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ON THE LAW OF THE ITERATED LOGARITHM FOR MARTINGALES

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The Kolmogorov law of the iterated logarithm fails when the boundedness condition on the increments is relaxed. In this paper, we consider this in the martingale setting and establish a lower bound, extending a result known in the independent case.

1. Introduction. Let $\{X_i,\ i \geq 1\}$ be a sequence of independent random variables with $EX_i = 0$ and $EX_i^2 < \infty$, for $i = 1, 2 \dots$. Define $s_n^2 = \sum_{i=1}^n EX_i^2$ and suppose that $s_n^2 \to \infty$ as $n \to \infty$. Kolmogorov's law of the iterated logarithm (LIL) [Kolmogorov (1929)] states that if

$$|X_n| \leq c_n s_n \left(\log_2 s_n^2\right)^{-1/2} \quad \text{a.s.,} \tag{1.1}$$

for constants $c_n \to 0$ as $n \to \infty$, then

$$\limsup_{n \to \infty} S_n / \left(2 s_n^2 \log_2 s_n^2\right)^{1/2} = 1 \quad \text{a.s.,} \tag{1.2}$$

where $S_n = \sum_{i=1}^n X_i$ and $\log_2 x = \log(\log x)$.

If the Kolmogorov condition (1.1) is weakened so that c_n is replaced by a constant $c > 0$, then the result (1.2) fails in general. This has been shown by Marcinkiewicz and Zygmund (1937), Feller (1943) and Weiss (1959).

Upper and lower bounds for $\limsup_{n \to \infty} S_n/(2s_n^2 \log_2 s_n^2)^{1/2}$ in this case have been derived by Tomkins (1978) and Teicher (1979). In particular, it follows from their results that

$$0 < \limsup_{n \to \infty} S_n / \left(2 s_n^2 \log_2 s_n^2\right)^{1/2} < \infty \quad \text{a.s.} \tag{1.3}$$

The second inequality in (1.3) was derived earlier by Egorov (1969).

A martingale analogue of the Kolmogorov law of the iterated logarithm was first established by Stout (1970). In the supermartingale case analogous to the weakened condition, a finite upper bound was derived by Fisher (1986), extending an earlier and more restricted result of Stout [(1974), Theorem 5.4.1].

In this paper we establish a lower bound in the martingale setting. A consequence is that (1.3) is extended to the martingale case.

Section 2 of this paper consists of a statement of the main result and a discussion of it. Section 3 consists of the proof of the theorem.

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2. Statement of theorem and remarks. Let $\{U_n, \mathscr{F}_n, n \geq 1\}$ be a martingale defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where $\{\mathscr{F}_n, n \geq 1\}$ is an increasing sequence of sub-σ-fields of $\mathscr{F}$. Let $\{X_i, i \geq 1\}$ be the martingale difference sequence defined by $X_i = U_i - U_{i-1}$ (define $U_0 = 0$). Suppose that $E[X_i^2 | \mathscr{F}_{i-1}] < \infty$, for $i \geq 1$ (let $\mathscr{F}_0 = \{\phi, \Omega\}$), and define $s_n^2 = \sum_{i=1}^n E[X_i^2 | \mathscr{F}_{i-1}]$, for $n \geq 1$. For convenience, we define $\varphi(x) = (2 \log_2(x^2 \vee e^2))^{1/2}$ and $\eta(x) = (2x \log_2(x \vee e^2))^{1/2}$, for $x > 0$.

THEOREM 1. *Let $\{U_n, \mathscr{F}_n, n \geq 1\}$ be a martingale described with the preceding notation. Assume that $s_n^2 \to \infty$ a.s. as $n \to \infty$ and that*

$$|X_i| \leq K_i s_i / \varphi(s_i) \quad a.s., \tag{2.1}$$

where K_i is an $\mathscr{F}_{i-1}$-measurable function for each integer $i \geq 1$ with

$$\limsup_{i \to \infty} K_i < K, \tag{2.2}$$

for $K > 0$ an arbitrary constant.

Then there exists a positive constant $\varepsilon(K)$ so that

$$\limsup_{n \to \infty} U_n / s_n \varphi(s_n) \geq \varepsilon(K) \quad a.s. \tag{2.3}$$

In particular, one can take $\varepsilon(K)$ as

$$\varepsilon(K) = h_K^{-1}(1) \wedge (1/81K), \tag{2.4}$$

where $h_K(x) = x^2 + 12K^{1/2}x^{5/2}$, $x > 0$.

REMARK 1. In the martingale analogue of the Kolmogorov LIL established by Stout (1970), condition (2.2) is replaced by the assumption $K_i \to 0$ as $i \to \infty$. The lower half of this result follows from Theorem 1 by observing that $h_K^{-1}(1) \uparrow 1$ as $K \to 0$.

REMARK 2. As noted in Stout (1970), the hypothesis that K_i is a random variable rather than simply a constant means a less restrictive hypothesis than the classical one when Theorem 1 is applied in the independent case.

REMARK 3. An immediate consequence of Theorem 1 is that

$$\limsup_{n \to \infty} U_n / s_n \varphi(s_n) > 0 \quad \text{a.s.} \tag{2.5}$$

This, in combination with Lemma 1, results in the conclusion that

$$0 < \limsup_{n \to \infty} U_n / s_n \varphi(s_n) < \infty \quad \text{a.s.}, \tag{2.6}$$

extending what has been proved in the independent case.

3. Proof of main result. The proof of Theorem 1 makes use of two results that we list as Lemma 1 and Lemma 2.

LEMMA 1. *Assume the hypothesis of Theorem* 1. *Then there exists a constant* $\lambda(K)$, $0 < \lambda(K) < \infty$, *so that*

$$\limsup_{n\to\infty} U_n/s_n\varphi(s_n) \le \lambda(K) \quad a.s. \tag{3.1}$$

PROOF. This is an immediate corollary of Fisher [(1986), Theorem 1]. □

Lemma 2 is a large deviation result for martingales derived by Freedman (1975). We adopt his notation for the following definitions.

Let a and b be positive numbers. Define $\sigma_b = \inf\{n: s_n^2 > b\}$ if such n exists and $\sigma_b = \infty$ otherwise. Let

$$L(b) = \operatorname*{ess\,sup}_{\omega} \sup_{n\le\sigma_b(\omega)} |X_n(\omega)|. \tag{3.2}$$

Let A and B be the events defined as

$$A = \{U_n \ge a \text{ for some } n \text{ such that } s_n^2 < b\}$$

and

$$B = \left\{\sup_n s_n^2 < b\right\}.$$

LEMMA 2. *Let* $0 < \delta \le \frac{1}{3}$. *Suppose* $L(b)$ *is finite and satisfies the conditions*

$$b/a > 9L(b)/\delta^2 \tag{3.3}$$

and

$$a^2/b > (16/\delta^2)\log(64/\delta^2). \tag{3.4}$$

Then

$$P(A\cup B) \ge \tfrac{1}{2}\exp\left[-\tfrac{1}{2}(a^2/b)(1+4\delta)\right]. \tag{3.5}$$

PROOF. See Freedman [(1975), Proposition 2.4]. □

PROOF OF THEOREM 1. Let $r > 1$. Define $t_k = \sup\{n: s_n^2 \le r^k\}$, where $k \ge 1$ is an integer. Since $s_n \to \infty$ a.s., t_k is a well-defined stopping time relative to $\{\mathscr{F}_i, i \ge 1\}$.

Consider the martingale $\{U_n^{(k)}, \mathscr{F}_n^{(k)}, n \ge 0\}$, where $U_n^{(k)} = U_{t_k+n} - U_{t_k}$ and $\mathscr{F}_n^{(k)} = \mathscr{F}_{t_k+n}$. [Recall that if τ is a stopping time relative to $\{\mathscr{F}_i, i \ge 1\}$, then by $\mathscr{F}_\tau$ is meant the σ-field of events $A \in \mathscr{F}$ such that $A \cap (\tau = l) \in \mathscr{F}_l$ for all integers $l \ge 1$.]

Let $X_n^{(k)} = U_n^{(k)} - U_{n-1}^{(k)}$ for $n \ge 1$ and

$$\left(s_n^{(k)}\right)^2 = \sum_{i=1}^{n} E\left[\left(X_i^{(k)}\right)^2\middle|\mathscr{F}_{i-1}^{(k)}\right].$$

Define

$$Y_n(k) = X_n^{(k)} I\left\{ \bigcap_{i=1}^{n} \left(K_i^{(k)} \le K\right)\right\},$$

for $n \ge 1$, where $K_i^{(k)} = K_{t_k+i}$. (The notation $I\{A\}$ denotes the indicator function of the event A.)

Define $V_n^{(k)} = \sum_{i=1}^n Y_i^{(k)}$. Then $\{V_n^{(k)}, \mathscr{F}_n^{(k)}, n \ge 1\}$ is a martingale. This follows from the fact that $\{X_n^{(k)}, \mathscr{F}_n^{(k)}, n \ge 1\}$ is a martingale difference sequence and $I\{\bigcap_{i=1}^n (K_i^{(k)} \le K)\}$ is $\mathscr{F}_{n-1}^{(k)}$-measurable. Let

$$\left(v_n^{(k)}\right)^2 = \sum_{i=1}^{n} E\left[Y_i^{(k)2} \middle| \mathscr{F}_{i-1}^{(k)}\right].$$

Assume the space $(\Omega, \mathscr{F})$ is sufficiently regular so that there exists a regular conditional probability $P_k(\omega, B)$ on $(\Omega, \mathscr{F}, P)$ given $\mathscr{F}_{t_k}$. That is, for each $\omega \in \Omega$, $P_k(\omega, \cdot\,)$ is a probability measure on $(\Omega, \mathscr{F})$ and, for each fixed $B \in \mathscr{F}$, $P_k(\omega, B) = P(B|\mathscr{F}_{t_k})(\omega)$ a.s. It follows as a consequence of a standard result on regular conditional probabilities that $\{V_n^{(k)}, \mathscr{F}_n^{(k)}, n \ge 1\}$ is a martingale defined on the space $(\Omega, \mathscr{F}, P_k(\omega, \cdot\,))$ for $\omega \in \Omega$ a.s. [see Loève (1978), Sections 29 and 30].

Following Freedman's approach [Freedman (1975), Section 6], we apply Lemma 2 to this martingale. Define the event A_k as

$$A_k = \left\{V_n^{(k)} > \varepsilon(K)(1 - r^{-1/2})\eta(r^{k+1}) \text{ for some } n \ge 1 \right.$$
$$\left. \text{such that } \left(v_n^{(k)}\right)^2 < r^{k+1} - r^k \right\},$$

where $\varepsilon(k)$ is defined by (2.4).

In the notation of Lemma 2, a and b are defined as

$$a = \varepsilon(K)(1 - r^{-1/2})\eta(r^{k+1})$$

and

$$b = r^{k+1} - r^k.$$

In addition, we have

$$L(b) \le \operatorname*{ess\,sup}_{\omega} \sup_{0 < n \le t_{k+1} - t_k + 1} \left|Y_n^{(k)}(\omega)\right|.$$

Let $m = t_{k+1}$ for k sufficiently large [e.g., so that $\varphi(r^{(k+1)/2}) > K$]. For $\omega \in \bigcap_{i=1}^{m+1-t_k}(K_i^{(k)} \le K)$, it follows that

$$s_{m+1}^2 \le s_m^2 + K^2 s_{m+1}^2/\varphi^2(s_{m+1}).$$

By the definition of t_{k+1},

$$s_{m+1}^2 \le r^{k+1}/\left(1 - K^2/\varphi^2(r^{(k+1)/2}\right).$$

Since $s_n/\varphi(s_n)$ increases as n increases we have

$$L(b) \le \left(K/\varphi(r^{(k+1)/2})\right) r^{(k+1)/2}\left(1 - K^2/\varphi^2(r^{(k+1)/2})\right)^{-1/2}.$$

Let $\delta = 3K^{1/2}\varepsilon^{1/2}(K)$. It is immediate from (2.4) that $0 < \delta \le \frac{1}{3}$. For k large, elementary calculations verify that the remaining hypotheses of Lemma 2 described by (3.3) and (3.4) are satisfied.

Define the event B_k as

$$B_k = A_k \bigcup \left\{ \sup_n \left(v_n^{(k)}\right)^2 < r^{k+1} - r^k \right\}.$$

Applying Lemma 2 we find from (3.5) and the definition of $h_K(\cdot)$ and $\varepsilon(K)$ that

$$P\left(B_k | \mathscr{F}_{t_k}\right) \ge \tfrac{1}{2}\exp\left[-\left(\log_2 r^{k+1}\right)h_K(\varepsilon(K))\right] \quad \text{a.s.}$$

Since $\varepsilon(K) \le h_K^{-1}(1)$ and $h_K(x)$ increases as x increases, the inequality $h_K(\varepsilon(K)) \le 1$ holds. Therefore the series $\sum_{k=1}^{\infty} P(B_k|\mathscr{F}_{t_k})$ diverges a.s. By Lévy's conditional form of the Borel–Cantelli lemma [see Stout (1974), page 55] we obtain

$$P(B_k \text{ i.o.}) = 1.$$

For k sufficiently large (depending on ω), it follows from (2.2) that $Y_n^{(k)}(\omega) = X_n^{(k)}(\omega)$, for $n \ge 1$. Therefore, for each ω outside a null set and for k sufficiently large (depending on ω), the equalities

$$V_n^{(k)}(\omega) = U_n^{(k)}(\omega) \quad \text{and} \quad \left(v_n^{(k)}\right)^2(\omega) = \left(s_n^{(k)}\right)^2(\omega)$$

hold for $n \ge 1$. Since $s_n^2 \to \infty$ a.s., it follows that $(v_n^{(k)})^2 \to \infty$ a.s. Therefore,

$$P(A_k \text{ i.o.}) = 1.$$

This implies that

$$P(C_k \text{ i.o.}) = 1, \tag{3.6}$$

where

$$C_k = \left\{ U_n^{(k)} > \varepsilon(K)(1 - r^{-1/2})\eta(r^{k+1}) \text{ for some } n \ge 1 \text{ such that } \left(s_n^{(k)}\right)^2 < r^{k+1} - r^k \right\}.$$

Applying Lemma 1 to the martingale $\{-U_n, \mathscr{F}_n, n \ge 1\}$ proves that there exists a finite constant $\lambda(K) > 0$ such that

$$P\left[U_{t_k} < -\lambda(K)\eta\left(s_{t_k}^2\right) \text{ i.o.}\right] = 0.$$

Since $\eta(s_{t_k}^2) \le \eta(r^k)$ and $\eta(r^{k+1}) \ge r^{1/2}\eta(r^k)$, then, for all $r > 1$, we obtain

$$P\left[U_{t_k} < -\lambda(K)r^{-1/2}\eta(r^{k+1}) \text{ i.o.}\right] = 0. \tag{3.7}$$

Combining (3.6) and (3.7) shows that, for all $r > 1$,

$$P\left[U_n > \eta(r^{k+1})\left(\varepsilon(K)(1 - r^{-1/2}) - \lambda(K)r^{-1/2}\right) \text{ for some } n \text{ satisfying } t_k < n \le t_{k+1} \text{ i.o.}\right] = 1. \tag{3.8}$$

Since $\eta(r^{k+1}) \geq \eta(s_n^2)$ for $t_k < n \leq t_{k+1}$, (3.8) implies that

$$\limsup_{n \to \infty} U_n / s_n \varphi(s_n) \geq \varepsilon(K) \quad \text{a.s.} \qquad \square$$

REFERENCES

EGOROV, V. A. (1969). On the law of the iterated logarithm. *Theory Probab. Appl.* **14** 693–699.

FELLER, W. (1943). The general form of the so-called law of the iterated logarithm. *Trans. Amer. Math. Soc.* **54** 373–402.

FISHER, E. (1986). An upperclass law of the iterated logarithm for supermartingales. *Sankhyā Ser. A* **3** 267–272.

FREEDMAN, D. A. (1975). On tail probabilities for martingales. *Ann. Probab.* **3** 100–118.

KOLMOGOROV, A. N. (1929). Über das Gesetz des iterierten Logarithmus. *Math. Ann.* **101** 126–135.

LOÈVE, M. (1978). *Probability Theory* **2**, 4th ed. Springer, New York.

MARCINKIEWICZ, J. and ZYGMUND, A. (1937). Remarque sur la loi du logarithme itéré. *Fund. Math.* **29** 215–222.

STOUT, W. F. (1970). A martingale analogue of Kolmogorov's law of the iterated logarithm. *Z. Wahrsch. Verw. Gebiete* **15** 279–290.

STOUT, W. F. (1974). *Almost Sure Convergence*. Academic, New York.

TEICHER, H. (1979). Generalized exponential bounds, iterated logarithm and strong laws. *Z. Wahrsch. Verw. Gebiete* **48** 293–307.

TOMKINS, R. J. (1978). On the law of the iterated logarithm. *Ann. Probab.* **6** 162–168.

WEISS, M. (1959). On the law of the iterated logarithm. *J. Math. Mech.* **8** 121–132.

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