribution as $\mathbf{v} \cdot \mathbf{Z}_1$ with $Z_0 = e_i$, the unit vector in the ith direction. Since $M\mathbf{v} = \mathbf{v}$, $EX_{ij} = v_i$ and we write $Y_r = \sum_{i=1}^{p} v_i Z_{(r-1)i} v_i^{-1} \overline{X}_{iZ(n-1)i}$. Now, noting that $Y_{r-1} = \mathbf{v} \cdot \mathbf{Z}_{r-1} = \sum_1^p v_i Z_{(r-1)i}$ and that the function $x \to \psi(x) \equiv x(\log(\rho^{-1}x))^+$ is convex in (ρ, ∞), we see that

$$Y_{r-1} \psi\left(\sum_{i=1}^{p} \frac{v_i Z_{(r-1)i}}{\mathbf{v} \cdot \mathbf{Z}_{r-1}} v_i^{-1} \overline{X}_{iZ(r-1)i} \right) \le \sum_{i=1}^{p} v_i Z_{(r-1)i} \psi\left(v_i^{-1} \overline{X}_{iZ(r-1)i} \right).$$

Taking expectations and using conditional independence of $\{X_{ij}\}$ and Z_{r-1}, we have

$$\begin{aligned} &E\left(Y_r \left(\log\left(\frac{Y_r}{\rho Y_{r-1}} \right) \right)^{+} : Y_{r-1} > 0 \right) \\ &\quad \le \sum_{1}^{p} E\left(v_i Z_{(r-1)i} \phi\left(Z_{(r-1)i}, F_i, \rho\right); Y_{r-1} > 0 \right), \end{aligned} \tag{9}$$

where F_i is the c.d.f. of $v_i^{-1} X_{i1}$ and ϕ is as in (4). From (8) and (9) it is clear that to establish (7) it is enough to show that, for each $1 \le i \le p$,

$$(10) \qquad (\log n)^{-1} \sum_{r=1}^{n} E\big(Z_{(r-1)i}\phi(Z_{(r-1)i}, F_i, \rho); Y_{r-1} > 0\big) \to 0.$$

Fix $\rho > 1$ and $\mathbf{i} \neq 0$. Now, using facts (3) and (4) and Lemma 3, we see that given $\varepsilon > 0$ we can find N and $\eta > 0$ such that, for $n > N$,

(a) $$n\phi(n, F_i, \rho) < \varepsilon, \quad \text{for all } i,$$

(b) $$\sup_x \left| P_{\mathbf{i}}(0 < Y_n < nx | Y_n > 0) - (1 - e^{-\lambda x}) \right| < \varepsilon,$$

(c) $$P_{\mathbf{i}}\left(\bigcup_{j=1}^{p} \left(\left| \frac{Z_{nj}}{Y_n} - u_j \right| > \frac{u_j}{2} \right) \middle| Y_n > 0 \right) < \varepsilon,$$

(d) $$1 - e^{-\lambda\eta} < \varepsilon.$$

Let $N_i = 2N/\eta u_i$ and $K_i \equiv \sup_n n\phi(n, F_i, \rho) < \infty$. Then

$$(11) \qquad \begin{aligned} \alpha_{ni} &\equiv \sum_{r=1}^{n} E\big(Z_{(r-1)i}\phi(Z_{(r-1)i}, F_i, \rho); Y_{r-1} > 0\big) \\ &= \sum_{r < N_i + 1} + \sum_{r \ge N_i + 1} \le K_i N_i + d_n, \quad \text{say}. \end{aligned}$$

But

$$\begin{aligned} d_n \le &\sum_{r \ge N_i + 1} E\big(Z_{(r-1)i}\phi(Z_{(r-1)i}, F_i\rho); 0 < Y_{r-1} < (r-1)\eta\big) \\ &+ \sum_{r \ge N_i + 1} E\left(Z_{(r-1)i}\phi(Z_{(r-1)i}, F_i\rho); Y_{r-1} \ge (r-1)\eta, Z_{(r-1)i} \ge \frac{u_i}{2} Y_{r-1}\right) \\ &+ \sum_{r \ge N_i + 1} E\left(Z_{(r-1)i}\phi(Z_{(r-1)i}, F_i\rho); Y_{r-1} \ge (r-1)\eta, Z_{(r-1)i} < \frac{u_i}{2} Y_{r-1}\right) \\ = &\; d_{n1} + d_{n2} + d_{n3}, \quad \text{say}. \end{aligned}$$

Now,

$$\begin{aligned} d_{n1} &\le K_i \sum_1^n P(0 < Y_{r-1} < (r-1)\eta) \\ &= K_i \sum_1^n \big(P(0 < Y_{r-1} < (r-1)\eta | Y_{r-1} > 0) - (1 - e^{-\lambda\eta})\big) P(Y_{r-1} > 0) \\ &\quad + K_i(1 - e^{-\lambda\eta}) \sum_1^n P(Y_{r-1} > 0) \le 2K_i\varepsilon \sum_1^n P(Y_{r-1} > 0), \quad \text{by (b) and (d)}. \end{aligned}$$

Also,

$$d_{n2} \le \varepsilon \sum_1^n P(Y_{r-1} > 0) \quad \text{by (a)}$$

and

$$d_{n3} \le K_i \varepsilon \sum_1^n P(Y_{r-1} > 0) \quad \text{by (c)}.$$

Thus, from (11), $\alpha_{ni} \le K_i N_i + (3K_i + 1)\varepsilon \Sigma_1^n P(Y_{r-1} > 0)$.

By fact 2(a), $(\log n)^{-1} \Sigma_1^n P_{\mathbf{i}}(Y_{n-1} > 0) \to \lambda_{\mathbf{i}}$ and hence

$$\limsup (\log n)^{-1} \alpha_{ni} = 0 \quad \text{for each } 1 \le i \le p.$$

This establishes (10) and the proof of Proposition 2 is hereby complete. □

REMARK 4. The use of Lemmas 1 and 2 has simplified the proof of the lower bound even in the one type case considered in [1], where the deeper results of Hsu and Robbins on complete convergence and of Kiefer and Wolfowitz on random walks were used.

REMARK 5. Some open questions remain:

(i) In the p type case, prove or disprove that

$$(\log n)^{-1} E_{\mathbf{i}} \max_{0 \le j \le n} Z_{ji} \to u_i \mathbf{i} \cdot \mathbf{v}, \quad \text{for } 1 \le i \le p,$$

(of course, assuming second moments).

(ii) In both the single type and multitype cases, show that even if the second moments are not finite, $\lim_n (\log a_n)^{-1} E_{\mathbf{i}} M_n \to \mathbf{i} \cdot \mathbf{v}$, where $a_n = (P(Z_n > 0))^{-1}$.

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